BOUNDS FOR PROJECTIVE CODES FROM SEMIDEFINITE PROGRAMMING

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Abstract. We apply the semidefinite programming method to derive bounds for projective codes over a finite field.

1. Introduction

In network coding theory, as introduced in [1], information is transmitted through a directed graph. In general this graph has several sources, several receivers, and a certain number of intermediate nodes. Information is modeled as vectors of fixed length over a finite field $\mathbb{F}_q$, called *packets*. To improve the performance of the communication, intermediate nodes should forward random linear $\mathbb{F}_q$-combinations of the packets they receive. This is the approach taken in the non-coherent communication case, that is, when the structure of the network is not known a priori [13]. Hence, the *vector space* spanned by the packets injected at the source is globally preserved in the network when no error occurs. This observation led Koetter and Kschischang [15] to model network codes as subsets of projective space $\mathcal{P}(\mathbb{F}_q^n)$, the set of linear subspaces of $\mathbb{F}_q^n$, or of Grassmann space $\mathcal{G}_q(n, k)$, the subset of those subspaces of $\mathbb{F}_q^n$ having dimension k. Subsets of $\mathcal{P}(\mathbb{F}_q^n)$ are called *projective codes* while subsets of the Grassmann space will be referred to as *constant-dimension codes* or *Grassmann codes*.

As usual in coding theory, in order to protect the system from errors, it is desirable to select the elements of the code so that they are pairwise as far as possible with respect to a suitable distance. The *subspace distance* between U and V

$$d_S(U, V) = \dim(U + V) - \dim(U \cap V) = \dim U + \dim V - 2\dim(U \cap V)$$

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was introduced in [15] for this purpose. It is natural to ask how large a code with a given minimal distance can be. Formally, we define

$$\begin{cases} A_q(n,d) := \max\{|\mathcal{C}| \;:\; \mathcal{C} \subset \mathcal{P}(\mathbb{F}_q^n),\; d_S(\mathcal{C}) \geq d\} \\ A_q(n,k,2\delta) := \max\{|\mathcal{C}| \;:\; \mathcal{C} \subset \mathcal{G}_q(n,k),\; d_S(\mathcal{C}) \geq 2\delta\} \end{cases}$$

where $d_S(\mathcal{C})$ denotes the minimal subspace distance among distinct elements of a code $\mathcal{C}$. In this paper we will discuss and prove upper bounds for $A_q(n,d)$ and $A_q(n,k,2\delta)$.

1.1. **Bounds for $A_q(n,k,2\delta)$.** Grassmann space $\mathcal{G}_q(n,k)$ is a homogeneous space under the action of the linear group $GL_n(\mathbb{F}_q)$. Moreover, the group acts *distance transitively* when we use the subspace distance; the orbits of $GL_n(\mathbb{F}_q)$ acting on pairs (U,V) of $\mathcal{G}_q(n,k)$ are characterized by the subspace distance $d(U,V)$. In other words, Grassmann space is *two-point homogeneous* under this action.

Due to this property, codes and designs in $\mathcal{G}_q(n,k)$ can be analyzed in the framework of Delsarte's theory, in the same way as other classical spaces in coding theory, such as Hamming space and binary Johnson space. In fact, $\mathcal{G}_q(n,k)$ is a *q-analog* of binary Johnson space; see [7]. The linear group plays the role of the symmetric group for the Johnson space, while the dimension replaces the weight function.

The classical bounds (anticode, Hamming, Johnson, Singleton) have been derived for the Grassmann codes [15, 26, 27]. The more sophisticated Delsarte linear programming bound was obtained in [7]. However, numerical computations indicate that it is not better than the anticode bound. Moreover, the Singleton and anticode bounds have the same asymptotic behavior which is attained by a family of Reed-Solomon-like codes constructed in [15] and closely related to the rank-metric Gabidulin codes.

1.2. **Bounds for $A_q(n,d)$.** In contrast to $\mathcal{G}_q(n,k)$, the projective space has a much nastier behavior, essentially because it is not two-point homogeneous. In fact it is not even homogeneous under the action of a group. For example, the size of balls in this space depends not only on their radius, but also on the dimension of their center. Consequently, bounds for projective codes are much harder to obtain. Etzion and Vardy in [10] provide a bound in the form of the optimal value of a linear program, which is derived by elementary reasoning involving packing issues. Up to now the Etzion-Vardy bound is the only successful generalization of the classical bounds to projective space.

In this paper we derive semidefinite programming bounds for projective codes and compare them with the above mentioned bounds.

In convex optimization, semidefinite programs generalize linear programs and one can solve them by efficient algorithms [23], [24]. They have numerous applications in combinatorial optimization. The earliest is due to Lovász [18] who found a semidefinite programming upper bound, the theta number, for the independence number of a graph.

Because a code with given minimal distance can be viewed as an independent set in a certain graph, the theta number also applies to coding theory. However, because the underlying graph is built on the space under consideration, its size grows exponentially with the parameters of the codes. So by itself the theta number is not an appropriate tool, unless the symmetries of the space are taken into account. A general framework for symmetry reduction techniques of semidefinite programs is provided in [3]. For the classical spaces of coding theory, after symmetry reduction, the theta number turns out to be essentially equal to the celebrated Delsarte linear programming bound. For projective spaces, the symmetry reduction was announced in [5] (see also [2]). The program remains a semidefinite program (it does not collapse to a linear program) but fortunately it has polynomial size in the dimension n.

The relationship between Delsarte's linear programming bound and the theta number was recognized long ago in [21] and [20]. Recently, more applications of semidefinite programming to coding theory have been developed, see [22], [4], [25], [12] and the survey [2].

1.3. **Organization of the paper.** In Section 2 we review the classical bounds for Grassmann codes and the Etzion-Vardy bound for projective codes. In Section 3 we present the semidefinite programming method in connection with the theta number. We show that most of the bounds for Grassmann codes can be derived from this method. In Section 4 we reduce the semidefinite program by the action of the group $GL_n(\mathbb{F}_q)$. In Section 5 we present numerical results obtained with this method and we compare them with the Etzion-Vardy method for $q = 2$ and $n \leq 16$. Another distance of interest on projective space, the *injection distance*, was introduced in [17]. We show how to modify the Etzion-Vardy bound as well as the semidefinite programming bound for this.

2. Elementary bounds for Grassmann and projective codes

2.1. **Bounds for Grassmann codes.** In this section we review the classical bounds for $A_q(n, k, 2\delta)$. We note that the subspace distance takes only even values on the Grassmann space and that one can restrict to $k \leq n/2$ by the relation $A_q(n, k, 2\delta) = A_q(n, n-k, 2\delta)$, which follows by considering orthogonal subspaces.

We recall the definition of the q-analog of the binomial coefficient that counts the number of k-dimensional subspaces of a fixed n-dimensional space over $\mathbb{F}_q$, i.e. the number of elements of $\mathcal{G}_q(n, k)$.

Definition 2.1. *The* q-ary binomial coefficient *is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)\dots(q^{n-k+1} - 1)}{(q^k - 1)\dots(q - 1)}.$$

2.1.1. *Sphere-packing bound.*

$$A_q(n,k,2\delta) \leq \frac{|\mathcal{G}_q(n,k)|}{|B_k(\delta-1)|} = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\sum_{m=0}^{\lfloor(\delta-1)/2\rfloor} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} n-k \\ m \end{bmatrix}_q q^{m^2}} \tag{1}$$

It follows from the well-known observation that balls of radius $\delta - 1$ centered at elements of a code $\mathcal{C} \subset \mathcal{G}_q(n,k)$ with minimal distance 2δ are pairwise disjoint and have the same cardinality $\sum_{m=0}^{\lfloor(\delta-1)/2\rfloor} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} n-k \\ m \end{bmatrix}_q q^{m^2}$.

