DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **3**, Number **1**, February **2003**

Website: http://math.smsu.edu/journal

pp. **45–68**

EXISTENCE RESULT FOR A MIXTURE OF NON NEWTONIAN FLOWS WITH STRESS DIFFUSION USING THE CAHN-HILLIARD FORMULATION

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Abstract. We consider a model of mixture of non-newtonian fluids described with an order parameter defined by the volume fraction of one fluid in the mixture, a mean-velocity field and an extra-stress tensor field. The evolution of the order parameter is given by a Cahn-Hilliard equation, while the velocity satisfies the classical Navier-Stokes equation with non constant viscosity. The non-newtonian extra-stress tensor, which is symmetric, evolves through a constitutive law with time relaxation of Oldroyd type. We derive at first a physical model for incompressible flows (with free-divergence property for the velocity). In fact, the model we consider contains an additional stress diffusion, which derives from a microscopic dumbbell model analysis. The main result of this paper concerns the existence and uniqueness of a local regular solution for this model.

Introduction. To study the behavior of a binary mixture polymer/solvent, we are interested in the specific case of an heterogeneous mixture. We propose a model which determines the evolution of the interface by diffusion and by movement. This approach was used in the 50's by J.W. Cahn and J.E. Hilliard [4]. Its main advantage is that it automatically yields to a continuous description of the surface tension, which plays an important role in topological transition [17]. Furthermore, to model the hydrodynamic properties of a mixture an equation for a mean velocity field is needed. The resulting equation is essentially a non-homogeneous Navier-Stokes equation. The Cahn-Hilliard and Navier-Stokes equations are coupled because of the surface tension source term (in Navier-Stokes) and the convective transport of the interface parameter in the Cahn-Hilliard equation. Moreover, we study the case where at least one of the two liquids in the mixture is a non-newtonian fluid (a viscoelastic fluid). This characteristic is given by a non-zero extra-stress tensor σ which is symmetric [10] and obeys the constitutive law of Jeffreys type [19].

In this paper, we first present a theoretical model to study the incompressible mixture of viscoelastic flows. For the mathematical study, we assume that the two phases have the same densities. In the second section, having introduced some supplementary hypotheses (non-degenerate mobility, "almost polynomial" potential...) and having defined the mathematical context, we state the main result: the local existence and the uniqueness of a strong solution for our problem. Two rather different cases appear depending on the covariant derivative chosen in the Jeffreys

1991 *Mathematics Subject Classification.* 35Q10, 76D05, 76A05, 76A10, 76T05.
Key words and phrases. Cahn-Hilliard equation, Navier-Stokes equation, Oldroyd model.

model. The main part of the paper is devoted to the proof of these results. In the third part, we recall some preliminary results on the order parameter (whose average is conserved), on the tensorial algebra, on the classical term of the transport in the Navier-Stokes equation and on various results of compactness. The fourth part is devoted to the proof of the existence in the general case, based on Galerkin approximations and energy estimates. The last two parts are devoted respectively to the proof of the uniqueness and the proof of the particular case where we have chosen the corotational invariant derivative in the constitutive law.

1. **Governing equations.**

1.1. **Derivation of the governing equations.** In this section, we develop a model for a binary fluid in which both constituents are incompressible (such a binary fluid is called quasi-incompressible, see [17]). In this way, we consider a mixture composed of two incompressible phases of mass densities ρ_1^0 and ρ_2^0. The temperature is supposed to be constant. To describe a continuous time-dependent concentration profile, we need a dynamical equation for a parameter describing the ratio between the phase 1 and the phase 2 at each point of the mixture. To derive such an equation, we define the volumic fraction of the first fluid in the mixture (or equivalently, its mass concentration) by

$$\Phi = \Phi(t,x) = dV_1/dV,$$

where dV_1 is the volume filled by the fluid 1.

Let us introduce the density and velocity of the mixture as follows: the apparent densities of each fluid in the mixture, denoted by ρ_1 and ρ_2, are defined by

$$\rho_1 = \Phi\rho_1^0, \qquad \rho_2 = (1-\Phi)\rho_2^0,$$

and the total density is equal to the sum $\rho = \rho_1 + \rho_2$. Each fluid possesses its own velocity field v_i. In this paper, we define the mean velocity v by $v = \Phi v_1 + (1-\Phi)v_2$. Note also that some authors (see for instance [17]) consider a mean velocity defined by $\rho v^* = \rho_1 v_1 + \rho_2 v_2$. The advantage of this first definition of the velocity is to preserve the divergence-free property.

In order to obtain the equations describing the mixture, we apply the usual mass and momentum conservation laws for the two fluids. The mass conservations read:

$$\partial_t \rho_i + \text{ div } (\rho_i v_i) = 0, \quad i = 1, 2.$$

Dividing each of the equations by ρ_i^0, adding the equations and using the definitions of ρ_1 and ρ_2, the mean velocity field satisfies the "incompressibility" property:

$$\text{div } (v) = 0.$$

Following A. Onuki [21] who introduces the relative velocity of the polymer and the solvent, we set $w = v_1 - v_2$. Moreover, to keep all the symmetry in the problem, we prefer to introduce the renormalized order parameter defined by

$$\varphi = 2\Phi - 1.$$

We have $-1 \leq \varphi \leq 1$ and $\varphi(x) = 1$ (resp. $\varphi(x) = -1$) if and only if the fluid 1 (resp. the fluid 2) is present at the point x.

