Modular forms, lattices and spherical designs

Christine Bachoc, Boris Venkov

March 9, 2000

Abstract

With the help of modular forms, we compute some Jacobi forms associated to n-dimensional extremal lattices of prime level l and determinant $l^{n/2}$. We show that the layers of these lattices support spherical designs. Moreover, the method is used to classify the extremal lattices of level 2, dimension 20 and determinant 2^{10}, and to show the non existence of extremal lattices of level 7, dimension 18 and determinant 7^9.

1 Introduction

To an even lattice $L \subset \mathbb{R}^n$ one can associate a family of modular forms, the so-called theta series with spherical coefficients. They are defined by:

$$\theta_{L,P} := \sum_{x \in L} P(x) q^{x.x/2}$$

where P is a spherical polynomial, i.e. a homogeneous polynomial in n variables, such that $\Delta P = 0$, where $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. The case $P = 1$ is the usual theta series of the lattice.

These modular forms have proved to be a powerful tool for the study of the spherical designs supported by the layers of extremal unimodular lattices, as shown in [V2], [CS, Chap7]. Another application is the a priori enumeration by the second author of the possible root systems of the Niemeier lattices ([CS, Chap18]).

In this paper, we study these modular forms for l-modular lattices in the sense of H.-G. Quebbemann ([Que95]), for prime levels l. Such a lattice L should be considered together with its rescaled dual $L' := \sqrt{l} L^*$; we show that the series $\theta_{L,P} \pm \theta_{L',P}$ are modular forms for the Fricke group $\Gamma^*(l)$ and for certain characters, just like in the case $P = 1$ studied in [Que95]. We derive the existence of spherical designs supported by the layers of extremal lattices, or of the union of an extremal lattice and its rescaled dual; we obtain 5- and 7-designs in the case of level 2 and 3 lattices, for certain congruences of dimensions, and also in the special case of 16-dimensional extremal level 5 lattices (see Corollary 4.1, Proposition 7.2, and Table 3).

These properties can be used further to get some information on the Jacobi theta series associated to the lattice. We recall that one can associate to a lattice L and a vector $\alpha \in \mathbb{R}L$ the so-called Jacobi theta series:

$$J_{L,\alpha}(\tau, z) := \sum_{x \in L} q^{x.x/2} \zeta^{x.\alpha}$$

which counts, as a formal series, the number of vectors in the lattice with prescribed norm and prescribed scalar product with α. When $q = e^{2i\pi\tau}$ and $\zeta = e^{2i\pi z}$, it is a Jacobi form in the sense of [EZ]. There is an injective map (denoted $\mathcal{D}$ in [EZ, ChapI.3]) from the space of Jacobi forms of given weight and index to the sum of spaces of modular forms of increasing weights, which, when restricted to a Jacobi theta series of a lattice L, associates to it some theta series with spherical coefficients $\theta_{L,P}$, where $P = P_d^{(\alpha)}$ are certain spherical polynomials constructed from Gegenbauer polynomials. Hence one can use the fact that these fall into certain explicitly described spaces of modular forms to derive some linear constraints on the coefficients of the Jacobi theta series $J_{L,\alpha}$ only depending on the norm of α. This method avoids to work directly in the space of Jacobi forms, and is described in more details in section 5.

There has been in recent years a lot of work done in the aim of classifying certain genera of lattices, especially for the genera of modular lattices (see [SSP] for a survey), the main method being Kneser neighbouring, combined with the computation of mass formula. But, when the dimension increases, the number of classes in a given genus becomes too large to allow a full classification; for example it is a well-known fact that there are at least 80 million classes of even unimodular lattices in dimension 32. Hence, one would like to restrict to the "best" ones, i.e. to classify only the extremal ones (i.e. the ones meeting a certain bound derived from the properties of θ_L as a modular form; see [Que95], [SSP]). This is what we do in section 6, for the genus of lattices of dimension 20, level 2, determinant 2^{10}, with the help of the method previously described; the information we get on the coefficients of certain $J_{L,\alpha}$ allows us to find a "path" from any extremal lattice to some lattice built up with lower dimensional ones. In the same spirit, we show the non existence of extremal lattices in the genus of dimension 18, level 7, determinant 7^9 lattices (which is not a surprise since such a lattice would have minimum 8 and would have been denser that the known ones; the non existence of hermitian unimodular lattices over $\mathbb{Q}(\sqrt{-7})$, in dimension 18, which would have had the same parameters, was already proved in [Sc]).

2 Prerequisites

In this section we recall some classical properties of the theta series associated to lattices. Let L be an even lattice of an euclidean space of dimension n, the scalar product of which we denote by $x.y$, and let L^* be its dual lattice. We assume n to be even.

The level l of L is by definition the smallest integral number such that the lattice $\sqrt{l}L^*$ is also even. We set

$$L' := \sqrt{l}L^*$$

and have the following easy properties: $\mathrm{Det}(L)$ divides l^n and $\mathrm{Det}(L)\mathrm{Det}(L') = l^n$. It should be noted that the lattice L' is again of level l, and that the map $L \to L'$ is an involution. We shall be mostly interested in the case $\mathrm{Det}(L) = \mathrm{Det}(L') = l^{n/2}$.

We assume now that $L \subset \mathbb{R}^n$ endowed with its usual euclidean structure. Let P be a n-variable homogeneous polynomial of even degree d. Let P be harmonic, i.e. verifies $\Delta P = 0$, where Δ is the Laplace operator. Let

$$\theta_{L,P} := \sum_{x \in L} P(x) q^{x.x/2}.$$

When τ belongs to the upper half complex plane $\mathfrak{H}$ and $q = e^{2i\pi\tau}$, this theta series is well-known to be modular. More precisely, let

$$\Gamma_0(l) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{S}l_2(\mathbb{Z}) \mid c \equiv 0 \mod l \}$$

and let $\chi_{n/2}$ be the quadratic (or trivial) character defined by

$$\chi_{n/2}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \left(\frac{(-1)^{n/2}\mathrm{Det}(L)}{d} \right)$$

Then:

Theorem 2.1 ([Miy], [Eb]) *If P is a degree d harmonic polynomial, $\theta_{L,P}$ is a modular form of weight $k = n/2 + d$ for the group $\Gamma_0(l)$ and the character $\chi_{n/2}$. If $d > 0$, it is parabolic.*

Moreover, when l is square free, the normaliser of $\Gamma_0(l)$ in $\mathcal{S}l_2(\mathbb{R})$ contains an abelian, 2-elementary group of rank the number of prime divisors of l, the so-called group of Fricke involutions. Together with $\Gamma_0(l)$, it generates the Fricke group $\Gamma_*(l)$, and, when $\mathrm{Det}(L) = l^{n/2}$, one can study its action on $\theta_{L,P}$.

It is a remarkable fact that, for certain values of l, the algebra of modular forms for $\Gamma_*(l)$ (with certain characters) has a very simple structure, as shown by H.-G. Quebbemann in [Que95], where it is proved to be a polynomial algebra, when the sum of the divisors of l divides 24.

In the next sections we assume l to be prime, although the method and the results adapt easily to the general case.