2.1.2. *Singleton bound* [15].

$$A_q(n,k,2\delta) \leq \begin{bmatrix} n-\delta+1 \\ k-\delta+1 \end{bmatrix}_q \tag{2}$$

It is obtained by the introduction of a "puncturing" operation on the code.

2.1.3. *Anticode bound* [26]. An *anticode* of diameter e is a subset of a metric space whose pairwise distinct elements are at distance less or equal than e. The general anticode bound (see [6]) states that, given a metric space X which is homogeneous under the action of a group G, for every code $\mathcal{C} \subset X$ with minimal distance d and for every anticode $\mathcal{A}$ of diameter $d-1$, we have

$$|\mathcal{C}| \leq \frac{|X|}{|\mathcal{A}|} \ .$$

Spheres of given radius r are anticodes of diameter $2r$. So if we take $\mathcal{A}$ to be a sphere of radius $\delta - 1$ in $\mathcal{G}_q(n,k)$ we recover the sphere-packing bound. Obviously, to obtain the strongest bound, we have to choose the largest anticodes of given diameter, which in our case are not spheres. Indeed, the set of all elements of $\mathcal{G}_q(n,k)$ which contain a fixed $(k-\delta+1)$-dimensional subspace is an anticode of diameter $2\delta - 2$ with $\begin{bmatrix} n-k+\delta-1 \\ \delta-1 \end{bmatrix}_q$ elements and which is in general larger than the sphere of radius $\delta - 1$. Moreover Frankl and Wilson proved in [11] that these anticodes have the largest possible size. Taking such $\mathcal{A}$ in the general anticode bound, we recover *the* (best) anticode bound for $\mathcal{G}_q(n,k)$:

$$\begin{aligned} A_q(n,k,2\delta) \leq \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-k+\delta-1 \\ \delta-1 \end{bmatrix}_q} &= \frac{\begin{bmatrix} n \\ k-\delta+1 \end{bmatrix}_q}{\begin{bmatrix} k \\ k-\delta+1 \end{bmatrix}_q} \\ &= \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+\delta}-1)}{(q^k-1)(q^{k-1}-1)\dots(q^\delta-1)} \end{aligned} \tag{3}$$

It follows from the previous discussion that the anticode bound improves the sphere-packing bound. Moreover, the anticode bound is usually stronger than the Singleton bound, with equality only in the cases $n = k$ or $\delta = 1$, see [27].

2.1.4. *First and second Johnson-type bound* [27].

$$A_q(n,k,2\delta) \le \left\lfloor \frac{(q^n-1)(q^k-q^{k-\delta})}{(q^k-1)^2-(q^n-1)(q^{k-\delta}-1)} \right\rfloor \tag{4}$$

as long as $(q^k-1)^2-(q^n-1)(q^{k-\delta}-1) > 0$, and

$$A_q(n,k,2\delta) \le \left\lfloor \frac{q^n-1}{q^k-1} A_q(n-1,k-1,2\delta) \right\rfloor. \tag{5}$$

These bounds were obtained in [27] through the construction of a binary constant-weight code associated to every constant-dimension code. Iterating the latter, one obtains

$$A_q(n,k,2\delta) \le \left\lfloor \frac{q^n-1}{q^k-1} \left\lfloor \frac{q^{n-1}-1}{q^{k-1}-1} \cdots \left\lfloor \frac{q^{n-k+\delta}-1}{q^\delta-1} \right\rfloor \cdots \right\rfloor \right\rfloor. \tag{6}$$

If the floors are removed from the right hand side of (6), the anticode bound is recovered, so (6) is stronger. In the particular case of $\delta = k$ and if $n \not\equiv 0 \bmod k$, (6) was sharpened in [10] to

$$A_q(n,k,2k) \le \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor - 1. \tag{7}$$

For $\delta = k$ and if k divides n, we have equality in (6), because of the existence of *spreads* (see [10])

$$A_q(n,k,2k) = \frac{q^n-1}{q^k-1}.$$

Summing up, the strongest upper bound for constant dimension codes reviewed so far comes by putting together (6) and (7):

Theorem 2.2. *If $n-k \not\equiv 0$ mod δ, then*

$$A_q(n,k,2\delta) \le \left\lfloor \frac{q^n-1}{q^k-1} \left\lfloor \cdots \left\lfloor \frac{q^{n-k+\delta+1}-1}{q^{\delta+1}-1} \left(\left\lfloor \frac{q^{n-k+\delta}-1}{q^\delta-1} \right\rfloor - 1 \right) \right\rfloor \cdots \right\rfloor \right\rfloor$$

otherwise

$$A_q(n,k,2\delta) \le \left\lfloor \frac{q^n-1}{q^k-1} \left\lfloor \cdots \left\lfloor \frac{q^{n-k+\delta+1}-1}{q^{\delta+1}-1} \left\lfloor \frac{q^{n-k+\delta}-1}{q^\delta-1} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor.$$

2.2. **A bound for projective codes.** Here we turn our attention to projective codes whose codewords have not necessarily the same dimension, and we review the bound obtained by Etzion and Vardy in [10]. The idea is to split a code $\mathcal{C}$ into subcodes $\mathcal{C}_k = \mathcal{C} \cap \mathcal{G}_q(n,k)$ of constant dimension, and then to derive linear constraints on the cardinality $|\mathcal{C}_k|$, coming from packing constraints.

Let $B(V,e) := \{U \in \mathcal{P}(\mathbb{F}_q^n) : d_S(U,V) \le e\}$ denote the ball with center V and radius e. If $\dim V = i$, we have

$$|B(V,e)| = \sum_{\ell=0}^{e} \sum_{j=0}^{\ell} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} n-i \\ \ell-j \end{bmatrix}_q q^{j(\ell-j)}.$$

We define $c(i,k,e) := |B(V,e) \cap \mathcal{G}_q(n,k)|$ for V of dimension i. It is not difficult to prove that

$$c(i,k,e) = \sum_{j=\lceil \frac{i+k-e}{2} \rceil}^{\min\{k,i\}} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} n-i \\ k-j \end{bmatrix}_q q^{(i-j)(k-j)}. \tag{8}$$

Theorem 2.3 (Linear programming bound for codes in $\mathcal{P}(\mathbb{F}_q^n)$, [10])**.**

$$A_q(n,2e+1) \leq \max\Big\{ \sum_{k=0}^{n} x_k \ : \ x_k \leq A_q(n,k,2e+2) \textit{ for } k=0,\dots,n, \\ \sum_{i=0}^{n} c(i,k,e)x_i \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \textit{ for } k=0,\dots,n \Big\}$$

Proof. For $\mathcal{C} \subset \mathcal{P}(\mathbb{F}_q^n)$ of minimal distance $2e+1$, and for $k=0,\dots,n$, we introduce $x_k = |\mathcal{C} \cap \mathcal{G}_q(n,k)|$. Then $\sum_{k=0}^n x_k = |\mathcal{C}|$ and each x_k represents the cardinality of a subcode of $\mathcal{C}$ of constant dimension k, so it is upper bounded by $A_q(n,k,2e+2)$. Moreover, balls of radius e centered at the codewords are pairwise disjoint, so the sets $B(V,e) \cap \mathcal{G}_q(n,k)$ for $V \in \mathcal{C}$ are pairwise disjoint subsets of $\mathcal{G}_q(n,k)$. So

$$\sum_{V \in \mathcal{C}} |B(V,e) \cap \mathcal{G}_q(n,k)| \leq |\mathcal{G}_q(n,k)|.$$

