ECE 188 – Homework #4

Writing part

1 Exercise 1 (Transpose of the 1d discrete gradient)

Consider the following matrix

$$\nabla = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix}$$

Show that

1. ∇x corresponds to the periodical 1d forward discrete derivative of x.
2. $-\nabla^T x$ corresponds to the periodical 1d backward discrete derivative of x.

2 Exercise 2 (Adjoint of the 2d gradient)

For a matrix $M \in \mathbb{R}^{m\times n}$, its transpose $M^T \in \mathbb{R}^{n\times m}$ is the matrix whose columns are the rows of M. The transpose matrix is also the unique matrix satisfying for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$:

$$\langle Mx,\, y\rangle = \langle x,\, M^T y\rangle \quad i.e, \quad \sum_{i=1}^{m} (Mx)_i y_i = \sum_{j=1}^{n} (M^T y)_j x_j$$

The 2d discrete gradient ∇ is not a matrix but an operator that takes in input an image (2d array) and returns a vector field (3d array)

$$(\nabla x)_{i,j,1} = (\nabla_1 x)_{i,j} \quad \text{and} \quad (\nabla x)_{i,j,2} = (\nabla_2 x)_{i,j}$$

∇ is no longer a matrix but it is still linear as $\nabla a + \nabla b = \nabla(a + b)$. Adjoint operators are for linear operators what transposes are for matrices. The adjoint of ∇ is the unique linear operator ∇^* such that for all images x and vector fields y:

$$\langle \nabla x,\, y\rangle = \langle x,\, \nabla^* y\rangle \quad i.e, \quad \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{2} (\nabla x)_{i,j,k} y_{i,j,k} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} x_{i,j} (\nabla^* y)_{i,j}$$

Consider that ∇ performs forward discrete differences in both directions $\nabla x = \begin{pmatrix} \nabla_1^{\text{forward}} x \\ \nabla_2^{\text{forward}} x \end{pmatrix}$ and show that under periodical boundary conditions:

1. $\text{div} = -\nabla^*$ performs backward differences as:

$$\operatorname{div} y = \nabla_1^{\text{backward}} y_1 + \nabla_2^{\text{backward}} y_2$$

2. the discrete Laplacian operator Δ defined as

$$(\Delta x)_{i,j} = x_{i+1,j} + x_{i-1,j} + x_{i,j+1} + x_{i,j-1} - 4x_{i,j}$$

satisfies $\Delta = \operatorname{div} \operatorname{grad}$

Practical part

The following exercises should be done after completing the Matlab codes of Homework #3 and committing them. Once done switch the the branch `homework4`. This version of the code contains updated and new files required for the following exercises. Merge them with your own progress from Homework #3. Read carefully the messages, you may have conflicts. Solve all the conflicts, and commit the solved version before doing anything else.

3 Exercise 3 (Heat equation)

Complete the file `imdiffuse.m` such that it implements the explicit and implicit schemes for the heat equation with different boundary conditions. Under periodical boundary conditions, it should also be able to compute the discretization of the continuous solution. The script `hw4_ex3_heat.m` should produce the following results:

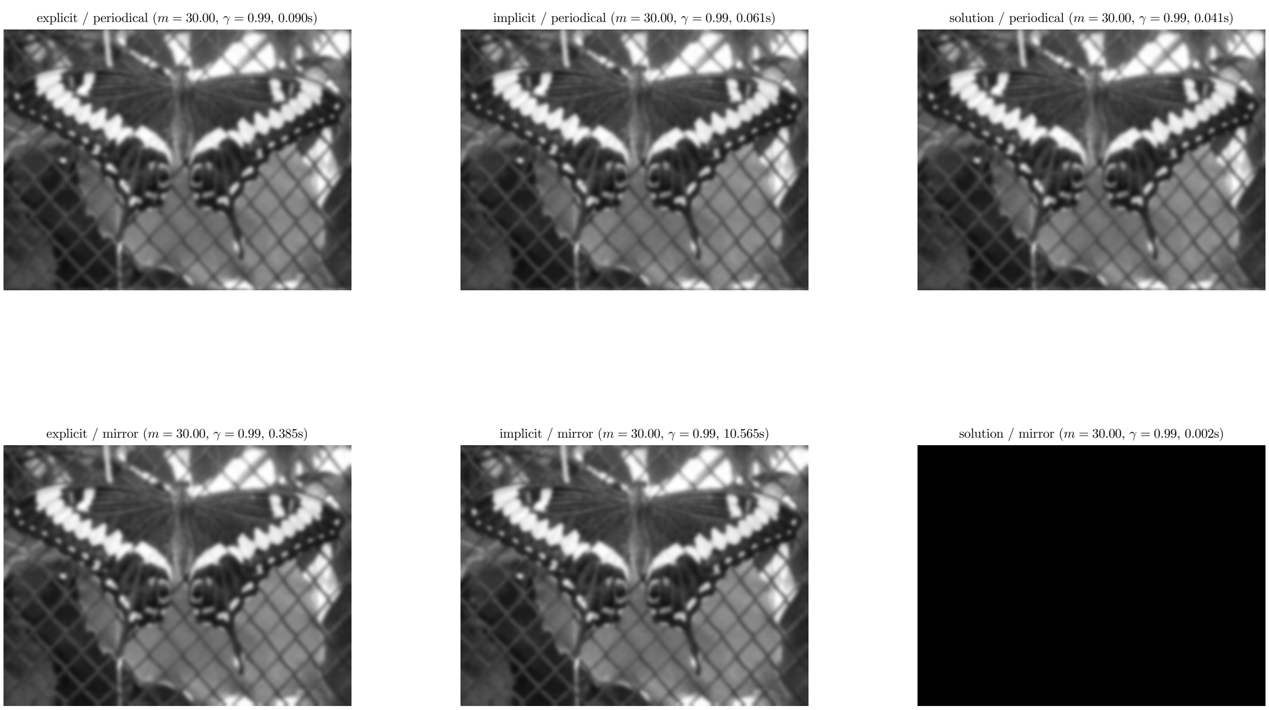

Figure 1: Don't forget to interpret the results!

Hint: only one line of code per section should be enough.

4 Exercise 4 (Anisotropic diffusion)

Complete the file `imdiffuse.m` such that it implements the original, regularized and truly anisotropic diffusions. The script `hw4_ex4_anis_diff.m` should produce the following results:

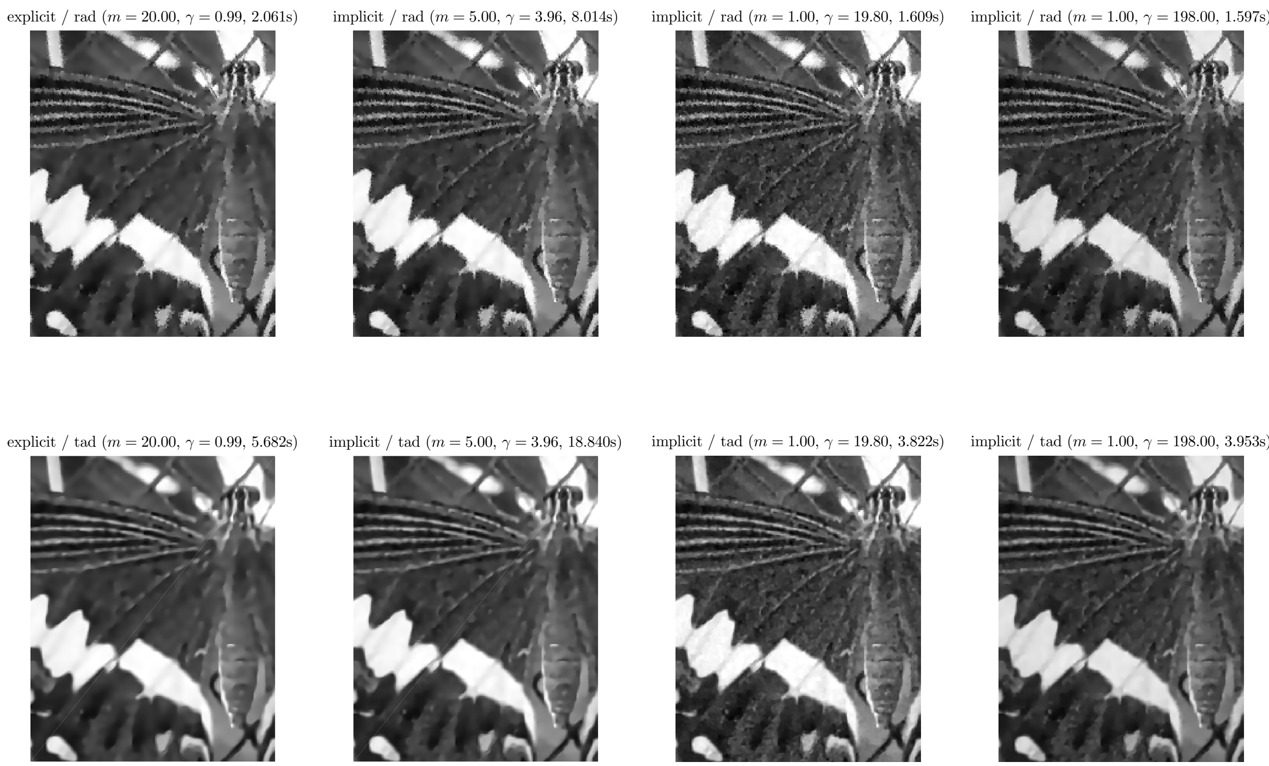

Figure 2: Don't forget to interpret the results!

Hint: only one line of code per section should be enough.