3 Prime levels

Let l be a prime The Fricke involution is

$$t_l = \begin{pmatrix} 0 & \frac{1}{\sqrt{l}} \\ -\sqrt{l} & 0 \end{pmatrix}$$

and its action on theta series with spherical coefficients is given by the following proposition, where $k := n/2 + d$:

Proposition 3.1 *If L is a n-dimensional level l lattice and if P is a degree d spherical polynomial,*

$$\theta_{L,P} \mid_k t_l = i^{n/2} \frac{l^{n/4}}{Det(L)^{1/2}} \theta_{L',P}$$

Proof. Apply [Eb, Proposition 3.1] which follows from the Poisson summation formula. □

For all integers s, we define a character χ_s of $\Gamma_*(l)$ by:

$$\begin{cases} \chi_s(A) & = \left(\frac{(-l)^s}{d}\right) \text{ if } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(l) \\ \chi_s(t_l) & = i^s \end{cases}$$

Remark 3.1 *The character χ_s depends only on the value of s modulo 4, and satisfies $\chi_s \chi_t = \chi_{s+t}$. Its restriction to $\Gamma_0(l)$ only depends on s modulo 2.*

Table 1: k_0, k_1, k_2

l	2	3	5	7	11	23
k_0	2	1	2	1	1	1
k_1	8	6	4	3	2	1
k_2	12	9	8	6	5	4

Corollary 3.1 *If L is a level l lattice of determinant $l^{n/2}$, then*

$$\forall A \in \Gamma_*(l) \quad \theta_{L,P} \mid_k A = \chi_{n/2}(A)\theta_{L',P}$$

Finally:

Theorem 3.1 *Under the same hypotheses, $\theta_{L,P} + \theta_{L',P} \in \mathcal{S}_k(\Gamma_*(l), \chi_{n/2})$ if $d > 0$ (respectively $\in \mathcal{M}_k(\Gamma_*(l), \chi_{n/2})$ if $P = 1$), and $\theta_{L,P} - \theta_{L',P} \in \mathcal{S}_k(\Gamma_*(l), \chi_{n/2+2})$ for all P.*

Proof. Straightforward from Theorem 2.1 and Corollary 3.1. When l is a prime and $\mathrm{Det}(L) = l^{n/2}$, since the lattices L and L' are in the same genus, the difference of their theta series is parabolic. □

In [Que95], H.-G. Quebbemann describes the structure of the algebra $\oplus_{p \geq 0}\mathcal{M}_p(\Gamma_*(l), \chi_p)$ when $1 + l$ divides 24. Set

$$\mathcal{A}_l := \oplus_{p\geq 0}\mathcal{M}_p(\Gamma_*(l), \chi_p) = \mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}]$$

with the same notations as in [Que95]: $k_0 = 1$ or 2 according to $l = 3$ or 1 modulo 4 and $\theta_{2k_0} = \theta_{L_0}$, where L_0 is a level l, determinant l^{k_0} and dimension $2k_0$ lattice, and $\Delta_{2k_1} = (\eta(\tau)\eta(l\tau))^{k_1}$, where $k_1 = 24/(1 + l)$. Hence θ_{2k_0} has weight k_0 and Δ_{2k_1} has weight k_1. In Table 1 we summarize the values of k_0, k_1, k_2 depending on l.

From the preceding theorem, we need to know

$$\mathcal{A}_l^- := \oplus_{p\geq 0}\mathcal{S}_p(\Gamma_*(l), \chi_{p+2})$$

which is, from the properties of the characters χ_s (Remark 3.1), an $\mathcal{A}_l$-module. We show that it is a free module:

Proposition 3.2 *For all prime number l such that $1 + l$ divides 24, there exists $\Phi_{2k_2} \in \mathcal{A}_l^-$ of weight k_2 such that $\mathcal{A}_l^- = \Phi_{2k_2}\mathcal{A}_l$.*

Table 2: Φ_{2k_2}

l	Φ_{2k_2}	Φ_{2k_2} (normalized)
2	$\theta_{D_4^2,P_8} - \theta_{(D_4^2)',P_8}$	$q - 88q^2 + 252q^3 + 64q^4 + O(q^5)$
3	$\theta_{A_2^3,P_6} + \theta_{(A_2^3)',P_6}$	$q - 42q^2 + 171q^3 - 248q^4 + O(q^5)$
5	$\theta_{L_0,P_6} + \theta_{L_0',P_6}$	$q - 14q^2 - 48q^3 + 68q^4 + O(q^5)$
7	$\theta_{L_0^2,P_4} - \theta_{(L_0^2)',P_4}$	$q - 10q^2 - 14q^3 + 68q^4 + O(q^5)$
11	$\theta_{L_0 \perp L_4,P_2} + \theta_{(L_0 \perp L_4)',P_2}$	$q - 6q^2 - 3q^3 - 14q^4 + O(q^5)$
23	$\theta_{L_0^{(1)} \perp L_0^{(2)},P_2} + \theta_{(L_0^{(1)} \perp L_0^{(2)})',P_2}$	$q - 2q^2 - 5q^3 - 4q^4 + O(q^5)$

Proof. Let Φ_{2k_2} be defined in Table 2. The notations are as follows: the index of the polynomial denotes its degree; the lattice L_0 is the unique modular lattice of level l and dimension $2k_0$ for $l = 5, 7, 11$; for $l = 23$, $L_0^{(1)} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 12 \end{smallmatrix}\right)$ and $L_0^{(2)} = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 6 \end{smallmatrix}\right)$; for $l = 11$, L_4 is the unique up to isometry 4-dimensional lattice of minimum 4 and determinant 11^2. From Theorem 3.1, the form Φ_{2k_2} has weight k_2 and belongs to $\mathcal{A}_l^-$. Hence $\Phi_{2k_2}\mathcal{A}_l \subset \mathcal{A}_l^-$ and it is enough to show equality of the dimensions of the subspaces of equal weight. The following holds:

$$\mathcal{S}_p(\Gamma_0(l), \chi_p) = \mathcal{S}_p(\Gamma_*(l), \chi_p) \oplus \mathcal{S}_p(\Gamma_*(l), \chi_{p+2})$$

and the dimensions of the left hand side of the equality can be found in [Miy, Table A], while the dimensions of the first space of the right handside are in [Que95, Theorem 6]. $\square$

4 Extremal lattices and spherical designs

We apply the preceding results to the extremal lattices of level l, when l is a prime number such that $1 + l$ divides 24. Let L be an n-dimensional lattice of level l. We set

$$n/2 = k_1\mu + k_0\rho$$

where $\rho < k_1/k_0$ and assume

$$\mathrm{Det}(L) = l^{n/2} \quad \text{and} \quad \theta_L = \theta_{L'} \tag{1}$$

Then ([Que95]) θ_L belongs to $\mathcal{A}_l$ which implies that the minimum of L cannot be greater than $2(\mu+1)$. A lattice meeting this bound is said to be extremal.

Proposition 4.1 *Let L be an n-dimensional lattice of level l, where l is a prime number such that $1+l$ divides 24, satisfying (1), and extremal. Let P be a spherical polynomial of even degree $d>0$. Then*

$$\theta_{L,P}+\theta_{L',P}=0 \ \textit{if} \left\{ \begin{array}{ll} & d\equiv 0 \mod 4 \ \textit{and}\ d<k_1-k_0\rho \\ \textit{or} & d\equiv 2 \mod 4 \ \textit{and}\ d<k_2-k_0\rho \end{array}\right.$$

$$\theta_{L,P}-\theta_{L',P}=0 \ \textit{if} \left\{ \begin{array}{ll} & d\equiv 0 \mod 4 \ \textit{and}\ d<k_2-k_0\rho \\ \textit{or} & d\equiv 2 \mod 4 \ \textit{and}\ d<k_1-k_0\rho \end{array}\right.$$

Proof. If $d\equiv 0 \mod 4$, then $\theta_{L,P}+\theta_{L',P}$ belongs to $\mathcal{A}_l$ and $\theta_{L,P}-\theta_{L',P}$ belongs to $\mathcal{A}_l^-$. From the hypotheses, the Fourier development of $\theta_{L,P}\pm\theta_{L',P}$ starts with a power of q at least equal to $\mu+1$. Hence

$$\theta_{L,P}+\theta_{L',P}=\Delta_{2k_1}^{\mu+1}\Psi$$

where $\Psi=0$ or has weight $n/2+d-k_1(\mu+1)=d-k_1+k_0\rho$, but the last case is only possible if $d-k_1+k_0\rho\geq 0$. In the same way,

$$\theta_{L,P}-\theta_{L',P}=\Phi_{2k_2}\Delta_{2k_1}^{\mu}\Psi'$$

where $\Psi'=0$ or has weight $n/2+d-k_1\mu-k_2=d-k_2+k_0\rho$, if $d-k_2+k_0\rho\geq 0$.