Because $|B(V,e) \cap \mathcal{G}_q(n,k)| = c(i,k,e)$ if $\dim(V) = i$ we obtain the second constraint

$$\sum_{i=0}^{n} c(i,k,e)x_i \leq \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

So $|\mathcal{C}|$ is at most the optimal value of the linear program above. □

Remark 2.4. *Of course, in view of explicit computations, if the exact value of $A_q(n,k,2e+2)$ is not available, it can be replaced in the linear program of Theorem 2.3 by an upper bound.*

3. The semidefinite programming method

3.1. **Semidefinite programs.** A (real) semidefinite program is an optimization problem of the form:

$$\sup\{ \ \langle C, Y \rangle \ : \ Y \succeq 0, \ \langle A_i, Y \rangle = b_i \text{ for } i=1,\dots,m \ \},$$

where

$C, A_1, \dots, A_m$ are given real symmetric matrices,
$b_1, \dots, b_m$ are given real values,
Y is a real symmetric matrix, which is the optimization variable,
$\langle A, B \rangle = \mathrm{trace}(AB)$ is the inner product between symmetric matrices,
$Y \succeq 0$ denotes that Y is symmetric and positive semidefinite.

This formulation includes linear programming as a special case when all matrices involved are diagonal matrices. When the input data satisfies some technical assumptions (which are fulfilled for our application) then there are polynomial time algorithms which determine an approximate optimal value. We refer to [23] and [24] for further details.

3.2. **Lovász' theta number.** In [18], Lovász gave an upper bound on the independence number $\alpha(\mathcal{G})$ of a graph $\mathcal{G} = (V, E)$ as the optimal value $\vartheta(\mathcal{G})$ of a semidefinite program:

Theorem 3.1 ([18])**.**

$$\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G}) := \max\Big\{ \sum_{(x,y)\in V^2} F(x,y) \ : \ F \in \mathbb{R}^{V\times V},\ F \succeq 0,\ \sum_{x\in V} F(x,x) = 1,\ F(x,y) = 0 \text{ if } xy \in E \Big\} \tag{9}$$

Here we can write max instead of sup because one can show using duality theory of semidefinite programming that the supremum is attained: the Slater condition [24, Theorem 3.1] is fulfilled.

In the above and all along this paper, we identify a matrix indexed by a given finite set V with a function defined on V^2. The program given in (9) is one of the many equivalent formulations of Lovász' original $\vartheta(\mathcal{G})$. If the constraint that F only attains nonnegative values is added, the optimal value gives a sharper bound for $\alpha(\mathcal{G})$. Traditionally this semidefinite program is denoted by $\vartheta'(\mathcal{G})$ [21].

This method applies to bound the maximal cardinality $\mathcal{A}(X, d)$ of codes in a metric space X with prescribed minimal distance d. Indeed $\mathcal{A}(X, d) = \alpha(\mathcal{G})$ where $\mathcal{G}$ is the graph with vertex set X and edges set $\{xy : 0 < d(x, y) < d\}$. So, we obtain:

Corollary 3.2 (The semidefinite programming bound)**.**

$$\mathcal{A}(X,d) \leq \max\Big\{ \sum_{(x,y)\in X^2} F(x,y) \ : \ F \in \mathbb{R}^{X\times X},\ F \succeq 0,\ F \geq 0,\ \sum_{x\in X} F(x,x) = 1,\ F(x,y) = 0 \text{ if } 0 < d(x,y) < d \Big\} \tag{10}$$

$$= \min\Big\{ t/\lambda \ : \ F \in \mathbb{R}^{X\times X},\ F - \lambda \succeq 0,\ F(x,x) \leq t \text{ for all } x \in X,\ F(x,y) \leq 0 \text{ if } d(x,y) \geq d \Big\} \tag{11}$$

The second semidefinite program (11) is the dual of (10). Furthermore, by weak duality, any feasible solution of the semidefinite program in (11) leads to an upper bound for $\mathcal{A}(X, d)$.

3.3. **Bounds for Grassmann codes.** In two-point homogeneous spaces, the semidefinite program in (11) collapses to the linear program of Delsarte, first introduced in [6] in the framework of association schemes. This fact was first recognized in the case of the Hamming space, independently in [20] and [21]. We refer to [3] for a general discussion on how (10) and (11) reduce to linear programs in the case of two-point homogeneous spaces.

Grassmann space $\mathcal{G}_q(n,k)$ is two-point homogeneous for the action of the group $G = GL_n(\mathbb{F}_q)$ and its associated zonal polynomials are computed in [7]. They belong to the family of q-Hahn polynomials, which are q-analogs of the Hahn polynomials related to the binary Johnson space.

Definition 3.3. *The q-Hahn polynomials associated to the parameters n, s, t with $0 \le s \le t \le n$ are the polynomials $Q_\ell(n, s, t; u)$ with $0 \le \ell \le \min(s, n-t)$ uniquely determined by the properties:*

(a) *Q_ℓ has degree ℓ in the variable $[u] = q^{1-u}\begin{bmatrix}u\\1\end{bmatrix}_q$*
(b) *They are orthogonal polynomials for the weights*

$$0 \le i \le \min(s, n-t) \quad w(n,s,t;i) = \begin{bmatrix}s\\i\end{bmatrix}_q \begin{bmatrix}n-s\\t-s+i\end{bmatrix}_q q^{i(t-s+i)}$$

(c) *$Q_\ell(0) = 1$.*

To be more precise, in the Grassmann space $\mathcal{G}_q(n,k)$, the zonal polynomials are associated to the parameters $s = t = k$. The other parameters will come into play when we analyze the full projective space in Section 4. The resulting linear programming bound is explicitly stated in [7]:

Theorem 3.4 (Delsarte's linear programming bound [7])**.**

$$A_q(n,k,2\delta) \le \min\Big\{1 + f_1 + \cdots + f_k \quad : \quad f_i \ge 0 \text{ for } i = 1,\ldots,k, \\ F(u) \le 0 \text{ for } u = \delta, \ldots, k\Big\},$$

where $F(u) = 1 + \sum_{i=1}^k f_i Q_i(u)$ and $Q_i(u) = Q_i(n,k,k;u)$ as in Definition 3.3.

In order to show the power of the semidefinite programming bound, we will verify that most of the bounds in Section 2 for Grassmann codes can be obtained from Corollary 3.2 or Theorem 3.4. In each case we construct a suitable feasible solution of (11).

3.3.1. *Singleton bound.* We fix an arbitrary subspace w of $\mathbb{F}_q^n$ of dimension $n-\delta+1$. We consider a function $\phi : \mathcal{G}_q(n,k) \to \{u \subset w : \dim(u) = k-\delta+1\}$ such that $\phi(x) \subset x$ for all x. Clearly $\dim(x \cap w) \ge k - \delta + 1$. In the case of equality, we set $\phi(x) = x \cap w$. If $\dim(x \cap w) > k - \delta + 1$, $\phi(x)$ is chosen arbitrarily among the $(k-\delta+1)$-dimensional subspaces of $x \cap w$.

