Geometry – M2R 2020
(course notes)

Christophe BAVARD

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Chapter I

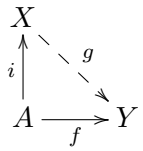
Homotopy and fundamental group

1 Basics of homotopy

1.1 Some motivations : extension, lifting and homotopy problems

A basic problem of topology is that of "factoring one continuous map through another". Many questions can be reduced to such problem (see for example Brouwer fixed point theorem 1.6 or Picard's theorem, see 1.7 below). We consider here particular cases of this problem : the extension problem, the lifting problem and the homotopy problem.

The extension problem. Given two topological spaces X, Y, a subspace A of X and a continuous map $f : A \to Y$, the question is the existence of a continuous maps $g : X \to Y$ which extends f, that is $f = g \circ i$ where $i : A \to X$ is the natural inclusion map (see diagram below).



Example 1.1. Assume that X and Y are metric spaces and that Y is complete. Then any *uniformly* continuous map $f : A \to Y$ has a (uniformly) continuous extension $g : X \to Y$.

Example 1.2 (Tietze's[1] Extension Theorem))**.** If X is normal and A is a closed subspace of X, then every continuous $f : A \to \mathbf{R}$ extends over X ([Dug78, p. 149]).

[1] Heinrich Tietze (1880–1964), Austrian mathematician.

Definition 1.3 (retract and retraction)**.** A subspace A of a topological space X is a *retract* of X if there exists a continuous map $r : X \to A$ (called a *retraction*) such that $r(x) = x$ for all $x \in A$.

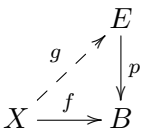
If $i : A \to X$ is the injection map, this condition reads $\mathrm{Id}_A = r \circ i$, meaning that the identity map $\mathrm{Id}_A : A \to A$ extends to X. Retracts are characterized by an stronger property.

Proposition 1.4. *A subspace $A \subset X$ is a retract if and only if , for any space Y, every continuous map $f : A \to X$ has an extension over X.*

Exercise 1.5. Let X, Y and $f : A \to Y$ be as above. Consider $Z_f = X \cup_f Y$ the space obtained by attaching X to Y by means of f. The space Y is identified with a subspace of Z_f, still denoted by Y. Prove that f extends over X if and only if Y is a retract of Z_f.

Exercise 1.6 (Brouwer[2] fixed point has a retract problem). Let $\mathbf{D}^n$ be the (usual) closed unit ball of $\mathbf{R}^n$ and let $\mathbf{S}^{n-1} = \partial\mathbf{D}^n$ be the unit sphere ($n \geqslant 1$). Assuming that $\mathbf{S}^{n-1}$ is not a retract of $\mathbf{D}^n$ (we will prove this fact later), prove the Brower fixed point theorem: *every continuous map from $\mathbf{D}^n$ to itself has a fixed point.*

The lifting problem. Let $p : E \to B$ be a continuous map (typicaly p is onto) and let $f : X \to B$ be a continuous map. Then the *lifting problem* is to look for a map $g : X \to E$ such that $f = p \circ g$ (see diagram below).



Reversing the arrows, we that this problem is dual to extension problem. Of course the lifting problem and the extension problem are special cases of the *factorization problem.*

Example 1.7 (from complex geometry)**.** Picard's[3] theorem states that every holomorphic map $f : \mathbf{C} \to \widehat{\mathbf{C}} \smallsetminus \{0, 1, \infty\}$ is constant. Here $\widehat{\mathbf{C}}$ denotes the Riemann sphere (or complex projective line). The main ingredient of the proof is the following lifting proprety. There exists a famous "modular invariant" $\lambda : \mathbf{H} \to \widehat{\mathbf{C}} \smallsetminus \{0, 1, \infty\}$, which is a holomorphic (onto) map, where $\mathbf{H}$ is the Poincaré upper halfplane. Then f admits a holomorphic lift to $\mathbf{H}$ (and the conclusion follows from Liouville[4]'s theorem).

[2] Luitzen Egbertus Jan Brouwer (1881–1966), Dutch mathematician.
[3] Émile Picard (1856–1904), French mathematician.
[4] Joseph Liouville (1809–1882), French mathematician.

The method of algebraic topology. Algegraic topology constructs algebraic ojects (groups, modules, ...) and morphisms associated with topologcal spaces and continuous maps in a functorial way. It means that if $f : X \to Y$ and $g : Y \to Z$ are continuous maps, we have similar morphisms $f_* : X_* \to Y_*$ and $g_* : Y_* \to Z_*$ in some algebraic category such that $(g \circ f)_* = g_* \circ f_*$. So an algebraic problem can be derived from the topological one. This provides *algebraic obstructions* to topological factorization problems. The primary method of proving nonexistense is to show that the corresponding algebraic problem has no solution.

For example, we will construct an algebraic functorial invariant $H_n(-)$ (the n-th homology group) which is $\mathbf{Z}$-module, such that $H_n(\mathbf{D}^{n+1}) = \{0\}$ and $H_n(\mathbf{S}^n) \neq \{0\}$. If there exists a retraction $r : \mathbf{D}^{n+1} \to \mathbf{S}^n$, i.e. $\mathrm{Id}_{\mathbf{S}^n} = r \circ i$ ($i : \mathbf{S}^n \to \mathbf{D}^{n+1}$ is the natural inclusion), we have in the algebraic level the diagram

$$\begin{array}{ccc} H_n(\mathbf{D}^{n+1}) & & \\ {\scriptstyle i_*}\uparrow & \searrow^{r_*} & \\ H_n(\mathbf{S}^n) & \xrightarrow[\mathrm{Id}_*]{} & H_n(\mathbf{S}^n) \end{array}$$

and the Brouwer fixed point theorem (see exercice 1.6) follows from a trivial algebraic fact: the identity of a non-trivial module to itself cannot be factorized through the zero morphism!

Homotopy problem and classification problem. A family of maps $h_u : X \to Y$ indexed by a real numbers $u \in [0, 1]$ is called a *homotopy* if the function $H : X \times [0, 1] \to Y$ defined by $H(x, u) = h_u(t)$ is continuous. Two continuous maps $f, g : X \to Y$ are said to be *homotopic* if there exists exists a homotopy $h_u : X \to Y$ ($u \in [0, 1]$) such that $h_0 = f$ and $h_1 = g$. The problem of the existence of a homotopy connecting f and g, called the *homotopy problem*, is a special case of the extension problem: a homotopy extends to $X \times [0, 1]$ the map defined on $X \times \{0, 1\}$ by $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Intuitively, f and g are homotopic if and only if each can be deformed continuously into the other.

Homotopy induces an equivalence relation on the set of continuous map from X to Y (see proposition 1.11). The *classification problem* is the description of the set of homotopy classes, denoted by $[X, Y]$. It is solved in many interesting cases. For example, according to Hopf's[5] theorem ([Mil65, p. 51]), two continous map $f, g : \mathbf{S}^n \to \mathbf{S}^n$ are homotopic if and only if $f_* = g_* : H_n(\mathbf{S}^n) \to H_n(\mathbf{S}^n)$. As $H_n(\mathbf{S}^n) \simeq \mathbf{Z}$, we get a bijection between $[\mathbf{S}^n, \mathbf{S}^n]$ and $\mathbf{Z}$. Thus continous maps $f : \mathbf{S}^n \to \mathbf{S}^n$ are characterized, up to homotopy, by an integer $d(f) \in \mathbf{Z}$ called the degree of f. The homotopic study of maps between spheres of various dimensions is a fundamental and very rich subject (see for example [Hat02, p. 339]).

[5] Heinz Hopf (1894–1971), German mathematician.

1.2 Homotopy, homotopy of pairs and relative homotopy

Let X and Y be topological spaces. The set of continuous maps from X to Y will be denoted by $\mathcal{C}(X,Y)$. This set can be equiped with a natural topology (see definition 1.17), bu we don't need it at the moment. It is often useful to consider *pairs* (X,A) where $A \subset X$ is a subspace. We denote by $\mathcal{C}((X,A),(Y,B))$ the set of $f \in \mathcal{C}(X,Y)$ such that $f(A) \subset B$. Throughout the rest of the chapter, we set

$$I = [0,1].$$

Definition 1.8 (Homotopy)**.** Let $f, g \in \mathcal{C}(X,Y)$. A *homotopy* from f to g is a continuous map $H : X \times I \to Y$ such that $H(\cdot,0) = f$ et $H(\cdot,1) = g$.

Definition 1.9 (Homotopy of pairs)**.** Let A and B be subspaces of X and Y respectively, and let $f, g \in \mathcal{C}((X,A),(Y,B))$. A *homotopy of pairs* from f to g is a continuous map $H : X \times I \to Y$ such that

(1) $H(\cdot,0) = f$ and $H(\cdot,1) = g$,
(2) $H(A,u) \subset B$ for all $u \in I$.

Definition 1.10 (Relative homotopy)**.** Let A be a subspace of X et and let $f, g \in \mathcal{C}(X,Y)$ such that $f(a) = g(a)$ for all $a \in A$. A *relative homotopy* from f to g is a continuous map $H : X \times I \to Y$ such that

(1) $H(\cdot,0) = f$ et $H(\cdot,1) = g$,
(2) $H(a,u) = f(a) = g(a)$ for all $(a,u) \in A \times I$.

Proposition 1.11.
1) The homopotopy is an equivalence relation in $\mathcal{C}(X,Y)$ (denoted by $\sim$); the set of homotopy classes is denoted by $[X,Y]$.
2) The homopotopy of pairs is an equivalence relation in $\mathcal{C}((X,A,(Y,B)$ (denoted by $\sim$); the set of homotopy classes is denoted by $[(X,A),(Y]$.
3) Let A be a subspace of X et let $\varphi : A \to Y$ be a fixed continuous map. We set $\mathcal{C}_\varphi(X,Y) = \{f \in \mathcal{C}(X,Y); f|A = \varphi\}$. Then the homotopy relative to A is an equivalence relation in $\mathcal{C}_\varphi(X,Y)$, denoted by $\sim_{\mathrm{rel}\, A}$.

The following lemma is very useful.

Lemma 1.12 (gluing continuous maps)**.** *Let X, Y be topological spaces and let $f : X \to Y$ a map. Let $X = \cup_{i \in I}$ be a localy finite covering of X by closed sets F_i $(i \in I)$ (for example a finite covering of X by closed sets). Then f is continuous if and only if all the restrictions $f|F_i : F_i \to Y$ $(i \in I)$ are continuous.*

Exercise 1.13. Prove the gluing lemma. Show that it is not necessarily true for an arbitrary covering by closed sets.

Example 1.14. For $X = \{*\}$ (the space reduced to a single point), we have that $\mathcal{C}(X,Y) \simeq Y$ and homotopies can be identified with (continuous) paths in Y. In this

case the set of homotopy classes of maps $\{*\} \to Y$ is the set of path components of Y, denoted by

$$\pi_0(Y) = [\{*\}, Y].$$

Example 1.15. If $Y = \mathbf{R}^n$, then for all X, the set of homotopy classes $[X, \mathbf{R}^n]$ is reduced to a single element (two arbitrary continuous maps $X \to \mathbf{R}^n$ are homotopic).

Proposition 1.16 (homotopy and composition)**.** *Let X, Y, Z by topological spaces and let $f_i \in C(X, Y)$ and $g_i \in C(Y, Z)$ $(i = 0, 1)$. If $f_0 \sim f_1$ et $g_0 \sim g_1$, then $g_0 \circ f_0$ et $g_1 \circ f_1$ are homotopic.*

Definition 1.17 (compact-open topology)**.** Let X and Y be two topological spaces and let $mcC(X, Y)$ the set of continuous mappings X *toY*. We call em compact-open topology the topology on $\mathcal{C}(X, Y)$ generated by the subsets of the form

$$W_{K,U} = \{f \in \mathcal{C}(X, Y); f(K) \subset U\},$$

where K is an arbitrary compact subset of X and U an arbitrary open subset of Y.

Exercise 1.18. If X is a compact space and (Y, d) is a metric space, show that the compact-open topology on $\mathcal{C}(X, Y)$ coincides with the topology of uniform convergence on X (metrized by $\delta(f, g) = \sup_{x \in X} d(f(x), g(x))$).

The idea that the set of homotopy classes $[X, Y]$ must correspond to path components of $\mathcal{C}(X, Y)$ (equiped with the compact-open topology) is correct in "good cases". For example, if X is a locally compact space, then we have (see [Dug78, XII, Thm. 3.1])

$$[X, Y] = \pi_0(\mathcal{C}(X, Y)). \tag{I.1}$$

1.3 Homotopy type

Definition 1.19 (homotopy equivalence)**.**
1) A *homotopy equivalence* is a continuous map $f : X \to Y$ invertible up to homotopy, i.e. there exists $g : Y \to X$ continuous such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$.
2) Two spaces X and Y have *the same homotopy type* (notation $X \sim Y$) if there exists a homotopy equivalence $f : X \to Y$.

Exercise 1.20. Show that $\mathbf{R}^{n+1} \backslash \{0\}$ and the unit sphere $\mathbf{S}^n$ $(n \in \mathbf{N})$ have the same homotopy type.

Definition 1.21 (contractible space)**.** A *contractible space* is a topological space which has the same homotopy type as a point.

Example 1.22. The space $\mathbf{R}^n$ is contractible. More generally, any star-shaped subspace of $\mathbf{R}^n$ is contractible.

Exercise 1.23. 1) Show that the following properties are equivalent :

(1) X is contractible,
(2) the identity map $\mathrm{Id}_X : X \to X$ is homotopic to a constant map,
(3) for any space Y, every continuous map $Y \to X$ is homotopic to a constant map,
(4) for any space Y, the set $[Y, X]$ is reduced to a single element.
(5) for any space Z, every continuous map $X \to Z$ is homotopic to a constant map,

2) Show that a contractible space is path-connected.
3) Is it true that if X is contractible, then the set $[X, Z]$ is reduced to a single element for any space Z ?

Exercise 1.24 (cone construction). Prove that for any toplogical space X, the cone $C(X) = X \times I/X \times \{1\}$ is contractible.

1.4 Retracts

Definition 1.25 (retract)**.** A subspace A of a topological space X is a *retract of* X if there exists a continuous map $r : X \to A$ (called a *retraction*) such that $r(x) = x$ for all $x \in A$ (i.e. $r \circ i = \mathrm{Id}_A$ if $i : A \to X$ is the natural inclusion).

Example 1.26. A product $X \times Y$ retracts on each factor $X \times \{y\}$ ($y \in Y$) or $\{x\} \times Y$ ($x \in X$). For example the circle $\mathbf{S}^1 \times \{1\}$ is a retract of the torus $\mathbf{S}^1 \times \mathbf{S}^1$.

Exercise 1.27. Let $A \subset X$ a retract of X. If X is Hausdorff, prove that A is *closed*.

Definition 1.28 (deformation retract)**.** Let A a subspace of X and let $i : A \to X$ be the natural inclusion.
1) A is a *(weak) deformation retract* of X if there exists a continuous map $r : X \to A$ (called a *deformation retraction*) such that $r \circ i = \mathrm{Id}_A$ and $i \circ r$ is homotopic to Id_X.
1) A is a *strong deformation retract* of X if there exists a continuous map $r : X \to A$ (called a *strong deformation retraction*) such that $r \circ i = \mathrm{Id}_A$ and $i \circ r$ is homotopic to Id_X *relative to* A.

For example, the sphere $\mathbf{S}^n$ is a (strong) deformation retract of $\mathbf{R}^{n+1} \setminus \{0\}$ (exercise).

Exercise 1.29. Prove that the following properties are equivalent :
(1) X is contractible,
(2) every point $\{x\}$ ($x \in X$) is a deformation retract of X,
(3) there exists a point $\{x_0\}$ ($x_0 \in X$) which is a deformation retract of X.

Exercise 1.30 (deformation retract but not strong deformation retract). The "comb" is the following subspace X of $\mathbf{R}^2$:

$$X = [0,1] \times \{0\} \cup \{0\} \times [0,1] \cup \{2^{-n}; n \in \mathbf{N}\} \times [0,1].$$

Let $x_0 = (0,1)$ and let $f_0 : X \to X$ be defined by $f_0(x) = x_0$.
1) Prove that f_0 est homotopic to Id_X (in particular X is contractible).
Prove that f_0 is not homotopic to Id_X relative to $\{x_0\}$.

1.5 Cell complexes (or CW-complexes)

A closed (resp. open) *cell* of dimension $n \in \mathbf{N}$, or closed (resp. open) n-cell, in a topological space X is a subspace homeomorphic to the closed (resp. open) unit ball. A cell-complex is a topological space which can be decomposed as a disjoint union of open cells in a particular way. More precesely, a cell-complex is a space X constructed inductively by the following procedure:
(1) Start with a discrete set X^0, whose points are regarded as $0-$cells,
(2) define inductively X^n from X^{n-1} by attaching n-cells e^n_α with maps $\varphi_\alpha : \mathbf{S}^{n-1} \to X^{n-1}$ ($\alpha \in I_n$, some indexing set), called the *attaching maps*,
(3) $X = \cup_{n\in E} X^n$ with the "weak topology": a set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n. Here $E = \{0, \dots, d\}$ for some $d \in \mathbf{N}$ (finite dimensional case) or $E = \mathbf{N}$ (infinite dimensional) case.

Condition (3) is not necessary if E is finite. Condition (2) means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with closed n-disks D^n_α by the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus we have

$$X^n = X^{n-1} \amalg_\alpha e^n_\alpha$$

where the e^n_α are closed n-cells ($\alpha \in I_n$). The subspace X^n is called the *n-skeleton* of the cell-complex X. For each cell e^n_β we define the *characteristic map* Φ_β as the composition

$$D^n_\beta \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \to X^n \hookrightarrow X.$$

This map ϕ_β is continuous and it induces a homeomorphism between the the interior of D^n_β a the open cell D^n_β.

Exercise 1.31. Let X be a cell-complex as above. Show that $A \subset X$ is open (or closed) if and only if $\phi^{-1}(A)$ is open (or closed) for every characteristic map $\phi_\alpha : D^n_\alpha \to X$.

The result of exercise 1.31 means that the toplogy of X is co-induced by the characteristic maps ϕ_α, $\alpha \in \amalg_n I_n$. So X is a quotient of the space $\amalg_{n,\alpha} D^n_\alpha$ (X is built by gluing cells with the characteristic maps).

Example 1.32 (spheres)**.** The sphere $\mathbf{S}^n$ has the structure of a cell-complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$.

Example 1.33 (real projective spaces)**.** Recall that the real projective space $\mathbf{RP}^n$ is the quotient obtained from a hemisphere by idenfication of antipodal points of its boundary. Thus $\mathbf{RP}^n = \mathbf{RP}^{n-1} \cup e_n$ and inductively we have that $\mathbf{RP}^n = e^0 \cup \dots e^n$ as the structure of a cell-complex with $n+1$ cells (one cell of dimension d for $d = 0, \dots, n+1$).

A cell-complex $X = \cup_n X^n$ is of *finite dimension* n ($n \in \mathbf{N}$) if $X = X^n$ and is *finite* if it has only a finite number of cells. A *subcomplex* of a cell-complex X is a subspace $A \subset X$ which is a union of cells of X, such that the closure of each cell in A is contained in A .

Proposition 1.34. *A compact subspace of a cell-complex is contained in a finite subcomplex.*

Lemma 1.35. *A finite union of cells (in a cell-complex) is contained in a finite subcomplex.*

Cell-complexes are also called *CW-complexes*. This terminology refers to the following properties of the cell-complexes:

- C = Closure-finiteness: the closure of each cell meets only finitely many other cells (this follows from the preceding proposition since the closure of a cell is compact, being the image of a characteristic map),
- W = Weak topology: a set is closed if and only if it meets the closure of each cell in a closed set.

Definition 1.36 (compactly generated space)**.** A topological space X is *compactly generated* if
(1) X is Hausdorff,
(2) $C \subset X$ is closed if and only if $C \cap K$ is closed for all compact subsets K of X.

Let $X = \cup_{i\in I} A_i$ be a topological space. The weak topology associated to the subspaces $(A_i)_{i\in I}$ on X (as a set) is the topology co-induced by the inclusion maps $A_i \to X$ (A_i space, X set). A subset $U \subset X$ is open (or closed) for this weak topology if and only if $U \cap A_i$ is open (or closed) in A_i for all $i \in I$. Thus compactly generated spaces are the Hausdorff spaces whose topology is the weak topology associated to the compacts subsets.