Using the "incompressibility" property (1.1) and the fact that $v_1 = v + \frac{1-\varphi}{2}w$, the equation of mass conservation for the first fluid reads

$$\partial_t\varphi + v.\nabla\varphi + \text{ div } \left(\frac{1-\varphi^2}{2}w\right) = 0. \tag{1}$$

The momentum conservation gives two additional equations: the equations of motion for the two components are

$$\partial_t(\rho_i v_i) - \text{ div } (\Sigma_i) + \text{ div } (\rho_i v_i \otimes v_i) = F_i, \quad i = 1,2 \tag{2}$$

where Σ_i and F_i respectively correspond to the stress tensors and the exterior forces. The stress tensors are split into two main parts, first the hydrodynamic partial pressures denoted by p_i, second the extra-stress tensors τ_i (which obeys a constitutive law that we will study further):

$$\Sigma_i = -p_i Id + \tau_i.$$

Moreover, we assume that the partial pressures are given from a total pressure p and that the extra-stress tensors are given from a global extra-stress tensor τ by the following relations

$$p_1 = \frac{1+\varphi}{2}p, \quad p_2 = \frac{1-\varphi}{2}p, \quad \tau_1 = \frac{1+\varphi}{2}\tau, \quad \tau_2 = \frac{1-\varphi}{2}\tau.$$

Finally, the forces are assumed to be of the following form:

$$F_1 = -\rho_1 \nabla \mu_1 + \rho_1 g - \xi(\varphi) w \quad , \quad F_2 = -\rho_2 \nabla \mu_2 + \rho_2 g + \xi(\varphi) w,$$

where the first terms correspond to the chemical potentials, the second terms to the gravity force and the last to the friction between the two components (ξ is called the coefficient of friction, see A. Onuki [21]). A thermodynamic description allows us to write the chemical potentials using the specific free energy $\mathcal{F}$ [20] as follows:

$$\mu_1 = \mu_0(\varphi) + \frac{1-\varphi}{2\rho(\varphi)}\frac{\partial \mathcal{F}}{\partial \varphi} \quad , \quad \mu_2 = \mu_0(\varphi) - \frac{1+\varphi}{2\rho(\varphi)}\frac{\partial \mathcal{F}}{\partial \varphi}.$$

Here $\mu_0(\varphi)$ is the chemical potential of a theoretical uniform alloy of composition φ and the other terms are exchange terms due to the non-uniformity of the alloy. Afterwards, we suppose that the temperature is constant and the specific free energy depends only on both the concentration (or equivalently the order parameter φ) and its spatial gradient (see for instance [5], [21]):

$$\mathcal{F}(\varphi) = \int_\Omega \frac{A}{2}|\nabla \varphi|^2 + EF(\varphi), \tag{3}$$

A and E being two physical parameters describing the interaction between the two phases, and F being the Cahn-Hilliard potential. For example, the Ginzburg-Landau-Wilson free energy functional for φ is given in the usual form (see [21]) by a polynomial expression of the following type

$$F(\varphi) = C_1\varphi^4 - C_2\varphi^2. \tag{4}$$

This polynomial form of the chemical potential is classically used in the literature but it is an approximation of a logarithmic expression given by M. Doi in [7]:

$$F(\varphi) = T_c(1-\varphi^2) + T((1+\varphi)\log(1+\varphi) + (1-\varphi)\log(1-\varphi)). \tag{5}$$

Adding equations (2) describing the motion and using the definitions of the mean velocity, the total density, the global stress tensor and the total pressure, we deduce:

$$\begin{aligned}\partial_t &\left(\rho v + (\rho_1^0 - \rho_2^0)\frac{1-\varphi^2}{4}w\right) - \operatorname{div}(-pId + \tau)\\ &+ \operatorname{div}\left(\rho v \otimes v + (\rho_1^0 + \rho_2^0 - \rho)\frac{1-\varphi^2}{4} w\otimes w + (\rho_1^0 - \rho_2^0)\frac{1-\varphi^2}{4}(v\otimes w + w \otimes v)\right)\\ &\qquad = -\rho\nabla\mu_0 + \rho g - (\rho_1^0 - \rho_2^0)\frac{1-\varphi^2}{4}\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right) + \frac{1}{2}\frac{\partial \mathcal{F}}{\partial\varphi}\nabla\varphi.\end{aligned} \tag{6}$$

To obtain a second equation coupling the two velocities v and w, we replace the equations of motion (2) by their non-conservative forms:

$$\rho_i\partial_t v_i + \rho_i v_i.\nabla v_i - \operatorname{div}(\Sigma_i) = F_i, \quad i = 1,2, \tag{7}$$

and from $\rho_2(7)_1 - \rho_1(7)_2$, the relative velocity w is governed by

$$\begin{aligned}\rho_1\rho_2\partial_t w + \rho_1\rho_2 &\left(v.\nabla w + w.\nabla v - \varphi(w.\nabla w) - \frac{1}{2}(w.\nabla\varphi)w\right) - \frac{\rho}{2}(\tau - pId).\nabla\varphi\\ &+ (\rho_1^0 - \rho_2^0)\frac{1-\varphi^2}{4}\operatorname{div}(\tau - pId) = -\rho_1\rho_2\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right) - \rho\xi(\varphi)w.\end{aligned} \tag{8}$$

