CODES AND DESIGNS IN GRASSMANNIAN SPACES.

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ABSTRACT. The notion of t-design in a Grassmannian space $\mathcal{G}_{m,n}$ was introduced by the first two authors and G. Nebe in a previous paper. In the present work, we give a general lower bound for the size of such designs. The method is inspired by Delsarte, Goethals and Seidel work in the case of spherical designs. This leads us to introduce a notion of f-code in Grassmannian spaces, for which we obtain upper bounds, as well as a kind of duality tight-designs/tight-codes. The bounds are in terms of the dimensions of the irreducible representations of the orthogonal group $O(n)$ occuring in the decomposition of the space $L^2(\mathcal{G}^\circ_{m,n})$ of square integrable functions on $\mathcal{G}^\circ_{m,n}$, the set of oriented grassmanianns.

1. INTRODUCTION.

There are various combinatorial problems related to finite sets of Euclidean spheres. Among those, two, in a sense dual to each other, have received much attention, namely the notions of spherical t-design (t an integer), and spherical A-codes (A a finite set in $[-1,1]$). The notion of spherical design was motivated by numerical integration: a spherical t-design is a finite subset X of a sphere S^{d-1}, such that the integral over S^{d-1} of a polynomial function up to degree t coincides with its average value at the points of X. It is thus important, for instance for applications, to find designs with smallest possible cardinality. So the question of finding a *lower* bound for the size of a spherical design is central. As for A-codes, it is natural conversely to ask for an *upper* bound of their size : an A-code is a finite set in a sphere S^{d-1} such that the scalar products of pairwise distinct elements belong to a fixed set $A \subset [-1,1]$. When $A = [-1,1/2]$, finding an upper bound is equivalent to the kissing number problem, known as the problem of the thirteen spheres when $n = 3$. In their landmark paper [5], Delsarte, Goethals and Seidel proposed a general method, based on harmonic analysis on the orthogonal group, to study both questions.

The problem of packings, and related combinatorial questions, in the Grassmanian spaces $\mathcal{G}_{m,n}$ of m-dimensional subspaces of $\mathbb{R}^n$ have been investigated in a series of recent papers (see [3], [4]). In [1], a theory of designs was developped in that framework. One task of the present paper is to define a notion of f-code in Grassmannian spaces, which reduces to A-code

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when $m = 1$ (the codes in the first Grassmannian $\mathcal{G}_{1,n}$ are in one-to-one correspondance with the antipodal codes of the unit sphere). Then, inspired by Delsarte, Goethals and Seidel's works, we establish lower/upper bounds for the size of such designs/codes, which involve the dimensions of some irreducible representations of $O(n)$.

2. Zonal functions on Grassmannian spaces.

Let $\mathcal{G}_{m,n} \simeq O(n)/O(m) \times O(n-m)$ be the Grassmannian space of m-dimensional subspaces of $\mathbb{R}^n$. Recall (see [1], [11]), that the orbits under $O(n)$ of pairs $(p, q) \in \mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$ are parametrized by the m-tuples

$$1 \geq t_1 \geq t_2 \geq \cdots \geq t_m \geq 0.$$

Namely, to a couple (p, q) of m-dimensional subspaces, one associates the m-tuple $t_1 = \cos\theta_1, \cdots, t_m = \cos\theta_m$, where $0 \leq \theta_1 \leq \cdots \leq \theta_m \leq \frac{\pi}{2}$ are the *principal angles* between p and q. One way to compute the t_i, is as follows : denoting by p_0 the subspace generated by the first m vectors of the canonical basis of $\mathbb{R}^n$, and writing $p = g \cdot p_0$, $q = h \cdot p_0$, with suitable g, h in $O(n)$, then the $y_i := t_i^2$ are the eigenvalues of the $m \times m$ symmetric matrix AA^t, where A is the m-size block appearing in the block-decomposition

$$g^{-1}h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

Moreover, g and h are defined up to multiplication by an element in $\mathrm{Stab}(p_0)$, and may be chosen so that $B = D = 0$ and

(2)

$$A = \begin{pmatrix} \cos\theta_1 & 0 & \dots & 0 \\ 0 & \cos\theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \cos\theta_m \end{pmatrix}, C = \begin{pmatrix} \sin\theta_1 & 0 & \dots & 0 \\ 0 & \sin\theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sin\theta_m \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Besides $\mathcal{G}_{m,n}$, we have to consider the set $\mathcal{G}_{m,n}^{\circ}$ of *oriented* m-dimensional subspaces of $\mathbb{R}^n$. We may view the elements of $\mathcal{G}_{m,n}^{\circ}$ as couples $\tilde{p} = (p, s)$, with p an m-dimensional subspace, and s an element in $\bigwedge^m p$. The action of $O(n)$ on these couples is given by

$$g.(p, s) := (gp, gs),$$

so that if we fix an orientation s_0 on p_0, the stabilizer of (p_0, s_0) identifies with $SO(m) \times O(n-m)$. Consequently

$$\mathcal{G}_{m,n}^{\circ} \simeq SO(n)/SO(m) \times SO(n-m)) \simeq O(n)/SO(m) \times O(n-m),$$

which is a 2 to 1 covering of $\mathcal{G}_{m,n}$. The orbits under $O(n)$ of pairs $(\tilde{p}, \tilde{q}) \in \mathcal{G}_{m,n}^{\circ} \times \mathcal{G}_{m,n}^{\circ}$ can be likewise parametrized by $(m+1)$-tuples $(\epsilon, t_1, \cdots, t_m)$, where $t_1, \cdots, t_m$ are defined as above, and $\epsilon = \frac{\det A}{|\det A|} \in \{\pm 1\}$, where A is as in (1). We still have a canonical block-decomposition like (2), but with top-left block

$$A = \begin{pmatrix} \epsilon \cos\theta_1 & 0 & \dots & 0 \\ 0 & \cos\theta_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \cos\theta_m \end{pmatrix}. \tag{2'}$$