Exercise 1.37. 1) Prove that a locally compact space is compactly generated.
2) Prove that a first countable Hausdorff space is compactly generated. In particular any metric space is compactly generated.

Proposition 1.38. *A cell-complex is compactly generated.*

Exercise 1.39 (the k-topology). Let X be a Hausdorff space. We define a space $k(X)$ as follows: $k(X)$ and X have the same underlying set, and a subset $A \subset X$ is closed in $k(X)$ if and only if $A \cap K$ is closed in X for every compact subset K of X. For any function $f : X \to Y$, let $k(f)$ be the same function, regarded as a map from $k(X)$ into $k(Y)$. Prove the following properties:
1) the identity map $k(X) \to X$ is continuous,
2) $k(X)$ is compactly generated,
3) X is compactly generated if and only if $k(X) = X$,
4) $k(X)$ and X have the same compact sets,
5) if $f : X \to Y$ is a function, then $k(f)$ is continuous if and only if $f_{|K} : K \to Y$ is continuous for every compact subset $K \subset X$.

1.6 Compact surfaces as CW-complexes

Definition 1.40. A *surface* is a topological space which is locally homeomorphic to some open subset of the plane $\mathbf{R}^2$ (with usual topology). A surface is *orientable* if it does not contain any subspace homeomophic to a (closed or open) Möbius[6] band.

As any open subset of the plane $\mathbf{R}^2$ is locally homeomorphic to $\mathbf{R}^2$ itself, *a surface is a toplogical space which is locally homeomorphic to* $\mathbf{R}^2$.

Definition 1.41 (wedge sum)**.** The *wedge sum* (or *bunche*) $X \vee Y$ of two pointed space (X, x) and (Y, y) is the quotient space of the topological (disjoint) sum $X \amalg Y$ obtained by identifying x and y to a single point.

Example 1.42. The bunche $\mathbf{S}^1 \vee \mathbf{S}^1$ of two circles is homeomorphic to the figure "8".

It is known (see for example [Mas91, ch. I]) that every compact connected surface is homeomorphic to the connected sum Σ_g of $g \geqslant 0$ torus (with $\Sigma_0 = \mathbf{S}^2$) or to the connected sum N_h of $h \geqslant 1$ real projective planes. The Σ_g are orientable while the N_h are non-orientable. Moreover, Σ_g, $g \geqslant 1$ (resp. N_h) can be obtained from a polygon with $4g$ sides (resp. $2h$ sides) and side-pairing identifications given by the pattern $a_1 b_1 \overline{a_1}\,\overline{b_1} \dots a_g b_g \overline{a_g}\,\overline{b_g}$ (resp. $a_1 a_1 \dots a_h a_h$).

This last description implies that each compact surface admits a 2-dimensional CW structure as follows:

- the 0-skeleton is just a single point x_0,
- the 1-skeleton X^1 is a bunche of $4g$ circles (resp. $2h$ circles),
- Σ_g (resp. $\mathbf{N}_h$) is obtained by attaching a single 2-cell to X^1, with attaching map $\mathbf{S}^1 \to X^1$ given by the identification pattern $a_1 b_1 \overline{a_1}\,\overline{b_1} \dots a_g b_g \overline{a_g}\,\overline{b_g}$ (resp. $a_1 a_1 \dots a_h a_h$).

2 The fundamental group

2.1 Paths and loops

Let X be a topological space. Remember that a *path* of X is a continuous map of $I = [0, 1] \to X$. A *loop* of a pointed space (X, x) is a continuous map $\alpha : (I, \partial I) \to (X, x)$ where $\partial I = \{0, 1\}$, that is a path α such that $\alpha(0) = \alpha(1) = x$. The space of loops, or *loop space*, of a pointed space (X, x) is denoted by $\Omega(X, x)$.

Definition 2.1 (strict homotopy)**.** Let c_0 and c_1 be two paths of X such that $c_0(0) = c_1(1)$ and $c_0(1) = c_1(1)$. A homotopy from c_0 to c_1 is said to be *strict* if it is a homotopy relative to ∂I (i.e. with fixed ends). In the case where c_0 and c_1 are loops in $x \in X$, we will also say *homotopy of loops*.

[6] August Ferdinand Möbius 1790–1868), German mathematician.

In the set of paths over a space X, we define the following operations:

- the inverse of a path c by $\overline{c}(t) = c(1-t)$ $(t \in I)$,
- the constant path in $x \in X$ by $e_x(t) = x$ $(t \in I)$,
- the "composition" (or concatenation) of two paths c and d such that $c(1) = d(0)$ by

$$(cd)(t) = \begin{cases} c(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ d(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}.$$

We will say that two paths c and d are *composable* if $c(1) = d(0)$.

2.2 The fundamental group

Lemma 2.2 (reparametrization)**.** *Let $\varphi : I \to I$ be a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then for any path $c : I \to X$, $c \circ \varphi$ is homotopic to c relatively to ∂I.*

Proposition 2.3. *Let $c : I \to X$ be a path. We set $x = c(0)$ and $y = c(1)$. Then*

(1) $[c\overline{c}] = [e_x]$ and $[\overline{c}c] = [e_y]$,
(2) $[ce_y] = [e_x c] = [c]$,
(3) if c, c' and c'' are composable, then $[c(c'c'')] = [(cc')c'']$.

Proposition 2.4. *If $[c_0] = [c_1]$ and $[c'_0] = [c'_1]$, then $[c_0 c'_0] = [c_1 c'_1]$.*

Proposition 2.5 (fundamental group)**.** *Let (X, x) be a pointed topological space. Then the composition of loops induces a group structure on the set*

$$\pi_1(X, x) = [(I, \partial I), (X, x)].$$

With this structure, $\pi_1(X, x)$ is called the fundamental group *of the pointed space (X, x).*

The fundamental group was introduced by H. Poincaré[7] in 1892.

Example 2.6 (fundamental group of the circle)**.** We will prove later (see examples 2.28) that $\pi_1(\mathbf{S}^1, 1) \simeq \mathbf{Z}$, generated by the loop $\alpha(t) = \exp(2i\pi t)$ $(t \in I)$.

Remark 2.7. The fundamental group $\pi_1(X, x)$ is not necessarily abelian (examples will be given later, see corollary 2.35 concerning surfaces).

2.3 Higher homotopy groups

Let $\alpha, \beta : (I^n, \partial I^n) \to (X, x)$ two (continuous) maps of pairs. We define the inverse of α as $\overline{\alpha}(t_1, t_2, \dots, t_n) = \alpha(1 - t_1, t_2, \dots, t_n)$ and the "composition" of α and β by

$$(\alpha\beta)(t_1, t_2, \dots t_n) = \begin{cases} \alpha(2t_1, t_2, \dots t_n) & \text{if } 0 \leqslant t_1 \leqslant 1/2 \\ \beta(2t_1 - 1, t_2, \dots t_n) & \text{if } 1/2 \leqslant t_1 \leqslant 1 \end{cases} \qquad ((t_1, t_2, \dots t_n) \in I^n).$$

[7] Henri Poincaré (1954–1912), French mathematician.

Proposition 2.8 (higher homotopy groups)**.** *Let (X, x) be a pointed topological space. Assume that $n \geqslant 2$. Then the composition above induces a group structure on the set*

$$\pi_n(X, x) = [(I^n, \partial I^n), (X, x)].$$

It is an abelian *group ($n \geqslant 2$) called the n-th homotopy group of the pointed space (X, x).*

Higher homotopy groups where introduced by W. Hurewicz[8] in 1935. There is a fundamental difference between $\pi_1(X, x)$ and higher homotopy groups: for $n \geqslant 2$, $\pi_n(X, x)$ is a $\mathbf{Z}$-module and belongs to commutative algebra, while $\pi_1(X, x)$ is a genuine group.

Let $\Omega(X, x)$ be the loop space of (X, x), equiped with the compact-open topology. As I is locally compact, we have from (I.1) that $\pi_1(X, x) = \pi_0(\Omega(X, x))$. Thus higher homotopy groups can also be defined inductively by

$$\pi_n(X, x) = \pi_{n-1}(\Omega(X, x), e_x)) \qquad (n \geqslant 1).$$

Thus $\pi_n(X, x)$ is the set of path components of the iterated loop space $\Omega^n(X, x) = \Omega(\Omega(\dots, \Omega(X, x)))$ $(n \geqslant 1)$.

Example 2.9 ($\pi_n(\mathbf{S}^n, *)$)**.** A reformulation af Hopf's theorem (see the end of 1.1 p. 3) gives a non-trivial example: $\pi_n(\mathbf{S}^n, *) \simeq \mathbf{Z}$.

Example 2.10 ($\pi_3(\mathbf{S}^2, *)$)**.** A less intuitive example is $\pi_3(\mathbf{S}^2, *) \simeq \mathbf{Z}$ (to be prove later) and this group is generated by the *Hopf's fibration* $\mathbf{S}^3 \to \mathbf{S}^2$. Note that $\dim \mathbf{S}^3 > \dim \mathbf{S}^2$. The existence of homotopically non trivial map from a object to a smaller dimension one was a great discovery in topology.

Exercise 2.11. 1) Show that every continuous map $\mathbf{S}^k \to \mathbf{S}^n$ $(k, n \in \mathbf{N})$ is homotopic to a smooth map [hint: use for example approximation by polynomials].
2) Using Sard's theorem prove that, if $n \geqslant 2$, then

$$\pi_k(\mathbf{S}^n, *) \simeq \{0\} \qquad (2 \leqslant k < n).$$

2.4 Dependence on the base point

Let $c : I \to X$ be a path and let $x = c(0)$, $x' = c(1)$. We pose

$$\theta_c([\alpha] = [\bar{c}\alpha c] \in \pi_1(X, x') \qquad [\alpha] \in \pi_1(X, x).$$

Proposition 2.12. *1) $\theta_c : \pi_1(X, x) \to \pi_1(X, x')$ is well-defined and it is a* isomorphism *of groups.*
2) θ_c only depends on $[c]$.
3) If d is a path from x to y, then $\theta_d = i_{[\bar{d}c]} \circ \theta_c$, where $i_{[\bar{d}c]}$ denote the conjugation by $[\bar{d}c]$ in $\pi_1(X, x')$.

[8] Witold Hurewicz (1904–1956), Polish mathematician (1904 - 1956).

Corollary 2.13. *If X is path-connected, then for any pair $(x, x') \in X \times X$ the groups $\pi_1(X, x)$ and $\pi_1(X, x')$ are isomorphic.*

Remark 2.14. As we see in proposition 2.12, there is no natural isomorphism between the groups $\pi_1(X, x)$ and $\pi_1(X, x')$ (it depends on the choice of some path from x to x'), except if $\pi_1(X, x)$ is abelian.

2.5 Functioriality. Homotopy invariance

Proposition 2.15. *1) Any continuous pointed map to $f : (X, x) \to (Y, y)$ induces a morphism of groups $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ defined by*

$$f_*([\alpha]) = [f \circ \alpha] \in \pi_1(Y, y) \qquad ([\alpha] \in \pi_1(X, x)).$$

2) f_* only depends on $[f] \in [(X, x), (Y, y)]$.
3) $(g \circ f)_* = g_* \circ f_*$.

Proposition 2.16 (unpointed case)**.** *Let $f, f' : X \to Y$ be continuous maps such that $[f] = [f']$ in $[X, Y]$ (unpointed spaces) and let H be a homotopy from f to f'. For $x \in X$, we set $c(u) = H(x, u)$ $(u \in I)$, c is a path from $y = f(x)$ to $y' = f'(x)$. Then $f'_* = \theta_c \circ f_* : \pi_1(X, x) \to \pi_1(Y, y')$.*

Corollary 2.17. *If $f : X \to Y$ is a homotopy equivalence, then for any $x \in X$, the morphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.*

Corollary 2.18. *The fundamental groups of two* path-connected *spaces having the same homotopy type are isomorphic (the base points can be chosen arbitrarily).*

Corollary 2.19. *If X is contractible, then $\pi_1(X, x) = \{1\}$ for all $x \in X$.*

2.6 Fundamental group of a product

Proposition 2.20. *Let X, Y be topological spaces and let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the natural projections. Then for all $(x, y) \in X \times Y$*

$$p_* \times q_* : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$$

is an isomorphism of groups.

Example 2.21. From $\pi_1(\mathbf{S}^1, 1) \simeq \mathbf{Z}$, we have $\pi_1(\mathbf{S}^1 \times \ldots \mathbf{S}^1, *) \simeq \mathbf{Z} \times \ldots \mathbf{Z}$. It follows for example that the circle $\mathbf{S}^1 \times \{*\}$ is a retract of the torus $\mathbf{S}^1 \times \mathbf{S}^1$, but it is not a deformation retract (these spaces have distinct fundamental groups).

2.7 Simply connected spaces

Definition 2.22. A space X is *simply connected* (we say also *1-connected*) if

(1) X is path-connected,
(2) $\pi_1(X, x) = \{1\}$ for all $x \in X$.

Example 2.23. Any contractile space is simply connected.

More generally, given $n \geqslant 0$, a space X is *n-connected* if X is path-connected ($\pi_0(X) = \{*\}$) and $\pi_k(X, *)$ is the trivial group for $k \leqslant n$. For example the sphere $\mathbf{S}^n$ is $(n-1)$-connected for $n \geqslant 1$, see exercise 2.11.

Exercise 2.24. If $n \geqslant 0$, we denote by $\mathbf{D}^n$ the closed unit ball of Euclidean space $\mathbf{R}^n$ and we set $\mathbf{S}^n = \partial\mathbf{D}^{n+1}$. Show that a space X is n-connected if and only if for any $k = 0, \ldots, n$, any continuous map $\mathbf{S}^k \to X$ extends as a continuous map $\mathbf{D}^{k+1} \to X$.

Theorem 2.25 (small van Kampen's theorem)**.** *Let X be a topological space. Assume that $X = U \cup V$ where U and V are open sets such that*

(1) U and V are simply connected,
(2) $U \cap V$ is nonempty and path-connected.

Then X is simply connected.

Corollary 2.26. *For $n \geqslant 2$, the sphere $\mathbf{S}^n$ is simply connected.*

Corollary 2.27. *For $n \geqslant 2$, the space $\mathbf{R}^{n+1}\backslash\{0\}$ is simply connected.*

Exercise 2.28. Show that the complex projective spaces $P^n_{\mathbf{C}}$ are simply connected [apply small van Kampen's theorem].

The proof of small van Kampen's theorem uses a very useful technical tool: the existence of "Lebesgue number", whose statement we recall here.

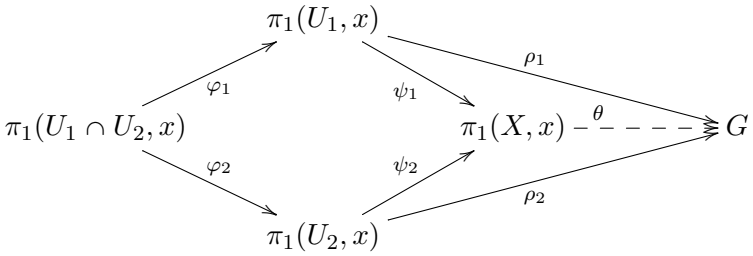
Lemma 2.29 (Lebesgue number)**.** *Let K be a compact metric space and let $(U_j)_{j\in J}$ be an open covering of K. Then there exists a real number $\rho > 0$ such that any subset of K of diameter $\leqslant \rho$ is contained in some U_j.*

Remark 2.30. If the hypothesis (1) of theorem 2.25 is weakened in (1'): U and V are path-connected, then the argument given in the proof of 2.25 proves that $\pi_1(X, x)$ is generated by the subgroups $i_*(\pi_1(U, x))$ and $j_*(\pi_1(V, x))$ ($x \in U \cap V$, $i : U \to X$ and $j : V \to X$ are the natural injections).

2.8 The theorem of Seifert and van Kampen

Let $X = U_1 \cup U_2$ such that $U_1 \cap U_2 \neq \varnothing$, and let $x \in U_1 \cap U_2$. The morphisms induced by the inclusion maps will be denoted by $\varphi_i : \pi_1(U_1 \cap U_2, x) \to \pi_1(U_i, x)$ and $\psi_i : \pi_1(U_i, x) \to \pi_1(X, x)$ ($i = 1, 2$).

Theorem 2.31 (Seifert[9] and van Kampen[10], 1933)**.** *Let $X = U_1 \cup U_2$ where U_1 and U_2 are open path-connected subsets such that $U_1 \cap U_2$ is nonempty and path-connected. Then for any group G and morphisms $\rho_i : \pi_1(U_i, x) \to G$ $(i = 1, 2)$ such that $\rho_1 \circ \varphi_1 = \rho_2 \circ \varphi_2$, there exists a unique morphism $\theta : \pi_1(X, x) \to G$ such that $\rho_i = \theta \circ \psi_i$ for $i = 1, 2$.*



The result means that $\pi_1(X, x)$ is *the freest possible* group G such that there exist morphisms $\rho_i : \pi_1(U_i, x) \to G$ $(i = 1, 2)$ with $\rho_1 \circ \varphi_1 = \rho_2 \circ \varphi_2$. Such a group is called the *amalgamated sum* of the diagram $(\varphi_i : \pi_1(U_1 \cap U_2, x) \to \pi_1(U_i, x))_{i=1,2}$, or the *amalgamated sum of $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$ along $\pi_1(U_1 \cap U_2, x)$* (it also depends on φ_1 and φ_2).

The existence of amalgamated sum for any diagram of group $(f_i : H \to G_i)_{i=1,2}$ is easily proved from the universal property of free groups. The amalgamated sum of such diagram is denoted by

$$G_1 *_H G_2.$$

This amalgam operation is the analogue of the operation of gluing topological spaces. If $H = \{1\}$ is the trivial group, the amalgamated sum is called *the free sum* of G_1 and G_2 (or sometimes the free product, but it's a sum in the categorical sense), denoted by $G_1 * G_2$.

Proposition 2.32 (fundamental group of a wedge sum)**.** *Let (X_1, x_1) and (X_2, x_2) be pointed spaces. Assume that the base point x_i is a deformation retract of some open neighborhood in X_i $(i = 1, 2)$. Then*

$$\pi_1(X_1 \vee X_2, *) \simeq \pi_1(X_1, x_1) * \pi_1(X_1, x_1).$$

Corollary 2.33. *The fundamental group of a bunche of n circles $(n \geqslant 1)$ is isomorphic to the free group F_n with n generators :*

$$\pi_1(\vee^n \mathbf{S}^1, *) \simeq F_n = \mathbf{Z} * \ldots * \mathbf{Z}.$$

Proposition 2.34 (attaching a 2-cell)**.** *Let X be a path-connected space, and let $X \cup_f D^2$ be the space obtained from X by attaching a 2-cell D^2 with attaching map $f : \partial D^2 \to X$. Choose $a \in \partial D^2$ and set $x = f(a) \in X$. Then, if $\mathcal{N}([f])$ denotes the normal subgroup of $\pi_1(X, x)$ generated by $[f] \in \pi_1(X, x)$, we have*

$$\pi_1(X \cup_f D^2, x) \simeq \pi_1(X, x)/\mathcal{N}([f]).$$

[9] Herbert Seifert (1907–1996), German mathematician.
[10] Egbert van Kampen (1908–1942), Belgian mathematician.

Thus the effect of attaching a 2-cell on the fundamental group is to "kill" the classe of the loop defined by the attaching map.

Corollary 2.35 (fondamental groups of compact surfaces)**.** *Let Σ_g be the connected sum of g torus ($g \geqslant 0$) and N_h be the connected sum of $h \geqslant 1$ real projective planes. Then*

(1) $\pi_1(\Sigma_g, *) \simeq \langle a_1, b_1, \dots, a_g, b_g | \prod_{i=1}^g [a_i, b_i] = 1 \rangle$,
(2) $\pi_1(N_h, *) \simeq \langle a_1, \dots, a_h | \prod_{i=1}^h a_i^2 = 1 \rangle$.