A characteristic of the mixture is the fact that the two phases are non-Newtonian. More precisely, we assume that the extra-stress tensor is given by a differential constitutive equation of Jeffreys-type (see [12]):

$$\frac{D\,\tau_i}{Dt} + \frac{\tau_i}{\lambda_{1,i}} = \frac{2\eta_i}{\lambda_{1,i}}D(v_i) + 2\eta_i\frac{\lambda_{2,i}}{\lambda_{1,i}}\frac{D}{Dt}D(v_i), \quad i = 1,2. \tag{9}$$

In this model, the deformation tensor $D(v)$, is defined as the symmetric part of the velocity gradient: $D(v) = (\nabla v + \nabla^t v)/2$ and the parameters η_i, $\lambda_{1,i}$ and $\lambda_{2,i}$ denote respectively the elastic viscosity, the relaxation time and the retardation time of the fluid i. The notation $\frac{D}{Dt}$ represents an invariant time derivative which can be defined by (see [12]):

$$\frac{D\,M}{Dt} = \partial_t M + v.\nabla M - W(v).M + M.W(v) - a(D(v).M + M.D(v)),$$

where $a \in [-1;1]$ is a rheologic parameter and $W(v)$ is the skew-symmetric part of the velocity gradient: $W(v) = (\nabla v - \nabla^t v)/2$ (called the vorticity tensor).

To obtain a constitutive equation for the total extra-stress tensor τ in the mixture, we assume that there exists a mean viscosity η such that

$$\eta_1 = \frac{1+\varphi}{2}\eta \quad , \quad \eta_2 = \frac{1-\varphi}{2}\eta,$$

and we define $\lambda_1 = \lambda_1(\varphi)$ and $\lambda_2 = \lambda_2(\varphi)$ by:

$$\frac{1}{\lambda_1} = \frac{1+\varphi}{2}\frac{1}{\lambda_{1,1}} + \frac{1-\varphi}{2}\frac{1}{\lambda_{1,2}}, \qquad \frac{\lambda_2}{\lambda_1} = \frac{1+\varphi}{2}\frac{\lambda_{2,1}}{\lambda_{1,1}} + \frac{1-\varphi}{2}\frac{\lambda_{2,2}}{\lambda_{1,2}}.$$

We can remark that the rate λ_2/λ_1 measures the Newtonian default of the mixture (in a Newtonian fluid, the retardation time and the relaxation time are equal and close to zero, see [12]). With these definitions, adding equations (9) we obtain:

$$\begin{aligned}\frac{D\,\tau}{Dt}+\frac{\tau}{\lambda_1}&=\frac{2\eta}{\lambda_1}D(v)+2\eta\frac{\lambda_2}{\lambda_1}\frac{D}{Dt}D(v)-\frac{\eta\lambda_2}{\lambda_1}\partial_t\varphi D(w)\\&+2\eta\frac{1-\varphi^2}{4}\left(\frac{1}{\lambda_{1,1}}-\frac{1}{\lambda_{1,2}}\right)D(w)-\frac{\eta}{\lambda_1}(\nabla\varphi\otimes w+w\otimes\nabla\varphi)\\&+2\eta\frac{1-\varphi^2}{4}\left(\frac{\lambda_{2,1}}{\lambda_{1,1}}-\frac{\lambda_{2,2}}{\lambda_{1,2}}\right)\frac{D}{Dt}D(w)-\frac{\eta\lambda_2}{\lambda_1}\frac{D}{Dt}(\nabla\varphi\otimes w+w\otimes\nabla\varphi).\end{aligned}\tag{10}$$

Following A. Onuki [21], we assume that the relative velocity of the fluid 1 and the fluid 2 is lower than the mean velocity of the mixture ($w \ll v$). Under the previous approximations, the equation describing the evolution of the relative velocity (8) can be written as $0=-\rho_1\rho_2\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right)-\rho\xi(\varphi)w$, and allows to deduce a convection-diffusion law for φ using (1):

$$\partial_t\varphi+v.\nabla\varphi-\rho_1^0\rho_2^0\operatorname{div}\left(\frac{(1-\varphi^2)^2}{8}\frac{1}{\rho\xi(\varphi)}\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right)\right)=0.$$

With these approximations, the equations (6) and (10) become (the terms $-\rho\nabla\mu_0$ and ρg are considered as pressure terms)

$$\begin{aligned}&\partial_t(\rho v)-\operatorname{div}(-pId+\tau)+\operatorname{div}(\rho v\otimes v)\\&\qquad\qquad=-(\rho_1^0-\rho_2^0)\frac{1-\varphi^2}{4}\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right)+\frac{1}{2}\frac{\partial\mathcal{F}}{\partial\varphi}\nabla\varphi,\\&\frac{D\,\tau}{Dt}+\frac{\tau}{\lambda_1}=\frac{2\eta}{\lambda_1}D(v)+2\eta\frac{\lambda_2}{\lambda_1}\frac{D}{Dt}D(v).\end{aligned}$$