Both $\mathcal{G}_{m,n}$ and $\mathcal{G}^\circ_{m,n}$ inherit from the Haar measure of $O(n)$, a measure denoted dp and $d\tilde{p}$ respectively. Since we will be mainly interested in non-oriented grassmanians, we normalize it so as $\int_{\mathcal{G}_{m,n}} dp = 1$ (whence $\int_{\mathcal{G}^\circ_{m,n}} d\tilde{p} = 2$). Accordingly, the space $L^2(\mathcal{G}^\circ_{m,n})$ of square integrable functions on $\mathcal{G}^\circ_{m,n}$ is endowed with the scalar product

$$\langle f, g\rangle = \frac{1}{2}\int_{\mathcal{G}^\circ_{m,n}} f(\tilde{p})\overline{g(\tilde{p})}d\tilde{p},$$

so that its restriction to $L^2(\mathcal{G}_{m,n})$ satisfies

$$\langle f, g\rangle = \int_{\mathcal{G}_{m,n}} f(p)\overline{g(p)}dp,\ f \in L^2(\mathcal{G}_{m,n}), g \in L^2(\mathcal{G}_{m,n}).$$

The group $O(n)$ acts isometrically on $L^2(\mathcal{G}^\circ_{m,n})$ by

$$\sigma \cdot f(\tilde{p}) = f(\sigma^{-1}\tilde{p}).$$

The structure of $L^2(\mathcal{G}^\circ_{m,n})$ as an $O(n)$-module is well-known, and is given for instance in [9], p. 546. To be precise, if we consider the subset $\mathcal{R}(\mathcal{G}^\circ_{m,n})$ of regular functions on $\mathcal{G}^\circ_{m,n}$ (*i.e.* the set of functions induced by regular functions on $O(n)$) which is a dense subset of $L^2(\mathcal{G}^\circ_{m,n})$, we have the following decomposition

$$\mathcal{R}(\mathcal{G}^\circ_{m,n}) = \bigoplus H^\mu_{m,n}$$

in pairwise orthogonal non isomorphic irreducible $O(n)$-submodules $H^\mu_{m,n}$, the sum being over partitions $\mu = \mu_1 \geq \mu_2 \geq \dots \mu_m \geq 0$, of depth at most m, with $\mu_i \equiv \mu_j \mod 2$ for all (i, j). We call these partitions *m-admissible*, or *admissible* for short. They split into *odd* and *even*, according to the parity of the μ_i.

Remark 2.1. *For a given even partition μ, the admissibility does not depend on m, as long as* $\mathrm{depth}(\mu) \leq m \leq \frac{n}{2}$*, whereas for μ odd it does, since in that case the μ_i have to be nonzero for all $1 \leq i \leq m$.*

It turns out that the $O(n)$-isomorphism class of $H^\mu_{m,n}$ is independent of m, provided that $\mathrm{depth}(\mu) \leq m \leq \frac{n}{2}$ and μ is m-admissible (see [1] for a more detailed description of $H^\mu_{m,n}$). The space $H^\mu_{m,n}$ is isomorphic to the irreducible representation of $O(n)$ canonically associated to the partition μ, and denoted V^μ_n in [9]. We shall denote the dimension of this space d_μ.

The sum over *even*, resp. *odd*, partitions corresponds respectively to $\mathcal{R}(\mathcal{G}_{m,n})$ and its orthogonal complement $\mathcal{R}(\mathcal{G}_{m,n})^{\perp}$. This also corresponds to the eigenspace decomposition of $\mathcal{R}(\mathcal{G}^{\circ}_{m,n})$ with respect to the canonical involution σ^* induced by orientation changing, namely

$$\mathcal{R}(\mathcal{G}_{m,n}) = \mathcal{R}(\mathcal{G}^{\circ}_{m,n})^{+} := \{f \in \mathcal{R}(\mathcal{G}^{\circ}_{m,n}) \mid \sigma^*(f) = f\},$$

and

$$\mathcal{R}(\mathcal{G}_{m,n})^{\perp} = \mathcal{R}(\mathcal{G}^{\circ}_{m,n})^{-} := \{f \in \mathcal{R}(\mathcal{G}^{\circ}_{m,n}) \mid \sigma^*(f) = -f\}.$$

Let $\mathcal{Z}^o$ (resp. $\mathcal{Z}$) be the set of $O(n)$-invariant functions F on $\mathcal{G}^{\circ}_{m,n} \times \mathcal{G}^{\circ}_{m,n}$ (resp. $\mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$), such that

$$F(\tilde{p}, .) \in \mathcal{R}(\mathcal{G}^{\circ}_{m,n}), F(., \tilde{q}) \in \mathcal{R}(\mathcal{G}^{\circ}_{m,n}) \text{ for all } (\tilde{p}, \tilde{q}) \in \mathcal{G}^{\circ}_{m,n} \times \mathcal{G}^{\circ}_{m,n}, \tag{3}$$

resp.