Exercise 2.36. Recall that the *abelianized* Γ^{ab} of a group Γ is defined as the quotient $\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$, where $[\Gamma, \Gamma]$ denotes the derived subgroup (subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$, $(x, y) \in \Gamma^2$).
1) Show the abelianized of the fundamental groups above are given by

(1) $\pi_1(\Sigma_g, *)^{\mathrm{ab}} \simeq \mathbf{Z}^{2g}$ $(g \geqslant 0)$,
(2) $\pi_1(N_h, *)^{\mathrm{ab}} \simeq \mathbf{Z}^{h-1} \times \mathbf{Z}/2\mathbf{Z}$ $(h \geqslant 1)$.

2) Deduce that the surfaces Σ_g $(g \geqslant 0)$ and N_h $(h \geqslant 1)$ are pairwise non homeomorphic.

Exercise 2.37. The space $\mathbf{R}^3$ is equiped with coordinates (x, y, z). Let K be the union of the vertical axis $x = y = 0$ with the unit circle C in the horizontal plane $z = 0$. Show that $\mathbf{R}^3 \smallsetminus K \sim \mathbf{S}^1 \times \mathbf{S}^1$. Then, using Seifert-van Kampen's theorem, compute $\pi_1(\mathbf{R}^3 \smallsetminus C, *)$ [Hint: $U = \mathbf{R}^3 \smallsetminus K$, $V = D \times \mathbf{R}$ where D is the unit open disc in the plane $z = 0$].

Note 2.38. The theorem of Seifert and van Kampen can be generalized (with essentially the same proof) to an arbirary decomposition $X = \cup_{i \in I} U_i$ of path-connected open sets, all containing the base point $x \in X$ as follows (see [Hat02, Thm. 1.20]). Assume that for all $(i, j, k) \in I^3$ the intersection $U_i \cap U_j \cap U_k$ is path-connected (in particular for $j = k$ we have that $U_i \cap U_j$ is path-connected). Then

$$\pi_1(X, x) \simeq (*_{i \in I} \pi_1(U_i, x))/N,$$

the quotient of the free sum $*_{i \in I} \pi_1(U_i, x)$ by the normal subgroup N generated by all elements of the form $\varphi_{ij}(\gamma)\varphi_{ji}(\gamma)^{-1}$ for $\gamma \in \pi_1(U_i \cap U_j, x)$, where $\varphi_{ij} : \pi_1(U_i \cap U_j, x) \to \pi_1(U_i, x)$ are induced by the injections, $(i, j) \in I^2$. In other words, there is no more relations between the elements of $\pi_1(U_i, x)$ $(i \in I)$ than the ones induced by the inclusions $U_i \to X$.

The proposition 2.34 extends as well for the attachment of an arbitrary set of 2-cells: the attachment kills all the classes which are defined by the attaching maps. As a corollary, we get that *every group is the fundmental group of some 2-dimensional CW-complex*. In fact, any group Γ admits some presentation $\Gamma = \langle \gamma_i ; i \in I | r_j ; j \in J \rangle$ (with generators $(\gamma_i)_{i \in I}$ and relations $(r_j)_{j \in J}$). We start with $X^1 = \bigvee_{i \in I} \mathbf{S}^1$, a bunche of circles indexed by I. Then attaching to X^1 a 2-cell for each relation r_j $(j \in J)$, we obtain a 2-dimensional CW-complexe X (with 1-skeleton X^1) such that

$$\pi_1(X, *) \simeq \Gamma.$$

Chapter II

Introduction to singular homology

1 Singular homology

1.1 Singular chains and homology groups

A n-simplex in the affine space $\mathbf{R}^N$ $(0 \leqslant n \leqslant N+1)$ is the convex hull of a *family* of $n+1$ affinely independant points $(x_0, \dots, x_n)$. In particular the *standard n-simplex* is the convex hull of the usual basis $(e_i)_{i=0,\dots n}$ of $\mathbf{R}^{n+1}$, $e_i = (\delta_{ij})_{j=1,\dots n}$, that is

$$\Delta_n = \{(t_0, \dots, t_n) \in [0,1]^{n+1}; \sum_{i=0}^{n} t_i = 1\} \subset \mathbf{R}^{n+1}.$$

For each n-simplex $(x_0, \dots, x_n)$ in $\mathbf{R}^n$, there is a unique affine map $\sigma : \Delta_n \to \sigma$ such that $\sigma(e_i) = x_i$, $i = 0, \dots, n$.

Definition 1.1 (singular simplices and chains)**.** Let X be a topological space.
1) Let $n \in \mathbf{N}$. A *singular n-simplex* in X is a continuous map $\phi : \Delta_n \to X$. The set of singular n-simplices will be denoted by $\mathcal{S}_n(X)$.
2) The abelian group (or $\mathbf{Z}$-module) $C_n(X) = \mathbf{Z}^{(\mathcal{S}_n(X))}$ is called the *group of singular n-chains of X*. A n-chain in X is an element of $C_n(X)$.

Example 1.2 (affine singular simplex)**.** Let $X \subset \mathbf{R}^N$ be non-emptpy convex set. Then for any finite family $(x_0, \dots, x_k) \in X^k$, there exists a unique affine map $\sigma : \Delta_k \to X$ such that $\sigma(e_i) = x_i$ $(0 \leqslant i \leqslant k)$. This map σ is defined by $\sigma(t_0, \dots, t_k) = \sum_{i=0}^{k} t_i x_i$ and called *affine singular k-simplexe.*

Recall that if A is a commutative ring and if I is any nonempty set, then $A^{(I)}$ is the A-module of functions $I \to A$ with finite support. It is a free A-module with basis $(e_i)_{i \in I}$, where $e_i \in A^{(I)}$ is defined by $e_i(j) = \delta_{ij}$ $(j \in I)$. Elements of $A^{(I)}$ can be written $\sum_{i \in I} a_i e_i$ (finite sum with $a_i \in A$ and $a_i = 0$ for almost all $i \in I$), or just $\sum_{i \in I} a_i i$. In this

last form, elements of $A^{(I)}$ are often called "formal combinations of elements of I with coefficients in A".

Note that the singular 0-simplices may be identified with the points of X and the singular 1-simplices with the paths in X.

We also define $C_n(X) = \{0\}$ (the trivial $\mathbf{Z}$-module) for $n < 0$ and we consider the graded $\mathbf{Z}$-module

$$C_*(X) = \bigoplus_{n\in\mathbf{Z}} C_n(X)$$

of singular chains of various degree on the space X. The graded module $C_*(X) = \bigoplus_{n\in\mathbf{Z}} C_n(X)$ is also denoted by $C_*(X) = (C_n(X))_{n\in\mathbf{Z}}$.

If ϕ is a singular p-simplex ($p \geqslant 1$) and i is an integer with $0 \leqslant i \leqslant p$, define a singular $(p-1)$-simplex $\partial^i\phi$ in X, by

$$\partial^i\phi(t_0, \dots, t_{p-1}) = \phi(t_0, t_{i-1}, 0, t_i, \dots, t_{p-1}),$$

called *the i-th face of* ϕ. This application extends to a graded morphism $\partial^i : C_*(X) \to C_*(X)$ of degree -1 (with $\partial^i = 0$ on $\oplus_{n\leqslant 0} C_n(X)$).

Definition 1.3. The boudary operator is the graded morphism of degree -1 defined by

$$\partial c = \begin{cases} \sum_{i=0}^{n} (-1)^i \partial^i c & \text{if } \ c \in C_n(X), n \geqslant 1, \\ 0 & \text{if } \ c \in \bigoplus_{n\leqslant 0} C_n(X). \end{cases}$$

Proposition 1.4. *The boundary operator* $\partial : C_*(X) \to C_*(X)$ *satisfies* $\partial \circ \partial = 0$.

So $(C_*(X), \partial) = (\oplus_{n\in\mathbf{Z}} C_n(X), \partial)$ is a so-called *(differential) chain complex*.

Definition 1.5. The *singular homology* of a space X is the homology of the chain complex $(C_*(X), \partial)$, that is the graded $\mathbf{Z}$-module

$$H_*^{sing}(X) = H_*(X) = \operatorname{Ker}\partial / \operatorname{Im}\partial = \bigoplus_{n\in\mathbf{Z}} H_n(X).$$

A element of $Z_n(X) = \operatorname{Ker}\partial \cap C_n(X)$ (resp. of $B_n(X) = \operatorname{Im}\partial \cap C_n(X)$) is called *a cycle of degree n* (resp. *a boundary of degree n*) and we have $H_n(X) = Z_n(X)/B_n(X)$ ($n \in \mathbf{Z}$).

Proposition 1.6 (homology of the point)**.** *We have that* $H_*(\{*\}) = H_0(\{*\}) \simeq \mathbf{Z}$.

Proposition 1.7 (H_0)**.** *If* X *is any space, then we have* $H_0(X) \simeq \mathbf{Z}^{(\pi_0(X))}$, *where* $\pi_0(X)$ *is the set of path-connected components of* X.

Exercise 1.8. Let X be a space and let $(X_\alpha)_{\alpha\in\pi_0(X)}$ be path-components of X. Prove that there is a graded isomorphism $H_*(X) \simeq \oplus_{\alpha\in\pi_0(X)} H_*(X_\alpha)$.

Proposition 1.9. *1) Let $f : X \to Y$ be a continuous map. The the composition of singular simplices by f induces a chain map $(C_*(X), \partial_X) \to (C_*(Y), \partial_Y)$ and a graded morphism $f_* : H_*(X) \to H_*(Y)$ of degree 0.*
2) If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $(g \circ f)_* = g_* \circ f_* : H_*(X) \to H_*(Z)$.

Proposition 1.10. *If $i : A \to X$ is the inclusion of a retract A of X, then $i_* : H_*(A) \to H_*(X)$ is an injective morphism an $i_*(H_*(A))$ is a direct summand of $H_*(X)$.*

1.2 Homotopy invariance of the homology

Definition 1.11 (algebraic homotopy)**.** Let (C, ∂) and (C', ∂') be two chain complexes. Then two chain maps $u, v : C \to C'$ of degree 0 are *chain homotopic* if there exists a graded morphism $T : C \to C'$ of degree one (but not necessarily a chain map) such that $u - v = T\partial + \partial' T$.

Proposition 1.12. *Let (C, ∂) and (C', ∂') be two chain complexes. Then two (algebraic) homotopic chain maps $u, v : C \to C'$ of degree 0 induce the same morphism $u_* = v_* : H(C) \to H(C')$ in homology.*

Theorem 1.13. *If $f, g : X \to Y$ are continuous homotopic maps, then we have $f_* = g_* : H_*(X) \to H_*(Y)$.*

Lemma 1.14. *Let $C \subset \mathbf{R}^N$ be a convex non-empty set. Then $H_*(C) = H_0(C) \simeq \mathbf{Z}$.*

Lemma 1.15. *Let X be any space. Denote by g_i the injection $X \hookrightarrow X \times [0,1]$ defined by $g_i(x) = (x, i)$ $(i = 0, 1)$. Then $(g_0)_\#, (g_1)_\# : C_*(X) \to C_*(X \times [0,1])$ are chain homotopic.*

Corollary 1.16. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_*(X) \to H_*(Y)$ is an isomorphism of graded $\mathbf{Z}$-modules.*

Corollary 1.17. *If X is a contractible space, then $H_*(X) = H_0(X) \simeq \mathbf{Z}$.*

1.3 H_1 and the fundamental group

Let (X, x) be a pointed space. Each loop $\alpha \in \Omega(X, x) = \mathcal{C}((I, \partial I), (X, x))$ defines also a singular simplex $\alpha \in S_1(X)$, which is a cycle because $\partial\alpha = \alpha(1) - \alpha(0) = 0$. We denote by $[\alpha] \in \pi_1(X, x)$ and $\{\alpha\} \in H_1(X)$ the corresponding elements of $\pi_1(X, x)$ and $H_1(X)$.

Theorem 1.18 (Hurewicz's theorem for π_1)**.** *The map $h : \pi_1(X, x) \to H_1(X)$ given by $h([\alpha]) = \{\alpha\}$ is well-defined and*
(1) h is a group morphism,
(2) if X is path-connected, then h is onto and $\operatorname{Ker} h = [\pi_1(X, x), \pi_1(X, x)]$. *Thus for path-connected spaces X, we have that*

$$H_1(X) \simeq \pi_1(X, x)^{\mathrm{ab}}$$

and h is the abelianization morphism.

Example 1.19 (bunche of circles)**.** We have $H_1(\bigvee^n \mathbf{S}^1) \simeq (F_n)^{\mathrm{ab}} \simeq \mathbf{Z}^n$ $(n \geqslant 1)$.

Example 1.20 (H_1 of compact surfaces)**.** For compact connected surfaces Σ_g and N_h we have (see exercise 2.36)

(1) $H_1(\Sigma_g) \simeq \mathbf{Z}^{2g}$ $(g \geqslant 0)$,
(2) $H_1(N_h) \simeq \mathbf{Z}^{h-1} \times \mathbf{Z}/2\mathbf{Z}$ $(h \geqslant 1)$.

Note 1.21. The theorem 1.18 can be generalized as follow (Hurewicz theorem): if $n \geqslant 2$, then for any $(n-1)$-connected space X we have $H_n(X) \simeq \pi_n(X, *)$.

1.4 Reduced homology

The *augmented chain complex* $\widetilde{C}_*(X)$ of a space X is defined as

$$\ldots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon_X} \mathbf{Z} \longrightarrow 0 \longrightarrow \ldots$$

where $\epsilon_X : C_0(X) \to \mathbf{Z}$ is given by $\epsilon(\sum_{i\in I} n_i\sigma_i) = \sum_{i\in I} n_i$ and called *the augmentation map*. We obviously have that $\epsilon_X \circ \partial_1 = 0$.

Definition 1.22. The *reduced homology* $\widetilde{H}_*(X)$ of X is the homology of the augmented chain complex $\widetilde{C}_*(X)$ of X.

Proposition 1.23. *Let X be any space. Then we have $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbf{Z}$ and $H_n(X) = \widetilde{H}_n(X)$ for all $n \neq 0$.*

Example 1.24. The reduced homology of a point is $\widetilde{H}_*(\{*\}) = \{0\}$.

The augmentation map ϵ_X induces a surjective morphism $\overline{\epsilon}_X : H_0(X) \to \mathbf{Z}$ and we have $\widetilde{H}_0(X) \simeq \operatorname{Ker} \overline{\epsilon}_X$. Any continuous map $f : X \to Y$ satisfies $f_*(\operatorname{Ker} \overline{\epsilon}_X) \subset \operatorname{Ker} \overline{\epsilon}_Y$, thus f induces a morphism in reduced homology still denoted by $f_* : \widetilde{H}_*(X) \to \widetilde{H}_*(Y)$.

1.5 Topological aspects of the homology of groups

Higher homotopy groups were introduced by Hurewicz in 1935. Then he tried to understand the spaces X with the simplest possible higher homotopy groups, i.e. such that $\pi_n(X, *) = \{0\}$ for all $n \geqslant 2$, that he called *aspherical spaces*.

Definition 1.25 ($K(\pi, 1)$ space)**.** Let π be a group. A path-connected CW-complex X is called a $K(\pi, 1)$ space if $\pi_1(X, *) \simeq \pi$ and $\pi_n(X, *) = \{0\}$ for $n \geqslant 2$.

Hurewicz proved in 1936 that two such $K(\pi, 1)$ spaces are homotopically equivalent. Thus if a CW-complex X is a $K(\pi, 1)$ space, the algebraic homotopical invariants of X become algebraic invariants of the group π. In particular the *homology of the group* π can be defined as

$$H_*(\pi) = H_*(X).$$

The homology of π is defined algebraically from free resolutions – and more generally projective resolutions – of the trivial $\mathbf{Z}\pi$-module $\mathbf{Z}$ by $\mathbf{Z}\pi$-modules (*via* the homology of co-invariants). The fact that all $K(\pi,1)$ spaces are homotopically equivalent has an algebraic analogue: given any module M (over an arbitrary ring), free resolutions of M are unique up to chain homotopy equivalence.

If X is a $K(\pi,1)$ CW-complex, then there is a cellular action of π on the universal covering $\widetilde{X}$ of X, and this action defines a free $\mathbf{Z}\pi$-resolution of $\mathbf{Z}$ (see [Bro82, I-4] for details). Thus the construction of $K(\pi,1)$ spaces is of fundamental importance and gives many informations on the homology $H_*(\pi)$.

2 Small chains and the Mayer-Vietoris sequence

2.1 Small chains

Let $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ be a cover of a space X, that is $X = \cup_{\alpha\in A} U_\alpha$. A singular chain $c = \sum_{i\in I} n_i\phi_i \in C_n(X)$ is called $\mathcal{U}$-*small* if for all $i \in I$, there exists some $\alpha \in A$ such that $\sigma_i(\Delta_n) \subset U_\alpha$. For any singular n-simplex ϕ, we have that $(\partial^i\phi)(\Delta_{n-1}) \subset \phi(\Delta_n)$ thus the subgroup $C_*^{\mathcal{U}}(X)$ generated by all $\mathcal{U}$-small singular simplices is stable by the boudary operator ∂. We call $C_*^{\mathcal{U}}(X), \partial)$ the *complex of $\mathcal{U}$-small chains of X*.

Theorem 2.1 (small-chains theorem)**.** *Let $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ be an cover of X by* open sets*. Then the injection $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces an isomorphism of graded $\mathbf{Z}$-modules $H_*(C_*^{\mathcal{U}}(X)) \simeq H_*(X)$.*

Remark 2.2. The small-chain theorem is true more generally for any cover $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ such that the family of interiors $(int(U_\alpha))_{\alpha\in A}$ cover X.

The key ingredient of the proof is the existence of a chain map $\mathcal{S} : C_*(X) \to C_*(X)$ of degree 0 (the *barycentric subdivision morphism*), which is a "size decreasing" and chain homotopic to the identity.

We first prove this key fact in the affine case. Let C be a non-empty convex set in the affine space $\mathbf{R}^N$. Each family of $(x_0,\dots,x_n) \in C^{n+1}$ defines a (singular) affine simplex $\sigma : \Delta_n \to C$, $\sigma(t_0,\dots,t_n) = \sum_{i=0}^n t_i x_i$. The sub-module $A_*(C)$ generated by affine singular simplices is stable by ∂.

Lemma 2.3 (barycentric subdivision)**.** *Let C be a non-empty convex subset of $\mathbf{R}^N$. Then there exists a chain map $\mathcal{S}^C : A_*(C) \to A_*(C)$ of degree 0 such that:*
(1) $\mathcal{S}^C$ is chain homotopic to the identity,
(2) for any n-singular affine simplex σ, we have

$$\operatorname{mesh}(\mathcal{S}^C(\sigma)) \leqslant \frac{n}{n+1}(\operatorname{diam}\sigma).$$

This map $\mathcal{S}^C : A_*(C) \to A_*(C)$ is called the barycentric subdivision morphism*.*

If X is a metric space, we define the *diameter* of a singular n-simplex ϕ as $\operatorname{diam}(\phi) = \operatorname{diam}(\phi(\Delta_n))$ and the *mesh* of a chain $\sum_i n_i\phi_i \in C_*(X)$ as $\operatorname{mesh}(\sum_i n_i\phi_i) = \max_i \operatorname{diam}(\phi_i)$.

Exercise 2.4. Prove the assertion (2) of lemma 2.3 above (this is elementary Euclidean geometry).