Some authors [10], [11] decompose the stress tensor into a viscous stress plus an elastic stress $\tau=2\eta\lambda_2/\lambda_1 D(v)+\sigma$ and introduce the retardation parameter $r=1-\lambda_2/\lambda_1$. Moreover, A. W. El-Kareh and L. G. Leal [8] have analyzed the microscopic dumbbell model and derive an additional stress diffusion term. In this way, we can add a term like $\tilde{\varepsilon}\Delta\sigma$ to obtain the equations describing the mixture:

$$\left\{\begin{aligned}&\partial_t\varphi+v.\nabla\varphi-\rho_1^0\rho_2^0\operatorname{div}\left(\frac{B(\varphi)}{\rho}\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right)\right)=0,\\&\rho(\partial_t v+v.\nabla v)-\operatorname{div}(2\eta(\varphi)(1-r(\varphi))D(v))+\nabla p-\operatorname{div}\sigma\\&\qquad\qquad=-(\rho_1^0-\rho_2^0)\frac{1-\varphi^2}{4}\nabla\left(\frac{1}{\rho}\frac{\partial\mathcal{F}}{\partial\varphi}\right)+\frac{1}{2}\frac{\partial\mathcal{F}}{\partial\varphi}\nabla\varphi,\\&\frac{D\,\sigma}{Dt}+\frac{\sigma}{\lambda_1(\varphi)}=2\frac{\eta(\varphi)r(\varphi)}{\lambda_1(\varphi)}D(v)+\tilde{\varepsilon}\Delta\sigma,\end{aligned}\right.$$

where $B(\varphi)=(1-\varphi^2)^2/8\xi(\varphi)$ (mobility coefficient).

1.2. **Dimensionless equations and final model.**

$$
\begin{cases}
\rho = 1 + \frac{\gamma}{2}(\varphi - 1), \qquad \operatorname{div} v = 0, \\[2ex]
\partial_t \varphi + v.\nabla\varphi - \frac{1}{\mathcal{P}e} \operatorname{div}\left(\frac{B(\varphi)}{\rho}\nabla\left(\frac{\mu}{\rho}\right)\right) = 0, \qquad \mu = -\alpha^2\Delta\varphi + F'(\varphi), \\[2ex]
\rho(\partial_t v + v.\nabla v) - \dfrac{1}{\mathcal{R}e} \operatorname{div}\Big(2\eta(\varphi)(1 - r(\varphi))D(v)\Big) + \nabla p - \dfrac{1}{\mathcal{R}e} \operatorname{div} \sigma \\
\qquad\qquad\qquad = -\gamma \dfrac{1-\varphi^2}{4}\mathcal{K}\nabla\left(\dfrac{\mu}{\rho}\right) + \dfrac{\mathcal{K}}{2}\mu\nabla\varphi, \\[2ex]
\frac{D\sigma}{Dt} + \frac{1}{\mathcal{W}e(\varphi)}\sigma = 2\eta(\varphi)\frac{r(\varphi)}{\mathcal{W}e(\varphi)}D(v) + \varepsilon\Delta\sigma.
\end{cases}
$$

In this system we have introduced eight independent dimensionless parameters for the model. The generalized chemical potential μ is defined with the Gibbs free energy $\mathcal{F}$ (see (3)) by $\mu = \frac{\partial\mathcal{F}}{\partial\varphi} = -A\Delta\varphi + EF'(\varphi)$. The characteristic size of the interface is given by $l = \sqrt{A/E}$. If we choose L as reference length, a first dimensionless parameter is introduced: $\alpha = l/L$. The reference density is $\bar\rho = max\{\rho_1^0, \rho_2^0\}$ ($\bar\rho = \rho_1^0$ for example) and the contrast between the two densities is described by another dimensionless parameter $\gamma = \frac{\rho_1^0 - \rho_2^0}{\bar\rho}$. Another reductions are obtained by using classical dimensionless variables, and introducing the Reynolds number $\mathcal{R}e = \bar\rho VL/\bar\eta$, the capillary coefficient $\mathcal{K} = ET/L\bar\rho V$, the Peclet number $\mathcal{P}e = LV/E(1-\gamma)$, the Weissenberg number $We = We(\varphi) = \lambda_1/T$, the retardation parameter $r(\varphi) = 1 - \frac{\lambda_2}{\lambda_1}$ and $\varepsilon = \tilde\varepsilon T/L$.