$$F(p, .) \in \mathcal{R}(\mathcal{G}_{m,n}), F(., q) \in \mathcal{R}(\mathcal{G}_{m,n}) \text{ for all } (p, q) \in \mathcal{G}_{m,n} \times \mathcal{G}_{m,n}. \tag{3'}$$

As usual, we call such functions *zonal*. Alternatively, if a base point $\tilde{p}$ is fixed, one can identify $\mathcal{Z}^o$ with $\mathcal{R}(\mathcal{G}^{\circ}_{m,n})^{\mathrm{Stab}(\tilde{p})}$, mapping $F \in \mathcal{Z}^o$ on $F(\tilde{p}, .) \in \mathcal{R}(\mathcal{G}^{\circ}_{m,n})^{\mathrm{Stab}(\tilde{p})}$, and similarly $\mathcal{Z}$ identifies with $\mathcal{R}(\mathcal{G}_{m,n})^{\mathrm{Stab}(p)}$

As explained in [1], [11], it follows from the Frobenius reciprocity theorem that $H^{\mu\ \mathrm{Stab}(\tilde{p})}_{m,n}$ is one-dimensional for any μ (if μ is even, then $H^{\mu}_{m,n} \subset \mathcal{R}(\mathcal{G}_{m,n}) = \mathcal{R}(\mathcal{G}^{\circ}_{m,n})^{+}$, so that $H^{\mu\ \mathrm{Stab}(\tilde{p})}_{m,n} = H^{\mu\ \mathrm{Stab}(p)}_{m,n}$). Consequently, to each summand $H^{\mu}_{m,n}$ is attached a unique (up to scaling) *zonal function* P_μ, which can be computed in the following way: denoting by d_μ the dimension of $H^{\mu}_{m,n}$, and $\{e_{\mu,i}\}_{1 \le i \le d_\mu}$ an orthonormal basis of it, one has

$$P_\mu(\tilde{p}, \tilde{p}') := \frac{1}{d_\mu} \sum_{i=1}^{d_\mu} e_{\mu,i}(\tilde{p}) \overline{e_{\mu,i}(\tilde{p}')}. \tag{4}$$

The results of the next section rely on the following properties of the P_μ:

Lemma 2.2.

(i) $P_\mu(\tilde{p}, \tilde{p}) = 1$, *for any* μ *and* $\tilde{p}$.

(ii) *For any* λ, μ *and* $\tilde{p}$, $\tilde{p}'$, *one has*

$$\langle P_\mu(\tilde{p}, .), P_\lambda(\tilde{p}', .)\rangle = \frac{\delta_{\lambda,\mu}}{d_\mu} P_\mu(\tilde{p}, \tilde{p}'), \tag{5}$$

if, for fixed $\tilde{p}$, *we view the map* $\tilde{q} \mapsto P_\mu(\tilde{p}, \tilde{q})$ *as a function in* $L^2(\mathcal{G}^{\circ}_{m,n})$. *In particular, for any finite set* $X \subset \mathcal{G}^{\circ}_{m,n}$, *the matrix* $(P_\mu(\tilde{p}, \tilde{p}'))_{\tilde{p}, \tilde{p}' \in X^2}$ *is positive semidefinite.*

(iii) *For any* λ, μ, *one has*

$$P_\lambda P_\mu = \sum_\tau c_{\lambda,\mu}(\tau) P_\tau, \tag{6}$$

with non-negative coefficients $c_{\lambda,\mu}(\tau)$. *In particular,* $c_{\lambda,\mu}(0) = \frac{\delta_{\lambda,\mu}}{d_\mu}$.

Proof. Since $P_\mu(\tilde{p},\tilde{p})$ does not depend on $\tilde{p}$, one has

$$P_\mu(\tilde{p},\tilde{p}) = \frac{1}{2}\int_{\mathcal{G}^\circ_{m,n}} P_\mu(\tilde{p},\tilde{p})d\tilde{p} = \frac{1}{2d_\mu}\sum_{i=1}^{d_\mu}\int_{\mathcal{G}^\circ_{m,n}} |e_{\mu,i}(\tilde{p})|^2 d\tilde{p}$$
$$= \frac{1}{d_\mu}\sum_{i=1}^{d_\mu}\langle e_{\mu,i}, e_{\mu,i}\rangle = 1,$$

which proves (i). As for (ii), this is clear using (4) and the orthogonality relations between the $e_{\mu,i}$. Finally, assertion (iii) is classical, see [10] Theorem 3.1.

□

The algebraic structure of $\mathcal{Z}$ and $\mathcal{Z}^o$ can be easily deduced from [9]. For lack of reference, we state it in the next proposition

Proposition 2.3.