Lemma 2.5. *The barycentric subdivision operator "extends" to any space X: there exists a degree 0 chain map $\mathcal{S} : C_*(X) \to C_*(X)$, which is chain homotopic to identity and such that $\mathcal{S}(\phi) = \phi_\#(\mathcal{S}^{\Delta_n}(\mathrm{Id}_{\Delta_n}))$ for all singular n-simplex $\phi \in S_n(X)$. If $\mathcal{U}$ is an open cover of X, then for any $\phi \in S_n(X)$ there exists $m_0 = m_0(\mathcal{U}, \phi) \in \mathbf{N}$ such that the chain $\mathcal{S}^m(\phi)$ is $\mathcal{U}$-small for $m \geqslant m_0$.*

2.2 The Mayer-Vietoris sequence

Theorem 2.6 (Mayer[1]-Vietoris[2] sequence)**.** *Let $X = U_1 \cup U_2$ where U_i $(i = 1, 2)$ are open subsets of X such that $U_1 \cap U_2 \neq \varnothing$. Then the following sequence*

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\delta} & H_n(U_1 \cap U_2) & \xrightarrow{\varphi_1 \oplus \varphi_2} & H_n(U_1) \oplus H_n(U_2) & \xrightarrow{\psi_1 - \psi_2} & H_n(X) & & \\ & & & & & \searrow^{\delta} & & & \\ & & H_{n-1}(U_1 \cap U_2) & \longrightarrow & H_{n-1}(U_1) \oplus H_{n-1}(U_2) & \xrightarrow{\psi_1 - \psi_2} & H_{n-1}(X) & \xrightarrow{\delta} & \ldots \end{array}$$

is exact, where φ_i, ψ_i are induced by the injections $U_1 \cap U_2 \to U_i$, $U_i \to X$ respectively $(i = 1, 2)$, and δ is a connecting map defined by $\delta([c_1 - c_2]) = [\partial(c_1)] \in H_{n-1}(U_1 \cap U_2)$ for an arbitrary cycle $c_1 - c_2$ $(c_i \in S_n(U_i),\ i = 1, 2)$.

The proof of Mayer-Vietoris long sequence involves a very useful result from homological algebra.

Proposition 2.7 (homology long sequence)**.** *Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of chain complexes (f and g are chain maps of degree 0). Then there exists a graded morphism $\delta : H_*(C) \to H_{*-1}(A)$ of degree -1 such that the following (long) sequence*

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\delta} & H_n(A) & \xrightarrow{f_*} & H_n(B) & \xrightarrow{g_*} & H_n(C) & & \\ & & & & & \searrow^{\delta} & & & \\ & & H_{n-1}(A) & \longrightarrow & H_{n-1}(B) & \xrightarrow{g_*} & H_{n-1}(C) & \xrightarrow{\delta} & \ldots \end{array}$$

is exact.

[1] Walther Mayer (1887–1948), Austrian mathematician,

[2] Leopold Vietoris (1891–2002), Austrian mathematician. He lived over 110 years!

The long Mayer-Vietoris sequence is derived with homological algebra from a short Mayer-Vietoris sequence

$$0 \longrightarrow H_*(X) \longrightarrow H_*(U_1) \oplus H_*(U_2) \longrightarrow H_*(X) \longrightarrow 0$$

where the surjectivity of $H_*(U_1) \oplus H_*(U_2) \to H_*(X)$ follows from the possibility of representing all homology classes by small chains (see the small chains theorem 2.1).

2.3 Homology of spheres

Theorem 2.8. *The (reduced) homology of the sphere $\mathbf{S}^n$ $(n \geqslant 0)$ is given by*

$$\widetilde{H}_*(\mathbf{S}^n) = H_n(\mathbf{S}^n) \simeq \mathbf{Z} \quad (n \geqslant 1) \qquad \text{and} \qquad \widetilde{H}_*(\mathbf{S}^0) = \widetilde{H}_0(\mathbf{S}^0) \simeq \mathbf{Z}.$$

Remark 2.9 (generator of $H_1(\mathbf{S}^1)$)**.** It follows from the expression of the connecting map δ that $H_1(\mathbf{S}^1) \simeq \mathbf{Z}$ is generated by the cycle $c(t) = \exp(2i\pi t)$ $(t \in [0,1])$.

Exercise 2.10 (topological invariance of the dimension). Show that if $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^m$ $(n, m \geqslant 0)$, then $n = m$ [Hint: consider $\mathbf{R}^n \backslash \{0\}$].

Definition 2.11. Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continous map $(n \geqslant 0)$. The *degree of f* is the integer $\deg(f) \in \mathbf{Z}$ defined by

$$f_*(z) = \deg(f) z$$

for all $z \in \widetilde{H}_*(\mathbf{S}^n) \simeq \mathbf{Z}$ (that is for $n \geqslant 1$, $f_*(z) = \deg(f)z$ for all $z \in H_n(\mathbf{S}^n)$).

Proposition 2.12. *Let $f, g : \mathbf{S}^n \to \mathbf{S}^n$ be continous maps $(n \geqslant 0)$. Then*

$$\deg(g \circ f) = \deg(f) \deg(g).$$

Remark 2.13. The degree $\deg(f)$ depends only on the class $[f] \in [\mathbf{S}^n, \mathbf{S}^n]$. By Hopf theorem, the degree is a bijection from $[\mathbf{S}^n, \mathbf{S}^n]$ to $\mathbf{Z}$.

Exercise 2.14 (suspension). Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continous map. The *suspension* of f is the continuous map $\Sigma f : \mathbf{S}^{n+1} \to \mathbf{S}^{n+1}$ defined by

$$(\Sigma f)(x,t) = \begin{cases} (|x| f(x/|x|), t) & \text{if } \ x \neq 0 \\ (0,t) & \text{if } \ x = 0 \end{cases}$$

for $(x,t) \in \mathbf{R}^{n+1} \times \mathbf{R}$ such that $|x|^2 + t^2 = 1$.
Prove that $\deg(\Sigma f) = \deg(f)$ [Hint: use induction on $n \geqslant 0$ and the Mayer-Vietoris sequence].

Exercise 2.15 (degree of the antipodal map). Let $f_i : \mathbf{S}^n \to \mathbf{S}^n$ $0 \leqslant i \leqslant n)$ be given by

$$f_i(x) = (x_0, x_1 \ldots, -x_i, x_{i+1}, \ldots, x_n) \qquad \text{for} \qquad x = (x_0, \ldots, x_n) \in \mathbf{S}^n \subset \mathbf{R}^{n+1}.$$

1) Show that $\deg(f_0) = -1$ (see exercise 2.14).
2) Dedude that $\deg(f_i) = -1$ for all $i = 0, \ldots, n$ [Hint: use a transposition map] and show that the degree of the antipodal map is $(-1)^{n+1}$.

2.4 Some applications

Theorem 2.16 (Brouwer fixed-point theorem)**.** *If $n \geqslant 1$, then for every continuous map $f : \mathbf{D}^n \to \mathbf{D}^n$ there exists $x \in \mathbf{D}^n$ with $f(x) = x$.*

Exercise 2.17 (Perron-Frobenius theorem). Let $A = (a_{ij})$ be a $n \times n$ square matrix with $a_{ij} > 0$ for all $1 \leqslant i, j \leqslant n$. Prove that there exists $\lambda > 0$ and a vector $u = (u_i)$ with $u_i \geqslant 0$ for all $i = 1, \ldots, n$ such that $Au = \lambda u$.

Definition 2.18. A continuous (tangent) vector field on the sphere $\mathbf{S}^n$ ($n \geqslant 1$) is a continuous map $X : \mathbf{S}^n \to \mathbf{R}^{n+1}$ such that $\langle x, X(x) \rangle = 0$ for all $x \in \mathbf{S}^n$ (where $\langle \cdot, \cdot \rangle$ denotes the usual euclidan inner product).

Theorem 2.19. *There exists no* non-vanishing *tangent vector field on the even-dimensional spheres $\mathbf{S}^{2k}$ ($k \geqslant 1$).*

Exercise 2.20 (non-vanishing vector fields on odd-dimensional spheres).
1) Construct a non-vanishing tangent vector field on $\mathbf{S}^{2k+1}$ for all $k \geqslant 0$.
2) Construct 3 linearly independent tangent vector fields on $\mathbf{S}^{4k+3}$ for all $k \geqslant 0$.

Note 2.21. A general reference for this chapter II is [Vic94, Chap. I].

Chapter III

Fiber bundles. Coverings

1 Fiber bundles

1.1 General definitions and vocabulary

Definition 1.1. A (locally trivial) *fiber bundle* with fiber F is a triple $\xi = (E, p, B)$ where $p : E \to B$ is a continuous projection map such that for each $b \in B$ there exists a neighborhood U of b and a homeomorphism $\Phi : p^{-1}(U) \to U \times F$ making the following diagram commutative

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & U & \end{array}$$

that is $p_{|p^{-1}(U)} = p_1 \circ \Phi$ where $p_1 : U \times F \to U$ is the projection to the first factor.

If $\xi = (E, p, B)$ is a fiber bundle as above, we use the following vocabulary:
- $E = E(\xi)$ is the total space of the bundle,
- $B = B(\xi)$ is the la base space of the bundle,
- $p = p(\xi)$ is the projection of the bundle,
- U is a trivializing neighborhood,
- Φ is a local trivialisation of $p^{-1}(U)$,
- $E_b = p^{-1}(b)$ is the fiber over $b \in B$,
- F is the fiber of the bundle.

We also say that $\xi = (E, p, B)$ is *a bundle structure with fiber* F on E. A fiber bundle can be thought of as a family of spaces $(p^{-1}(b))_{b \in B}$ parametrized by B, all homeomorphic to a fixed space F, and glued together by the topology of the total space E.

Fiber bundles can also be thought of as "short exact sequence of spaces", denoted by

$$F \hookrightarrow E \to B.$$

Example 1.2 (trivial bundle)**.** The trivial bundle with base B and fiber F is the first projection $p : E = B \times F \to B$.

Example 1.3 (induced bundle)**.** If $\xi = (E, p, B)$ is a bundle with fiber F, then for all subspace $A \subset B$, the restriction p_A of p to $E_{|A} = p^{-1}(A)$ ("E over A") defines a bundle $\xi_{|A} = (E_{|A}, p_A, A)$ over A with the same fiber F, called the *induced bundle over* A.

Example 1.4 (Möbius strip)**.** The group $\mathbf{Z}$ acts by homeomorphisms in $\mathbf{R}^2$ and $\mathbf{R}$ in the following way:

$$k \cdot (x, y) = (x + k, (-1)^k y) \qquad \text{and} \qquad k \cdot x = x + k \qquad (x, y) \in \mathbf{R}^2.$$

The first projection $\mathbf{R}^2 \to \mathbf{R}$ is equivariant with respect to these actions (from the definition) and it induces a fiber bundle map between the orbit spaces $p : M = \mathbf{R}^2/\mathbf{Z} \to \mathbf{R}/\mathbf{Z} \simeq \mathbf{S}^1$. Thus the Möbius strip M has a fiber bundle structure over the circle $\mathbf{S}^1$ with fiber $\mathbf{R}$.

Example 1.5 (and exercise: tautological bundle over the projective space)**.**
For $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$, let $\mathbf{P}^n_k$ be the space of lines of $\mathbf{k}^{n+1}$ (or projective space) equiped with the topology co-induced by the natural projection $\mathbf{k}^{n+1}\backslash\{0\} \to \mathbf{P}^n_k$. Let

$$E = \{(\ell, u) \in \mathbf{P}^n_k \times \mathbf{k}^{n+1}; u \in \ell\} \subset \mathbf{P}^n_k \times \mathbf{k}^{n+1}.$$

and let p be defined by $p(\ell, u) = \ell$ for $(\ell, u) \in E$. Prove that $\gamma_{n,\mathbf{k}} = (E, p, \mathbf{P}^n_k)$ is a bundle with fiber $\mathbf{k}$ [Hint: consider the usual affine charts $U_i = \pi(x_i \neq 0)$ on $\mathbf{P}^n_k$, $i = 0, \ldots, n$]. This bundle $\gamma_{n,\mathbf{k}}$ is called the *tautological bundle* over the projective space, and also often denoted by $\mathcal{O}(-1)$ in the complex case.

Example 1.6 (Hopf bundle)**.** Let $\mathbf{S}^3 \subset \mathbf{C}^2$ be the unit sphere for the usual hermitian structure. Then the restriction of the projection $\pi : \mathbf{C}^2\backslash\{0\} \to \mathbf{P}^1_{\mathbf{C}}$ to $\mathbf{S}^3$ defines a fiber bundle $h = \pi_{|\mathbf{S}^3} : \mathbf{S}^3 \to \mathbf{P}^1_{\mathbf{C}} \simeq \mathbf{S}^2$ with fiber $\mathbf{S}^1$ called *the Hopf bundle*. We have a similar Hopf bundle $\mathbf{S}^1 \hookrightarrow \mathbf{S}^{2n+1} \to \mathbf{P}^n_{\mathbf{C}}$ for any $n \geqslant 1$.

Definition 1.7. (1) Let (E, p, B) and (E', p', B') be two bundles. A *bundle morphism* (or bundle map) $(H, h) : (E, p, B) \to (E', p', B')$ is a pair of continuous maps $H : E \to E'$ and $h : B \to B'$ such that $p' \circ H = h \circ p$.
(2) If $B = B'$ and $h = \mathrm{Id}_B$, a bundle morphism is called a *bundle morphism over* B (or B-morphism). A *bundle isomorphism over* B is a B-morphism which is invertible as a B-morphism (it is equivalent to require that H is a homeomorphism).

The condition $p' \circ H = h \circ p$ means that (H, h) is fiber-preseving : $H(p^{-1}(b)) \subset p'^{-1}(h(b))$ for each $b \in B$. As p is surjective, the map h is uniquely determined by H. Of course composition of morphisms (or B-morphisms) is a morphism (or a B-morphism), so we have a category of bundles and for fixed B a category of bundles over B.

Definition 1.8 (trivial bundle)**.** A fiber bundle (E, p, B) is *trivializable* (or trivial) if it is B-isomorphic to the trivial bundle $(B \times F, p_1, B)$.

Example 1.9. The Hopf bundle $\mathbf{S}^1 \hookrightarrow \mathbf{S}^3 \to \mathbf{S}^2$ is not trivial because $\mathbf{S}^3$ cannot be homeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$ ($\mathbf{S}^3$ is simply connected while $\mathbf{S}^2 \times \mathbf{S}^1$ is not).

Exercise 1.10. Prove that the tautological bundle $\gamma_{n,\mathbf{R}}$ ($n \geqslant 1$) over the real projective space (see example 1.5) is not trivial [Hint : if $\gamma_{n,\mathbf{R}}$ were trivial, show that there is a continuous map $\lambda : \mathbf{S}^n \to \mathbf{R}^\times$ such that $\lambda(-x) = -\lambda(x)$ for all $x \in \mathbf{S}^n$ and get a contracdition].

1.2 Cross-sections and lifts. Pullback bundles

Definition 1.11 (cross-section)**.** A *cross-section* (or section) of a fiber bundle $p : E \to B$ is a continuous maps $s : B \to E$ such that $p \circ s = \mathrm{Id}_B$.

$$\begin{array}{ccc} E & & \\ {\scriptstyle p}\downarrow & \circlearrowright & {\scriptstyle s} \\ B & & \end{array}$$

In other words, a section is a continuous data of a point $s(b)$ above b for all $b \in B$. A *local section* of a bundle $p : E \to B$ defined on an open U of B is a section of the bundle $E_{|U}$ (such a section always exists for U trivializing). The set of section of a fiber bundle $\xi = (E, p, B)$ is usually donoted by

$$\Gamma(E) = \Gamma(\xi) = \{s \in \mathcal{C}(B, E); p \circ s = \mathrm{Id}_B\}.$$

Example 1.12. The sections of a trivial bundle $p : B \times F \to B$ are the graphs of continuous maps $B \to F$. Thus $\Gamma(B \times F) \simeq \mathcal{C}(B, F)$.

As the above example shows, The concept of "section" generalizes that of "function" because functions correspond to sections of a trivial bundle. Sections are the right objects to consider when working on global problems.

Definition 1.13 (lift)**.** Let $p : E \to B$ be a fiber bundle and let $f : Y \to B$ be a continuous map. A *lift* of f is a continuous map $\tilde{f} : Y \to E$ such that $f = p \circ \tilde{f}$.

$$\begin{array}{ccc} & & E \\ & {\scriptstyle \tilde{f}} \nearrow & \downarrow {\scriptstyle p} \\ Y & \xrightarrow{f} & B \end{array}$$

Definition 1.14 (fiber product)**.** The *fiber product* of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow {\scriptstyle p} \\ Y & \xrightarrow{f} & B \end{array}$$

of sets and maps is $Y \times_f E = \{(y, x) \in Y \times E; f(y) = p(x)\}$, also denoted by f^*E.

In the topological setting where sets are space and maps are continuous, the fiber product f^*E will be equiped with the induced topology from the topological product $Y \times E$. The restriction of the projections $Y \times E \to Y, E$ will be denoted by p_Y and p_E.

Proposition 1.15 (pullback – or induced – bundle)**.** *Let $\xi = (E, p, B)$ be a bundle of fiber F and let $f : Y \to B$ be a continuous map. Then*

(1) $f^*(\xi) = (f^*E, p_Y, Y)$ is a bundle with the same fiber F, called pullback bundle *(or* induced bundle*) from ξ by f; moreover $(p_E, f) : f^*(\xi) \to \xi$ is a bundle map,*
(2) the lifts of the map f correspond bijectively to the sections of the pullback bundle $f^*(\xi)$.

$$\begin{array}{ccc} f^*E & \xrightarrow{p_E} & E \\ {\scriptstyle p_Y}\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & B \end{array}$$

Example 1.16. Let $p : E \to B$ be a fiber bundle and let A be a sub-space of B with injection $i_A : A \to B$. Then the pullback bundle i_A^*E is isomorphic to $E_{|A}$.

1.3 Bundles and homotopy. Homotopy long sequence

Lemma 1.17. *Let $\alpha < \beta < \gamma$ be real numbers and let ξ be a bundle over $B \times [\alpha, \gamma]$. If ξ is trivial over $B \times [\alpha, \beta]$ and over $B \times [\beta, \gamma]$, then ξ is trivial.*

By induction, the lemma is valid for a finite number of intervals. Let $\alpha_0, \alpha_1, \dots \alpha_n$ be an increasing sequence of real numbers and let ξ be a bundle over $B \times [\alpha_0, \alpha_n]$. If ξ is trivial over $B \times [\alpha_i, \alpha_{i+1}]$ for all $i = 0, \dots, n-1$, then ξ is trivial.

Theorem 1.18. *Any fiber bundle over $I^n = [0,1]^n$ ($n \in \mathbf{N}$) is a trivial bundle.*

Corollary 1.19 (homotopy lifting properties for cubes)**.** *Let A be a retract of $I^n \times I$ and let $p : E \to B$ be a fiber bundle. Given a continuous map $\widetilde{f} : A \to E$ and a homotopy $H : I^n \times I \to B$ such that $H_{|A} = p \circ \widetilde{f}$, then there exists a homotopy $\widetilde{H} : I^n \times I \to E$ lifting the homotopy H (that is $p \circ \widetilde{H} = H$) such that $\widetilde{H}_{|A} = \widetilde{f}$.*

Main examples of retracts A are:
• $A = I^n \times \{0\}$ (lifting homotopies),
• $A = I^n \times \{0\} \cup \partial I^n \times I$ (lifting relative homotopies).

Theorem 1.20 (the homotopy long sequence of a fiber bundle)**.** *Let $p : (E, x) \to (B, b)$ be a* pointed *fiber bundle with fiber $F = p^{-1}(b)$. Then there exists connecting morphisms*

$\delta_n : \pi_n(B,b) \to \pi_{n-1}(F,x)$ $(n \geqslant 1)$ such that the following (long) sequence

$$\begin{array}{ccccccc}
 & & \dots & \longrightarrow & \pi_3(B,b) & & \\
 & \overset{\delta_3}{\swarrow} & & & & & \\
\pi_2(F,x) & \xrightarrow[i_*]{} & \pi_2(E,x) & \xrightarrow{p_*} & \pi_2(B,b) & & \\
 & \overset{\delta_2}{\swarrow} & & & & & \\
\pi_1(F,x) & \xrightarrow[i_*]{} & \pi_1(E,x) & \xrightarrow{p_*} & \pi_1(B,b) & & \\
 & \overset{\delta_1}{\swarrow} & & & & & \\
\pi_0(F,x) & \xrightarrow[i_*]{} & \pi_0(E,x) & \xrightarrow{p_*} & \pi_0(B,b) & \longrightarrow & \{*\}
\end{array}$$

is exact, where i_* is induced by the injection $F \hookrightarrow E$.

The 0-level of the sequence just a sequence of pointed sets, exactness of

$$(A,a) \xrightarrow{f} (B,b) \xrightarrow{g} (C,c)$$

at (B,b) meaning that $f(A) = g^{-1}(c)$.