To obtain a well-posed problem, we define the initial conditions and the boundary conditions. We assume that the initial conditions $\varphi_0(x)$, $v_0(x)$ and $\sigma_0(x)$ are given on a bounded domain Ω. For the order parameter, we assume

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{and} \quad \frac{\partial\mu}{\partial n} = 0,$$

where n is the outward normal on the boundary of the domain Γ. The first condition imposes locally the interface to be orthogonal to the boundary. The second imposes no-diffusion through this boundary. For the velocity, we assume that

$$v|_\Gamma = h \quad \text{with} \quad h.n = 0.$$

As we have an diffusion term $\varepsilon\Delta\sigma$ in the constitutive equation for σ, we have give a boundary condition for the stress tensor:

$$\frac{\partial\sigma}{\partial n} = 0 \quad \text{on } \Gamma.$$

In the following, we restrict the study to the incompressible model: we suppose that the density are perfectly matched ($\rho_1 = \rho_2$ or simply $\gamma = 0$). For the mathematical study, all the physical constants (Reynolds number, Peclet number...) are equal to 1 and our model reduces to the following:

$$
\begin{cases}
\partial_t\varphi + v.\nabla\varphi - \operatorname{div}(B(\varphi)\nabla\mu)) = 0, \text{ with } \quad \mu = -\alpha^2\Delta\varphi + F'(\varphi), \\[2ex]
\partial_t v + v.\nabla v - 2\operatorname{div}(\eta(\varphi)D(v)) + \nabla p - \operatorname{div}\sigma = \mu\nabla\varphi, \text{ with } \quad \operatorname{div} v = 0, \\[2ex]
\partial_t\sigma + v.\nabla\sigma + g_a(\sigma, v) + l(\varphi)\sigma - \varepsilon\Delta\sigma = f(\varphi)\eta(\varphi)D(v), \\[2ex]
\text{with boundary conditions and initial values,}
\end{cases}
\tag{11}
$$

where we define g_a, l, f and h as follows: the application g_a is bilinear with values in the set of 2-tensors :

$$g_a : (\ \sigma\ ,\ v\ ) \longmapsto -W(v).\sigma + \sigma.W(v) + a(D(v).\sigma + \sigma.D(v)),$$

the functions $f, l : \mathbb{R} \to \mathbb{R}$, of C^∞ class with all derivatives bounded and in particular with $0 < f_1 \leq f \leq f_2$, $0 < l_1 \leq l \leq l_2$.

The first two equations are known as Cahn-Hilliard equation (see [4], [17]) with variable mobility. The parameter α measures the thickness of the interface between the two phases (a narrow transition layer across which the fluids may mix). Although the Cahn-Hilliard equation has been intensively studied, little mathematical analysis has been done for a diffusional mobility B which depends on φ. This concentration-dependent mobility appeared in the original derivation of the Cahn-Hilliard equation (see [4]). The mathematical study of the degenerate case of the Cahn-Hilliard equation is performed in [9]. Moreover, we can easily see that the convexity of the Cahn-Hilliard potential has an important effect on the comportment of the interface: if F is non-convex, the mixture is unstable (the diffusivity is negative in the Cahn-Hilliard equation) and the fluids are only partially miscible.

The next two are the classical Navier-Stokes equation for an incompressible fluid. Following [3], the viscosity η is generally non constant and depends on the local composition of the mixture. The chemical interaction $\mu\nabla\varphi$ and the extra-stress tensor σ are the two force terms which naturally appear.

The fifth equation represents the constitutive law of the extra-stress tensor. Contrary to Newtonian fluids, there is no universal equation for viscoelastic fluids (the various models are given for different values of the rheological coefficient a).

In the literature, various articles are devoted to these problems. Both uncoupled equations: Cahn-Hilliard [9], Navier-Stokes [6], [13] or transport equation, and coupled equations like Cahn-Hilliard/Navier-Stokes [2] or Navier-Stokes/transport [11] and [16].

2. **Assumptions and main results.**

2.1. **Assumptions.** The functions B, η and F are supposed to be regular enough (C^1 class with bounded derivatives for instance). We suppose here that the mobility is non-degenerate and that the viscosity is bounded:

$$\exists\ B_1, B_2 > 0, \quad B_1 \leq B \leq B_2, \tag{12}$$

$$\exists\ \eta_1, \eta_2 > 0, \quad \eta_1 \leq \eta \leq \eta_2. \tag{13}$$

It remains to make assumptions on the Cahn-Hilliard potential F. A physical-meaningful potential is always bounded from below and adding a constant to the potential does not change the equations. Then we can assume that

$$F \geq 0. \tag{14}$$

Usual potentials, as the potentials described before (see equations (4) and (5)) contain at the same moment stable and unstable regions. From the perspective to handle such cases, we shall suppose that

$$\exists F_5 \geq 0 \quad / \quad \forall x \in \mathbb{R} \quad F''(x) \geq -F_5. \tag{15}$$

To obtain the existence of solutions, we shall add more general hypotheses on the growth of the potential in the neighborhood of the points $\varphi = 1$ and $\varphi = -1$.

$$\exists\ F_1, F_2 > 0 \text{ such that } \ |F'''(x)| \leq F_1|x|^q + F_2, \tag{16}$$

where $1 \le q < +\infty$. To obtain a global solution (theorem 2.2) we must suppose

$$1 \le q \le 3 \text{ if } d = 3 \text{ and } 1 \le q < +\infty \text{ if } d = 2. \tag{17}$$

It is clear that polynomial functions of even order for $d = 2$, or of order 4 for $d = 3$ (see for instance (4)) with a positive dominant coefficient can be used as a function F. The Cahn-Hilliard equation is usually studied with such non-linearities in the literature. Finally, the last assumption is a generalization of the convexity:

$$\forall \gamma \in \mathbb{R} \quad \exists F_3(\gamma), F_4(\gamma) > 0 \quad / \quad \forall x \in \mathbb{R} \quad (x-\gamma)F'(x) \ge F_3(\gamma)F(x) - F_4(\gamma). \tag{18}$$

This hypothesis is satisfied for example by any convex function by using $F_3(\gamma) = 1$ and $F_4(\gamma) = F(\gamma)$. However, under the critical point the Cahn-Hilliard potential considered have typically a double-well structure and are obviously not convex. They verify nevertheless this property.