(i) *There is an isomorphism*

$$\mathcal{Z} \simeq \mathbb{C}[Y_1,\cdots,Y_m]^{S_m},$$

the ring of symmetric polynomials in m variables, mapping Y_i to $y_i = y_i(p,q)$. Similarly, one has

$$\mathcal{Z}^o \simeq \mathbb{C}[Y_1,\cdots,Y_m]^{S_m}[\theta], \text{ with } \theta^2 = Y_1\cdots Y_m,$$

mapping θ to $\epsilon t_1\cdots t_m$, where $\epsilon = \epsilon(\tilde{p},\tilde{q})$, $t_i = t_i(\tilde{p},\tilde{q})$. Moreover, the eigenspace decomposition of $\mathcal{Z}^o$ with respect to the involution σ^* is given by

$$\mathcal{Z}^{o+} = \mathcal{Z} \text{ and } \mathcal{Z}^{o-} = \epsilon t_1\cdots t_m\mathcal{Z} \simeq \theta\mathbb{C}[Y_1,\cdots,Y_m]^{S_m}$$

(ii) *The P_μ corresponding to even partitions may be expressed as*

$$P_\mu(p,q) = p_\mu(y_1(p,q),\cdots,y_m(p,q))$$

with $p_\mu(Y_1,\cdots,Y_m)$ a symmetric polynomial of total degree $\frac{|\mu|}{2}$, and those corresponding to odd partitions as

$$P_\mu(\tilde{p},\tilde{q}) = (\epsilon t_1\cdots t_m)p_\mu(y_1,\cdots,y_m),$$

with $p_\mu(Y_1,\cdots,Y_m)$ a symmetric polynomial of degree $\frac{|\mu|-m}{2}$.

Proof. (i) As explained above, we can identify $\mathcal{Z}$ with $\mathcal{R}(\mathcal{G}_{m,n})^{\mathrm{Stab}(p)}$ (resp. $\mathcal{Z}^o$ with $\mathcal{R}(\mathcal{G}^\circ_{m,n})^{\mathrm{Stab}(\tilde{p})}$), p (resp. $\tilde{p}$) being any fixed base point. From the isomorphism $\mathrm{Stab}(p) \simeq O(m)\times O(n-m)$, it is easily seen, using (2), that an element $F = F(p,\cdot) \in \mathcal{R}(\mathcal{G}_{m,n})^{\mathrm{Stab}(p)}$ is of the form

$$F = P(\cos\theta_1,\cdots,\cos\theta_m,\sin\theta_1,\cdots\sin\theta_m),$$

where $P(T_1,\cdots,T_m,Z_1,\cdots Z_m)$ is a polynomial, symmetric in $T_1,\cdots,T_m$ and $Z_1,\cdots Z_m$ respectively. Now the Stab(p)-invariance also implies that all the exponents are even, so that F is indeed a symmetric polynomial in $Y_1 = T_1^2, \cdots, Y_m = T_m^2$, which is the first part of assertion (i). As for the second part of the assertion, one first shows in the same way that a $\mathrm{Stab}(\tilde{p})$-invariant element in $\mathcal{R}(\mathcal{G}_{m,n}^{\circ})$ is of the form

$$F = F(\tilde{p}, \cdot) = P(\epsilon \cos\theta_1, \cdots, \cos\theta_m, \sin\theta_1, \cdots \sin\theta_m),$$

where $P(T_1, \cdots, T_m, Z_1, \cdots Z_m)$ is a polynomial, symmetric in $T_1, \cdots, T_m$ and $Z_1, \cdots Z_m$ respectively. Since $\mathrm{Stab}(\tilde{p}) \simeq SO_m \times O_{n-m}$, the $\mathrm{Stab}(\tilde{p})$-invariance also implies that the exponents in the last m variables are even, whereas the exponents in the first m ones are only restricted to have the same parity. Consequently, P is the sum of a polynomial in $Y_1 = T_1^2, \cdots, Y_m = T_m^2$ plus $T_1 \cdots T_m$ times a polynomial in $Y_1, \cdots, Y_m$, as asserted. The eigenspace decomposition is clear.

As for assertion (ii), we only need to observe that the P_μ belong to $\mathcal{Z}^{o+}$ or $\mathcal{Z}^{o-}$ according to as μ is even or odd, and that the p_μ have total degree $|\mu|$ in $T_1, \cdots, T_m$. □

3. Bounds on codes and designs.

Among the various equivalent definitions of a t-design given in [1] we recall the following one (see [1] Proposition 4.2.)

Definition 3.1. *A finite subset $\mathcal{D}$ of $\mathcal{G}_{m,n}$ is a $2k$-design if*

$$\forall \varphi \in H_{2k}^{+},\ \langle \varphi, 1\rangle = \frac{1}{|\mathcal{D}|}\sum_{p\in\mathcal{D}} \varphi(p). \tag{7}$$

As for spherical codes, the natural generalization to our context is as follows:

Definition 3.2. *Let $f(Y_1, \cdots, Y_m)$ be a symmetric polynomial such that $f(1, \cdots, 1) = 1$. A finite subset $\mathcal{D}$ of the Grassmannian space $\mathcal{G}_{m,n}$ is a f-code, if for any pair (p, q) of distinct elements in $\mathcal{D}$ one has*

$$f(y_1(p,q), \cdots, y_m(p,q)) = 0.$$

On the other hand, one can associate canonically to a symmetric polynomial $f(Y_1, \cdots, Y_m)$ as above, an $O(n)$-invariant function F on $\mathcal{G}_{m,n} \times \mathcal{G}_{m,n}$, satisfying $F(p,p) = 1$, by the formula:

$$F(p,q) := f(y_1(p,q), \cdots, y_m(p,q)),$$

and the definition of an f-code now reads

$$F(p,q) = \delta_{p,q},\ (p,q) \in \mathcal{D}^2. \tag{8}$$

The following notion of *type* is consistent with [6] Definition 5.4.:

Definition 3.3. *The type of an f-code is 1 if $Y_1 \cdots Y_m$ divides the polynomial $f(Y_1, \cdots, Y_m)$, and 0 otherwise.*

For any integer k, we define

$$H_k = \bigoplus_{\substack{|\mu|\leq k\\ \mu \text{ admissible}}} H^{\mu}_{m,n}.$$

It decomposes under σ^* as $H_k = H_k^+ \oplus H_k^-$, and we have, for the respective dimensions $d_k^{\pm}$ of $H_k^{\pm}$,

$$d_k^+ := \sum_{\substack{|\mu|\leq k\\ \mu \text{ even}\\ \text{depth}(\mu)\leq m}} d_\mu, \text{ resp. } d_k^- := \sum_{\substack{|\mu|\leq k\\ \mu \text{ odd}\\ \text{depth}(\mu)\leq m}} d_\mu.$$

It's worth noticing that, from remark 2.1, d_k^+ does not depend on m, while d_k^- does. The next two theorems establish bounds for t-designs and f-codes in terms of these numbers. Some explicit values of d_k^+ and d_k^- are collected in the Appendix (the d_μ are computed from the formulas in [7] §24.2, pp 407-410):

Remark 3.4. *In* [1], *we considered only non-oriented grassmanians, and what was denoted H_k there, corresponds to what is denoted H_k^+ here.*

Theorem 3.5. *Let $\mathcal{D} \subset \mathcal{G}_{m,n}$ be a $2k$-design. Then*

$$|\mathcal{D}| \geq \max\{d_k^+, d_k^-\}. \tag{9}$$

If equality holds in (9), then $\mathcal{D}$ is an f-code for $f = \frac{1}{d_k^+}\sum_{\substack{|\mu|\leq k\\ \mu \text{ even}}} d_\mu p_\mu$ or $f = \frac{Y_1\cdots Y_m}{d_k^-}\sum_{\substack{|\mu|\leq k\\ \mu \text{ odd}}} d_\mu p_\mu$, depending on whether $d_k^+ \geq d_k^-$ or not.

Proof. Let s be a section of the canonical surjection $\mathcal{G}^\circ_{m,n} \to \mathcal{G}_{m,n}$, and $\widetilde{\mathcal{D}} = s(\mathcal{D})$. Fix $\tilde{p}$ and $\tilde{q}$ in $\mathcal{G}^\circ_{m,n}$. If μ and λ are two partitions of degree $\leq k$, the formula

$$\varphi(\tilde{p}') = P_\mu(\tilde{p}, \tilde{p}')P_\lambda(\tilde{q}, \tilde{p}')$$

defines an element in H_{2k}. If moreover μ and λ are both *even* (resp. *odd*), then φ is σ^*-invariant, so it belongs to H_{2k}^+. Consequently (7) applies in both cases and reads:

$$\begin{aligned}
\frac{1}{|\mathcal{D}|}\sum_{\tilde{p}'\in\widetilde{\mathcal{D}}}\varphi(\tilde{p}') &= \frac{1}{|\mathcal{D}|}\sum_{p'\in\mathcal{D}}\varphi(p') \\
&= \langle \varphi, 1\rangle_{L^2(\mathcal{G}_{m,n})} \\
&= \langle P_\mu(p,.)P_\lambda(q,.), 1\rangle_{L^2(\mathcal{G}_{m,n})} \\
&= \langle P_\mu(\tilde{p},.), P_\lambda(\tilde{q},.)\rangle_{L^2(\mathcal{G}^\circ_{m,n})} \\
&= \frac{\delta_{\lambda,\mu}}{d_\mu}P_\mu(\tilde{p},\tilde{q}).
\end{aligned}$$

In other words, the matrices $S_\mu := (d_\mu P_\mu(\tilde{p}, \tilde{p}'))_{\tilde{p},\tilde{p}' \text{ in } \widetilde{\mathcal{D}}}$, $|\mu| \le k$, satisfy the relations

$$S_\mu S_\lambda = \delta_{\lambda,\mu} |\widetilde{\mathcal{D}}| S_\mu = \delta_{\lambda,\mu} |\mathcal{D}| S_\mu,$$

as long as μ and λ are both *even* (resp. *odd*). Setting $S^+ := \sum_{\substack{|\mu|\le k\\ \mu \text{ even}}} S_\mu$, resp. $S^- := \sum_{\substack{|\mu|\le k\\ \mu \text{odd}}} S_\mu$, it follows that

$$S^{\pm 2} = |\mathcal{D}| S^\pm. \tag{10}$$

On the other hand, one has $\operatorname{Tr} S_\mu = d_\mu |\mathcal{D}|$, from Lemma 2.2, so that $\operatorname{Tr} S^+ = \sum_{\substack{|\mu|\le k\\ \mu \text{ even}}} \operatorname{Tr} S_\mu = d_k^+ |\mathcal{D}|$, and likewise $\operatorname{Tr} S^- = d_k^- |\mathcal{D}|$. Therefore,

$$d_k^\pm = \frac{\operatorname{Tr} S^\pm}{|\mathcal{D}|} = \operatorname{rank} S^\pm \le |\mathcal{D}|,$$

whence the conclusion.