We define the function

$$F(x,y) = \begin{cases} 1 \text{ if } \phi(x) = \phi(y) \\ 0 \text{ otherwise} \end{cases}$$

$$= \sum_{\substack{u \subset w \\ \dim(u)=k-\delta+1}} \mathbf{1}(\phi(x) = u)\mathbf{1}(\phi(y) = u)$$

where $\mathbf{1}(\phi(x) = u)$ denotes the characteristic function of the set $\{x \in \mathcal{G}_q(n,k) : \phi(x) = u\}$. Then, F is obviously positive semidefinite, and (F, t, λ) is a feasible solution of (11) where $t = 1$ and

$$\lambda = \begin{bmatrix} n \\ k \end{bmatrix}_q^{-2} \sum_{(x,y)\in\mathcal{G}_q(n,k)^2} F(x,y)$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_q^{-2} \sum_{\substack{u \subset w \\ \dim(u)=k-\delta+1}} \Big(\sum_{x \in \mathcal{G}_q(n,k)} \mathbf{1}(\phi(x) = u) \Big)^2.$$

It follows from Cauchy-Schwarz inequality that $\lambda \geq \begin{bmatrix} n-\delta+1 \\ k-\delta+1 \end{bmatrix}_q$ so the Singleton bound (2) is recovered from (11).

3.3.2. *Sphere-packing and anticode bounds.* The sphere-packing bound and the anticode bound in $\mathcal{G}_q(n,k)$ can also be obtained directly, with

$$F(x,y) = \sum_{\dim(z)=k} \mathbf{1}_{B(z,\delta-1)}(x)\mathbf{1}_{B(z,\delta-1)}(y),$$

and

$$F(x,y) = \sum_{\dim(z)=k-\delta+1} \mathbf{1}(z \subset x)\mathbf{1}(z \subset y) \ .$$

In general, the anticode bound $|\mathcal{C}| \leq |X|/|\mathcal{A}|$ can be derived from (11), using the function $F(x,y) = \sum_{g\in G} \mathbf{1}_{\mathcal{A}}(gx)\mathbf{1}_{\mathcal{A}}(gy)$.

3.3.3. *First Johnson-type bound.* We want to apply Delsarte's linear programming bound of Theorem 3.4 with a function F of degree 1, i.e. $F(u) = f_0Q_0(u) + f_1Q_1(u)$. According to [7] the first q-Hahn polynomials are

$$Q_0(u) = 1 \quad , \quad Q_1(u) = \left(1 - \frac{(q^n-1)(1-q^{-u})}{(q^k-1)(q^{n-k}-1)}\right).$$

In order to construct a feasible solution of the linear program, we need $f_0, f_1 \geq 0$ for which $F(u) = f_0 + f_1Q_1(u)$ is non-positive for $u = \delta, \ldots, k$. Then $1 + f_1/f_0$ will be an upper bound for $A_q(n,k,2\delta)$. As $Q_1(u)$ is decreasing, the optimal choice of (f_0, f_1) satisfies $F(\delta) = 0$. So $f_1/f_0 = -1/Q_1(\delta)$ and we need $Q_1(\delta) < 0$. We obtain (4):

$$A_q(n,k,2\delta) \leq 1 + \frac{f_1}{f_0} = 1 - \frac{1}{Q_1(\delta)} = \frac{(q^n-1)(q^k-q^{k-\delta})}{(q^k-1)^2-(q^n-1)(q^{k-\delta}-1)}.$$

3.3.4. *Second Johnson-type bound.* Here we find an inequality for the optimal value $B_q(n,k,2\delta)$ of the semidefinite program (11) in the case $X = \mathcal{G}_q(n,k)$ (with the subspace distance) which resembles (5):

$$B_q(n,k,2\delta) \leq \frac{q^n-1}{q^k-1} B_q(n-1,k-1,2\delta).$$

Let (F,t,λ) be an optimal solution for the program (11) in $\mathcal{G}_q(n-1,k-1)$ relative to the minimal distance 2δ, i.e. F satisfies the conditions: $F \succeq \lambda$, $F(x,x) \leq t$, $F(x,y) \leq 0$ if $d(x,y) \geq 2\delta$, and $t/\lambda = B_q(n-1,k-1,2\delta)$. We consider the function G on $\mathcal{G}_q(n,k) \times \mathcal{G}_q(n,k)$ given by

$$G(x,y) = \sum_{\dim(D)=1} \mathbf{1}(D \subset x)\mathbf{1}(D \subset y) F(x \cap H_D, y \cap H_D),$$

where, for every one-dimensional space D, H_D is an arbitrary hyperplane such that $D \oplus H_D = \mathbb{F}_q^n$. It can be verified that the triple (G,t',λ') is a feasible solution of the program (11) in $\mathcal{G}_q(n,k)$ for the minimal distance 2δ, where $t' = t\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ and $\lambda' = \lambda \begin{bmatrix} k \\ 1 \end{bmatrix}_q^2 / \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, thus leading to the upper bound

$$B_q(n,k,2\delta) \leq \frac{t'}{\lambda'} = \frac{t}{\lambda}\frac{q^n-1}{q^k-1} = \frac{q^n-1}{q^k-1} B_q(n-1,k-1,2\delta).$$

Remark 3.5. *In* [10] *another Johnson-type bound is given:*

$$A_q(n,k,2\delta) \leq \frac{q^n-1}{q^{n-k}-1} A_q(n-1,k,2\delta),$$

which follows easily from the second Johnson-type bound combined with the equality $A_q(n,k,2\delta) = A_q(n,n-k,2\delta)$. Similarly to above, an analogous inequality holds for the semidefinite programming bound $B_q(n,k,2\delta)$.

4. Semidefinite programming bounds for projective codes

In this section we perform a symmetry reduction of the semidefinite programs (10) and (11) in the case of projective space, under the action of the group $G = GL_n(\mathbb{F}_q)$. We follow the general method described in [3]. The key point is that these semidefinite programs are left invariant under the action of G so the set of feasible solutions can be restricted to G-invariant functions F. The main work is to compute an explicit description of the G-invariant positive semidefinite functions on the projective space.

4.1. **G-invariant positive semidefinite functions on projective spaces.** In order to compute these functions, we use the decomposition of the space of real-valued functions under the action of G. We take the following notations:

$$X = \mathcal{P}(\mathbb{F}_q^n), \quad X_k = \mathcal{G}_q(n,k), \quad \mathbb{R}^X = \{f : X \to \mathbb{R}\}.$$

The space $\mathbb{R}^X$ is endowed with the inner product $(,)$ defined by:

$$(f,g) = \frac{1}{|X|}\sum_{x \in X} f(x)g(x).$$

For $k = 0, \dots, n$, an element of $\mathbb{R}^{X_k} := \{f : X_k \to \mathbb{R}\}$ is identified with the element of $\mathbb{R}^X$ that takes the same value on X_k and the value 0 outside of X_k. In this way, we see the spaces $\mathbb{R}^{X_k}$ as pairwise orthogonal subspaces of $\mathbb{R}^X$.