Same proof in the case $d\equiv 2 \mod 4$. □

These results have an interpretation in terms of spherical designs. Recall that a subset X of the unit sphere of $\mathbb{R}^n$ is a spherical t-design if and only if $\sum_{x\in X}P(x)=0$ for all harmonic polynomials P of degree at most t ([DGS], [V1]). If $X=-X$, it is enough to consider the even degree polynomials. If L is a lattice, the coefficient of $q^{m/2}$ in $\theta_{L,P}$ for $\deg(P)\leq t$ is zero if and only if the set L_m of the norm m vectors is (if non empty) a t-design, while the coefficient of $q^{m/2}$ in $\theta_{L,P}+\theta_{L,P'}$ for $\deg(P)\leq t$ is zero if and only if the set $L_m\cup L'_m$ of the norm m vectors in $L\cup L'$ is (if non empty) a t-design. We get similar results as in the case of even unimodular lattices. This case is known from [V2], and is included here.

Corollary 4.1 *Let L be a level l lattice, for which (1) holds, extremal. For all $m > 0$ such that L_m is non empty,*

- $l = 1$
  - *if* $n \equiv 0 \mod 24$, L_m *is a spherical 11-design.*
  - *if* $n \equiv 8 \mod 24$, L_m *is a spherical 7-design.*
  - *if* $n \equiv 16 \mod 24$, L_m *is a spherical 3-design.*
- $l = 2$
  - *if* $n \equiv 0 \mod 16$, L_m *is a spherical 7-design.*
  - *if* $n \equiv 4 \mod 16$, L_m *is a spherical 5-design and* $L_m \cup L'_m$ *is a spherical 7-design.*
  - *if* $n \equiv 8 \mod 16$, L_m *is a spherical 3-design.*
  - *if* $n \equiv 12 \mod 16$, $L_m \cup L'_m$ *is a spherical 3-design.*
- $l = 3$
  - *if* $n \equiv 0, 2 \mod 12$, L_m *is a spherical 5-design, and* $L_m \cup L'_m$ *is a spherical 7-design.*
  - *if* $n \equiv 4, 6 \mod 12$, L_m *is a spherical 3-design.*
  - *if* $n \equiv 8, 10 \mod 12$, $L_m \cup L'_m$ *is a spherical 3-design.*
- $l = 5$
  - *if* $n \equiv 0 \mod 8$, L_m *is a spherical 3-design.*
  - *if* $n \equiv 4 \mod 8$, $L_m \cup L'_m$ *is a spherical 3-design.*
- $l = 7$
  - *if* $n \equiv 0 \mod 6$, L_m *is a spherical 3-design.*
  - *if* $n \equiv 2, 4 \mod 6$, $L_m \cup L'_m$ *is a spherical 3-design.*
- $l = 11$
  - *if* $n \equiv 0, 2 \mod 4$, $L_m \cup L'_m$ *is a spherical 3-design.*
- $l = 23$
  - *if* $n \equiv 0 \mod 2$, $L_m \cup L'_m$ *is a spherical 3-design.*

5 Further use of the $\theta_{L,P}$

In this section, L is still an even lattice of level l, determinant $l^{n/2}$ and $L' = \sqrt{l}L^*$. Let α be a fixed vector of L. Then $x.\alpha \in \mathbb{Z}$ for all $x \in L$ and $y.\alpha \in \sqrt{l}\mathbb{Z}$ for all $y \in L'$. We set, for $t, i \in \mathbb{Z}_{\geq 0}$,

$$n_{2t,i}(\alpha) := \mathrm{Card}\{x \in L \mid x^2 = 2t, x.\alpha = \pm i\}$$

and

$$n^*_{2t,i}(\alpha) := \mathrm{Card}\{y \in L' \mid y^2 = 2t, y.\alpha = \pm\sqrt{l}i\}.$$

From the results of the preceding sections, some linear relations between these parameters can be obtained, depending only on the norm a of α, in the following way:

To any $\alpha \in \mathbb{R}^n$ is associated a sequence of degree d spherical polynomials $(P_d^{(\alpha)})_{d\geq 1}$ which are constructed from the Gegenbauer polynomials $G_d^{(n/2-1)}$:

$$P_d^{(\alpha)}(x) := F_d((x.\alpha), ((x.x)(\alpha.\alpha))^{1/2})$$

where

$$F_d(t, 1) = G_d^{(n/2-1)}(t).$$

For more details, see [DGS], [V1]: one shows that a polynomial F_d of degree d, such that, for all $\alpha \in \mathbb{R}^n$ the polynomial $F_d((x.\alpha), ((x.x)(\alpha.\alpha))^{1/2})$ is harmonic in the variable x, is unique (up to a scalar) and coincides with the well-known orthogonal Gegenbauer polynomials with parameter $n/2-1$, i.e. satisfy the second equality.

Gegenbauer polynomials are as follows ([DGS], [Vi]):

$$G_m^{(p)}(t) = 2^m \binom{p+m-1}{m} \sum_{k=0}^{[m/2]} (-1)^k \frac{m(m-1)\ldots(m-2k+1)}{2^{2k}k!(p+m-1)(p+m-2)\ldots(p+m-k)} t^{m-2k}$$

and also verify the following iterative relations:

$$\begin{cases} G_1^{(p)}(t) & = 2pt \\ G_2^{(p)}(t) & = 2p(p+1)t^2 - p \\ G_m^{(p)}(t) & = (\frac{2(m+p-1)}{m})tG_{m-1}^{(p)}(t) - \frac{(m+2p-2)}{m}G_{m-2}^{(p)}(t) \end{cases}$$

We consider polynomials of even degree. For all t, the coefficient of q^t in $\theta_{L,P_d^{(\alpha)}} \pm \theta_{L',P_d^{(\alpha)}}$ is a linear form in the $\{n_{2t,i}(\alpha), n^*_{2t,j}(\alpha)\}_{i,j}$. More precisely, this coefficient is equal to:

$$\sum_{i\geq 0} F_d(i,\sqrt{2ta})n_{2t,i}(\alpha) \pm \sum_{i\geq 0} F_d(li,\sqrt{2ta})n^*_{2t,i}(\alpha).$$

Let L be an extremal lattice satisfying (1). Set $n/2 = k_1\mu + k_0\rho$, so that $2(\mu+1)$ is the minimum of L and L'. Let $\alpha \in L$ a vector of fixed norm a. Let s be a number of "layers" of vectors of L and L', our goal is to find linear relations between the

$$N_s(\alpha) := \{n_{2t,i}(\alpha), n^*_{2t,j}(\alpha)\}_{\substack{2(\mu+1)\leq 2t\leq 2(\mu+s)\\ 0\leq i\leq\sqrt{2ta}\\ 0\leq j\leq\sqrt{2ta/l}}}$$

only depending on the norm a of α.

We first look for some trivial relations of the type $n_{2t,i}(\alpha) = 0$ (respectively $n^*_{2t,i}(\alpha) = 0$), which arise from the value of the minimum of the lattices, or from the fact that the minimum of the class $\alpha + 2L$ is greater than the minimum of L and congruent to a modulo 4. Also $n_{a,a}(\alpha) = 2$ (resp $n^*_{la,a}(la) = 2$).