When equality holds, then (10) implies that $S^+ = |\mathcal{D}| I_{|\mathcal{D}|} = d_k^+ I_{|\mathcal{D}|}$ resp. $S^- = d_k^- I_{|\mathcal{D}|}$, depending on whether $d_k^+ \ge d_k^-$ or not, where $I_{|\mathcal{D}|}$ stands for the identity matrix in dimension $|\mathcal{D}|$. This means that $F(p,q) = \delta_{p,q}$, for all $(p,q) \in \mathcal{D}^2$, where $F = \frac{1}{d_k^+} \sum_{\substack{|\mu|\le k\\ \mu \text{ even}}} d_\mu P_\mu$, resp. $\frac{1}{d_k^-} \sum_{\substack{|\mu|\le k\\ \mu \text{ odd}}} d_\mu P_\mu$. In the first case, this is clearly equivalent to the assertion that $\mathcal{D}$ be an f-code, according to (8) and the definition of f. This is also the case in the second case, since each P_μ, μ odd, is divisible by the product $t_1 \cdots t_m$, so that

$$\frac{1}{d_k^-} \sum_{\substack{|\mu|\le k\\ \mu \text{ odd}}} d_\mu P_\mu(p,q) = \delta_{p,q} \Leftrightarrow \frac{1}{d_k^-} \sum_{\substack{|\mu|\le k\\ \mu \text{ odd}}} (t_1 \cdots t_m) d_\mu P_\mu(p,q) = \delta_{p,q}.$$

□

Theorem 3.6. *Any f-code $\mathcal{D}$ in $\mathcal{G}_{m,n}$ satisfies*

$$|\mathcal{D}| \le d_k^+ \tag{11}$$

where $k = 2\deg f$. If moreover f is of type 1, then

$$|\mathcal{D}| \le d_k^- \tag{12}$$

where $k = 2\deg f - m$. Whenever equality holds in (11), resp. (12), then

$$f = \frac{1}{d_k^+} \sum_{\substack{|\mu|\le k\\ \mu \text{ even}}} d_\mu p_\mu,$$

resp.

$$f = \frac{Y_1 \cdots Y_m}{d_k^-} \sum_{\substack{|\mu|\le k\\ \mu \text{ odd}}} d_\mu p_\mu,$$

and $\mathcal{D}$ is a $2k$-design.

Proof. Setting $k = 2\deg f$, we first see that the functions $F(p,.),\ p \in \mathcal{D}$ are in H_k^+. We claim that they form a linearly independent system. Indeed, if $\sum_{p\in\mathcal{D}} \lambda_p F(p,.) = 0$, then evaluating the left-hand side successively on each $p \in \mathcal{D}$, and using (8), we see that $\lambda_p = 0$ for all $p \in \mathcal{D}$. Hence $|\mathcal{D}| \leq \dim H_k^+ = d_k^+$, which is the first assertion. As for the second one, if f is divisible by $Y_1 \cdots Y_m$, we write it as $f(Y_1, \cdots, Y_m) = (Y_1 \cdots Y_m) g(Y_1, \cdots, Y_m)$. Then the functions $t_1(p,.) \cdots t_m(p,.) G(p,.),\ p \in \mathcal{D}$ are linearly independent elements in H_k^-, with $k = 2\deg f - m$, and the inequality $|\mathcal{D}| \leq d_k^-$ follows, as in the first case.

To see when equality is achieved, let us assume, for instance that $|\mathcal{D}| = d_k^+$, $k = 2\deg f$ (the case $|\mathcal{D}| = d_k^-, k = 2\deg f - m$ for $\mathcal{D}$ of type 1 is dealt with similarly). Under this assumption, the family $\{F(p,.)\ ,\ p \in \mathcal{D}\}$ is now a basis of H_k^+. Moreover, it is readily checked that the following formula holds for any φ in H_k^+:

$$\varphi = \sum_{p\in\mathcal{D}} \varphi(p) F(p,.). \tag{13}$$

On the other hand, we know from Proposition 2.3 that F (resp. f) may be written as a linear combination of the P_μ (resp. p_μ),

$$F = \sum_{\substack{|\mu|\leq k \\ \mu \text{ even}}} f_\mu P_\mu \text{ (resp. } f = \sum_{\substack{|\mu|\leq k \\ \mu \text{ even}}} f_\mu p_\mu). \tag{14}$$

Let $P := \sum_{\substack{|\mu|\leq k \\ \mu \text{ even}}} d_\mu P_\mu$. What we want to show is that $F = \frac{1}{d_k^+} P$, or in other words that

$$f_\mu = \frac{d_\mu}{d_k^+}, \text{ for } |\mu| \leq k, \mu \text{ even}. \tag{15}$$