Delsarte [7] showed that the irreducible decomposition of the $\mathbb{R}^{X_k}$ under the action of G is given by the *harmonic subspaces* $H_{k,i}$:

$$\mathbb{R}^{X_k} = H_{0,k} \oplus H_{1,k} \oplus \cdots \oplus H_{\min\{k,n-k\},k} \tag{12}$$

Here, $H_{k,k}$ is the kernel of the *differentiation operator*

$$\begin{array}{cccl} \delta_k : & \mathbb{R}^{X_k} & \longrightarrow & \mathbb{R}^{X_{k-1}} \\ & f & \longrightarrow & [\ x \to \sum\{f(y) : \dim(y) = k, x \subset y\}\ ] \end{array}$$

and $H_{k,i}$ is the image of $H_{k,k}$ under the *valuation operator*

$$\begin{array}{cccl} \psi_{ki} : & \mathbb{R}^{X_k} & \longrightarrow & \mathbb{R}^{X_i} \\ & f & \longrightarrow & [\ x \to \sum\{f(y) : \dim(y) = k, y \subset x\}\ ] \end{array}$$

for $k \le i \le n-k$. Because δ_k is surjective, we have $h_k := \dim(H_{k,k}) = \begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$. Moreover, ψ_{ki} commutes with the action of G, so $H_{k,i}$ is isomorphic to $H_{k,k}$. Putting together the spaces $\mathbb{R}^{X_k}$ one gets the global picture:

$$\begin{array}{ccccccccccccccc}
\mathbb{R}^X & = & \mathbb{R}^{X_0} & \oplus & \mathbb{R}^{X_1} & \oplus & \cdots & \oplus & \mathbb{R}^{X_{\lfloor \frac{n}{2} \rfloor}} & \oplus & \cdots & \oplus & \mathbb{R}^{X_{n-1}} & \oplus & \mathbb{R}^{X_n} \\
 & & & & & & & & & & & & & & \\
\mathcal{I}_0 & = & H_{0,0} & \oplus & H_{0,1} & \oplus & \cdots & \oplus & H_{0,\lfloor \frac{n}{2} \rfloor} & \oplus & \cdots & \oplus & H_{0,(n-1)} & \oplus & H_{0,n} \\
\mathcal{I}_1 & = & & & H_{1,1} & \oplus & \cdots & \oplus & H_{1,\lfloor \frac{n}{2} \rfloor} & \oplus & \cdots & \oplus & H_{1,(n-1)} & & \\
\mathcal{I}_2 & = & & & & & \cdots & \oplus & H_{2,\lfloor \frac{n}{2} \rfloor} & \oplus & \cdots & & & & \\
\vdots & = & & & & & \ddots & & & & & & & & \\
\vdots & = & & & & & & \ddots & & & & & & & \\
\mathcal{I}_{\lfloor \frac{n}{2} \rfloor} & = & & & & & & & H_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor} & \oplus & \cdots & & & &
\end{array}$$

Here, the columns give the irreducible decomposition (12) of the spaces $\mathbb{R}^{X_k}$. The irreducible components which lie in the same row are all isomorphic, and together they form the *isotypic components*

$$\mathcal{I}_m := H_{m,m} \oplus H_{m,m+1} \oplus \cdots \oplus H_{m,n-m} \simeq H_{m,m}^{n-2m+1}.$$

Starting from this decomposition, one builds the *zonal matrices* $E_k(x, y)$ [3, Section 3.3] in the following way. We take an isotypic component $\mathcal{I}_k$ and we fix an orthonormal basis $(e_{kk1}, \dots, e_{kkh_k})$ of $H_{k,k}$. Let $e_{ksi} := \psi_{ks}(e_{kki})$. It follows from [7, Theorem 3] that $(e_{ks1}, \dots, e_{ksh_k})$ is an orthogonal basis of $H_{k,s}$ and that

$$(e_{ksi}, e_{ksi}) = \begin{bmatrix} n-2k \\ s-k \end{bmatrix}_q q^{k(s-k)}. \tag{13}$$

Then we define $E_k(x,y) \in \mathbb{R}^{(n-2k+1)\times(n-2k+1)}$ entrywise by

$$E_{kst}(x,y) = \sum_{i=1}^{h_k} e_{ksi}(x)e_{kti}(y). \tag{14}$$

We note that [3, Theorem 3.3] requires orthonormal basis in every subspace, while in the definition (14) of E_{kst} we do not normalize the vectors e_{ksi}. Because the norms (13) do not depend on i, but only on k, s, the matrix $(E'_k(x,y))_{s,t}$ associated to the normalized basis is obtained from $(E_k(x,y))_{s,t}$ by left and right multiplication by a diagonal matrix. So the characterization of the G-invariant positive semidefinite functions given in [3, Theorem 3.3] holds aswell with (14):

Theorem 4.1. *$F \in \mathbb{R}^{X\times X}$ is positive semidefinite and G-invariant if and only if it can be written as*

$$F(x,y) = \sum_{k=0}^{\lfloor n/2\rfloor} \langle F_k, E_k(x,y)\rangle \tag{15}$$

where $F_k \in \mathbb{R}^{(n-2k+1)\times(n-2k+1)}$ and $F_0, \ldots, F_{\lfloor n/2\rfloor}$ are positive semidefinite.

Now we compute the E_k's explicitly. They are zonal matrices: in other words, for all $k \leq s, t \leq n-k$, for all $g \in G, E_{kst}(x,y) = E_{kst}(gx, gy)$. This means that E_{kst} is a function of the variables which parametrize the orbits of G on $X \times X$. It is easy to see that the orbit of the pair (x,y) is characterized by the triple $(\dim(x), \dim(y), \dim(x\cap y))$.

The next theorem gives an explicit expression of E_{kst}, in terms of the polynomials Q_k of Definition 3.3.

Theorem 4.2. *If $k \leq s \leq t \leq n-k$, $\dim(x) = s$, $\dim(y) = t$,*

$$E_{kst}(x,y) = |X| h_k \frac{\begin{bmatrix} t-k \\ s-k \end{bmatrix}_q \begin{bmatrix} n-2k \\ t-k \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} q^{k(t-k)} Q_k(n,s,t; s-\dim(x\cap y))$$

If $\dim(x) \neq s$ or $\dim(y) \neq t$, $E_{kst}(x,y) = 0$.

We note that the weights involved in the orthogonality relations of the polynomials Q_k have a combinatorial meaning:

Lemma 4.3 ([8])**.** *Given $x \in X_s$, the number of elements $y \in X_t$ such that $\dim(x\cap y) = s - i$ is equal to $w(n,s,t;i)$.*

Proof of Theorem 4.2. By construction, $E_{kst}(x,y) \neq 0$ only if $\dim(x) = s$ and $\dim(y) = t$, so in this case E_{kst} is a function of $(s - \dim(x\cap y))$. Accordingly, for $k \leq s \leq t \leq n-k$, we introduce P_{kst} such that $E_{kst}(x,y) = P_{kst}(s - \dim(x\cap y))$. Now we want to relate P_{kst} to the q-Hahn polynomials. We start with two lemmas: one obtains the orthogonality relations satisfied by P_{kst} and the other computes $P_{kst}(0)$.

Lemma 4.4. *With the above notations,*

$$P_{kst}(0) = |X| h_k \frac{\begin{bmatrix} t-k \\ s-k \end{bmatrix}_q \begin{bmatrix} n-2k \\ t-k \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} q^{k(t-k)}. \tag{16}$$

Proof. We have $P_{kst}(0) = E_{kst}(x, y)$ for all x, y with $\dim(x) = s$, $\dim(y) = t$, $x \subset y$. Hence,

$$\begin{aligned} P_{kst}(0) &= \frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{\substack{\dim(x)=s \\ \dim(y)=t \\ x \subset y}} E_{kst}(x, y) \\ &= \frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{\substack{\dim(x)=s \\ \dim(y)=t \\ x \subset y}} \sum_{i=1}^{h_k} e_{ksi}(x) e_{kti}(y) \end{aligned}$$

$$\begin{aligned} P_{kst}(0) &= \frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{i=1}^{h_k} \sum_{\dim(y)=t} \Big(\sum_{\substack{\dim(x)=s \\ x \subset y}} e_{ksi}(x) \Big) e_{kti}(y) \\ &= \frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{i=1}^{h_k} \sum_{\dim(y)=t} \psi_{s,t}(e_{ksi})(y) e_{kti}(y). \end{aligned}$$