Next, the theta series of L and L' being given, for all t,

$$\sum_i n_{2t,i}(\alpha) = \mathrm{Card}(L_{2t})$$

and

$$\sum_j n^*_{2t,j}(\alpha) = \mathrm{Card}(L'_{2t})$$

Then, we proceed as in the proof of Proposition 3.1: if $2d \equiv 0 \mod 4$, and $2d > 0$, from Theorem 3.1, there exists $\Psi \in \mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}]$ such that

$$\theta_{L,P_d^{(\alpha)}} + \theta_{L',P_d^{(\alpha)}} = \Delta_{2k_1}^{\mu+1}\Psi$$

where Ψ has weight $2d - k_1 + k_0\rho$. The subspace of $\mathbb{C}[\theta_{2k_0}, \Delta_{2k_1}]$ of the forms of weight $2d - k_1 + k_0\rho$ has dimension $\delta(d) := 1 + [(2d - k_1 + k_0\rho)/k_1]$, and an explicit basis is known; as long as $\delta(d) \leq s$, one gets $s - \delta(d)$ linear conditions on the $N_s(\alpha)$. In the same way,

$$\theta_{L,P_d^{(\alpha)}} - \theta_{L',P_d^{(\alpha)}} = \Phi_{2k_2}\Delta_{2k_1}^{\mu}\Psi'$$

where Ψ' has weight $2d - k_2 + k_0\rho$. Hence $\theta_{L,P_d^{(\alpha)}} - \theta_{L',P_d^{(\alpha)}}$ belongs to a space of dimension $\delta'(d) := 1 + [(2d - k_2 + k_0\rho)/k_1]$ a basis of which is known, which leads to $s - \delta'(d)$ linear conditions on the $N_s(\alpha)$, as long as $\delta'(d) \leq s$. The case $2d \equiv 2 \mod 4$ works the same.

Finally we get, for all s, a system of linear equations $\mathcal{S}_{s,a}$ in the unknowns $N_s(\alpha)$. The number of these equations grows with s quicker than the number of the unknowns.

Example 1: 32-dimensional even unimodular lattices of minimum 4.

Let L be a 32-dimensional even unimodular lattice of minimum 4. It is enough to take $s = 3$; the systems $\mathcal{S}_{3,4}$, $\mathcal{S}_{3,6}$ have a unique solution, while the system $\mathcal{S}_{3,8}$ has a set of solutions depending on one parameter $x := n_{4,4}(\alpha)$. We give the first rows in the following Table:

	$n_{4,0}(\alpha)$	$n_{4,1}(\alpha)$	$n_{4,2}(\alpha)$	$n_{4,3}(\alpha)$	$n_{4,4}(\alpha)$
$\alpha^2 = 4$	80910	63488	2480	0	2
$\alpha^2 = 6$	66060	71280	9396	144	0
$\alpha^2 = 8$	$35x + 57040$	$-56x + 72480$	$28x + 16368$	$-8x + 992$	x
$\alpha^2 = 10$	51150	70928	22196	2544	62
$\alpha^2 = 12$	46530	69120	26352	4608	270

There is a nice interpretation of this parameter x in terms of the "Nachbareffekt" of the lattice ([KV]): when $\alpha^2 = 8$, the neighbour lattice $L^\alpha := L_\alpha + \mathbb{Z}\alpha/2$ contains pairwise orthogonal roots, and it is easy to see that the total number of roots in V^α is $2 + n_{4,4}(\alpha)/2$. Hence the number $1 + n_{4,4}(\alpha)/4$ is an integer between 1 and 32. See [KV] for a computation of the occurence of these values in the five lattices constructed from extremal type II codes (i.e. with Nachbareffekt 32). However, the mean value of this parameter is a constant and is equal to 2.22 from [KV] (equivalently, the second Siegel theta series of an extremal lattice is uniquely determined). We can go further to the systems $\mathcal{S}_{3,10}$, $\mathcal{S}_{3,12}$; with the additional hypotheses that α is a minimal vector in its class $\alpha + 2L$, (allways true when $\alpha^2 \leq 8$), which gives the additional equations: $n_{2t,i}(\alpha) = 0$ for all $i > t$, we find again a unique solution. When $\alpha^2 = 14$, we get a system $\mathcal{S}_{3,14}$ with an empty set of solutions, which calls for the fact that the classes $L/2L$ have representants with norm bounded by 12.

Proposition 5.1 *Let L be a 32-dimensional even unimodular lattice of minimum 4. All the classes of $L/2L$ contain vectors of norm at most* 12.

Proof. For all $\alpha \in L$, we denote by $(\alpha + 2L)_m$ the set of vectors of norm m in the class $\alpha + 2L$, and by $|(\alpha + 2L)_m|$ its cardinality. We also denote by N_i the number of classes of $L/2L$ of minimum i. If $\beta \in \alpha + 2L$, we have $\beta = \alpha + 2u$, $u \in L$, and $\beta^2 = \alpha^2 + 4(u^2 + u.\alpha)$; it should be noted that $\beta^2 \equiv \alpha^2 \mod 4$. Equivalently, the map $x \to x.x/2$ induces a (non degenerated) quadratic form over the $\mathbb{F}_2$-vector space $L/2L$. It is then easy to check the following assumptions:

- If $\alpha^2 = 4$, $|(\alpha + 2L)_4| = 2$, $|(\alpha + 2L)_8| = 0$, $|(\alpha + 2L)_{12}| = n_{4,2}(\alpha) = 1240$.
- If $\alpha^2 = 6$, $|(\alpha + 2L)_6| = 2$, $|(\alpha + 2L)_{10}| = n_{4,3}(\alpha) = 72$.
- If $\alpha^2 = 8$, $|(\alpha + 2L)_8| = 2 + n_{4,4}(\alpha)/2$, $|(\alpha + 2L)_{12}| = n_{4,3}(\alpha) = 992 - 8n_{4,4}(\alpha)$.
- If $\alpha^2 = 10$, and $\mathrm{Min}(\alpha + 2L) = 10$, $|(\alpha + 2L)_{10}| = 2 + n_{4,4}(\alpha) = 64$.
- If $\alpha^2 = 12$, and $\mathrm{Min}(\alpha + 2L) = 12$, $|(\alpha + 2L)_{12}| = 2 + n_{6,6}(\alpha)/2 + n_{4,4}(\alpha) = 1024$.

It is worth noticing that we are here simply computing the theta series of the classes $\alpha + 2L$. From these values, we have: $N_4 = |L_4|/2$, $N_6 = |L_6|/2$, and $N_{10} = (|L_{10}| - 36|L_6|)/64$. We find that $N_6 + N_{10} = 2^{15}(2^{16} - 1)$ which is exactly the number of non isotropic vectors in a 32-dimensional non singular quadratic $\mathbb{F}_2$ vector space. Hence the non isotropic classes are represented by the vectors of norm 6 and 10. We compute in the same way N_{12}: $N_{12} = (|L_{12}| - 620|L_4| - \sum_{\alpha+2L,\alpha^2=8}(992 - 8n_{4,4}(\alpha)))/1024$ and use the equality: $\sum_{\alpha+2L,\alpha^2=8}(2 + n_{4,4}(\alpha)/2) = |L_8|$ to get $N_{12} = 2147442975 - N_8$ which shows that $1 + N_4 + N_8 + N_{12} = 2^{15}(1 + 2^{16})$ which is the number of isotropic vectors.

$\square$

Example 2: 48-dimensional extremal even unimodular lattices.

Extremal lattices have minimum 6, and only three such lattices are known [CS]. The sets of solutions for the systems $\mathcal{S}_{3,m}$ behave just like in the case of dimension 32: there is a unique solution for $\alpha^2 = 6, 8, 10$, and, for $\alpha^2 = 12$, a set of solutions depending on the parameter $n_{4,6}(\alpha)$, or equivalently on the number of pairwise orthogonal norm 3 vectors in the neighbour L^α of α. The dimension of the vector space spanned by these norm 3 vectors is $1 + n_{4,6}(\alpha)/4$ and its average value is independent of the lattice and equals 1.7... Hence this calls for a definition of a "Nachbareffekt" also in that case, and for a connection with extremal ternary codes (which remain to be classified..). With the additional hypotheses that α is minimal in its class modulo $2L$, there is again a unique solution when $\alpha^2 = 14, 16$. The same computation as in dimension 32 shows that the classes $L/2L$ are represented by vectors of norm at most 16, and counts the number of classes of each minimum. We omit the numerical details.