Finally, we need of one hypothesis on the velocity boundary condition h. It's defined only on the boundary and is rather regular:

$$h \in H^s(\Gamma) \text{ where } s = (d+1)/2, \quad h.n = 0. \tag{19}$$

In the three dimensional case we add an assumption on h which correspond, for instance, to the physical case of a channel where the velocity is constant on each plane (see section 4.1 and remark 4).

2.2. **Function spaces.** Throughout this paper, the same letter C stands for a constant which may be different each time it appears. Let Ω be a smooth bounded domain in $\mathbb{R}^d$, $d = 2$ or 3. We denote by Γ its boundary and n the outward unit normal vector. We will use the following spaces: the Lebesgue space $L^p(\Omega)$ (or simply L^p), $1 \le p \le \infty$, with norms $|.|_p$ (for $p = 2$ its inner product is $(.,.)$); the Sobolev spaces $H^s(\Omega)$, (or simply H^s), $s \in \mathbb{R}$, with norms $\|.\|_s$ and inner products $((.,.))_s$. To simplify the notations we will also denote the vector spaces $(L^p)^d$, $(H^s)^d$, $(L^p)^{d\times d}$, $(H^s)^{d\times d}$ ... by $\mathbb{L}^p$ and $\mathbb{H}^s$ respectively , their norms being denoted in the same way as above. If I is an interval of $\mathbb{R}^+$ and X a Banach space, we will also use the function space $L^p(I;X)$, $1 \le p \le \infty$, which consists of p-integrable functions with values in X. We now introduce the natural spaces linked to the problem (for $s \ge 0$):

$$\Phi = \{\phi \in \mathcal{D}(\overline{\Omega}) \quad / \quad \frac{\partial \phi}{\partial n}|_\Gamma = \frac{\partial \Delta\phi}{\partial n}|_\Gamma = 0\} \quad \text{and} \quad \Phi_s = \overline{\Phi}^{H^s} \text{ with the norm } \|.\|_s,$$

$$\mathcal{V} = \{w \in \mathcal{D}(\Omega)^d \quad / \quad \text{div } w = 0\} \quad \text{and} \quad V_s = \overline{\mathcal{V}}^{\mathbb{H}^s} \text{ with the norm } \|.\|_s \ (s \le 3/2),$$

$$\Sigma = \{\tau \in \mathcal{D}(\overline{\Omega})^{d\times d} \quad / \quad \frac{\partial \tau}{\partial n}|_\Gamma = 0\} \quad \text{and} \quad \Sigma_s = \overline{\Sigma}^{\mathbb{H}^s} \text{ with the norm } \|.\|_s.$$

For $s' < 0$, we also use the notation $\Phi_{s'}$, $V_{s'}$ and $\Sigma_{s'}$ for the dual spaces of $\Phi_{-s'}$, $V_{-s'}$ and $\Sigma_{-s'}$ respectively. Finally, we denote by A the classical Stokes operator: $Au = -\Delta u + \nabla \pi$ ($\pi \in L^2_0$), with domain $V_1 \cap H^2$.

2.3. **Main results.** Since h verifies the assumption (19), we know that there exists a vector field, being still noted h, which extend h on Ω and such that $h \in H^2(\Omega)$ and div $h = 0$ on Ω. With this result, we can express the following theorems

THEOREM 2.1 (The general case). *Assume that* (12), (13), (14), (15), (16) *and* (18) *hold, and let* $\varphi_0 \in \Phi_2$, $v_0 \in h + V_1$ *and* $\sigma_0 \in \Sigma_1$. *Then there exists* $T^* > 0$ *and* *an unique solution* (φ, v, σ) *of* (11) *on* $[0, T^*[$ *such that*

$$\begin{aligned}
&\varphi \in L^2(0,T^*;\Phi_4) \cap L^\infty(0,T^*;\Phi_2),\\
&v - h \in L^2(0,T^*;V_1 \cap H^2) \cap L^\infty(0,T^*;V_1),\\
&\sigma \in L^2(0,T^*;\Sigma_2) \cap L^\infty(0,T^*;\Sigma_1).
\end{aligned}$$

THEOREM 2.2 (The case $\boldsymbol{a} = \mathbf{0}$). *Assume that* (12), (13), (14), (15), (16) *with* (17) *and* (18) *hold, and let* $a = 0$ *in* (11), $\varphi_0 \in \Phi_1$, $v_0 \in h + V_0$ *and* $\sigma_0 \in \Sigma_0$. *Then there exists a global solution* (φ, v, σ) *of* (11) *such that*

$$\begin{aligned}
&\varphi \in L^2_{loc}(0,+\infty;\Phi_3) \cap L^\infty(0,+\infty;\Phi_1),\\
&v - h \in L^2_{loc}(0,+\infty;V_1) \cap L^\infty(0,+\infty;V_0),\\
&\sigma \in L^2_{loc}(0,+\infty;\Sigma_1) \cap L^\infty(0,+\infty;\Sigma_0).
\end{aligned}$$