With the relation $\psi_{st} \circ \psi_{ks} = \begin{bmatrix} t-k \\ s-k \end{bmatrix}_q \psi_{kt}$,

$$\psi_{st}(e_{ksi}) = \psi_{st} \circ \psi_{ks}(e_{kki}) = \begin{bmatrix} t-k \\ s-k \end{bmatrix}_q \psi_{kt}(e_{kki}) = \begin{bmatrix} t-k \\ s-k \end{bmatrix}_q e_{kti},$$

and we obtain

$$\begin{aligned} P_{kst}(0) &= \frac{1}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{i=1}^{h_k} \sum_{\dim(y)=t} \begin{bmatrix} t-k \\ s-k \end{bmatrix}_q e_{kti}(y) e_{kti}(y) \\ &= \frac{\begin{bmatrix} t-k \\ s-k \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} \sum_{i=1}^{h_k} |X| (e_{kti}, e_{kti}) = |X| h_k \frac{\begin{bmatrix} t-k \\ s-k \end{bmatrix}_q \begin{bmatrix} n-2k \\ t-k \end{bmatrix}_q}{\begin{bmatrix} n \\ t \end{bmatrix}_q \begin{bmatrix} t \\ s \end{bmatrix}_q} q^{k(t-k)}. \end{aligned}$$

□

Lemma 4.5. *With the above notation,*

$$\sum_{i=0}^{s} w(n, s, t; i) P_{kst}(i) P_{\ell st}(i) = \delta_{k,\ell} |X|^2 h_k \frac{\begin{bmatrix} n-2k \\ s-k \end{bmatrix}_q \begin{bmatrix} n-2k \\ t-k \end{bmatrix}_q q^{k(s+t-2k)}}{\begin{bmatrix} n \\ s \end{bmatrix}_q}. \tag{17}$$

Proof. We compute $\Sigma := \sum_{y\in X} E_{kst}(x,y)E_{\ell,s',t'}(y,z)$.

$$\begin{aligned}
\Sigma &= \sum_{y\in X}\sum_{i=1}^{h_k}\sum_{j=1}^{h_\ell} e_{ksi}(x)e_{kti}(y)e_{\ell s'j}(y)e_{\ell t'j}(z)\\
&= \sum_{i=1}^{h_k}\sum_{j=1}^{h_\ell} e_{ksi}(x)e_{\ell t'j}(z)\Big(\sum_{y\in X} e_{kti}(y)e_{\ell s'j}(y)\Big)\\
&= \sum_{i=1}^{h_k}\sum_{j=1}^{h_\ell} e_{ksi}(x)e_{\ell t'j}(z)|X|(e_{kti}, e_{\ell s'j})\\
&= \sum_{i=1}^{h_k}\sum_{j=1}^{h_\ell} e_{ksi}(x)e_{\ell t'j}(z)|X|\delta_{k\ell}\delta_{ts'}\delta_{ij}\begin{bmatrix} n-2k\\ t-k\end{bmatrix}_q q^{k(t-k)}\\
&= \delta_{k\ell}\delta_{ts'}|X|\begin{bmatrix} n-2k\\ t-k\end{bmatrix}_q q^{k(t-k)}\sum_{i=1}^{h_k} e_{ksi}(x)e_{kt'i}(z)\\
&= \delta_{k\ell}\delta_{ts'}|X|\begin{bmatrix} n-2k\\ t-k\end{bmatrix}_q q^{k(t-k)}E_{kst'}(x,z).
\end{aligned}$$

We obtain, with $t = s'$, $t' = s$, $x = z \in X_s$, and taking $E_{\ell ts}(y,x) = E_{\ell st}(x,y)$ into account,

$$\sum_{y\in X_t} E_{kst}(x,y)E_{\ell st}(x,y) = \delta_{k\ell}|X|\begin{bmatrix} n-2k\\ t-k\end{bmatrix}_q q^{k(t-k)}E_{kss}(x,x).$$

The above identity becomes in terms of P_{kst}

$$\sum_{y\in X_t} P_{kst}(s-\dim(x\cap y))P_{\ell st}(s-\dim(x\cap y)) = \delta_{k\ell}|X|\begin{bmatrix} n-2k\\ t-k\end{bmatrix}_q q^{k(t-k)}P_{kss}(0).$$

Now we obtain (17) by (16) and Lemma 4.3. □

We showed that the functions P_{kst} satisfy the same orthogonality relations as the q-Hahn polynomials. So we are done if P_{kst} is a polynomial of degree at most k in the variable $[u] = [\dim(x\cap y)]$. This property is proved in the case $s = t$ in [7, Theorem 5] and extends to $s \le t$ with a similar line of reasoning. The multiplicative factor between $P_{kst}(u)$ and $Q_k(n,s,t;u)$ is then given by $P_{kst}(0)$ and the proof of Theorem 4.2 is completed. □

4.2. **Symmetry reduction of the semidefinite program** (10) **for projective codes.** Clearly, (10) is G-invariant: this means that for every feasible solution F and for every $g \in G$, also gF is feasible with the same objective value. Hence, we can average every feasible solution over G. In particular, the optimal value of (10) is attained by a function F which is G-invariant and so we can restrict the optimization variable in (10) to be a G-invariant function.

A function $F(x,y) \in \mathbb{R}^{X\times X}$ is G-invariant if it depends only on $\dim(x)$, $\dim(y)$, and $\dim(x\cap y)$. So we introduce $\tilde{F}$, such that $F(x,y)=\tilde{F}(s,t,i)$ for $x,y\in X$ with $\dim(x)=s, \dim(y)=t, \dim(x\cap y)=i$. Let

$$N_{sti} := |\{(x,y)\in X\times X : \dim(x)=s, \dim(y)=t, \dim(x\cap y)=i\}|$$

and

$$\begin{aligned}(18)\qquad \Omega(d) := \{(s,t,i) : 0\le s,t\le n,\ 0\le i\le \min(s,t),\ s+t\le n+i,\\ \text{either } s=t=i \text{ or } s+t-2i\ge d\}.\end{aligned}$$

Then, (10) becomes:

$$\begin{aligned}A_q(n,d) \le \max\Big\{ \sum_{s,t,i} N_{sti}\tilde{F}(s,t,i) \quad : \quad & \tilde{F}\in\mathbb{R}^{[n]^3},\ \tilde{F}\succeq 0,\ \tilde{F}\ge 0,\\ & \sum_{s=0}^{n} N_{sss}\tilde{F}(s,s,s)=1,\\ & \tilde{F}(s,t,i)=0 \text{ if } (s,t,i)\notin\Omega(d) \Big\},\end{aligned}$$

where $\tilde{F}\succeq 0$ means that the corresponding F is positive semidefinite.

Then, we introduce the variables $x_{sti} := N_{sti}\tilde{F}(s,t,i)$. It is straightforward to rewrite the program in terms of these variables, except for the condition $\tilde{F}\succeq 0$. From Theorem 4.1, this is equivalent to the semidefinite conditions $F_k\succeq 0$, where the matrices F_k are given by the scalar product of F and E_k:

$$\begin{aligned}(F_k)_{st} &= \frac{1}{|X|^2 h_k \begin{bmatrix} n-2k \\ s-k\end{bmatrix}_q q^{k(s-k)} \begin{bmatrix} n-2k \\ t-k\end{bmatrix}_q q^{k(t-k)}} \sum_{(x,y)\in X^2} F(x,y)E_{kst}(x,y)\\ &= \frac{1}{|X|^2 h_k \begin{bmatrix} n-2k \\ s-k\end{bmatrix}_q q^{k(s-k)} \begin{bmatrix} n-2k \\ t-k\end{bmatrix}_q q^{k(t-k)}} \sum_{u,v,i} x_{uvi}\tilde{E}_{kst}(u,v,i)\end{aligned}$$