Example 3: Dimension 72. The solutions of the systems $\mathcal{S}_{3,m}$ don't show any contradiction with the existence of an extremal even unimodular lattice in dimension 72. One parameter appears already for $\alpha^2 = 12$, and there are two parameters for $\alpha^2 = 16$. The second Siegel theta series is also uniquely determined, as already noticed in [P]. It seems that the classes modulo 2 should be represented by vectors of norm at most 24.

Example 4: 20-dimensional extremal lattices of level 2.

The set of equations $\mathcal{S}_{3,4}$ has the following unique solution:

i	0	1	2	3	4
$n_{4,i}$	1638	2048	272	0	0
$n_{6,i}$	60928	79872	26112	2048	0
$n_{8,i}$	615468	1001472	394752	79872	3276
$n^*_{4,i}$	2412	1536	12	0	0
$n^*_{6,i}$	83712	79872	5376	0	0
$n^*_{8,i}$	898662	1038848	155792	1536	0

while the system $\mathcal{S}_{3,8}$ with the same hypothesis on L has a set of solutions of rank 1, depending on the parameter $x = n^*_{4,3}(\alpha)$, given by:

i	0	1	2	3
$n_{4,i}$	$10x + 1132$	$-8x + 1920$	$-8x + 768$	$8x + 128$
$n_{6,i}$	$-48x + 43264$	$16x + 67840$	$64x + 40960$	$-24x + 13952$
$n_{8,i}$	$-16x + 434790$	$144x + 811264$	$-120x + 503040$	$-120x + 256128$
$n^*_{4,i}$	$-10x + 1728$	$15x + 1920$	$-6x + 312$	x
$n^*_{6,i}$	$90x + 58928$	$-128x + 82816$	$40x + 25024$	2176
$n^*_{8,i}$	$-236x + 636268$	$298x + 969600$	$-32x + 408576$	$-43x + 76928$

i	4	5	6	7	8
$n_{4,i}$	$-2x+12$	0	0	0	0
$n_{6,i}$	$-16x+2816$	$8x+128$	0	0	0
$n_{8,i}$	$144x+74896$	$-24x+13952$	$-8x+768$	0	2
$n^*_{4,i}$	0	0	0	0	0
$n^*_{6,i}$	$-2x+16$	0	0	0	0
$n^*_{8,i}$	$12x+3468$	x	0	0	0

(although the number of equations is bigger than the number of unknowns; increasing s doesn't change the rank). Since the entries of this table are even and positive numbers, a look at $n^*_{4,3}$ and $n^*_{6,4}$ shows that x can only take the values 0, 2, 4, 6, 8. We shall see in the next section how these values can be interpreted.

Remark 5.1 *More generally, this method applies to non extremal lattices, and even to lattices not satisfying $\theta_L = \theta_{L'}$ by setting $2(\mu+1)$ to be the smallest of the minimum of L and L', and by fixing the two series θ_L and $\theta_{L'}$.*

6 20-dimensional level 2 lattices of determinant 2^{10}

The genus of the level 2 lattices of dimension n and determinant $2^{n/2}$ is classified up to dimension $n = 16$ [S-V1]. In dimension 16, there are 24 classes, among which only one has minimum 4, which is the one of the Barnes-Wall lattice (this last characterisation of the Barnes-Wall lattice was shown in [Que95]). In dimension 20, there is no complete classification, but three lattices of minimum 4 are known. Two of them appear in [N-P] as associated to the maximal finite matrix groups $2.M_{12}.2$ and $SU_5(2) \circ SL_2(3)$; the second one is generated by an hermitian root system of the Hurwitz quaternions, and its automorphism group is essentially a quaternionic reflection group ([Co]). The third lattice was found by R. Scharlau and B. Hemkemeier using an extension of the computer program of neighbouring search of [SH], and has an automorphism of order $2^{14}.3^2.5$. G. Nebe and R. Scharlau have found already some 500 classes in that genus. Moreover, these three lattices are modular in the sense of [Que95], which means that they are isometric to their rescaled duals. We show here that there isn't any other extremal lattice. To this aim, we show that such a lattice is necessarily reached through certain chains of level 2 lattices, starting from a particular lattice, each lattice being related to the previous one by an index 2.

In dimension 20, a level 2 lattice of determinant 2^{10} has the same theta series as its rescaled dual. The reason is that $\theta_L - \theta'_L$ has weight 10 and belongs to $\mathcal{A}_2^- = \Phi_{24}\mathcal{A}_2$ from Proposition 3.2. Since Φ_{24} has weight 12, $\theta_L - \theta_{L'} = 0$. So $\theta_L \in \mathcal{A}_l$ and $\min(L) \leq 4$; moreover, if $\min(L) = 4$, $\theta_L = 1 + 3960x^2 + 168960x^3 + 2094840x^4 + O(x^5)$.

We start with a lemma describing the sublattices and superlattices of level 2 and index 2 of a given level 2 lattice L. If $x \in L^*$, we denote by

$$L_x := \{y \in L \mid x.y \equiv 0 \mod 2\}$$

Recall that the index 2 sublattices of L are in one-to-one correspondance with the non zero classes $\overline{x}$ of $L^*/2L^*$ by the map $\overline{x} \rightarrow L_x$. Moreover, if L has level 2, the following inclusions hold:

$$2L^* \subset L \subset L^*$$

Lemma 6.1 *Let L be a level 2 lattice.*

1. *The level 2 sublattices of index 2 of L are the previously defined L_x with $x \in L \setminus 2L^*$ and $x^2 \equiv 0 \mod 4$. The lattice L_x only depends on the class of x in $L/2L^*$, and its isometry class only depends on the orbit of the class of x under the action of the automorphism group of L.*

2. *The level 2 lattices containing L with index 2 are the $L + \mathbb{Z}x$ with $x \in L^* \setminus L$ and $x^2 \equiv 0 \mod 2$. The lattice $L + \mathbb{Z}x$ only depends on the class of x in L^*/L, and its isometry class only depends on the orbit of the class of x under the action of the automorphism group of L.*

Proof. Let $x \in L^*$. The lattice L_x has level 2 if and only if $z^2 \in \mathbb{Z}$ for all $z \in (L_x)^*$. But $(L_x)^* = L^* + \mathbb{Z}x/2$, so we need $x^2/4 \in \mathbb{Z}$. Then, for all $y \in L^*$, $(y + x/2)^2 \equiv y.x \mod \mathbb{Z}$, since L is itself of level 2. Hence $x \in L$.

Let $x \in L^*$; the level of the lattice $L + \mathbb{Z}x$ is 2 if and only if it is even. Hence $x^2 \equiv 0 \mod 2$; then $(y + x)^2 \equiv 0 \mod 2$ for all y in L since L is even. $\square$

Remark 6.1 *Hence, if the search is restricted to level 2 lattices, the quotient $L^*/2L^*$ is replaced by one of the partial quotients L^*/L or $L/2L^*$.*

Proposition 6.1 *Let L be a level 2, 20-dimensional lattice, of determinant 2^{10}, and extremal. There exists level 2 lattices L_i such that L is in one of the following situations, where all the inclusions have index 2:*

1. $L \subset L_1 \subset L_2 \supset L_3 \supset L_4$ *with:*
   - $R(L_1) = 4A_1$
   - $R(L_2) = R(L_3) = D_4$
   - $L_4 \simeq D_4 \perp BW_{16}$
2. $L \subset L_1 \supset L_2 \subset L_3 \supset L_4$ *with:*
   - $R(L_1) = R(L_2) = 4A_1$
   - $R(L_3) = D_4$
   - $L_4 \simeq D_4 \perp BW_{16}$
3. $L \subset L_1 \supset L_2 \subset L_3 \supset L_4 \supset L_5$ *with:*
   - $R(L_1) = R(L_2) = 4A_1$
   - $R(L_3) = R(L_4) = D_4$
   - $L_5 \simeq D_4 \perp L_{16}$, *where L_{16} is the 16-dimensional lattice of minimum 4 and determinant 2^{10} obtained by construction A from the Reed-Muller code R(2,4) of parameters [16,11,4].*