Corollary 1.21. *We have $\pi_k(\mathbf{S}^1, *) = \{0\}$ for $k \geqslant 2$.*

Corollary 1.22. *We have $\pi_2(\mathbf{S}^2, *) \simeq \mathbf{Z}$ and $\pi_k(\mathbf{S}^2, *) \simeq \pi_k(\mathbf{S}^3, *)$ for $k \geqslant 3$.*

Remark 1.23. From Hopf's theorem ($\pi_n(\mathbf{S}^n, *) \simeq \mathbf{Z}$), it follows that $\pi_3(\mathbf{S}^2, *) \simeq \mathbf{Z}$.

1.4 Bundles defined by transformation groups

Let $p : E \to B$ be a bundle with fiber F and let $(U_i)_{i\in I}$ some open cover of the base B by trivializing open subsets U_i ($i \in I$) with local trivializations $\Phi_i : p^{-1}(U_i) \to U_i \times F$ ($i \in I$). If $U_i \cap U_j$ is nonempty, we get a *transition map* $\Phi_i \circ \Phi_j^{-1} : U_i \cap U_j \times F \to U_i \cap U_j \times F$ given by

$$\Phi_i \circ \Phi_j^{-1}(x,u) = (x, g_{ij}(x) \cdot u) \qquad (x,u) \in U_i \cap U_j \times F \qquad (*)$$

where $g_{ij}(x) \in \mathrm{Homeo}(F)$ and $g_{ij}(x) \cdot u$ is continuous with respect to $(x,u) \in U_i \cap U_j \times F$.

To introduce more structure, it is useful to impose that the transitions maps are given by some precise type of *continuous* action on the fiber. For example, if the fiber F is a finite dimensional (real or complex) vector space and if we require that each $g_{ij}(x)$ is a linear map, we get the notion of *vector bundle*.

Definition 1.24 (topological group)**.** A topological group G is a group G equiped with a topology such that the map $\phi : G \times G \to G$ given by $\phi(g,h) = gh^{-1}$ is continuous.

Examples 1.25. Classical groups $\mathbf{GL}_n(\mathbf{R}), \mathbf{GL}_n(\mathbf{C}), \mathbf{SL}_n(\mathbf{R}), \mathbf{SL}_n(\mathbf{C}), \mathbf{O}(n), \mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n), \mathbf{Sp}_{2g}(\mathbf{R})$ are all topological groups.

Definition 1.26 (G-space)**.** Let G be a topological group. A *(left) G-space* is a topological space F equiped with an action $G \times F \to F$ which is continuous. A G-space F is *faithful* if the action is faithful (that is $g \cdot u = u$ for all $u \in F$ implies $g = 1$).

Examples 1.27. Let H be closed subgroup of $\mathbf{GL}_n(\mathbf{R})$ (resp. $\mathbf{GL}_n(\mathbf{C})$). Then $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) is a faithful H-space. It is the case for all the classical groups of example 1.25.

Exercise 1.28. Let F be compact Hausdorff space and let $G = \mathrm{Homeo}(F)$ equiped with the compact-open topology. Prove that G is a topological group and that F is a left G-space for the natural action of G on F by homeomorphisms.

Let $p : E \to B$ be a bundle with fiber F. We define a *bundle chart of* (E, p, B) as a local trivialization $\Phi : p^{-1}(U) \to U \times F$ where U is a trivializing open subset of B (see defintion 1.1). Until the end of this subsection, we fix a topological group G and a faithful (left) G-space F.

Definition 1.29 (G-bundle atlas)**.** A *G-bundle atlas with fiber F* for (E, p, B) is a set $\mathcal{A} = \{(p^{-1}(U_i), \Phi_i); i \in I\}$ of bundle charts such that

(1) $B = \cup_{i \in I} U_i$,
(2) for all $(i, j) \in I^2$ with $U_i \cap U_j \neq \varnothing$, the transition map (*) is given by the action of G on the fiber F *via* a continuous map $g_{ij} : U_i \cap U_j \to G$.

Definition 1.30 (equivalence of G-bundle atlases)**.** Two G-bundle atlases $\mathcal{A}$ and $\mathcal{A}'$ with fiber F for (E, p, B) are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is G-bundle atlas with fiber F for (E, p, B).

Exercise 1.31. Prove that this relation is transitive. Show that any G-bundle atlas with fiber F for (E, p, B) is contained in a maximal one.

Definition 1.32 (G-bundle structure)**.** A G-bundle structure (with fiber F) on (E, p, B) is an equivalence class of G-bundle atlases for (E, p, B).

A G-bundle structure on (E, p, B) can also be defined as a *maximal* G-bundle atlas, see exercise 1.31. We will say the a bundle (E, p, B) "is a G-bundle with fiber F" if it admits some G-bundle structure with fiber F, that is some G-atlas with fiber F. A G-bundle is also called a *bundle with structural group G*.

Example 1.33 (trivial G-bundle)**.** The trivial bundle $B \times F \to B$ is a G-bundle, defined by the atlas consisting of the single bundle chart $\mathrm{Id}_{B \times F}$. It is called the "trivial G-bundle".

In the special case where $F = G$ and G acts on itself by left translations, then a G-bundle is called a *principal G-bundle*.

Example 1.34 (and exercise)**.** The Hopf bundle $\mathbf{S}^1 \hookrightarrow \mathbf{S}^3 \to \mathbf{S}^2$ is a principal $\mathbf{S}^1$-bundle.

Exercise 1.35. Show that the Möbius strip (see example 1.4) is a $\mathbf{GL}_1(\mathbf{R})$-bundle with fiber $\mathbf{R}$ equiped with the linear action of $\mathbf{GL}_1(\mathbf{R})$.

Exercise 1.36. Show that a *principal* G-bundle admits a natural (global) right G-action [Hint: define this right action *via* the G-bundle charts].

Definition 1.37. Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two G-bundles with fiber F, equiped with G-atlases $\mathcal{A} = \{(p^{-1}(U_i), \Phi_i); i \in I\}$ and $\mathcal{A}' = \{(p'^{-1}(V_j), \Psi_j); j \in J\}$ respectively. A bundle map $(H, h) : (\xi, \mathcal{A}) \to (\xi', \mathcal{A}')$ is a *G-bundle morphism* (or a *G-bundle map*) if for all $(i, j) \in I \times J$, there exists a continuous map $h_{ij} : U_i \cap h^{-1}(V_j) \to G$ such that

$$\Psi_j \circ H \circ \Phi_i^{-1}(x, u) = (h(x), h_{ij}(x) \cdot u) \qquad (x, u) \in (U_i \cap h^{-1}(V_j)) \times F.$$

A G-bundle map $(H, h) : \xi \to \xi'$ will be often just denoted by $H : \xi \to \xi'$, because h is determined by H. The composition of two G-bundle morphisms is clearly a G-bundle morphism. Note also that the equivalence of two atlases $\mathcal{A}$ and $\mathcal{A}'$ on a bundle ξ means that the identity map $Id : (\xi, \mathcal{A}) \to (\xi, \mathcal{A}')$ is a G-bundle map. So we get the following result.

Proposition 1.38 (G-bundle morphisms for G-bundle structures)**.** *Let $(\xi, \mathcal{A})$ and $(\xi', \mathcal{A}')$ be two G-bundles with fiber F, and let $H : (\xi, \mathcal{A}) \to (\xi', \mathcal{A}')$ be a G-bundle morphism. Then for any atlases $\mathcal{A}_1$ on ξ and $\mathcal{A}'_1$ on ξ' equivalent to $\mathcal{A}$ and $\mathcal{A}'$ respectively, the bundle map $H : (\xi, \mathcal{A}_1) \to (\xi', \mathcal{A}'_1)$ is a G-bundle morphism.*

Thus, for G-bundle structures, there are well-defined notions of G-bundle morphism and G-bundle isomorphism (G-bundle morphism invertible as a G-bundle morphism).

Example 1.39. Le $\xi = (E, p, B)$ be a G-bundle with fiber F and let $f : Y \to B$ be a continuous map. Then the induced bundle $f^*(\xi)$ is a G-bundle over Y and $(p_E, f) : f^*(\xi) \to \xi$ is a G-bundle morphism.

Remark 1.40. If $\mathcal{A} = \{(p^{-1}(U_i), \Phi_i); i \in I\}$ is a G-bundle atlas for some bundle ξ, then the condition (2) of defintion 1.29 means that Φ_i is a G-bundle isomphism over U_i from the induced G-bundle $\xi_{|U_i}$ to the trivial G-bundle $U_i \times F$.

Proposition 1.41. *Let $(H, h) : \xi \to \xi'$ be G-bundle morphism (where ξ and ξ' are G-bundles with the same fiber). If $h : B(\xi) \to B(\xi')$ is a homeomorphism, then (H, h) is an* isomorphism *of G-bundles. In particular any G-morphism over B (case where $B(\xi) = B(\xi') = B$ and $h = \mathrm{Id}_B$) is a G-isomorphism over B.*

Definition 1.42 (trival G-bundle)**.** A G-bundle ξ with base B and fiber F is *trivializable* (or *trivial*) if it is G-isomorphic to the G-bundle $B \times F \to B$.

Exercise 1.43. Prove that a *principal* G-bundle is trivial if and only if it admits a section.

Proposition 1.44. *Let $(H, h) : \eta \to \xi$ be a G-bundle map. Then η is G-isomorphic with $h^*(\xi)$ over $B(\eta)$.*

1.5 G-bundles and cocycles

Let ξ a G-bundle defined by a G-atlas $\mathcal{A} = \{(p^{-1}(U_i), \Phi_i); i \in I\}$. We set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Then, if $U_{ijk} \neq \varnothing$, the maps $g_{ij} : U_{ij} \to G$ defined by the transition maps (*) satisfy the following condtion :

$$g_{ij}(x)g_{jk}(x)g_{ki}(x) = 1 \qquad (x \in U_{ijk}), \tag{C}$$

called the *cocyle condition*. Note that it implies $g_{ii}(x) = 1$ for $x \in U_i$ $(k = j = i)$ and $g_{ji}(x) = (g_{ij}(x))^{-1}$ for $x \in U_{ij}$ $(k = j)$.

Definition 1.45 (cocycle)**.** Let $\mathcal{U} = \{U_i; i \in I\}$. A G-valued *cocycle* on $\mathcal{U}$ is a family of continuous maps $g_{ij} : U_{ij} \to G$ $((i, j) \in I^2$ such that $U_{ij} \neq \varnothing)$ which satisfy the above condition (C).

Theorem 1.46 (gluing construction of bundles)**.** *Fix a faithful (left) G-space F. Let B be topological space and let $\mathcal{U} = \{U_i; i \in I\}$ be some open cover of B.*

(1) For any G-valued cocycle (g_{ij}) on $\mathcal{U}$ there exists a G-bundle over B with fiber F and transitions maps given by the (g_{ij}). Any two such bundles are G-isomorphic.

(2) Let (g'_{ij}) be another G-valued cocycle on $\mathcal{U}$. Then the G-bundles associated to (g_{ij}) and to (g'_{ij}) are G-isomorphic if and only if there exists continuous maps $h_i : U_i \to G$ such that

$$h_i(x)g_{ij}(x) = g'_{ij}(x)h_j(x) \qquad (x \in U_{ij}).$$

Example 1.47. The bundle associated with the trivial cocycle $g_{ij}(x) = 1$ is the trivial G-bundle. Thus the bundle associated with (g_{ij}) is trivial if and only if there exists continuous maps $h_i : U_i \to G$ such that $g_{ij}(x) = h_i(x)^{-1}h_j(x)$ for all (i, j).

Example 1.48. Let ξ be the bundle obtained in part (1) of theorem 1.46. If $f : Y \to B$ is any continuous map, then the bundle $f^*(\xi)$ is associated with the cocycle $(g_{ij} \circ f)$ defined on the cover $\{f^{-1}(U_i); i \in I\}$ of Y.

Example 1.49. Let $G = \mathbf{GL}_1(\mathbf{C})$ acting on the line $\mathbf{C}$. The complex projective space $\mathbf{P}^n_{\mathbf{C}}$ is equiped with usual affine charts $U_i = \{[z_0, \ldots, z_n] \in \mathbf{P}^n_{\mathbf{C}}; z_i \neq 0\}$ $(0 \leqslant i \leqslant n)$. For $k \in \mathbf{Z}$, we denote by $\mathcal{O}_n(k)$ the bundle over $\mathbf{P}^n_{\mathbf{C}}$ defined by the cocycle $g_{ij}([z]) \cdot u = (z_j/z_i)^k u$, $(u \in \mathbf{C},\ i, j = 0, \ldots, n)$.

Exercise 1.50. Let $G = \mathbf{GL}_1(\mathbf{C})$ acting on the line $\mathbf{C}$ as above. Show that the tautological bundle $\gamma_{n,\mathbf{C}}$ (see example 1.5) is a G-bundle G-isomorphic to $\mathcal{O}_n(-1)$.

Let ξ be the G-bundle defined by an atlas $\mathcal{A} = \{(p^{-1}(U_i), \Phi_i); i \in I\}$ with cocycle (g_{ij}). Then the sections of ξ are given locally by $s(x) = \Phi_i^{-1}(x, s_i(x))$ $(x \in U_i)$ where $s_i : U_i \to F$ are continuous map which satisfy the gluing condition

$$s_i(x) = g_{ij}(x) \cdot s_j(x) \qquad (x \in U_{ij}, (i, j) \in I^2).$$

Definition 1.51 (associated principal bundle)**.** Let ξ be a G-bundle with cocycle (g_{ij}). The *principal* G-bundle defined by the same cocycle (g_{ij}) is called *the principal G-bundle associated to ξ*.

Exercise 1.52. Show that two G-bundles are G-isomorphic if and only if their associated principal G-bundles are isomorphic. Thus the classification problem of G-bundle reduces to the classification of principal G-bundles.

1.6 Homotopy invariance for G-bundles

First we start with some useful notions in topology : paracompactness and partitions of unity (see for example [Dug78, Chap. VIII]). Recall that *a refinement* of cover $(u_i)_{i\in I}$ is a cover $(V_j)_{j\in J}$ such that any V_j is contained in some U_i.

Definition 1.53 (paracompact space)**.** A space B is *paracompact* if

(1) B is Hausdorff,
(2) any open cover of B admits a *locally finite* open refinement.

Example 1.54 (metric spaces)**.** According to Stone[1]'s theorem, every metric space is paracompact (see for example [Dug78, Chap. IX]).

Exercise 1.55. Prove that the product of a paracompact space with a compact Hausdorff space is paracompact [hint: adapt the proof that a product of two compact spaces is compact].

Definition 1.56 (partition of unity)**.** Let B be a Hausdorff space. A *partition of unity* is a family of continuous map $\varphi_i : B \to I = [0,1]$ $(i \in I)$ such that

(1) the family of supports $(\operatorname{supp}\varphi_i)_{i\in I}$ is locally finite,
(2) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in B$.

The support $\operatorname{supp}\varphi_i$ of φ_i is the closure of the set $\{x \in B; \varphi_i(x) \neq 0\}$. Note that the sum $\sum_{i\in I} \varphi_i(x)$ is finite, due to condition (1).

It is know that paracompact spaces admits partitions of unity. More precisely, any open cover $(U_i)_{i\in I}$ of a paracompact space admit a partition of unity *subordinated* to it, that is a partition of unity $(\varphi_i)_{i\in I}$ indexed by the same set I and such that $\operatorname{supp}\varphi_i \subset U_i$ (see [Dug78, Chap. VIII]). We will show that Hausdorff smooth manifolds with countable atlas admit (smooth) partitions of unity.

Theorem 1.57. *Let B be a paracompact space and let ξ be a G-bundle over $B \times [0,1]$. Then the map $r : B \times [0,1] \to B \times [0,1]$ defined by $r(x,t) = (x,1)$ is covered by a G-bundle morphism $(H,r) : \xi \to \xi$.*

Lemma 1.58. *If ξ is G-bundle over $B \times [0,1]$, then there exists an open cover $(U_\alpha)_{\alpha\in A}$ of B such that $\xi_{|U_\alpha\times[0,1]}$ is G-trivial for all $\alpha \in A$.*

[1] Marshall Harvey Stone (1903–1989), American mathematician.

Theorem 1.59. *Let ξ be a G-bundle over B and let Y be a paracompact space. If two continuous maps $f_0, f_1 : Y \to B$ are homotopic, then $f_0^*(\xi)$ and $f_1^*(\xi)$ are G-isomorphic over Y.*

Corollary 1.60. *Let B be a paracompact space and let $i_u : B \to B \times [0,1]$ be defined by $i_u(x) = (x,u)$ for $u = 0, 1$. Then for any G-bundle ξ over $B \times [0,1]$, the induced bundles $i_0^*(\xi)$ and $i_1^*(\xi)$ are G-isomorphic over B.*

Corollary 1.61. *Every G-bundle over a paracompact* contractible *space is trivial.*

Note 1.62. The above theorem 1.59 is the starting point of the classification of G-bundle over a given paracompact base B. Recall that this classification reduces to the classification of principal G-bundle, see exercise 1.52. Now, there exists a universal principal G-bundle $\xi_G = (E_G, p_G, B_G)$, in the sense that any principal G-bundle over a paracompact base B is induced from ξ_G by some map $B \to B_G$. Then the G-isomorphism classes of principal G-bundles over B correspond bijectively to the set of homotopy classes $[B, B_G]$. For example the $\mathbf{S}^1$-bundle $\mathbf{S}^\infty = \lim_{n\to\infty} \mathbf{S}^{2n+1} \to \mathbf{P}^\infty_{\mathbf{C}} = \lim_{n\to\infty} \mathbf{P}^n_{\mathbf{C}}$ (direct limits) is universal for $G = \mathbf{S}^1$. Moreover, given $d \geqslant 1$, if B is a CW-complex of finite dimension $\leqslant 2d$, then any map $B \to \mathbf{P}^\infty_{\mathbf{C}}$ is homotopic to some map $f : B \to \mathbf{P}^d_{\mathbf{C}}$ (from the cellular approximation theorem, see for example [Hat02, Sect. 4.1]), thus $\mathbf{S}^{2d+1} \to \mathbf{P}^d_{\mathbf{C}}$ is universal for principal $\mathbf{S}^1$-bundles over CW-complexes of dimension $\leqslant 2d$.

1.7 Vector bundles

Definition 1.63 (vector bundle)**.** Let $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. A vector bundle of rank $r \geqslant 1$ is a $\mathbf{GL}_r(\mathbf{k})$-bundle with fiber $\mathbf{k}^r$, equiped with the linear action of $\mathbf{GL}_r(\mathbf{k})$. If $\mathbf{k} = \mathbf{R}$ (resp. $\mathbf{C}$), ξ is a *real* (resp. *complex*) vector bundle. If $r = 1$, ξ is *a line bundle*.

Exercise 1.64. Let $\xi = (E, p, B)$ be a vector bundle of rank $r \geqslant 1$. Show that each fiber E_x ($x \in B$) admits a natural structure of $\mathbf{k}$-vector space of dimension r.

Exercise 1.65. Prove that a vector bundle $\xi = (E, p, B)$ of rank $r \geqslant 1$ is trivial if and only if there exists r sections $s_1, \dots, s_r : B \to E$ such that for all $x \in B$, $(s_1(x), \dots, s_r(x))$ is a basis of the fiber E_x (for the natural vector space structure of E_x, see exercise 1.64).

Examples of vector bundles are the tautological bundle $\gamma_{n,\mathbf{k}}$ over $\mathbf{P}^n_{\mathbf{k}}$ or the complex vector bundle $\mathcal{O}(k)$ over $\mathbf{P}^n_{\mathbf{C}}$ (and of course trivial vector bundles $B \times \mathbf{k}^r$).

We consider here vector bundles of arbitrary rank. So unlike the previous sections, the fiber and the structural group are not fixed, and we need to extend the notion of morphism.

Definition 1.66 (vector bundles morphism)**.** Let ξ and ξ' be vector bundles (non necessarily of the same rank). A *vector bundle morphism*, or *vector bundles map*, is a bundle map $(H, f) : \xi \to \xi'$ which is linear in each fiber (recall that there is a well-defined vecor space structure in each fiber, see exercise 1.64).

Thanks to the description by cocycles (theorem 1.46), the usual operations between vector spaces can easily be extended to vector bundles.