We can substitute the value of $\tilde{E}_{kst}(u,v,i)$ using Theorem 4.2; in particular it vanishes when $(u,v)\neq(s,t)$, and, when $(u,v)=(s,t)$ and $s\le t$:

$$(19)\qquad (F_k)_{st} = \frac{1}{|X|}\sum_i x_{sti} \frac{\begin{bmatrix} t-k \\ s-k\end{bmatrix}_q}{\begin{bmatrix} n \\ t\end{bmatrix}_q \begin{bmatrix} t \\ s\end{bmatrix}_q \begin{bmatrix} n-2k \\ s-k\end{bmatrix}_q} q^{-k(s-k)} Q_k(n,s,t;s-i).$$

Theorem 4.6.

$$\begin{aligned}A_q(n,d) \le \max\Big\{ \sum_{(s,t,i)\in\Omega(d)} x_{sti} \quad : \quad & (x_{sti})_{(s,t,i)\in\Omega(d)},\ x_{sti}\ge 0,\\ & \sum_{s=0}^{n} x_{sss}=1,\\ & F_k\succeq 0 \ \ \textit{for all } k=0,\dots,\lfloor n/2\rfloor \Big\},\end{aligned}$$

where $\Omega(d)$ *is defined in* (18) *and the matrices* F_k *are given in (19).*

Remark 4.7. *A projective code $\mathcal{C}$ with minimal distance d provides a feasible solution of the above program, given by:*

$$x_{sti} = \frac{1}{|\mathcal{C}|}|\{(x,y) \in \mathcal{C} : \dim(x) = s, \dim(y) = t, \dim(x \cap y) = i\}.$$

In particular, we have

$$\sum_{t,i} x_{sti} = |\mathcal{C} \cap \mathcal{G}_q(n,s)|,$$

so, we can add the valid inequality

$$\sum_{t,i} x_{sti} \leq A_q(n, s, 2\lceil d/2 \rceil)$$

to the semidefinite program of Theorem 4.6 in order to tighten it.

Following the same line of reasoning, we could also add the linear inequalities

$$\sum_{s=0}^{n} c(s,k,e)\big(\sum_{t,i} x_{sti}\big) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q, \; k = 0, \dots, n$$

where $e = \lfloor (d-1)/2 \rfloor$, to the semidefinite program of Theorem 4.6, so that the resulting semidefinite program contains all the constraints of the linear program of Theorem 2.3. It turns out that this semidefinite program behaves numerically badly, and that, when it can be computed, its optimal value is equal to the minimum of the optimal values of the initial semidefinite program and of the linear program.

5. Numerical results

In this section we report the numerical results obtained for the binary case $q = 2$. Table 1 contains upper bounds for $A_2(n,d)$ for the subspace distance d_S while Table 2 contains upper bounds for $A_2^{inj}(n,d)$ for the injection distance d_i recently introduced in [17].

5.1. **Subspace distance.** The first column of Table 1 displays the upper bound obtained from Etzion-Vardy's linear program, Theorem 2.3. Observing that the variables x_k in this program represent integers, its optimal value as an integer program gives an upper bound for $A_q(n, 2e+1)$ that may improve on the optimal value of the linear program in real variables. However, we observed a difference with the optimal value of the linear program in real variables of at most 1. In Table 1, we display the bound obtained with the optimal value of the linear program in real variables, and indicate with a superscript $*$ the cases when the integer program gives a better bound (of one less).

The second column contains the upper bound from the semidefinite program of Theorem 4.6, strengthened by the inequalities (see Remark 4.7):

$$\sum_{t,i} x_{sti} \leq A_2(n, s, 2\lceil d/2 \rceil) \;\text{ for all } s = 0, \dots, n.$$

In both programs, $A_2(n,k,2\delta)$ was replaced by the upper bound from Theorem 2.2.

parameter	**E-V LP**	**SDP**
$A_2(4,3)$	6	6
$A_2(5,3)$	20	20
$A_2(6,3)$	*124	124
$A_2(7,3)$	832	776
$A_2(7,5)$	36	35
$A_2(8,3)$	9365	9268
$A_2(8,5)$	361	360
$A_2(9,3)$	*114387	107419
$A_2(9,5)$	*2531	2485
$A_2(10,3)$	*2543747	2532929
$A_2(10,5)$	*49451	49394
$A_2(10,7)$	*1224	1223
$A_2(11,5)$	693240	660285
$A_2(11,7)$	9120	8990
$A_2(12,7)$	323475	323374
$A_2(12,9)$	*4488	4487
$A_2(13,7)$	4781932	4691980
$A_2(13,9)$	*34591	34306
$A_2(14,9)$	2334298	2334086
$A_2(14,11)$	*17160	17159
$A_2(15,11)$	*134687	134095
$A_2(16,13)$	*67080	67079

TABLE 1. Bounds for the subspace distance

5.2. **Additional inequalities.** Etzion and Vardy [10] found additional valid inequalities for their linear program in the special case of $n = 5$ and $d = 3$. With this, they could improve their bound to the exact value $A_2(5,3) = 18$. In this section we establish analogous inequalities for other parameters (n,d).

Theorem 5.1. *Let $\mathcal{C} \subset \mathcal{P}(\mathbb{F}_q^n)$, of minimal subspace distance d, and let $D_k := |\mathcal{C} \cap \mathcal{G}_q(n,k)|$. Then, if*

$$d + 2\lceil d/2\rceil + 2 < 2n < 2d + 2\lceil d/2\rceil + 2,$$

we have:

- $D_{2n-d-\lceil d/2\rceil-1} \leq 1$;
- *if* $D_{2n-d-\lceil d/2\rceil-1} = 1$ *then*

$$D_{\lceil d/2\rceil} \leq \frac{q^n - q^{2n-d-\lceil d/2\rceil-1}}{q^{\lceil d/2\rceil} - q^{n-d-1}}.$$

Proof. It is clear that $D_i \leq 1$ for $0 \leq i < \lceil d/2 \rceil$. Moreover, for all $x, y \in \mathcal{C} \cap \mathcal{G}(n, \lceil d/2 \rceil)$, $x \neq y$, $\dim(x \cap y) = 0$. We want to show that $D_{2n-d-\lceil d/2 \rceil -1} \leq 1$. Indeed assume by contradiction $x \neq y \in \mathcal{C} \cap \mathcal{G}(n, 2n-d-\lceil d/2 \rceil - 1)$, we have

$$\begin{cases} 4n - 2d - 2\lceil d/2 \rceil - 2 \leq n + \dim(x \cap y) \\ d \leq 4n - 2d - 2\lceil d/2 \rceil - 2 - 2\dim(x \cap y) \end{cases}$$

leading to

$$\begin{cases} 2\dim(x \cap y) \geq 6n - 4d - 4\lceil d/2 \rceil - 4 & (*) \\ 2\dim(x \cap y) \leq 4n - 3d - 2\lceil d/2 \rceil - 2 & (**) \end{cases}$$

To obtain a contradiction, we must have $(*) > (**)$ which is equivalent to the hypothesis $2n > d + 2\lceil d/2 \rceil + 2$.

With a similar reasoning, we prove that, for all $x \in \mathcal{C} \cap \mathcal{G}(n, \lceil d/2 \rceil)$ and $w \in \mathcal{C} \cap \mathcal{G}(n, 2n-d-\lceil d/2 \rceil - 1)$, $\dim(x \cap w) = n - d - 1$. Indeed,

$$\begin{cases} 2n - d - 1 \leq n + \dim(x \cap w) \\ d \leq 2n - d - 1 - 2\dim(x \cap w) \end{cases}$$

so

$$\begin{cases} \dim(x \cap w) \geq n - d - 1 \\ \dim(x \cap w) \leq n - d - 1/2 \end{cases}$$

which yields the result.