To that end, it is sufficient to show that

$$0 \leq f_\mu \leq \frac{d_\mu}{d_k^+}, \text{ for } |\mu| \leq k, \mu \text{ even}. \tag{16}$$

since applying (14) to any p in $\mathcal{D}$, we see that $1 = F(p,p) = \sum f_\mu P_\mu(p,p) = \sum f_\mu$, so the right inequality in (16) is an equality. First we note, using (5) and the above decomposition, that

$$\langle F(q,.), \sum_{|\mu|\leq k} d_\mu P_\mu(p,.)\rangle = F(p,q),$$

so that the condition $F(p,q) = \delta_{p,q},\ (p,q) \in \mathcal{D}^2$ implies that the family $\{\sum_{|\mu|\leq k} d_\mu P_\mu(p,.),\ p \in \mathcal{D}\}$ is a basis, dual to $\{F(p,.),\ p \in \mathcal{D}\}$ with respect to the scalar product $\langle , \rangle$. Consequently, the matrix $S = (P(p,q))_{p,q\in\mathcal{D}^2} = (\langle P(p,.), P(q,.)\rangle)_{p,q\in\mathcal{D}^2}$ is invertible and its inverse is given by

$$S^{-1} = (\langle F(p,.), F(q,.)\rangle)_{p,q\in\mathcal{D}^2}\,.$$

One easily checks, using lemma 2.2 that

$$\langle F(p,.), P_\mu(q,.)\rangle = \frac{f_\mu}{d_\mu} P_\mu(q,p),$$

for p, q in $\mathcal{D}^2$, and $|\mu| \leq k$, μ even. But according to (13), this means that the functions $P_\mu(q,.)$, $q \in \mathcal{D}$, $|\mu| \leq k$, μ even, are eigenfunctions of the matrix $(\langle F(p,.), F(q,.)\rangle)_{p,q\in\mathcal{D}^2} = S^{-1}$, with corresponding eigenvalue $\frac{f_\mu}{d_\mu}$. Thus the $\frac{f_\mu}{d_\mu}$ are eigenvalues of the Gram matrix of a basis of H_k^+, hence positive. Now, writing (14) for all $(p,q) \in \mathcal{D}^2$, and adding up we obtain

$$|\mathcal{D}| = \sum_{(p,q)\in\mathcal{D}^2} F(p,q) = \sum_{\substack{|\mu|\leq k\\ \mu \text{ even}}} f_\mu \sum_{(p,q)\in\mathcal{D}^2} P_\mu(p,q)$$

whence

$$|\mathcal{D}|(1 - f_0|\mathcal{D}|) = \sum_{\substack{|\mu|\leq k\\ \mu \text{ even}, \mu\neq 0}} f_\mu \sum_{(p,q)\in\mathcal{D}^2} P_\mu(p,q) \geq 0, \tag{17}$$

because of the positivity of the matrix $(P_\mu(p,p'))_{p,p'\in\mathcal{D}^2}$ (lemma (2.2)(ii)), so that $f_0 \leq \frac{1}{|\mathcal{D}|} = \frac{1}{d_k^+}$. If we now consider the annihilator polynomial

$$F_\lambda := P_\lambda F = \sum_{\substack{|\mu|\leq k\\ \mu \text{even}}} g_{\lambda,\mu} P_\mu,$$

we contend that the $g_{\lambda,\mu}$ are nonnegative and that $g_{\lambda,0} = \frac{f_\lambda}{d_\lambda}$: this is an easy consequence of lemma 2.2 (6). Consequently, the argument used to get (17) still holds, and we obtain $g_{\lambda,0} = \frac{f_\lambda}{d_\lambda} \leq \frac{1}{d_k^+}$, as desired.

It remains to prove that $\mathcal{D}$ is a $2k$-design. From [1] Proposition 4.2., it amounts to prove that

$$\forall \varphi \in H_{2k}^+, \ \langle \varphi, 1\rangle = \frac{1}{|\mathcal{D}|}\sum_{p\in\mathcal{D}} \varphi(p) = \frac{1}{d_k^+}\sum_{p\in\mathcal{D}} \varphi(p).$$

It's enough to check this for functions of the form gh, with g, h in H_k^+, since they generate H_{2k}^+. Using the expansion (13) of g and h, we see that

$$\begin{aligned}\langle 1, gh\rangle &= \langle g, \overline{h}\rangle\\ &= \sum_{p,q\in\mathcal{D}^2} g(p)h(q)\langle F(p,.), F(q,.)\rangle\\ &= \frac{1}{|\mathcal{D}|}\sum_{p\in\mathcal{D}} g(p)h(p),\end{aligned}$$

whence the conclusion. □

4. Examples.

4.1. **The case $m = 1$.** This is the case of the projective space over the real numbers, the codes of which are studied in [6]. The $2k$-designs in the real projective space can be viewed as *antipodal* $(2k+1)$-designs on the unit sphere of the Euclidean space for which absolute bounds are given in [5]. We recover here these bounds, since for $\mu = \mu_1 \geq 0$ the space $H_{1,n}^{\mu}$ is isomorphic to the space of harmonic polynomials in n variables of degree μ_1. One has $d_k^+ = \binom{n+k-1}{n-1}$ and $d_k^- = \binom{n+k-2}{n-1}$ if k is even, and *vice versa* if k is odd.

A t-design is called tight if its cardinality attains this lower bound. Tight t-designs are only known for $(n,t) = (7,4), (8,6), (23,4), (23,6), (24,10)$. Moreover, it is known that tight t-designs cannot exist when $t \geq 8$, apart from the $(24,10)$ given by the lines supporting the minimal vectors of the Leech lattice (see [2]).