COROLLARY 2.1 (The case $\boldsymbol{a} = \mathbf{0}$ in dimension $\mathbf{2}$). *Assume that* $d = 2$ *and that* (12), (13), (14), (15), (16) *and* (18) *hold. Let* $a = 0$ *in* (11), $\varphi_0 \in \Phi_2$, $v_0 \in h + V_1$ *and* $\sigma_0 \in \Sigma_1$. *Then there exists an unique global solution* (φ, v, σ) *of* (11) *such that*

$$\begin{aligned}
&\varphi \in L^2_{loc}(0,+\infty;\Phi_4) \cap L^\infty(0,+\infty;\Phi_2),\\
&v - h \in L^2_{loc}(0,+\infty;V_1 \cap H^2) \cap L^\infty(0,+\infty;V_1),\\
&\sigma \in L^2_{loc}(0,+\infty;\Sigma_2) \cap L^\infty(0,+\infty;\Sigma_1).
\end{aligned}$$

REMARK 1. *- In the second theorem, we can suppose that* F *is of* C^2*-class only.*

- The value $a = 0$ *corresponds to the corotational convected derivative (the values* $a = 1$ *and* $a = -1$ *correspond respectively to the upper convected and lower convected derivative).*

- In the general case, we show the existence and the uniqueness result for a local solution. In the case $a = 0$, *we obtain a global weak (strong in dimension* 2*) solution for a general initial condition, but we do not have the uniqueness of such solutions in the three dimensional case.*

3. **Preliminary results.** First of all, for any $f \in L^1(\Omega)$, we define the average of f by $m(f) = \frac{1}{|\Omega|}\int_\Omega f$. In particular when φ verifies the Cahn-Hilliard equation (with a regular velocity field for the transport) and the boundary conditions: $\varphi \in \Phi_1$, its average is preserved along the time. Moreover, we have the following lemma, proved in [24]:

LEMMA 3.1. *If* $\phi \in \Phi_1$ *then* $\|\phi - m(\phi)\|_1 \le C|\nabla\phi|_2$, *and more generally if* $\phi \in \Phi_{s+2}$ *then* $\|\phi - m(\phi)\|_{s+2} \le C\|\Delta\phi\|_s$.

From this lemma, if we use the embedding $H^{3/2} \subset L^\infty$ for $d = 2$, or Agmon's inequality if $d = 3$, and the interpolation between L^2 and H^3, we have in both cases

LEMMA 3.2. *If* $\phi \in \Phi_4$ *then* $|\nabla\phi|_\infty \le C|\nabla\phi|_2^{1/2}|\Delta^2\phi|_2^{1/2}$.

3.1. **Elementary tensorial analysis.** To obtain energy bounds, we recall some tensorial results. At the end of this part, we prove two equalities on the application g_a which will be usefull in the sequel.

DEFINITION 3.1. *The* s*-contracted product of a* p*-tensor* A *and a* q*-tensor* B *(for* $s \le min\{p, q\}$*) is a* $(p + q - 2s)$*-tensor defined by:*

$$A_p \overset{(s)}{:} B_q = \left(\sum_{k_1,\dots,k_s} a_{i_1,\dots,i_{p-s},k_1,\dots,k_s} b_{k_1,\dots,k_s,j_{s+1},\dots,j_q} \right)_{i_1,\dots,i_{p-s},j_{s+1},\dots,j_q}.$$

REMARK 2. *For more simplifications, we shall note* $A \overset{(0)}{:} B = AB$, $A \overset{(1)}{:} B = A.B$ *and* $A \overset{(2)}{:} B = A : B$. *Moreover the operator* ∇ *represents the* 1*-tensor* $(\partial_k)_k$ *and we have* $\Delta = \nabla.\nabla$, $div\ M = \nabla.M$. *With these notations, when* A *and* B *are two* p*-tensors, we define directly:* $(A,B) = \int_\Omega A \overset{(p)}{:} B$.

PROPOSITION 3.1. *For* $1 \leq s \leq min\{p, q+1\}$, *let* A_p *be a* C^∞ p*-tensor and* B_q *a* C^∞ q*-tensor. If* A_p *is symmetric (or if* $p = s$*) then:*

$$\int_\Omega A_p \overset{(s)}{:} (\nabla B_q) = -\int_\Omega (\nabla.A_p) \overset{(s-1)}{:} B_q + \int_\Gamma (n.A_p) \overset{(s-1)}{:} B_q.$$

REMARK 3. *Remember that, for two* p*-tensors* A *and* B, *we have by definition* $(A,B) = \int_\Omega A \overset{(p)}{:} B$, *and in the case* $p = 2$ *(matricial case), since* $A : B = Tr(A.{}^tB)$, *we obtain:* $(A,B) = \int_\Omega Tr(A.{}^tB)$. *This remark allows us to obtain two interesting equalities in which we can see the particularity of the case* $a = 0$.

