ECE 188 – Homework #5

Writing part

1 Exercise 1 (Bias-variance decomposition)

Let $\hat{x} \in \mathbb{R}^n$ be a random vector representing an estimate of a (deterministic) vector x. Denote by $\mathrm{Var}[\hat{x}]$ and $\mathrm{Bias}^2(\hat{x}, x)$ its variance/covariance matrix and square bias respectively as

$$\mathrm{Var}[\hat{x}] = \mathbb{E}[(\hat{x} - \mathbb{E}[\hat{x}])(\hat{x} - \mathbb{E}[\hat{x}])^T] \quad \text{and} \quad \mathrm{Bias}^2(\hat{x}, x) = \|x - \mathbb{E}[\hat{x}]\|^2.$$

Define the mean square error as

$$\mathrm{MSE}(x, \hat{x}) = \mathbb{E}[\|x - \hat{x}\|^2].$$

Show that

1. $\mathrm{Bias}^2(\hat{x}, x) = \|x\|^2 + \|\mathbb{E}[\hat{x}]\|^2 - 2\langle x, \mathbb{E}[\hat{x}]\rangle$
2. $\mathrm{tr}\,\mathrm{Var}[\hat{x}] = \mathbb{E}[\|\hat{x}\|^2] - \|\mathbb{E}[\hat{x}]\|^2$
3. $\mathrm{MSE}(x, \hat{x}) = \|x\|^2 + \mathbb{E}[\|\hat{x}\|]^2 - 2\langle x, \mathbb{E}[\hat{x}]\rangle$
4. Conclude that $\mathrm{MSE}(x, \hat{x}) = \mathrm{Bias}^2(\hat{x}, x) + \mathrm{tr}\,\mathrm{Var}[\hat{x}]$

Hint: Recall that $x^T y = \langle x, y\rangle$, $\|x\|^2 = x^T x = \langle x, x\rangle$ *and* $\mathrm{tr}[AB] = \mathrm{tr}[BA]$.

2 Exercise 2 (MVUE of linear combinations)

Let $y_1, \ldots, y_m$ be iid with $\mathbb{E}[y_k] = x$ and $\mathrm{Var}[y_k] = \sigma^2 \mathrm{Id}$. Consider the class of estimators performing a linear combinations of the y_k

$$\mathcal{C} = \left\{ \hat{x} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \setminus \exists \alpha, \beta \in \mathbb{R}, \quad \hat{x}(y_1, \ldots, y_m) = \sum_{k=1}^m \alpha_k y_k \right\}$$

Show that

1. An unbiased estimator in $\mathcal{C}$ must read
$$\hat{x} = \sum_{k=1}^m \alpha_k y_k \quad \text{with} \quad \sum_{k=1}^m \alpha_k = 1$$
2. In order to find the MVUE, we need to find the coefficients α_k that minimizes $\mathrm{tr}\,\mathrm{Var}[\hat{x}]$ under the constraint $\sum_{k=1}^m \alpha_k = 1$. Such a constrained optimization problem can be solved via the technique of Lagrange multipliers. It states that the minima of this problem can be in fact obtained by solving the following system of equations
$$\begin{cases} \dfrac{\partial \,\mathrm{tr}\,\mathrm{Var}[\hat{x}] + \lambda((\sum_{k=1}^m \alpha_k) - 1)}{\partial \alpha_k} & = 0, \quad \forall k = 1, \ldots, m \\ \dfrac{\partial \,\mathrm{tr}\,\mathrm{Var}[\hat{x}] + \lambda((\sum_{k=1}^m \alpha_k) - 1)}{\partial \lambda} & = 0 \end{cases}.$$
where λ is called a Lagrange multiplier. Show that the solution of this system are given by
$$\alpha_k^\star = \frac{1}{m} \quad \text{and} \quad \lambda^\star = \frac{2n\sigma^2}{m}$$
and give the final expression of the MVUE for this class.

3 Exercise 2 (Maximum likelihood Estimator)

Let $y_1, \ldots, y_m$ iid observations of some quantity of interest x. Show that the sample mean is the MLE for x, find its efficiency and determine of it is the MVUE, in the case where

1. $y_k \in \mathbb{R}$ have a Gaussian distribution around $x \in \mathbb{R}$ with variance σ^2,
2. $y_k > 0$ have a gamma distribution around $x > 0$ with variance x^2/L.

Writing + Practical part

The following exercises should be done after completing the Matlab codes of Homework #4, committing them, and switching to the the branch `homework5`.

4 Exercise 4 (Comparing moving average filter)

For comparing filters, it is relevant to look at their performance when they achieve equal smoothing level. The rate of noise reduction can be defined at pixel i as

$$\text{rate} = \frac{\mathrm{Var}[y_i]}{\mathrm{Var}[\hat{x}_i]} .$$

Consider the estimator $\hat{x} = \kappa * y$ obtained by assuming periodical boundary conditions, and consider that $\mathrm{Var}[y_i] = \sigma^2$ and y_i and y_j are independant, for all $i \neq j$.

1. Show that the rate of noise reduction does not depend on the location i and is given by

   $$\text{rate} = 1/\|\kappa\|^2 \quad \text{where} \quad \|\kappa\|^2 = \sum_k |\kappa_k|^2$$

   Complete the corresponding section of `imconvolve.m` accordingly.

2. Show that

   $$\text{rate} = n/\|\lambda\|^2 \quad \text{where} \quad \lambda = \mathcal{F}(\kappa)$$

3. Determine what should be the smallest radius τ of a boxcar filter (its width is $2\tau + 1$) in order to achieve a rate of at least r. Complete the corresponding section of `imconvolve.m` accordingly.

The script `hw5_ex4_imconvolve.m` should produce the following results:

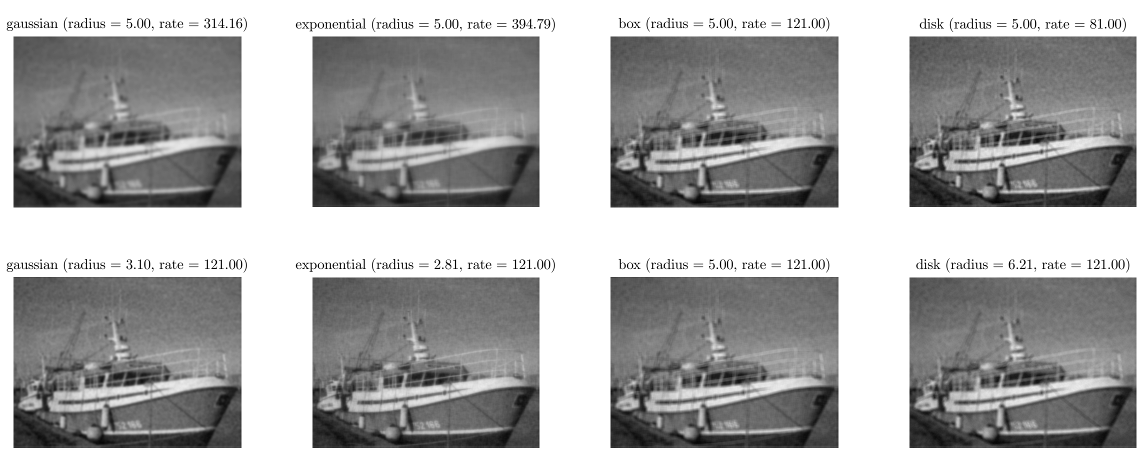

Figure 1: Don't forget to interpret the results!