Definition 1.67 (dual, sum, tensor product)**.** Let ξ and ξ' be vector bundles over X of rang r et r', associated with cocyles (g_{ij}) et (g'_{ij}) defined on the same trivializing cover $\mathcal{U} = \{U_i; i \in I\}$. We define the following vector bundles over X by cocycles on $\mathcal{U}$:

(1) ξ^* associated to ${}^t g_{ij}^{-1} : U_{ij} \to \mathbf{GL}_r(\mathbf{k})$ (*dual bundle* of ξ),
(2) $\xi \oplus \xi'$ associated to $g_{ij} \oplus g'_{ij} : U_{ij} \to \mathbf{GL}_{r+r'}(\mathbf{k})$ (*sum* of ξ and ξ'),
(3) $\xi \otimes \xi'$ associated to $g_{ij} \otimes g'_{ij} : U_{ij} \to \mathbf{GL}_{rr'}(\mathbf{k})$ (*tensor product* of ξ and ξ').

It is easily checked that ξ^*, $\xi \oplus \xi'$ and $\xi \otimes \xi'$ are well-defined up to isomorphism, independently of the choice of trivializations of ξ and ξ'. In the same way, symmetric and exterior powers $\odot^k \xi$ and $\bigwedge^k \xi$ ($k \in \mathbf{N}^*$) are also defined for any vector bundle ξ.

Example 1.68. If $\mathcal{O}_n(k)$ is the complex line bundle defined at exemple 1.49, then $\mathcal{O}_n(k)^* \simeq \mathcal{O}_n(-k)$ ($k \in \mathbf{Z}$) and $\mathcal{O}_n(k) \simeq \mathcal{O}_n(1)^{\otimes k}$ ($k \in \mathbf{N}^*$).

1.8 Vector bundles over spheres

The unit (Euclidean) sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ is covered by two closed half-spheres $S_+ = \mathbf{S}^n \cap \mathbf{R}^n \times [0, \infty[$ and $S_- = \mathbf{S}^n \cap \mathbf{R}^n \times]-\infty, 0]$. Let

$$f : \mathbf{S}^{n-1} = S_+ \cap S_- \to \mathbf{GL}_r(\mathbf{k}) \qquad (r \geqslant 1, \mathbf{k} = \mathbf{R} \text{ or } \mathbf{C})$$

be a continuous map. Then the gluing construcion of theorem 1.46 applies as well to the (closed) cover $\mathbf{S}^n = S_+ \cup S_-$ and we get a well-defined vector bundle ξ_f of rank r over $\mathbf{S}^n$. More precisely, the total space $E(\xi_f)$ is the quotient of $(S_+ \times \mathbf{k}^r) \amalg (S_- \times \mathbf{k}^r)$ obtained by identifying $(x, u) \in \partial S_- \times \mathbf{k}^r$ with $(x, f(x) \cdot u) \in \partial S_+ \times \mathbf{k}^r$.

Theorem 1.69 (vector bundles over the spheres)**.** *Let $n \in \mathbf{N}^*$. Then*
(1) any real (resp. complex) vector bundle of rank $r \geqslant 1$ over $\mathbf{S}^n$ is isomorphic to the vector bundle ξ_f defined by some continuous map $f : \mathbf{S}^{n-1} \to \mathbf{GL}_r(\mathbf{R})$ (resp. $\mathbf{GL}_r(\mathbf{C})$) as above,
(2) [real case] the real vector bundles ξ_f and ξ_g are isomorphic if and only if there exists two constants $\alpha_+, \alpha_- \in \mathbf{GL}_r(\mathbf{R})$ such that $\alpha_+ f$ and $g\alpha_-$ are homotopic,
(3) [complex case] the complex vector bundles ξ_f and ξ_g are isomorphic if and only if f and g are homotopic. Thus isomorphism classes of rank $r \geqslant 1$ complex vector bundles over $\mathbf{S}^n$ correspond bijectively to $\pi_{n-1}(\mathbf{GL}_r(\mathbf{C}), *)$.

Theorem 1.70 (vector bundles on the circle)**.**
(1) [real case] *For each integer $r \geqslant 1$, there exists exactly two isomorphism classes of real vector bundles of rank r on the circle $\mathbf{S}^1$. More precisely, any rank r vector bundle over $\mathbf{S}^1$ is isomorphic either to the trivial bundle denoted by ϵ^r, either to $\epsilon^{r-1} \oplus \gamma_{1,\mathbf{R}}$ which is non-trivial.*
(2) [complex case] *Any complex vector bundle over $\mathbf{S}^1$ is trivial.*

Theorem 1.71 (complex vector bundles on $\mathbf{S}^2$)**.**
Any rank $r \geqslant 1$ complex vector bundle over $\mathbf{S}^2$ is isomorphic to $\epsilon^{r-1} \oplus \mathcal{O}_1(k)$ for some $k \in \mathbf{Z}$, where ϵ^r is the trivial bundle of rank r and $\mathcal{O}_1(k)$ is defined in example 1.49.

Exercise 1.72. An *orientable* real vector bundle of rank $r \geqslant 1$ over B is a vector bundle defined by a $\mathbf{GL}_r^+(\mathbf{R})$-valued cocycle. Prove that isomorphism classes of rank r orientable vector bundles over the sphere $\mathbf{S}^n$ correspond bijectively to $\pi_{n-1}(\mathbf{GL}_r^+\mathbf{R}), *)$.

Exercise 1.73. Show that any real line bundle over $\mathbf{S}^2$ is trivial.

Note 1.74 (Picard group). An interesting algebraic invariant arises from line bundles. Given a space X, the tensor product of (real or complex) line bundles over X induces an abelian group structure on the isomorphism classes of line bundles (the neutral element is the trivial bundle of rank 1 and the inverse map is the duality). This abelian group is called the (topological) Picard group of X, denoted by $\mathrm{Pic}^{\mathbf{k}}_{\mathrm{top}}(X)$ ($\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$). For example, the theorem 1.71 implies that

$$\mathrm{Pic}^{\mathbf{C}}_{\mathrm{top}}(\mathbf{S}^2) \simeq \mathbf{Z},$$

generated by $\mathcal{O}_1(-1)$ or $\mathcal{O}_1(1)$. The Picard group is the analogue of the ideal class group of a Dedekind ring. The setting of schemes in arithmetic geometry provides a unified definition of these two groups.

Note 1.75 (K-theory). Another very intersting algebraic invariant is the (topological) K-theory. If X is any space, the set $\mathrm{Vect}_{\mathbf{k}}(X)$ of isomorphism classes of real ($\mathbf{k} = \mathbf{R}$) or complex ($\mathbf{k} = \mathbf{C}$) vectors bundles over X, equiped with the direct sum $\oplus$, is an abelian monoïd. The Grothendieck[2] group associated to $(\mathrm{Vect}_{\mathbf{k}}(X), \oplus)$ is denoted by $KO(X)$ if $\mathbf{k} = \mathbf{R}$ and $K(X)$ if $\mathbf{k} = \mathbf{C}$. Recall that this (abelian) group is defined as the quotient of $\mathrm{Vect}_{\mathbf{k}}(X) \times \mathrm{Vect}_{\mathbf{k}}(X)$ by the following relation: $(E, F) \sim (E', F')$ if there exists $G \in \mathrm{Vect}_{\mathbf{k}}(X)$ such that $E \oplus F' \oplus G = E' \oplus F \oplus G$ (G is necessary because the cancellation law does not hold in $\mathrm{Vect}_{\mathbf{k}}(X)$). Moreover, the tensor product $\otimes$ of vector bundles induces a (commutative) ring structure on $KO(X)$ and $K(X)$. For example, from theorem 1.70, it is not difficult to see that:

$$KO(\mathbf{S}^1) \simeq \mathbf{Z}[t]/((t-1)^2, 2(t-1)) \simeq \mathbf{Z}[\eta]/(\eta^2, 2\eta), \qquad K(\mathbf{S}^1) \simeq \mathbf{Z}$$

(isomorphisms of rings), where $t = \gamma_{1,\mathbf{R}}$ is the tautological bundle over $\mathbf{P}^1_{\mathbf{R}} \simeq \mathbf{S}^1$ (and $\eta = t-1$). The Picard group $\mathrm{Pic}^{\mathbf{k}}_{\mathrm{top}}(X)$ embeds as a subgroup of the group of units $KO(X)^\times$ (if $\mathbf{k} = \mathbf{R}$) or $K(X)^\times$ (if $\mathbf{k} = \mathbf{C}$), for example $\mathrm{Pic}^{\mathbf{R}}_{\mathrm{top}}(\mathbf{S}^1) = \{1, t\} \subset KO(\mathbf{S}^1)^\times$. There is also an algebraic version of the K-theory. If A is a ring, then $K_0(A)$ is the Grothendieck group associated to the monoïd of (isomorphism classes of) finitely generated projective modules over A.

[2] Alexandre Grothendieck (1928–2014), French mathematician

2 Coverings

2.1 Definition and examples

Definition 2.1. A *covering* is a bundle $p : X \to B$ with *discrete* fiber F (i.e. the topology of F is $\mathcal{P}(F)$).

The terminology and results about general bundles apply as well in this particular case (see subsections 1.1, 1.2 and 1.3). In particular, the homotopy long exact sequence (theorem 1.20) gives the following result.

Theorem 2.2. *Let $p : (X, x) \to (B, b)$ be a pointed covering. Then*
(1) $p_* : \pi_1(X, x) \to \pi_1(B, b)$ is an injective morphism,
(2) for all $n \geqslant 2$, $p_* : \pi_n(X, x) \to \pi_n(B, b)$ is an isomorphism.

Remarks 2.3. (1) A covering is always a local homeomorphism (and surjective).
(2) A local homeomorphism (even surjective) is not necessarily a covering. For example $f :]0.2[\to \mathbf{S}^1$ defined by $f(t) = \exp(2i\pi t)$ is a surjective local homeomorphism but the fibers are not all homeomorphic.

Here are some examples of covering (see exercise 2.14 for a proof):

(1) Trivial coverings $p_1 : B \times F \to B$.
(2) $\exp : \mathbf{C} \to \mathbf{C}^*$.
(3) $p : \mathbf{R}^n \to T^n = \mathbf{S}^1 \times \ldots \times \mathbf{S}^1$ (n factors) defined by $p(t_1, \ldots, t_n) = (e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})$.
(4) $p : \mathbf{S}^1 \to \mathbf{S}^1$ defined by $p(z) = z^n$ ($n \in \mathbf{N}^*$).

Remark 2.4. If $\mathrm{card}(F) \geqslant 2$, then the total space X of a trivial covering with fiber F is not connected. Thus the examples (2), (3) and (4) above *are not trivial.*

Exercise 2.5. Prove that a continuous map $p : X \to B$ is a covering with fiber F if and only if the following condition is satisfied: for all $b \in B$, there exists an open neighborhood U of b and a family $s_f : U \to X$ ($f \in F$) of sections of p over U such that
(i) $s_f(U)$ is open for all $f \in F$,
(ii) $s_f(U) \cap s_g(U) = \varnothing$ if $f \neq g$,
(iii) $p^{-1}(U) = \bigcup_{f \in F} s_f(U)$.

Proposition 2.6. *Let $p : X \to B$ be a covering with B connected and locally connected (resp. B connected and locally path-connected). Then for any connected (resp. path-connected) component C of X, the restriction $p_{|C} : C \to B$ is a covering.*

The sections of a covering have very special properties, due to the discreteness of the fiber.

Proposition 2.7. *If $s : B \to X$ is a section of a covering $p : X \to B$, then $s(B)$ is open and closed in X, and s induces a homeomorphism of B to $s(B)$.*

Corollary 2.8. *Let $p : X \to B$ be a covering with* connected *total space X. Then p admits a section if and only if p is a homeomorphism.*

Corollary 2.9. *Let $s_1, s_2 : B \to X$ be two sections of a covering $p : X \to B$. Then the set $\{b \in B; s_1(b) = s_2(b)$ is open and closed in B. In particular, if B is connected, then two sections which coincide in some point are equal.*

2.2 Free and discontinuous actions

Definition 2.10. Let X be a space and let Γ be a subgroup of group $\mathrm{Homeo}(X)$ (the group of homeomorphisms of X). Consider the following properties:

(1) for all $x \in X$, there exists an open neighborhood Ω of x such that $\gamma(\Omega) \cap \Omega = \varnothing$ for all $\gamma \neq \mathrm{Id}_X$.
(2) for all $(x, x') \in X \times X$, $x' \notin \Gamma \cdot x$, there are open neighborhoods Ω, Ω' of x, x' such that $\gamma(\Omega) \cap \Omega' = \varnothing$ for all $\gamma \in \Gamma$.

We say that the action of Γ on X is *free and discontinuous* (resp. *free and properly discontinuous*) if it satisfies (1) (resp. (1) and (2)).

Proposition 2.11 (fundamental example of covering)**.** *Let X be a space and let Γ be a subgroup of* $\mathrm{Homeo}(X)$*. If the action of Γ is free and discontinuous (property (1)), then the projection $X \to X/\Gamma$ is a covering. If the action of Γ further satisfies (2), then X/Γ is Hausdorff.*

Examples (2), (3) and (4) of §2.1 are obtained by the fundamental proposition 2.11 (see exercise 2.14).

Note that by definition Γ acts transitively on the fiber of the covering $X \to X/\Gamma$, thus this covering is of a particular type, called *Galois (or normal, or regular)* covering (see definition ??). But this fundamental example is very useful, because if B is a "good" space (B is connected and locally simply connected), then it can be proved that the total space of any covering of B is obtained by this construction (see ??).

Remark 2.12. If X is locally compact and $\Gamma \subset \mathrm{Homeo}(X)$ acts properly (for all compact K of X the set $\{\gamma \in \Gamma; \gamma(K) \cap K \neq \varnothing\}$ is finite) and freely (the stabilizers of Γ are trivial), then we can then verify that (1) and (2) are satisfied and the proposition 2.11 applies.

Remark 2.13. Hypothesis (2) only serves to ensure the separation of the quotient (see exercise 2.15).

Exercise 2.14. Let X be a space and let Γ be a subgroup of $\mathrm{Homeo}(X)$ acting freely on X. If $f : X \to B$ is a surjective (continuous) map such that $f(x) = f(x')$ if and only if $x' \in \Gamma \cdot x$, then f induces a continuous bijection $\overline{f} : X/\Gamma \to B$.
1) Assuming that f is a local homeomorphism, prove that $\overline{f}$ is a homeomorphism and that $f : X \to B$ is a covering.
2) Deduce that examples (2–4) of §2.1 are coverings (give X and Γ).

Exercise 2.15. We consider the following matrix group

$$\Gamma = \left\{ \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} ; n \in \mathbf{Z} \right\}$$

acting on $\mathbf{R}^2\backslash\{0\}$ by matrix multiplication. Show that this action satisfies condition (1) of definition 2.10, but not condition (2). Thus $\mathbf{R}^2\backslash\{0\}/\Gamma$ is a non Hausdorff surface.

Exercise 2.16 (quasi-circle). Set $A = \{x + i\sin(\pi/x); x \in]0,1]\} \subset \mathbf{C}$ and define

$$X = \bigcup_{n\in\mathbf{Z}} (n + \overline{A}) \subset \mathbf{C}$$

equiped with the induced topology (where $\overline{A}$ is the closure of A in $\mathbf{C}$).
1) Draw a picture of X. Show that X is connected but not path-connected.
2) Let $\gamma \in \mathrm{Homeo}(X)$ by defined by $\gamma(x) = x + 1$ $(x \in X)$. Show that the action of $\Gamma = \{\gamma^n; n \in \mathbf{Z}\}$ on X is free and properly discontinuous. The base $B = X/\Gamma$ of the covering $X \to X/\Gamma$ is the *quasi-circle*. Prove that B is path-connected.

2.3 Lifting loops: action on the fiber

Due to the discreteness of the fiber, the lifting of loops and homotopies has special properties for coverings. Let $p : X \to B$ be a covering. Given $b \in B$ and $x \in p^{-1}(b)$, then

- for every path $\alpha : I = [0,1] \to B$ with $\alpha(0) = b$ (in particular for every loop $\alpha \in \Omega(B,b)$), there exists a *unique* lift $\tilde{\alpha} : [0,1] \to X$ of α starting at x (that is $\tilde{\alpha}(0) = x$),
- for every homotopy H of paths with fixed endpoints $b = H(0,u)$ and $b' = H(1,u)$ $(u \in I)$, there exists a *unique* homotopy $\widetilde{H}$ lifting H with $x = \widetilde{H}(0,u)$ $(u \in I)$ and $\widetilde{H}$ *has still fixed endpoints.*

Proposition 2.17. *Let $p : X \to B$ be a covering. If $b \in B$, then*

(1) the lifting of loops induces a right action of $\pi_1(B,b)$ on the fiber $p^{-1}(b)$;
(2) the stabilizer of any $x \in p^{-1}(b)$ is $p_*(\pi_1(X,x))$;
(3) the orbit of $x \in p^{-1}(b)$ is $p^{-1}(b) \cap C_x$, where C_x is the path component of x in X.

Corollary 2.18. *Let $p : X \to B$ be a covering with X path-connected. Then p is a homeomophism if and only if $p_* : \pi_1(X,x) \to \pi_1(B,b)$ is surjective (that is p_* is an isomorphism).*

Theorem 2.19 (lifting criterion)**.** *Let $p : (X,x) \to (B,b)$ be a pointed covering and let $f : (Y,y) \to (B,b)$ be some pointed (continuous) map where Y is* connected and locally path-connected*. Then there exists a continous pointed map $\tilde{f} : (Y,y) \to (X,x)$ such that $f = p \circ \tilde{f}$ if and only if*

$$f_*(\pi_1(Y,y)) \subset p_*(\pi_1(X,x)).$$

Corollary 2.20. *Let $p : X \to B$ be a covering. If Y is simply connected and locally path-connected, then any continuous map $f : Y \to B$ lifts to X.*

For example, Picard's theorem (see example 1.7) in complex geometry follows from the lifting criterion because the modular invariant $\lambda : \mathbf{H} \to \widehat{\mathbf{C}} \smallsetminus \{0, 1, \infty\}$ is a (holomorphic) covering. Thus, as $\mathbf{C}$ is simply connected, we can lift any holomorphic map $f : \mathbf{C} \to \widehat{\mathbf{C}} \smallsetminus \{0, 1, \infty\}$ to $\tilde{f} : \mathbf{C} \to \mathbf{H}$ such that $f = \lambda \circ \tilde{f}$ and $\tilde{f}$ is holomorphic. But the upper halfplane $\mathbf{H}$ is bi-holomorphic to the unit disk, so $\tilde{f}$ is constant according to Liouville's theorem.

Now we have a strong version of corollary 1.61 (triviality of G-bundle over a paracompact contractible base) for coverings.

Theorem 2.21. *Let B be simply connected and locally path-connected. Then any covering $p : X \to B$ is trivial.*

2.4 Automorphisms

Definition 2.22 (deck group)**.** An *automorphism* (or *deck transformation*) of a covering $p : X \to B$ is an isomorphism $X \to X$ over B. The automorphism group of $p : X \to B$ is denoted by $\mathrm{Aut}_B(X)$, and sometimes called the *deck group* of the covering.

Theorem 2.23 (automorphism group of a covering)**.** *Let $p : (X, x) \to (B, b)$ be a pointed covering with X connected and locally path-connected. If N is the normalizer of $p_*(\pi_1(X, x))$ in $\pi_1(B, b)$, then the lifting of loops of (B, b) induces an exact sequence*

$$1 \longrightarrow p_*(\pi_1(X, x)) \longrightarrow N \longrightarrow \mathrm{Aut}_B(X) \longrightarrow 1 \, .$$

In particular if $\mathrm{Aut}_B(X) \simeq N/p_*(\pi_1(X, x))$.

Definition 2.24 (Galois covering)**.** A covering $p : X \to B$ is *a Galois covering* (or is *normal or regular*) if

(1) X is connected,
(2) $\mathrm{Aut}_B(X)$ acts transitevely in each fiber of the covering.

Example 2.25. Let X be a connected space. If Γ is a subgroup of $\mathrm{Homeo}(X)$ acting freely and discontinuously on X, then the projection $X \to X/\Gamma$ is a Galois covering with deckgroup Γ.

Proposition 2.26. *Let $p : (X, x) \to (B, b)$ be a pointed covering with X connected and locally path-connected. Then p is a Galois covering if and only if $p_*(\pi_1(X, x)$ is a normal subgroup of $\pi_1(B, b)$.*

The following result provides a useful way to compute the fundamental group in many intersting cases.