Proof. Let $\alpha \in L'$ such that $\alpha.\alpha = 4$. Let $\beta = \alpha/\sqrt{2} \in L^*$; since the norm of β is 2, it doesn't belong to L, and $L_1 := L + \mathbb{Z}\beta$ has level 2 and contains L with index 2, from lemma 6.1. If $\beta' \neq \pm\beta$ is another root of L_1, then $\beta \pm \beta' \in L \setminus \{0\}$ so $(\beta \pm \beta')^2 \geq 4$ which implies $\beta.\beta' = 0$. Hence the root system of L_1 has type rA_1. To compute r, we use, as in [Que95, Theorem 5] the fact that $\mathcal{S}_{10}(\Gamma_0(2))$ is 1-dimensional. Hence we can write $\theta_{L_1} = \theta_{D_4^3 \perp D_8} + a\theta_{D_4}\Delta_{16}$ which leads to $\mathrm{Card}(L_1)_2 = 184 + a$ and, applying the Fricke involution, to $\mathrm{Card}(L_1')_2 = 88 + a/2$. Since L_1' has no roots, $a = -2 * 88$ and L_1 contains exactly eight roots.

Let r_1, r_2, r_3, r_4 be four independent roots in L_1 and let $\gamma = r_1 + r_2 + r_3 + r_4$. Then $\gamma \in L$ and $\gamma.\gamma = 8$. Two cases must be considered: either $\gamma \in 2L_1^*$, or $\gamma \notin 2L_1^*$.

- If $\gamma \in 2L_1^*$, since $\gamma/2 \notin L_1$, the lattice $L_2 := L_1 + \mathbb{Z}\gamma/2$ has level 2 and contains L_1 with index 2 from lemma 6.1. Clearly $R(L_2) \supset D_4$.

- If $\gamma \notin 2L_1^*$, let $L_2 = (L_1)_\gamma$ which is of level 2 from lemma 6.1, and contains the vectors r_i since $\gamma.r_i = 2$. Then $L_3 := L_2 + \mathbb{Z}\gamma/2$ has level 2 and contains L_2 with index 2 ($\gamma/2 \in L_2^*$ from the construction and, since the norm of $\gamma/2$ is 2, it is not in L_2). Clearly $R(L_3) \supset D_4$.

In both cases, we have constructed a sublattice of $L_1 + \mathbb{Z}\gamma/2$ containing D_4; let us show that this one does not contain more than 24 roots. It requires to count the norm 8 vectors in:

$$2L_1 + \mathbb{Z}\gamma = 2L \cup (2L + \sqrt{2}\alpha) \cup (2L + \gamma) \cup (2L + (\sqrt{2}\alpha + \gamma)).$$

Notice first that the argument which allowed to count the number of roots of L_1 cannot be applied to $L + \mathbb{Z}\gamma/2$ since $\gamma/2$ doesn't necessarily belong to L^*. Indeed, γ belongs to $2L^*$ if and only if it belongs to $2L_1^*$, which stands for the first case. The level of this lattice is 2 or 4.

The vectors $\sqrt{2}\alpha$, γ, $\sqrt{2}\alpha + \gamma$ belong to the lattice L and have norm 8 (if the last one is not of norm 8, it is congruent to a norm 8 vector, since $\sqrt{2}\alpha = \pm 2r_i$ for an index i). We need some information on the vectors of smallest norm in the classes modulo $2L$ of the norm 8 vectors of L.

Lemma 6.2 *If $x \in L_8$, Card$\{y \in x + 2L \mid y.y = 8\} = 2 + n_{4,4}(x)/2 \leq 8$.*

Proof. If y is a norm 8 vector of $x + 2L$, distinct from $\pm x$, then $(y \pm x)/2 \in L \setminus \{0\}$, hence its norm is at least 4, which shows that $y.x = 0$. If $y = x + 2z$, then $z.x = -x.x/2 = -4$ and $z.z = 4$, which shows that $\mathrm{Card}\{y \in x + 2L \mid y.y = 8\} = 2 + n_{4,4}(x)/2$. The results in Example 3 of Section 5 show that $n_{4,4}(x) \leq 12$ for all x of norm 8. $\square$

Since $2L$ doesn't contain any vector of norm 8, the lemma shows that the number of norm 8 vectors in $2L_1 + \mathbb{Z}\gamma$ is at most 24.

Lemma 6.3 *Let F be a level 2 lattice with root system D_4. Let M be the orthogonal of D_4 in F. Then M is also of level 2, and Det$(M) \leq 2^2$Det(F)*

Proof. Let V be the subspace spanned by D_4. Then $V \cap F = D_4$ and $V^\perp \cap F = M$, $M^* = p_{V^\perp}(F^*)$. The lattice M is of level 2 if and only if $y^2 \in \mathbb{Z}$ for all $y \in M^*$. if x belongs to F^*, $x^2 = (p_V(x))^2 + (p_{V^\perp}(x))^2$; since F and D_4 have level 2, the same holds for M.

In order to prove the inequality, we use:

$$[M^* : M] = [p_{V^\perp}(F^*) : V^\perp \cap F^*][V^\perp \cap F^* : V^\perp \cap F]$$

and the isomorphism:

$$p_{V^\perp}(F^*)/V^\perp \cap F^* \simeq p_V(F^*)/V \cap F^*.$$

□

To conclude the proof of the proposition, we apply the last lemma to the lattice $F = L_2$ in the first case (respectively $F = L_3$ in the second case). The lattice M is a 16-dimensional level 2 lattice of determinant at most 2^8 (respectively 2^{10}), and of minimum 4. From [Que95, Theorem 5], $\mathrm{Det}(M) = 2^8$ or 2^{10}; from [Que95], it is in the first case isometric to the Barnes-Wall lattice, and, from [S-V2], it is in the second case unique up to isometry. The lattice constructed from the Reed-Muller code as described in the proposition has the same properties, and hence coincides with it. □

Theorem 6.1 *There are, up to isometry, exactly three* 20*-dimensional level* 2 *lattices of determinant* 2^{10} *and minimum* 4*.*

Proof. We go back along the chain of lattices described in Proposition 6.1, starting from the lattices $D_4 \perp BW_{16}$ and $D_4 \perp L_{16}$. The number of orbits of lattices L_{i-1} under the action of $\mathrm{Aut}(L_i)$ having the expected root system is:

Case 1 and 2: one orbit for L_3, L_2, L_1 and two orbits for L_0, which are lattices with automorphism groups of cardinality $2^{14}.3^6.5.11$ (which is the cardinality of $SU_5(2) \circ Sl_2(3)$) and $2^{14}.3^2.5$.

Case 3: two orbits for L_4 which are: $D_4 \perp BW_{16}$ and an indecomposable lattice. From the last one, we find two orbits for L_3, and three orbits for L_2, four orbits for L_1 and eight for L_0. Among these eight lattices are lattices isometric to the three known extremal lattices; finally we check isometry between lattices having automorphism groups of the same cardinality.

The computations have been executed in the Magma system. □

7 Further results

One may ask whether the sets L_m of vectors of a given norm m in an extremal lattice L can be designs of level greater than the ones claimed in

Corollary 4.1. It can be checked easily when the set of equations $\mathcal{S}_{s,m}$ has a unique solution for a big enough s, by using a criterion due to B. Venkov for a set X to be a design, which only involves the mutual scalar products of the vectors. We recall this criterion here, with the notations of section 4.