4.2. **The case $k = 2$.** In [3] and [4], packings in the Grassmannian spaces are considered, with respect to the so-called *chordal distance*, given in our notations by

$$d^2(p,q) = m - \sum_{i=1}^{m} y_i.$$

In [3], a *simplex bound* is settled for the sets $\mathcal{D}$ for which $d(p,q) \geq d$ (using an isometric embedding into the Euclidean sphere of $\mathbb{R}^{(n-1)(n+2)/2}$). Equality holds if and only if $|\mathcal{D}| = \frac{n(n+1)}{2}$ and $d(p,q)$ is constant.

In [4, §5], an infinite family of packings in $\mathcal{G}_{\frac{p-1}{2},p}$ meating this bound, is constructed. Here p is a prime, which is either equal to 3 or congruent to -1 modulo 8. Let us denote it by $\mathcal{D}_p$. Then one has:

Proposition 4.1. *$\mathcal{D}_p$ is a tight 4-design in $\mathcal{G}_{\frac{p-1}{2},p}$.*

Proof. According to [4] Theorem 3, $\mathcal{D}_p$ consists on $\frac{p(p+1)}{2} = d_{[0,\cdots,0]} + d_{[2,0,\cdots,0]} = d_2^+$ subspaces with same pairwise chordal distance $d^2 = \frac{(p+1)^2}{4(p+2)}$. Since $d^2 = \sum \sin^2\theta_i = \frac{p-1}{2} - \sum y_i$, the conclusion follows, applying Theorem 3.6 to the polynomial $f = \frac{4(p+2)(\sum Y_i)-(p^2-5)}{p^2-5}$. $\square$

4.3. **The case $k = 3$.** From the definitions, one has $d_3^+ = d_2^+ = \frac{n(n+1)}{2}$, and d_3^- equals 0 unless $m = 1$ in which case $d_3^- = \binom{n+2}{3}$, or $m = 3$ in which case $d_3^- = d_{(1,1,1)} = \binom{n}{3}$. Therefore, it is very unlikely that tight 6-designs exist for $m \neq 1, 3$.

A family of packings in the Grassmannian $\mathcal{G}_{2^k,2^m}$ is constructed in [4, Theorem 1], each of them are orbits under the Clifford group $\mathcal{C}_m$. We have checked that, for $m = 2,3,4$, and for $(m,k) = (5,4)$, these packings are 6-designs. For each m, the smallest of these sets corresponds to $k = m-1$ and its cardinality equals $2^{2m} + 2^m - 2 = 2(d_2^+ - 1)$.

Remark 4.2. *It is known that the orbits of the Clifford group* on the first Grassmannian *provide* 6*-designs, because the first non trivial invariant polynomial of this group has degree* 8 *(and corresponds to the Hamming code, see* [12] *and the earlier work of B. Runge). We conjecture that the orbits of the Clifford group on all the Grassmannians provide* 6*-designs. This, according to* [1, Th 4.5, Rem 4.6]*, is equivalent to the fact that the* $\mathcal{C}_m$*-invariants of the* $(n = 2^m)$ Gl_n*-irreducible modules canonically associated to the partitions* $(4,2)$ *and* $(2,2,2)$ *(denoted* F_n^{μ} *in* [9]*) have dimension* 1.

5. Appendix.

We list below some values of d_k^+ and d_k^- for $m = 2, 3$ and 4.

m=2				m=3				m=4			
n	k	d_k^+	d_k^-	n	k	d_k^+	d_k^-	n	k	d_k^+	d_k^-
4	1	1	0	6	1	1	0	8	1	1	0
4	2	10	3	6	2	21	0	8	2	36	0
4	3	10	3	6	3	21	10	8	3	36	0
4	4	40	18	6	4	210	10	8	4	630	35
4	5	40	18	6	5	210	136	8	5	630	35
5	1	1	0	7	1	1	0	9	1	1	0
5	2	15	10	7	2	28	0	9	2	45	0
5	3	15	10	7	3	28	35	9	3	45	0
5	4	105	91	7	4	378	35	9	4	990	126
5	5	105	91	7	5	378	651	9	5	990	126
6	1	1	0	8	1	1	0	10	1	1	0
6	2	21	15	8	2	36	0	10	2	55	0
6	3	21	15	8	3	36	56	10	3	55	0
6	4	210	190	8	4	630	56	10	4	1485	210
6	5	210	190	8	5	630	1352	10	5	1485	210
7	1	1	0	9	1	1	0	11	1	1	0
7	2	28	21	9	2	45	0	11	2	66	0
7	3	28	21	9	3	45	84	11	3	66	0
7	4	378	351	9	4	990	84	11	4	2145	330
7	5	378	351	9	5	990	2541	11	5	2145	330
8	1	1	0	10	1	1	0	12	1	1	0
8	2	36	28	10	2	55	0	12	2	78	0
8	3	36	28	10	3	55	120	12	3	78	0
8	4	630	595	10	4	1485	120	12	4	3003	495
8	5	630	595	10	5	1485	4432	12	5	3003	495
9	1	1	0	11	1	1	0	13	1	1	0
9	2	45	36	11	2	66	0	13	2	91	0
9	3	45	36	11	3	66	165	13	3	91	0
9	4	990	946	11	4	2145	165	13	4	4095	715
9	5	990	946	11	5	2145	7293	13	5	4095	715

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