Theorem 2.27. *Let (X, x) be a path-connected pointed space and let $\Gamma \subset \mathrm{Homeo}(X)$ acting freely and discontinuously on X. By proposition 2.11, the projection $p : X \to B = X/\Gamma$ is a covering. Set $b = p(x) \in B$. Then the lifting of loops of (B, b) induces an exact sequence*

$$1 \longrightarrow p_*(\pi_1(X, x)) \longrightarrow \pi_1(B, b) \longrightarrow \Gamma \longrightarrow 1 \ .$$

In particular if X is simply connected, then we have an isomorphism $\pi_1(B, b) \simeq \Gamma$.

Examples 2.28. 1) $X = \mathbf{R}^n$, $\Gamma = \mathbf{Z}^n$ acting by integer translations. We get $\pi_1(T^n, *) \simeq \mathbf{Z}^n$, in particular $\pi_1(\mathbf{S}^1, 1) \simeq \mathbf{Z}$.

2) $X = \mathbf{S}^n \subset \mathbf{R}^{n+1}$, $\Gamma = \{\mathrm{Id}, \alpha\}$ were α is the antipodal map. We get $\pi_1(P^n_{\mathbf{R}}, *) \simeq \mathbf{Z}/2\mathbf{Z}$.

Note 2.29 (universal covering and Galois correspondance). Let B be a connected and locally simply connected space (for example a differentiable manifold). It can be shown that such B admits a *simply connected covering* $p : \widetilde{B} \to B$, unique up to isomorophism, called *universal covering* of B. The space B is identified with the quotient of $\widetilde{B}$ by the action of the deck group $\Gamma = \mathrm{Aut}_B(\widetilde{B})$. Now, the choice of a base point $\tilde{b} \in \widetilde{B}$ gives an isomorphism $\pi_1(B, b) \simeq \Gamma$ as in theorem 2.27 (where $b = p(\tilde{b})$) and we have two bijective correspondances:

- between *connected pointed coverings* of (B, b) (up to pointed isomorophism) and the subgroups of Γ (every subgroup H of Γ is assoiated to the pointed quotient $(\widetilde{B}, \tilde{b})/H$),
- between *connected (non pointed) coverings* of (B, b) (up to isomorophism) and the conjugation classes of subgroups of Γ.

Moreover, Galois coverings correspond to normal subgroups of Γ. So the theory of coverings is a Galoisian theory in which the univeral covering is the analogue of the algebraic closure in field theory.

Chapter IV

Differential manifolds (short introduction)

A *real differential manifold* is a space obtained by gluing open subsets of $\mathbf{R}^n$ with local diffeormorphisms. A general reference for differential topology is [Hir94], in particular Chapter I for background.

1 Real differential manifolds

1.1 Definition and examples

Definition 1.1 (chart)**.** Let X be a space. A *chart* of dimension $n \geqslant 0$ (or $\mathbf{R}^n$-valued chart) is a couple (U, φ) where $\varphi : U \to \varphi(U) \subset \mathbf{R}^n$ is a homeomorphism from an open subset u of X onto an open subset $\varphi(U)$ of $\mathbf{R}^n$. The inverse map $\varphi^{-1} : \varphi(U) \to U$ is called a *local coordinate*. A chart (U, φ) is *centered* at $x \in U$ if $\varphi(x) = 0$.

Let Ω (resp. Ω') be an open set in $\mathbf{R}^n$ (resp. in $\mathbf{R}^{n'}$). Recall that a map $f : \Omega \to \Omega'$ is of class $\mathcal{C}^r$ ($r \in \mathbf{N}$) if it is differentiable up to order r with $D^{(r)}f$ continuous. This is equivalent to require that f admits continuous partial derivatives up to order r. The class $\mathcal{C}^0$ is just the class of continuous maps. A map is of class $\mathcal{C}^\infty$, or *smooth*, if it is of class $\mathcal{C}^r$ for all $r \in \mathbf{N}$.

Definition 1.2 (atlas)**.** Let X be a space and let $r \in \{0, 1, \ldots, \infty\}$. A $\mathcal{C}^r$-atlas (of dimension $n \in \mathbf{N}$) on X is a set $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ of $\mathbf{R}^n$-valued charts on X such that

(1) all the transition maps $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are $\mathcal{C}^r$-diffeomorphisms (for $(i, j) \in I^2$ with $U_i \cap U_j \neq \varnothing$),

(2) $X = \cup_{i \in I} U_i$.

Exercise 1.3. 1) Two $\mathcal{C}^r$-atlases (of dimension n) $\mathcal{A}$ and $\mathcal{B}$ on X are equivalent if $\mathcal{A} \cup \mathcal{B}$ is a $\mathcal{C}^r$-atlas on X (we write $\mathcal{A} \sim \mathcal{B}$). Show that equivalence of atlases is an equivalence

relation.
2) Prove that any $\mathcal{C}^r$-atlases of dimension n on X is contained in a unique *maximal* $\mathcal{C}^r$-atlases of dimension n on X.

Definition 1.4 (manifold structure)**.** Let X be a space and let $r \in \{0, 1, \dots, \infty\}$, $n \in \mathbf{N}$. A structure of $\mathcal{C}^r$-manifold of dimension n on X is an equivalence class of $\mathcal{C}^r$-atlases (of dimension n) on X, or equivalently a maximal $\mathcal{C}^r$-atlas (of dimension n) on X (see exercice 1.3). A $\mathcal{C}^r$-manifold of dimension n is a space equiped with a $\mathcal{C}^r$-manifold structure of dimension n.

Any given atlas $\mathcal{A}$ on a space X defines a manifold structure on X (the equivalence class of $\mathcal{A}$, or the unique maximal atlas containing $\mathcal{A}$). Manifold structures are usualy defined in this way. Of course all notions that are independant of the choice of a particular atlas in the equivalence class (for example the notion of morphism, see below) will be notions associated to the manifold structure.

Remark 1.5 (analytic regularity)**.** A map $g : \Omega \to \mathbf{R}$ (Ω open set of $\mathbf{R}^n$) is *analytic* if for all $a \in \Omega$ there exists a power series $\sum_{\alpha \in \mathbf{N}^n} a_\alpha x^\alpha$ which converges for $|x - a| < \epsilon$ (for some ϵ depending on a) such that $g(x) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha x^\alpha$ for $|x - a| < \epsilon$. A map $f : \Omega \to \Omega'$ as above is *analytic* (or of class $\mathcal{C}^\omega$) if all its components are analytic. The above definitions of atlases and manifold structure work with analytic regularity. So we also have a class of *analytic real manifolds*.

Here are some examples of $\mathcal{C}^\infty$ (smooth) manifolds.

The model space $\mathbf{R}^n$. The space $\mathbf{R}^n$ has of course a natural structure of smooth manifold defined by the atlas with a single chart $\mathcal{A} = \{(\mathbf{R}^n, \mathrm{Id})\}$.

Open subsets of manifolds. Let $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ be a $\mathcal{C}^r$ atlas of dimension n on a space X. Then any open subset U of X has a (induced) manifold structure of the same type (dimension and regularity) defined by $\mathcal{A}_{|U} = \{(U \cap U_i, \varphi_{i|U \cap U_i}); i \in I\}$.

0-dimensional manifolds. They are exactly the discrete spaces (exercise).

Spheres. Let $\mathbf{S}^n$ be the unit sphere of the Euclidean space $\mathbf{R}^{n+1}$ (equiped with the usual inner product), that is $\mathbf{S}^n = \{(x, t) \in \mathbf{R}^n \times \mathbf{R}; |x|^2 + t^2 = 1\}$. Let ϕ, ψ be the stereographic projections defined outside the north pole $N = (0, 1)$ and the south pole $S = (0, -1)$ respectively.

Proposition 1.6. *The stereographic projections ϕ and ψ from the north pole $N = (0, 1)$ and from the south pole $S = (0, -1)$ respectively define a smooth manifold structure of dimension n on $\mathbf{S}^n$.*

Recall that $\phi : \mathbf{S}^n \smallsetminus \{N\} \to \mathbf{R}^n$ and $\psi : \mathbf{S}^n \smallsetminus \{S\} \to \mathbf{R}^n$ are given by

$$\phi(x,t) = \frac{x}{1-t} \quad (t \neq 1) \qquad \psi(x,t) = \frac{x}{1+t} \quad (t \neq -1).$$

Real projective spaces. In the real projective space $\mathbf{P}^n_{\mathbf{R}}$ consider the affine open subsets $U_i = \{[x_0, \dots, x_n] \in \mathbf{P}^n_{\mathbf{R}}; x_i \neq 0\}$ $(i = 0, \dots, n)$. Then the usual affine charts on $\mathbf{P}^n_{\mathbf{R}}$ are defined by

$$\varphi_i(x) = (\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}) \qquad (x = [x_0, \dots, x_n] \in U_i).$$

Proposition 1.7. *The set of affine charts $\{(U_i, \varphi_i); i = 0, \dots, n\}$ is a smooth atlas of dimension n on the projective space $\mathbf{P}^n_{\mathbf{R}}$.*

Tori. Consider the n-dimensional torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ $(n \geqslant 1)$. Recall that the projection map $p : \mathbf{R}^n \to T^n$ is a covering. Let $(U_i)_{i \in I}$ be a trivializing cover of the torus T^n. For each $i \in I$, choose a section s_i defined over U_i. Then s_i induces a homeomorphism $s_i : U \to s_i(U_i)$, where $s_i(U_i)$ is open in $\mathbf{R}^n$.

Proposition 1.8. *The set $\{(U_i, s_i); i \in I\}$ is a smooth atlas of dimension n on the torus T^n (the transition maps are locally given by the action of $\mathbf{Z}^n$).*

Product of two manifolds. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be $\mathcal{C}^r$-manifolds of dimension n and m respectively, where $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ and $\mathcal{B} = \{(V_j, \psi_j); j \in J\}$. Then $\mathcal{A} \times \mathcal{B} = \{(U_i \times V_j, \varphi_i \times \psi_j); (i,j) \in I \times J\}$ is a $\mathcal{C}^r$-atlas of dimension $n + m$ on the product $X \times Y$.

Exercise 1.9 (the line with two origins). Let X be the space obtained from $\mathbf{R} \times \{1, 2\}$ by identifying $(x, 1)$ with $(x, 2)$ for all $x \neq 0$.
1) Prove that X is connected and non Hausdorff.
2) Let $\pi : \mathbf{R} \times \{1, 2\} \to X$ be the projection map. For $i = 1, 2$, set $U_i = \pi(\mathbf{R} \times \{i\})$ and define $\varphi_i : U_i \to \mathbf{R}$ by $\varphi_i(\pi(x, i)) = x$. Prove that $\mathcal{A} = \{(U_i, \varphi_i); i = 1, 2\}$ is a smooth atlas of dimension 1 on X.

Exercise 1.10 (the long line). Let ω_1 be the least non countable ordinal. The set $Y = \omega_1 \times [0, 1[$ is ordered with the lexicograhic order and equiped with the order topology.
1) Prove that $X = Y \smallsetminus \{(0, 0)\}$ is a 1-dimensional smooth manifold.
2) Show that X does not admit any countable atlas.
The space Y is called *the long ray* (or half long line). Gluing two copies of Y on the point $(0, 0)$ gives the *long line*.

Convention. Up to now, all manifolds are assumed to be Hausdorff and defined by a countable atlas.

Remark 1.11. Let $0 \leqslant s < r \leqslant \infty$. Then any $\mathcal{C}^r$-atlas $\mathcal{A}$ over X is contained in a unique maximal $\mathcal{C}^s$-atlas $\mathcal{B}$ on X ($\mathcal{B}$ is the set of charts $\mathcal{C}^s$ -compatible with $\mathcal{A}$). In other words, every $\mathcal{C}^r$ manifold is a $\mathcal{C}^s$ manifold for all $s < r$.

Notes 1.12 (comment about the regularity). There is a partial converse property (see [Hir94, thm. 2.9]): if $1 \leqslant s < r \leqslant \infty$, then any $\mathcal{C}^s$-structure contains (as an atlas) a $\mathcal{C}^r$-structure, unique up to $\mathcal{C}^r$-diffeoorphism (see below Definiton 1.19 for the notion of diffeomorphism).

This last property implies that the study of $\mathcal{C}^1$-manifolds reduces to that of $\mathcal{C}^\infty$ or smooth manifolds (see again [Hir94, thm. 2.10]). The following non-elementary results show that the theory of topological manifolds (TOP) differs from that of smooth manifolds (DIFF), except in low dimensions:

- there are topological manifolds admitting no differential structure [M. Kervaire[1], 1960]; this phenomena already occurs in dimension 4 [M. Freedman[2], 1982],
- there exists topological manifolds which admit several (non diffeomorphic) smooth structures [J. Milnor[3], 1956]. For example, the sphere $\mathbf{S}^7$ admits exactly 28 (non diffeomorphic) smooth structures,
- if $n \leqslant 3$, any topological manifold (Hausdorff and with countable atlas) admits a $\mathcal{C}^\infty$-structure, unique up to C^∞-diffeomorphism (see [Moi77]).

1.2 Submanifolds

Definition 1.13 (submanifold)**.** Let X be a $\mathcal{C}^r$-manifold of dimension $n \geqslant 0$. A subset (non empty) $Y \subset X$ is a *$\mathcal{C}^r$-submanifold of dimension d* of X if for every point $y \in Y$ there exists some chart (U, φ) (called a *submanifold chart*) such that

$$\varphi(y) = 0, \qquad \varphi(U \cap Y) = \varphi(U) \cap \{0\} \times \mathbf{R}^{n-d} \subset \mathbf{R}^n = \mathbf{R}^d \times \mathbf{R}^{n-d}.$$

The integer $q = n - d$ is called the *codimension* of Y in X.

Example 1.14. The unit sphere $\mathbf{S}^n$ is a smooth submanifold of dimension n of $\mathbf{R}^{n+1}$ (the stereographic projections extend to submanifold charts for $\mathbf{S}^n$, exercise).

Many examples will be given later (see Corallary 1.26).

Let Y be a $\mathcal{C}^r$-submanifold (of dimension d) of a $\mathcal{C}^r$-manifold X (of dimension n). Denote by $\pi : \mathbf{R}^d \times \mathbf{R}^{n-d} \to \mathbf{R}^d$ the first projection. If (U, φ) is a submanifold chart for Y, we call $(Y \cap U, \pi \circ \varphi_{|Y \cap U})$ the "restriction" of (U, φ) to Y (it is a chart of Y).

Proposition 1.15. *Let X be a $\mathcal{C}^r$-manifold and let Y be a $\mathcal{C}^r$-submanifold of dimension $d \leqslant \dim X$ of X. Then the set of restrictions of all submanifold chart is an $\mathcal{C}^r$-atlas on Y. Thus Y has a manifold structure of dimension d.*

[1] Michel André Kervaire (1927–2007), French mathematician.
[2] Michael Hartley Freedman (born in 1951): American mathematician.
[3] John Willard Milnor (born in 1931): American mathematician.

1.3 Morphisms and diffeomorphisms

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be $\mathcal{C}^r$-manifolds of dimension n and m respectively, where $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ and $\mathcal{B} = \{(V_j, \psi_j); j \in J\}$. Let $f : X \to Y$ be some continuous map. If $U_i \cap f^{-1}(V_j) \neq \varnothing$, the map

$$\psi_j \circ f \circ \varphi_i^{-1} : U_i \cap f^{-1}(V_j) \to \psi_j(V_j)$$

is called a *local expression* of f.

Definition 1.16 (morphism)**.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be $\mathcal{C}^r$-manifolds of dimension n and m respectively. A map $f : X \to Y$ is *of class* $\mathcal{C}^r$ (or is a $\mathcal{C}^r$*-morphism*) if all its local expression as above are of class $\mathcal{C}^r$.

Proposition 1.17. *Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a $\mathcal{C}^r$-morphism. If $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, then $f : (X, \mathcal{A}') \to (Y, \mathcal{B}')$ is a $\mathcal{C}^r$-morphism. Thus we have a well-defined notion of $\mathcal{C}^r$-morphism for $\mathcal{C}^r$-manifolds.*

Proposition 1.18 (chain rule)**.** *Let X, Y, Z be manifods of class $\mathcal{C}^r$. Il $f : X \to Y$ and $g : Y \to Z$ are $\mathcal{C}^r$-morphisms, then $g \circ f : X \to Z$ is a $\mathcal{C}^r$-morphism.*

Definition 1.19 (diffeomorphism)**.** Let X, Y be $\mathcal{C}^r$-manifolds. A map $f : X \to Y$ is a *diffeomorphism* if it is bijective with f and f^{-1} of class $\mathcal{C}^r$. Two $\mathcal{C}^r$-manifolds are $\mathcal{C}^r$-diffeomorphic if there exists a $\mathcal{C}^r$-diffeomorphism $f : X \to Y$.

Note that there exists bijective $\mathcal{C}^r$-morphisms f such that f^{-1} is not $\mathcal{C}^r$ (for example $f(x) = x^3$, $f : \mathbf{R} \to \mathbf{R}$ is $\mathcal{C}^\infty$ but f^{-1} is just continuous). Thus a bijective $\mathcal{C}^r$-morphism need not to be a diffeormorphism.

Proposition 1.20 (invariance of the dimension)**.** *Let X, Y be $\mathcal{C}^r$-manifolds with $r \geqslant 1$. If X and Y are $\mathcal{C}^r$-diffeomorphic then they have the same dimension.*

Remark 1.21. This result is still true for $r = 0$ but more difficult (it is Brouwer domain invariance theorem, see [Dug78]).

The *rank* at some point $p \in X$ of a $\mathcal{C}^1$-morphism $f : X \to Y$ is the rank of one (or any) local expression of f, at the corresponding point. Recall that the rank of a usual differentiable map f (between open sets of $\mathbf{R}^n$ and $\mathbf{R}^m$) a some point x is the rank of the linear map $D_x f$.

Definition 1.22 (morphisms of maximal rank)**.** Let $f : X \to Y$ be a $\mathcal{C}^1$-morphism. Let $p \in X$, denote by $\mathrm{rk}_x f$ the rank of f at p. We say that

(1) f is an *immersion at* $p \in X$ if $\mathrm{rk}_p f = \dim X$ (thus $\dim X \leqslant \dim Y$),
(2) f is a *submersion at* $p \in X$ if $\mathrm{rk}_p f = \dim Y$ (thus $\dim Y \leqslant \dim X$),
(3) f is *etale at* $p \in X$ if f is a immersion and a submersion at p (thus $\dim X = \dim Y$).

The local structure of maximal rank morphisms (at some point) follows easily from the local inversion theorem.

Proposition 1.23 (local structure of immersions)**.** *Let X, Y be $\mathcal{C}^r$-manifolds of dimension n, m respectively, with $r \geqslant 1$. If $f : X \to Y$ is a $\mathcal{C}^r$-immersion at some point $p \in X$, then for any chart (U, φ) at p there exists a chart (V, ψ) at $f(p)$ such that $\psi \circ f \circ \varphi^{-1}(x) = (x, 0) \in \mathbf{R}^n \times \mathbf{R}^{m-n}$ for all $x \in \varphi(U \cap f^{-1}(V))$.*

Proposition 1.24 (local structure of submersions)**.** *Let X, Y be $\mathcal{C}^r$-manifolds of dimension n, m respectively, with $r \geqslant 1$. If $f : X \to Y$ is a $\mathcal{C}^r$-submersion at some point $p \in X$, then for any chart (V, ψ) at $f(p)$ there exists a chart (U, φ) at p such that $\varphi(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}(x, y) = x$ for all $(x, y) \in \varphi(U) \subset \mathbf{R}^m \times \mathbf{R}^{n-m}$.*

Corollary 1.25. *If $f : X \to Y$ is a $\mathcal{C}^r$-morphism with $r \geqslant 1$, then the set of points $p \in X$ where f is an immersion (resp. f is a submersion) is open in X.*

Corollary 1.26 (fibers of submersion)**.** *Let $f : X \to Y$ be a $\mathcal{C}^r$-submersion (that is a submersion at all $p \in X$) with $r \geqslant 1$. Then for all $q \in f(X)$, the (non empty) fiber $f^{-1}(q)$ is a $\mathcal{C}^r$-submanifold of of X of codimension* $\dim Y$.