Proposition 7.1 *Let X be a set of vectors of equal norms.*

1. *For all $d > 0$, $\sum_{x,y\in X} F_d((x.y),((x.x)(y.y))^{1/2}) \geq 0$*
2. *$\sum_{x,y\in X} F_d((x.y),((x.x)(y.y))^{1/2}) = 0$ for all $d \leq t$ if and only if X is a $(2t+1)$-design*

In the case $X = L_m$, we have

$$\sum_{x,y\in X} F_d((x.y),((x.x)(y.y))^{1/2}) = \sum_{i\geq 0} F_d(i,m)(\sum_{x\in L_m} n_{m,i}(x)).$$

Remark 7.1 *The previous criterion shows that the property that the layers of a lattice form a design only depends on its second Siegel theta series.*

It turns out that in most cases, the designs are not better than what is expected from 4.1, except for one remarkable case:

Proposition 7.2 *Let L be an extremal lattice of level 5 and dimension 16, satisfying (1). For all m such that L_m is nonempty, L_m is a spherical 5-design and $L_m \cup L'_m$ is a spherical 7-design.*

Proof. From Proposition 4.1, we know that $\theta_{L,P}+\theta_{L',P} = 0$ if $\deg(P) = 2, 6$ and that $\theta_{L,P} - \theta_{L',P} = 0$ if $\deg(P) = 2, 4$. Hence we only need to prove that $\theta_{L,P}+\theta_{L',P} = 0$ if $\deg(P) = 4$. Moreover, since this modular form has weight 12, since the minimum of L is 6, and from 3.1 and 3.2, $\theta_{L,P_4} + \theta_{L',P_4} = c\Delta_8^3$ where c is a constant. Hence it is enough to prove that the first term of this series is zero, i.e. that $L_6 \cup L'_6$ is a 4-design.

But the set of equations $\mathcal{S}_{2,6}$ (15 unknowns, 16 equations) has a unique solution given by

i	0	1	2	3	4	5	6
$n_{6,i}$	630	1008	504	256	0	0	2
$n_{8,i}$	2340	4176	2772	1008	504	0	0
$n^*_{6,i}$	1374	1008	18	0	0	0	0
$n^*_{8,i}$	5292	5184	324	0	0	0	0

The first two Gegenbauer polynomials being $16x-1$ and $120x^2-36x+1$, we easily check that L_6 is a 5-design from Proposition 7.1. $\square$

Remark 7.2 *Only one lattice satisfying the hypothesis of Proposition 7.2 is in fact known, and appears in [N-P]. The question of their classification remains open.*

The inequalities of Proposition 7.1(1) give restrictive conditions on the set $n_{2t,i}(\alpha)$ which, joined with the fact that the $n_{2t,i}(\alpha)$ are even positive numbers, may lead to the non existence of some extremal lattices. This is the case in dimension 18 and level 7.

Theorem 7.1 *In the genus of* 18*-dimensional, level* 7*, determinant* 7^9 *lattices, all lattices have minimum at most 6.*

Proof. Let L be a lattice in this genus. Assume first that $\theta_L - \theta_{L'}$ is non zero, and that $\mathrm{Min}(L) \leq \mathrm{Min}(L')$. Since it is a weight 9-element of $\Phi_{12}\mathbb{C}[\theta_2, \Delta_6]$ with the notations of section 2, $\theta_L - \theta_{L'} = \Phi_{12}(a\theta_2^3 + b\Delta_6) = aq+(b-4a)q^2+(-13b-50a)q^3+\ldots$. Then, easily, $\mathrm{Min}(L) \leq 4$ and $\mathrm{Min}(L') \leq 6$.

If $\theta_L = \theta_{L'}$, from [Que95], $\mathrm{Min}(L) \leq 8$. Now we assume that $\mathrm{Min}(L) = 8$; we consider $\alpha \in L_8$. The set of equations $\mathcal{S}_{5,8}$ (with 55 unknowns and 55 equations) has a set of solutions of rank 2, but there are only two even positive solutions for which we give the vectors $[n_{8,i}(\alpha) : i \in [0..8]]$:

$$[1540, 2686, 1834, 810, 562, 0, 0, 0, 2]$$

and

$$[1624, 2560, 1904, 768, 576, 0, 0, 0, 2]$$

In both cases, the condition 7.1(1) doesn't hold for the second Gegenbauer polynomial $440x^2 - 120x + 3$, so no combination of these two possibilities can hold. $\square$

8 Cardinality of the t-designs

Table 3: Cardinality of the t-designs, $t \geq 5$ constructed from known extremal lattices in dimension $n \leq 36$, on the minimal vectors of the lattice or on the union of the minimal vectors of the lattice and of its rescaled dual.

Dim.	Lattice	$5-$ designs	$7-$ designs	$11-$ designs
4	D_4	24	48	
8	E_8		240	
12	K_{12}	756	1512	
14	$3-modular$	756	1512	
16	BW_{16}		4320	
	$5-modular$	2400	4800	
20	$2-modular$	3960	7920	
24	$unimodular$			$196560(Leech)$
	$3-modular$	26208	52416	
26	$3-modular$	21840	43680	
32	$unimodular$		146880	
	$2-modular$		261120	
36	$2-modular$		328320	

Appendix

Unicity of the extremal 5-modular lattice of dimension 16.

C. Bachoc, G. Nebe, B. Venkov

An extremal 5-modular lattice of dimension 16 has been found in [N-P] during the classification of maximal finite rational matrix groups. Its automorphism group is the maximal finite irreducible subgroup $[2.Alt_{10}]_{16}$ of $GL_{16}(\mathbb{Q})$. In this appendix we show that this lattice is the unique lattice in its genus, that has minimum 6 and hence answer the question raised in Remark 7.2. It is worth noticing that the mass of this genus is greater than 30325, so a complete enumeration of its classes is hopeless.

Theorem 8.1 *Let L be an even lattice of level 5, determinant 5^8 and dimension 16 such that $min(L) \geq 6$. Then $L \cong [2.Alt_{10}]_{16} =: L_0$ is unique.*

Proof. Let L be such a lattice. Then by Theorem 3.1 $\theta_L - \theta_{L'} \in \mathcal{S}_8(\Gamma_*(5), \chi_2)$. By Proposition 3.2 there is a constant c such that

$$\theta_L - \theta_{L'} = c(q - 14q^2 + \ldots).$$

Since $min(L) \geq 6$ this implies that $\theta_{L'}$ starts with $-c(q - 14q^2)$, which is only possible if $c = 0$. Therefore $\theta_L = \theta_{L'}$ and the hypotheses of Proposition 7.2 are satisfied and clearly $min(L) = 6$. Choose a minimal vector $x \in L$, i.e. $(x, x) = 6$. Then by Proposition 7.2, there are 9 vectors $y_1, \ldots, y_9 \in L^*$ with $(y_i, y_i) = 6/5$ and $(y_i, x) = 2$ for all $1 \leq i \leq 9$. Since the y_i are minimal vectors of L^*, the scalar products (y_i, y_j) are $\leq 3/5$ if $i \neq j$. Therefore the square length

$$a := (y_1 + \ldots + y_9, y_1 + \ldots + y_9) \leq 9 \cdot \frac{6}{5} + 9 \cdot 8 \cdot \frac{3}{5} = 54.$$

One calculates

$$(y_1 + \ldots + y_9 - 3x, y_1 + \ldots + y_9 - 3x) = 9 \cdot 6 - 2 \cdot 9 \cdot 3 \cdot 2 + a = a - 54.$$

Therefore $a = 54$, $(y_i, y_j) = 3/5$ for all $i \neq j$ and $y_1 + \ldots + y_9 = 3x$.