Now we assume $D_{2n-d-\lceil d/2 \rceil -1} = 1$. Let $w \in \mathcal{C} \cap \mathcal{G}(n, 2n-d-\lceil d/2 \rceil - 1)$. Let $\mathcal{U}$ denote the union of the subspaces x belonging to $\mathcal{C} \cap \mathcal{G}(n, \lceil d/2 \rceil)$. We have $|\mathcal{U}| = 1 + D_{\lceil d/2 \rceil}(q^{\lceil d/2 \rceil} - 1)$ and $|\mathcal{U} \cap w| = 1 + D_{\lceil d/2 \rceil}(q^{n-d-1} - 1)$. On the other hand, $|\mathcal{U} \backslash (\mathcal{U} \cap w))| \leq |\mathbb{F}_q^n \backslash w|$, leading to

$$D_{\lceil d/2 \rceil}(q^{\lceil d/2 \rceil} - q^{n-d-1}) \leq q^n - q^{2n-d-\lceil d/2 \rceil -1} .$$

□

In several cases, adding these inequalities led to a lower optimal value, however we found that only in one case other than $(n, d) = (5, 3)$, the final result, after rounding down to an integer, is improved. It is the case $(n, d) = (7, 5)$, where $D_3 \leq 17$ and, by Theorem 5.1, if $D_5 = 1$ then $D_3 \leq 16$. So we can add $D_3 + D_5 \leq 17$ and $D_2 + D_4 \leq 17$, leading to: $A_2(7, 5) \leq 34$. This bound can be obtained with both the linear program of Theorem 2.3 and the semidefinite program of Theorem 4.6.

5.3. **Injection distance.** Recently, a new metric has been considered in the framework of projective codes, the *injection* metric, introduced in [17]. The *injection distance* between two subspaces $U, V \in \mathcal{P}(\mathbb{F}_q^n)$ is defined by

$$d_i(U, V) = \max\{\dim(U), \dim(V)\} - \dim(U \cap V).$$

When restricted to the Grassmann space, i.e. when U, V have the same dimension, the new distance coincides with the subspace distance (up to multiplication by 2). In general we have the relation (see [17])

$$d_i(U, V) = \frac{1}{2} d_S(U, V) + \frac{1}{2} |\dim(U) - \dim(V)|,$$

where d_S denotes the subspace distance.

It is straightforward to modify the programs in order to produce bounds for codes on this new metric space $(\mathcal{P}(\mathbb{F}_q^n), d_i)$. Let

$$A_q^{inj}(n,d) = \max\{|\mathcal{C}| \ : \ \mathcal{C} \subset \mathcal{P}(\mathbb{F}_q^n), \ d_i(\mathcal{C}) \geq d\}.$$

For constant dimension codes, we have $A_q^{inj}(n,k,d) = A_q(n,k,2d)$.

To modify the linear program of Etzion and Vardy for this new distance, we need to write down packing-constraints. The cardinality of balls in $\mathcal{P}(\mathbb{F}_q^n)$ for the injection distance can be found in [14]. Let $B^{inj}(V,e)$ be the ball with center V and radius e. If $\dim(V) = i$, we have

$$\begin{aligned} |B^{inj}(V,e)| &= \sum_{r=0}^{e} q^{r^2} \begin{bmatrix} i \\ r \end{bmatrix}_q \begin{bmatrix} n-i \\ r \end{bmatrix}_q \\ &+ \sum_{r=0}^{e} \sum_{\alpha=1}^{r} q^{r(r-\alpha)} \left(\begin{bmatrix} i \\ r \end{bmatrix}_q \begin{bmatrix} n-i \\ r-\alpha \end{bmatrix}_q + \begin{bmatrix} i \\ r-\alpha \end{bmatrix}_q \begin{bmatrix} n-i \\ r \end{bmatrix}_q \right). \end{aligned}$$

We define $c^{inj}(i,k,e) := |B^{inj}(V,e) \cap \mathcal{G}_q(n,k)|$ where $\dim(V) = i$. We set $\alpha := |i-k|$.

$$c^{inj}(i,k,e) = \begin{cases} \sum_{r=0}^{e} q^{r(r-\alpha)} \begin{bmatrix} i \\ r \end{bmatrix}_q \begin{bmatrix} n-i \\ r-\alpha \end{bmatrix}_q & \text{if } i \geq k \\ \sum_{r=0}^{e} q^{r(r-\alpha)} \begin{bmatrix} i \\ r-\alpha \end{bmatrix}_q \begin{bmatrix} n-i \\ r \end{bmatrix}_q & \text{if } i \leq k \end{cases}$$

Theorem 5.2 (Linear programming bound for codes in $\mathcal{P}(\mathbb{F}_q^n)$ with injection distance)**.**

$$A_q^{inj}(n,d) \leq \max\Big\{ \sum_{k=0}^{n} x_k \ : \ x_k \leq A_q^{inj}(n,k,d) \ \forall \ k = 0,\dots,n \\ \sum_{i=0}^{n} c^{inj}(i,k,e) x_i \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \ \forall \ k = 0,\dots,n \Big\}$$

For the semidefinite programming bound, we only need to change the definition of $\Omega(d)$ by

$$\text{(20)} \qquad \Omega^{inj}(d) := \{(s,t,i) : 0 \leq s,t \leq n, \ i \leq \min(s,t), \ s+t \leq n+i, \\ \text{either } s = t = i \text{ or } \max(s,t) - i \geq d\}.$$

Theorem 5.3.

$$A_q^{inj}(n,d) \leq \max\Big\{ \sum_{(s,t,i)\in\Omega^{inj}(d)} x_{sti} \ : \ (x_{sti})_{(s,t,i)\in\Omega^{inj}(d)}, \ x_{sti} \geq 0, \\ \sum_{s=0}^{n} x_{sss} = 1, \\ F_k \succeq 0 \ \text{ for all } k = 0,\dots,\lfloor n/2 \rfloor \Big\}$$

where $\Omega^{inj}(d)$ *is defined in* (20) *and the matrices* F_k *are given in (19).*

Table 2 displays the numerical computations we obtained from the two programs.

parameter	E-V LP	SDP
$A_2^{inj}(7,3)$	37	37
$A_2^{inj}(8,3)$	362	364
$A_2^{inj}(9,3)$	2533	2536
$A_2^{inj}(10,3)$	49586	49588
$A_2^{inj}(10,4)$	1229	1228
$A_2^{inj}(11,4)$	9124	9126
$A_2^{inj}(12,4)$	323778	323780
$A_2^{inj}(12,5)$	4492	4492
$A_2^{inj}(13,5)$	34596	34600
$A_2^{inj}(14,6)$	17167	17164
$A_2^{inj}(15,6)$	134694	134698
$A_2^{inj}(16,7)$	67087	67084

TABLE 2. Bounds for the injection distance

Remark 5.4. *We observe that the bound obtained for $A_2^{inj}(n, 2e+1)$ is most of the time slightly larger than the one obtained for $A_2(n, 4e+1)$. In* [14]*, the authors noticed that their constructions led to codes that are slightly better for the injection distance that for the subspace distance. So both experimental observations indicate that $A_2(n, 4e+1)$ is larger than $A_2^{inj}(n, 2e+1)$.*

The computational part of this research would not have been possible without the use of free software: We computed the values of the linear programs with Avis' lrs 4.2 available from http://cgm.cs.mcgill.ca/˜avis/C/lrs.html. The values of the semidefinite programs we computed with SDPA or SDPT3, available from the NEOS website http://www.neos-server.org/neos/.

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