Example 1.27 (unit sphere)**.** The map $f : \mathbf{R}^{n+1} \smallsetminus \{0\} \to \mathbf{R}$ given by $f(x) = \sum_{i=1}^{n+1} x_i^2$ is a smooth submersion. Thus the unit sphere $\mathbf{S}^n = f^{-1}(1)$ is a smooth submanifold of $\mathbf{R}^{n+1}$.

Exercise 1.28 (classical groups). Show that the classical groups $\mathbf{SL}_n(\mathbf{R})$, $\mathbf{O}(n)$, $\mathbf{Sp}_{2g}(\mathbf{R})$ (the symplectic group) are submanifolds of the vector space of square matrices [hint: use Corallary 1.26]. What are their dimensions?

Definition 1.29 (embedding)**.** Let X, Y be smooth manifolds. An *smooth embedding* is a map $f : X \to Y$ such that

(1) f is a smooth immersion,
(2) f is injective,
(3) f induces a homeomorphism between X and $f(X)$ (equiped with the induced topology).

Remark 1.30. If X a compact smooth manifold, then conditions (1) and (2) imply (3).

Exercise 1.31. 1) Show that the map $f : \mathbf{R}^n \to \mathbf{C}^n \simeq \mathbf{R}^{2n}$ given by $f(t_1, \dots, t_n) = (e^{2i\pi t_1}, \dots, e^{2i\pi t_n})$ induces a smooth embedding of the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ into $\mathbf{R}^{2n}$ $(n \geqslant 1)$.
2) Construct a smooth embedding of the 2-torus T^2 in $\mathbf{R}^3$.

Exercise 1.32. For $u \in \mathbf{S}^n \subset \mathbf{R}^{n+1}$, denote by $f(u) \in M_{n+1}(\mathbf{R})$ the matrix of the orthogonal (for the usual inner product) projection of $\mathbf{R}^{n+1}$ onto the line $\mathbf{R}u$ spanned by u. Prove that the map $f : \mathbf{S}^n \to M_{n+1}(\mathbf{R})$ induces a smooth embedding of the real projective space $\mathbf{P}^n_{\mathbf{R}}$ into $M_{n+1}(\mathbf{R})$ [hint: $f(u) = uu^t$].

2 Differential forms

From now on, all the manifolds are smooth, that is $\mathcal{C}^\infty$ (and of course Hausdorff with a countable atlas).

2.1 Tangent bundle. Vector fields

Let Σ be a submanifold of $\mathbf{R}^n$ and let $p \in \Sigma$. Then the tangent subspace of Σ at the point p is the set $T_p\Sigma$ of all velocity vectors $c'(0)$ of (small) $\mathcal{C}^1$-curves $c :]-\epsilon, \epsilon[\to \Sigma$ with $c(0) = p$. This set $T_p\Sigma$ is a linear subspace of $\mathbf{R}^n$.

Now let X be a n-dimensional smooth manifold. If $p \in X$, we denote by $\mathcal{C}_p$ the set of (small) $\mathcal{C}^1$-curves $c :]-\epsilon, \epsilon[\to X$ with $c(0) = p$ ($\epsilon > 0$ is not fixed). Two curves $c_1, c_2 \in \mathcal{C}_p$ are *equivalent* is there exists some chart (U, φ) of X such that $p \in U$ and

$$(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0).$$

This condition, written $c_1 \sim c_2$, is clearly independant of the choice of the chart (U, φ) and the relation $\sim$ is an equivalence relation on $\mathcal{C}_p$. We define T_pX as the quotient set $T_pX = \mathcal{C}_p/\sim$ and TX as the disjoint union of all T_pX for $p \in X$:

$$TX = \coprod_{p \in X} T_pX = \bigcup_{p \in X} \{p\} \times T_pX.$$

Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha); \alpha \in A\}$ be a smooth atlas of X. For each $\alpha \in A$ we have a bijection $\Psi_\alpha : U_\alpha \times \mathbf{R}^n \to TX_{|U_\alpha} = \amalg_{p \in U_\alpha} T_pX$ defined by

$$\Psi_\alpha(p, u) = (p, [t \mapsto \varphi_\alpha^{-1}(\varphi_\alpha(p) + tu)]),$$

where $[c]$ denotes the class of a curve $c \in \mathcal{C}_p$. Now, the set TX can be topologized by the $(\Psi_\alpha)_{\alpha \in A}$: a subset $\Omega \subset TX$ is open if $\Psi_\alpha^{-1}(\Omega \cap TX_{|U_\alpha})$ is open for all $\alpha \in A$.

Proposition 2.1 (construction of the tangent bundle)**.** *(1) With the topology above, the natural projection $TX \to X$ is a vector bundle of rank n associated to the (smooth) cocycle $g_{\alpha\beta}(p) = D_{\varphi_\beta(p)}(\varphi_\alpha \circ \varphi_\beta^{-1})$ ($p \in U_\alpha \cap U_\beta$).*

(2) The set of charts $\{(TX_{|U_\alpha}, (\varphi_\alpha \times \mathrm{Id}) \circ \Psi_\alpha^{-1}; \alpha \in A\}$ is a smooth atlas on TX, for which the projection $TX \to X$ is smooth. Thus TX is a smooth manifold of dimension $2n$.

(3) Equivalent altases $\mathcal{A} \sim \mathcal{A}'$ on X define the same toplogy and $\mathcal{C}^\infty$-structure on TX.

This (smooth) bundle $TX \to X$ is called the *tangent bundle of X*. The fiber T_pX is the *tangent space at $p \in X$*. Recall that the bundle structure of TX induces a well-defined vector space structure on T_pX.

Example 2.2. Let $\mathbf{S}^n$ be the unit sphere of $\mathbf{R}^{n+1}$. Then TX is isomorphic (as a smooth vector bundle) with $\{p, u) \in \mathbf{S}^n \times \mathbf{R}^{n+1}; \langle p, u \rangle = 0\} \subset \mathbf{S}^n \times \mathbf{R}^{n+1}$.

The (smooth) sections of TX are called (smooth) *vector fields on* X. The space of smooth vector fields on X is denoted by $\mathcal{X}(X) = \Gamma^\infty(TX)$. It is a module over the ring $\mathcal{C}^\infty(X) = \mathcal{C}^\infty(X, \mathbf{R})$ of smooth functions on X.

Example 2.3 (vector fields on a open subset of $\mathbf{R}^n$)**.** Let U be an open subset of $\mathbf{R}^n$, equiped with the usual basis $(e_1, \dots, e_n)$. Then $TU = U \times \mathbf{R}^n$ is the trivial bundle of rank n and every $\xi \in \mathcal{X}(U)$ can be written as

$$\xi = \sum_{i=1}^{n} X_i \partial_i,$$

where $\partial_i \in \mathcal{X}(U)$ is the "constant" vector field defined by $\partial_i(x) = (x, e_i)$ for $x \in U$ and $X_i \in \mathcal{C}^\infty(U)$ $(i = 1, \dots, n)$. This means that $\mathcal{X}(U)$ is a free $\mathcal{C}^\infty(U)$-module with basis $(\partial_1, \dots, \partial_n)$.

In general, any vector field on a manifold X with atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha); \alpha \in A\}$ is given by local vectors fields $\xi^\alpha \in \mathcal{X}(\varphi_\alpha(U_\alpha))$ $(\alpha \in A)$ with the gluing condition (see p. 32 the general gluing condition for sections of G-bundles):

$$\xi_\alpha(x_\alpha) = D_{x_\beta}(\varphi_\alpha \circ \varphi_\beta^{-1}) \cdot \xi_\beta(x_\beta), \qquad x_\alpha = \varphi_\alpha \circ \varphi_\beta^{-1}(x_\beta) \in \varphi_\alpha(U_\alpha \cap U_\beta).$$

For any smooth map $f : X \to Y$, we have a map $Tf : TX \to TY$ between tangent spaces defined in the obvious way:

$$T_p f([c]) = [f \circ c] \in T_{f(p)}, \qquad (p \in X, c \in \mathcal{C}_p(X).$$

Proposition 2.4 (tangent map)**.** *The map $Tf : TX \to TY$ is a smooth vector bundles morphism, called the* tangent map *of the smooth map $f : X \to Y$.*

2.2 Cotangent bundle. Differential forms

Definition 2.5 (cotangent bundle and 1-forms)**.** If X is a smooth manifold, we define:

- the *cotangent bundle* of X as the dual bundle T^*X of TX (thus T^*X is defined by the cocycle $((g^t_{\alpha\beta})^{-1})$, see prop. 2.1),
- the $\mathcal{C}^\infty(X)$-module of *differential forms of degree* 1 as $\Omega^1(X) = \Gamma^\infty(T^*X)$.

Example 2.6 (differential 1-forms on a open subset of $\mathbf{R}^n$)**.** Let U be an open subset of $\mathbf{R}^n$, equiped with the usual basis $(e_1, \dots, e_n)$. Then $T^*U = U \times (\mathbf{R}^n)^*$ is the trivial bundle of rank n and every $\omega \in \Omega^1(U)$ can be written as

$$\omega = \sum_{i=1}^{n} a_i dx_i,$$

where $dx_i \in \Omega^1(U)$ is the "constant" 1-form defined by $dx_i(x) = (x, e_i^*)$ for $x \in U$ (that is the graph of the usual differential of the linear form e_i^* restricted to U), and $a_i \in \mathcal{C}^\infty(U)$ $(i = 1, \dots, n)$. Thus $\Omega^1(U)$ is a free $\mathcal{C}^\infty(U)$-module with basis $(dx_1, \dots, dx_n)$. In particular, if $f \in \mathcal{C}^\infty(U)$, then the graph of the usual differential of f, still denoted by df, is a 1-form given by $df = \sum_{i=1}^n f'_{x_i} dx_i$.

By definition, the fiber T_pX^* is the dual space of T_pX (for any $p \in X$). Thus 1-forms $\omega \in \Omega^1(X)$ act on vector fields $\xi \in \mathcal{X}(X)$ and we get a smooth map $\omega(\xi) \in \mathcal{C}^\infty(X$ defined by $(\omega(\xi))(p) = \omega_p(\xi_p) \in \mathbf{R}$ for all $p \in X$.

Proposition 2.7 (inverse image)**.** *If $f : X \to Y$ is a smooth map, then the map $f^* : \Omega^1(Y) \to \Omega^1(X)$ defined by*

$$(f^*\omega)_p \cdot u = \omega_{f(p)} \cdot T_p f(u) \qquad (p \in X, \omega \in \Omega^1(Y), u \in T_pX)$$

is a morphism of $\mathcal{C}^\infty$-module. The 1-form $f^*\omega \in \Omega^1(X)$ is called the inverse image, or pullback *of $\omega \in \Omega^1(Y)$ by f.*

In general, any 1-form $\omega \in \Omega^1(X)$ on a manifold X with atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha); \alpha \in A\}$ is given by local 1-forms $\omega^\alpha \in \Omega^1(\varphi_\alpha(U_\alpha))$ $(\alpha \in A)$, with the gluing condition

$$\omega^\beta = (\varphi_\alpha \circ \varphi_\beta^{-1})^* \omega^\alpha \qquad \text{on} \qquad \varphi_\beta(U_\alpha \cap U_\beta).$$

We now recall some facts from linear algebra. Let V be a vector space over some field $\mathbf{k}$, and let $p \in \mathbf{N}$. The space of p-linear alternate maps from $V \times \ldots \times V \to \mathbf{k}$ is denoted by $\bigwedge^p V^*$ (with $\bigwedge^0 V^* = \mathbf{k}$). The *wedge product*, or *exterior product* $l_1 \wedge \ldots \wedge l_p \in \bigwedge^p V^*$ of p linear forms $l_1, \ldots, l_p \in V^*$ is defined by

$$(l_1 \wedge \ldots \wedge l_p)(u_1, \ldots, u_p) = \det((l_i(u_j))_{1 \leqslant i,j \leqslant n}) \qquad (u_1, \ldots, u_p) \in V^p.$$

The wedge products of linear forms span $\bigwedge^p V^*$ and the wedge product extends to a bilinear map $\wedge : \bigwedge^p V^* \times \bigwedge^q V^* \to \bigwedge^{p+q} V^*$ $(p, q \in \mathbf{N})$, still called *wedge product*. Thus we get a graded k-algebra, $\bigwedge^*(V^*) = \bigoplus_{p \in \mathbf{N}} \bigwedge^p V^*$ which is anticommutative (meaning that $\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$ for $\alpha \in \bigwedge^p V^*$ and $\beta \in \bigwedge^q V^*$).

If $\varphi : V \to W$ is some $\mathbf{k}$-linear map, there is an *inverse image, or pullback* operation $\varphi^* : \bigwedge^*(W^*) \to \bigwedge^*(V^*)$, defined by

$$(\varphi^*\beta)(u_1, \ldots, u_p) = \beta(\varphi(u_1), \ldots, \varphi(u_p)) \qquad (\beta \in \bigwedge^p W^*, (u_1, \ldots, u_p) \in V^p),$$

for $p \geqslant 1$ and by identity for $p = 0$. If $p = 1$, $\varphi^* : W^* \to V^*$ is the transposed of φ. Moreover, the map φ^* is a morphim of graded $\mathbf{k}$-algebra (φ^* is $\mathbf{k}$-linear, degree preserving and is commuting with the wedge product $\wedge$).

Finally, is V is of finite dimension n with basis $(v_i)_{1 \leqslant i \leqslant n}$, then $\bigwedge^p V^* = \{0\}$ for $p > n$ and $\bigwedge^p V^*$ is of dimension $\binom{n}{p}$ with basis $(v_{i_1}^* \wedge \ldots \wedge v_{i_p}^*)_{1 \leqslant i_1 < \ldots < i_p \leqslant n}$ for $p \leqslant n$.

Definition 2.8 (differential p-forms)**.** Let X be a smooth manifold of dimension n. For $p \geqslant 1$, we set $\Omega^p(X) = \Gamma^\infty(\bigwedge^p TX^*)$ (thus $\Omega^p(X) = \{0\}$ for $p > n$). We also set $\Omega^0(X) = \mathcal{C}^\infty(X) \simeq \Gamma^\infty(X \times \mathbf{R})$ and $\Omega^p(X) = \{0\}$ for $p < 0$. Elements of the $\mathcal{C}^\infty(X)$-module $\Omega^p(X)$ are called ($\mathbf{R}$-valued) *differential forms of degree p (or p-forms)* on X.

Example 2.9 (differential p-forms on a open subset of $\mathbf{R}^n$)**.** Let U be an open subset of $\mathbf{R}^n$ and let $p \in \{1, \dots, n\}$. For any increasing sequence $I = (1 \leqslant i_1 < i_2 < \dots < i_p \leqslant n)$ we set $dx_I = dx_{i_1} \wedge \dots dx_{i_p} \in \Omega^p(U)$ (see example 2.6 for the definition of $dx_i \in \Omega^1(U)$) Then $\bigwedge^p T^*U = U \times \bigwedge^p(\mathbf{R}^n)^*$ is the trivial bundle of rank $\binom{n}{p}$ and every $\omega \in \Omega^p(U)$ can be written as

$$\omega = \sum_I a_I dx_I,$$

where and $a_I \in \mathcal{C}^\infty(U)$ for all multi-indices $I = (1 \leqslant i_1 < i_2 < \dots < i_p \leqslant n)$. Thus $\Omega^p(U)$ is a free $\mathcal{C}^\infty(U)$-module with basis $(dx_I)_I$.

In general, any diffential p-form $\omega \in \Omega^p(X)$ on a manifold X with atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha); \alpha \in A\}$ is given by local p-forms $\omega^\alpha \in \Omega^p(\varphi_\alpha(U_\alpha))$ $(\alpha \in A)$, which satisfy a gluing condition analogous to the above gluing condition for 1-forms.

The operations of wedge product and inverse image (from linear algebra) extend easily to the bundles setting: the $\mathcal{C}^\infty(X)$-module

$$\Omega^*(X) = \bigoplus_{p=0}^{n} \Omega^p(X)$$

equiped with the $\mathcal{C}^\infty(X)$-bilinear map $\wedge : \Omega^p(X) \times \Omega^p(X) \to \Omega^{p+q}(X)$ is a graded $\mathcal{C}^\infty(X)$-algebra. Moreover, for any smooth map $f : X \to Y$ we have a map $f^* : \Omega^*(Y) \to \Omega^*(X)$ which is a morphism of graded $\mathcal{C}^\infty(X)$-algebras of degree 0 (f^* is $\mathcal{C}^\infty(X)$-linear, degree preserving and is commuting with the wedge product).

2.3 de Rham complex and cohomology

The most interesting fact about $\Omega^*(X)$ is the existence of a third operation $d : \Omega^*(X) \to \Omega^*(X)$, which does not come from linear algebra but from the differential structure of X.

Proposition 2.10 (exterior differential)**.** *Let X be a smooth manifold. Then there exists a map $d : \Omega^*(X) \to \Omega^{*+1}(X)$, called the* exterior differential*, such that*

(1) d is $\mathbf{R}$-linear,
(2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$,
(3) $d \circ d = 0$,
(4) if $f \in \Omega^0(X)$, then df is the differential of f,
(5) d is commuting with inverse images.

Moreover, the exterior differential d is the unique map $\Omega^*(X) \to \Omega^{*+1}(X)$ satisfying these properties.

If $\omega \in \Omega^p(X)$ is locally given by some $\omega^\alpha = \sum_I a_I^\alpha dx_I \in \Omega^p(\varphi_\alpha(U_\alpha))$ $(\alpha \in A$, see example 2.9), then the exterior differential d is locally given by

$$d\omega^\alpha = \sum_I da_I^\alpha \wedge dx_I.$$

Definition 2.11 (de Rham cohomology)**.** The complex $(\Omega^*(X), d)$, where d is the the exterior differential, is called the *de Rham*[4] *complexe of the manifold* X. Its homology

$$H^*_{dR}(X) = \operatorname{Ker} d/\operatorname{Im} d = \oplus_{p=0}^{n} H^p_{dR}(X)$$

is called *the (***R***-valued) de Rham cohomology of* X.

A p-form $\omega \in \Omega^p(X)$ is *closed* (resp. *exact*) if $\omega \in \operatorname{Ker} d$ (resp. $\omega \in \operatorname{Im} d$).

Example 2.12. For any manifold X, we have $H^0_{dR}(X) \simeq \mathbf{R}^{\pi_0(X)}$, where $\pi_0(X)$ is the set of path components of X (for a smooth map $f \in \mathcal{C}^\infty(X)$ such that $df = 0$ is locally constant).

Note 2.13. Here are the first basics facts about de Rham cohomology.

(1) *There is no local de Rham cohomology.* This is a special case of Poincaré's lemma: if B is an open ball in $\mathbf{R}^n$, then $H^*_{dR}(B) = H^0_{dR}(B) \simeq \mathbf{R}$.

(2) There exists a Mayer-Vietoris short exact sequence for de Rham cohomology. If $X = U_1 \cup U_2$, where U_i $(i = 1, 2)$ are open subsets of X such that $U_1 \cap U_2 \neq \varnothing$, then the sequence

$$0 \longrightarrow \Omega^*(X) \xrightarrow{\psi_1^* \oplus \psi_2^*} \Omega^*(U_1) \oplus \Omega^*(U_2) \xrightarrow{\varphi_1^* - \varphi_2^*} \Omega^*(U_1 \cap U_2) \longrightarrow 0$$

is exact, where φ_i, ψ_i $(i = 1, 2)$ are the injections $U_1 \cap U_2 \to U_i$, $U_i \to X$ respectively (φ_i^* and ψ_i^* are just the restriction maps). The surjectivity of $\varphi_1^* - \varphi_2^*$ follows from the existence of smooth partitions of unity. Now the derived Mayer-Vietoris long exact sequence is a good tool for computing $H^*_{dR}(X)$ in some cases. For example, it implies that

$$H^*_{dR}(\mathbf{S}^n) = H^0_{dR}(\mathbf{S}^n) \oplus H^n_{dR}(\mathbf{S}^n) \simeq \mathbf{R} \oplus \mathbf{R}.$$

[4]Georges de Rham (1903–1990), Swiss mathematician and mountainclimber.

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