Let $y := y_9$. Then interchanging the roles of L and L^*, one equally finds 9 vectors $x_1, \ldots, x_9 = x \in L$ with $(x_i, x_i) = 6$ and $(x_i, y) = 2$ for all $1 \leq i \leq 9$. The same calculation as above shows that

$$x_1 + \ldots + x_9 = 15y \text{ and } (x_i, x_j) = 3 \text{ for all } i \neq j.$$

We can even calculate the scalar products (x_i, y_j) for $i, j \neq 9$: Let $1 \leq i \leq 8$. Then

$$\sum_{j=1}^{8} (x_i, y_j) + 2 = \sum_{j=1}^{9} (x_i, y_j) = (x_i, 3x) = 9.$$

Hence $\sum_{j=1}^{8} (x_i, y_j) = 7$. We now claim that $(x_i, y_j) \leq 1$ for all $1 \leq i, j \leq 8$: Clearly the scalar products $(x_i, y_j) < 3$. Seeking a contradiction we therefore assume that $(x_i, y_j) = 2$ for some $1 \leq i, j \leq 8$. Then $(x_i, y_j) = (x_9, y_j) = (x_i, y_9) = (x_9, y_9) = 2$ and the vector $(x_i + x_9 - 2y_j - 2y_9) \in L^*$ has square length $2/5$ contradicting the fact that $min(L') = 6$. Therefore $(x_i, y_j) = 1$ for seven values $j \leq 8$ and $(x_i, y_j) = 0$ for the other j. Analogously one has $(y_i, x_j) = 1$ for seven values $j \leq 8$ and $(y_i, x_j) = 0$ for the other j for all $1 \leq i \leq 8$. This shows that one can enumerate the x_i and y_i such that $(x_i, y_j) = 1$ for $1 \leq i \neq j \leq 8$ and $(x_i, y_i) = 0$ for $1 \leq i \leq 8$. Now $x_1 = 15y_9 - \sum_{i=2}^{9} x_i$ and $y_1 = 3x_9 - \sum_{i=2}^{9} y_i$. Therefore the lattice Λ generated by the y_i and x_j has Gram matrix (with respect to the basis $(y_2, \ldots, y_9, x_2, \ldots, x_9)$):

$$\frac{1}{5}\begin{pmatrix}
6 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 5 & 5 & 5 & 5 & 5 & 5 & 10 \\
3 & 6 & 3 & 3 & 3 & 3 & 3 & 3 & 5 & 0 & 5 & 5 & 5 & 5 & 5 & 10 \\
3 & 3 & 6 & 3 & 3 & 3 & 3 & 3 & 5 & 5 & 0 & 5 & 5 & 5 & 5 & 10 \\
3 & 3 & 3 & 6 & 3 & 3 & 3 & 3 & 5 & 5 & 5 & 0 & 5 & 5 & 5 & 10 \\
3 & 3 & 3 & 3 & 6 & 3 & 3 & 3 & 5 & 5 & 5 & 5 & 0 & 5 & 5 & 10 \\
3 & 3 & 3 & 3 & 3 & 6 & 3 & 3 & 5 & 5 & 5 & 5 & 5 & 0 & 5 & 10 \\
3 & 3 & 3 & 3 & 3 & 3 & 6 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & 0 & 10 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
0 & 5 & 5 & 5 & 5 & 5 & 5 & 10 & 30 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \\
5 & 0 & 5 & 5 & 5 & 5 & 5 & 10 & 15 & 30 & 15 & 15 & 15 & 15 & 15 & 15 \\
5 & 5 & 0 & 5 & 5 & 5 & 5 & 10 & 15 & 15 & 30 & 15 & 15 & 15 & 15 & 15 \\
5 & 5 & 5 & 0 & 5 & 5 & 5 & 10 & 15 & 15 & 15 & 30 & 15 & 15 & 15 & 15 \\
5 & 5 & 5 & 5 & 0 & 5 & 5 & 10 & 15 & 15 & 15 & 15 & 30 & 15 & 15 & 15 \\
5 & 5 & 5 & 5 & 5 & 0 & 5 & 10 & 15 & 15 & 15 & 15 & 15 & 30 & 15 & 15 \\
5 & 5 & 5 & 5 & 5 & 5 & 0 & 10 & 15 & 15 & 15 & 15 & 15 & 15 & 30 & 15 \\
10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 30
\end{pmatrix}$$

and determinant $4^6 \cdot 5^{-8}$. Let $\Lambda' := \Lambda + \Lambda^*$. Then $\Lambda' \supset L^* \supset \Lambda$ with $[\Lambda' : L^*] = [L^* : \Lambda] = 2^6$. With the computer one now easily checks that all overlattices M of Λ with $[\Lambda' : M] = [M : \Lambda] = 2^6$ such that $\sqrt{5}M$ is even and M has minimum $6/5$ are isometric to L_0: Let M be such a lattice. Since $\Lambda + 2\Lambda^*$ contains vectors of square length $4/5$ one has $2M \not\subseteq \Lambda$. Therefore M/Λ contains a totally isotropic vector $v+\Lambda$ in Λ'/Λ with $2v \notin \Lambda$. There is only one orbit (of length 1120) of such vectors $v+\Lambda$ under the automorphism group of Λ. Let $\Lambda_1 := \Lambda + \mathbb{Z}v$. Then we may assume that $M \supseteq \Lambda_1$. Again $2M \not\subseteq \Lambda_1$, since $\Lambda_1 + 2\Lambda_1^*$ contains vectors of square length $4/5$. Replacing Λ by Λ_1, we again find a unique orbit (of length 72) of totally isotropic vectors under the action of the automorphism group of Λ_1 on Λ_1'/Λ_1 and hence construct a lattice Λ_2 of determinant $4^2 \cdot 5^{-8}$. Since $\Lambda_2 + 2\Lambda_2^*$ has minimum $4/5$, we find $M/\Lambda_2 \cong \mathbb{Z}/4\mathbb{Z}$. The four totally isotropic vectors in Λ_2'/Λ_2 yield 4 isometric lattices M. These lattices M are all isometric to L_0. □

References

[Co] A. M. Cohen, *Finite quaternionic reflection groups,* J. Algebra 64:2 (1980) 293-324

[CS] J. Conway, N.J.A. Sloane, “Sphere packings, Lattices and Groups”, Springer-Verlag, 1988

[DGS] Ph. Delsarte, J.M. Goethals, J.J. Seidel, *Spherical codes and designs,* Geom. Dedicata **6** (1977), 363-388

[Eb] W. Ebeling, “Lattices and Codes”, Vieweg Editor, 1994

[EZ] M. Eichler, D. Zagier, *The theory of Jacobi forms,* Progress in Mathematics **55**, Birkhaüser

[KV] H. Koch, B. Venkov *Über gerade unimodulare Gitter der Dimension 32, III*, Math. Nachr. **152** (1991) 191-213

[Miy] T. Miyake, “Modular forms”, Springer, Berlin, 1989

[N-P] G. Nebe, W. Plesken, *Finite rational matrix groups of degree 16,* AMS-Memoirs, vol. 116, No 556 (1995)

[P] M. Peters, *Siegel theta series of degree* 2 *of extremal lattices*, J. Numb. Th. **35** (1990) 58-61

[Que95] H.-G. Quebbemann, *Modular Lattices in Euclidean Spaces*, J. Number Theory **54** (1995), 190-202

[Que97] H.-G. Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices*, L'Enseignement Mathématique **43** (1997), 55-65

[SH] R. Scharlau, B. Hemkemeier, *Classification of integral lattices with large class number*, Math. Comp. **67** (1998), no. 222, 737-749

[SSP] R. Scharlau, R. Schulze-Pillot *Extremal Lattices*, Algorithmic Algebra and Number Theory (Heidelberg 1997), 139-170, Springer, Berlin, 1999

[S-V1] R. Scharlau, B. Venkov, *The genus of the Barnes-Wall lattice*, Comment. Math. Helvetici **69** (1994), 322-333

[S-V2] R. Scharlau, B. Venkov, *Classification of 2-elementary lattices*, to appear

[Sc] A. Schiemann, *Classification of hermitian forms with the neighbour method*, J. Symbolic Comput. **26 (4)** (1998), 487-508

[V1] B. Venkov, *Réseaux et designs sphériques*, Notes taken by J. Martinet of lectures by B. Venkov (1996-1997, Bordeaux), this issue.

[V2] B. Venkov, *Even unimodular extremal lattices*, Proc. Steklov Inst. Math. **165** (1984), 47-52

[Vi] N.J. Vilenkin, "Special Functions and the Theory of Group Representations", Translations of Mathematical Monographs 22, AMS editors, 1968