LEMMA 3.3. *If* g_a *is defined as in* (11) *with* v *and* σ *regular, then we have:*

$$(g_a, \sigma) = 2a(\sigma.\sigma, \nabla v) \quad \text{and} \quad (g_a, \Delta\sigma) = (a+1)(\nabla v.\sigma, \Delta\sigma) + (a-1)(\sigma.\nabla v, \Delta\sigma).$$

PROOF: By writing the definition of g_a, a direct computation gives us the result (the hypothesis of symmetry on σ is very important here)

$$\begin{aligned} g_a : \sigma &= Tr\Big(-W(v).\sigma.{}^t\sigma + \sigma.W(v).{}^t\sigma\Big) + aTr\Big(D(v).\sigma.{}^t\sigma + \sigma.D(v).{}^t\sigma\Big), \\ &= 2aTr(D(v).\sigma.\sigma) = aTr(\nabla v.\sigma.\sigma + \nabla^t v.\sigma.\sigma) = 2aTr(\sigma.\sigma.\nabla^t v). \end{aligned}$$

The proof of the second statement is very similar: we use the fact that $\Delta\sigma$ is symmetric too. ☐

3.2. Result for the inertial term in the Navier-Stokes equation.

DEFINITION 3.2. *Let* u *be a divergence-free vector field of* $\mathbb{R}^d$. *We define, for* σ *and* τ *two* p*-tensors:* $\underline{b}(u,\sigma,\tau) = \int_\Omega (u.\nabla\sigma) \overset{(p)}{:} \tau$. *For two vectors* v *and* w, *we find the classical definition (see* [6]*):* $b(u,v,w) = \sum_{i,j}\int_\Omega u_i(\partial_i v_j)w_j$.

The trilinear form b, as well as $\underline{b}$, has many continuity properties (see [6], [23]). Among these, we will mainly use the ones given below:

LEMMA 3.4. *If* $u \in V_1$, $\sigma \in \mathbb{H}^1$ *and* $\tau \in \mathbb{H}^1$ *then* $\underline{b}(u,\sigma,\tau) = -\underline{b}(u,\tau,\sigma)$, *in particular*

$$\underline{b}(u,\tau,\tau) = 0. \tag{20}$$

If $u \in \mathbb{H}^1$, $v \in \mathbb{H}^1$ *and* $w \in V_{d/2}$ *then* $|b(u,v,w)| \leq C|u|_2^{1/2}\|u\|_1^{1/2}|v|_2^{1/2}\|v\|_1^{1/2}\|w\|_{d/2}$.

4. **Proofs of the existence results.** All these proofs are based on a Galerkin approximations associated to energy estimates [15], [23]. In section 4.1, we reduce the non-homogeneous Navier-Stokes problem for the velocity v to a homogeneous problem for a new velocity $u = v - h_\delta$. In a second step, we construct approximate solutions of the problem (11) by the Galerkin method. In section 4.3 and 4.4, we give some estimates for (φ, u, σ) and their derivatives. These estimates allow us to perform the limit in the non-linear terms (using compactness methods, and some strong convergence properties).

4.1. **The non-homogeneous boundary conditions.** First, we recall the assumption (19): $h \in H^s(\Gamma)$, $s = \frac{d+1}{2}$ and $h.n = 0$ on Γ. In the three-dimensional case, we should add the following hypothesis: div $h \circ \Pi = 0$ on a neighborhood of Γ where Π is the projection on the boundary Γ. This assumption is verified in the physical case of a channel where the velocity is constant on each plane (see remark 4). We saw before expressing the results (see section 2.3) that we needed to extend the application h to Ω. For the proofs, we shall need such a extension verifying other properties. More precisely we have the following lemma

LEMMA 4.1. *For any $\delta > 0$, there exists h_δ, a vector field on Ω satisfying*

$$\begin{aligned} &h_\delta \in H^2(\Omega), \quad div\, h_\delta = 0 \ on\ \Omega, \quad h_\delta = h\ on\ \Gamma, \\ &|b(w, h_\delta, w)| \le C_1(\Omega)\delta \|w\|_1^2, \quad \forall\, w \in V_1, \\ &|h_\delta|_4 \le C_1(\Omega)\delta. \qquad\qquad (*) \end{aligned}$$

This result (without $(*)$) is a classical result for the non-homogeneous Stokes problem [23]. During the proofs of the existence results, we shall choose δ more precisely to obtain good estimates.

REMARK 4. *If we consider the channel $\Omega = \{-1 \le z \le 1\} \subset \mathbb{R}^d$ and if the shear velocity is chosen to be $U\ e_x$ on the upper boundary $\{z = 1\}$ and $-U\ e_x$ on the lower boundary $\{z = -1\}$, for any $\delta > 0$, we have an explicit function h_δ which verifies the properties of the lemma 4.1 (see* [18], [2]*).*

PROOF:

- At first, since Ω is regular with a regular boundary Γ (in fact, the hypothesis of C^2-class is sufficient) we have the existence of a tubular neighborhood of Γ in which the projection on Γ is well defined (see for instance [1], p.106). Let $Tub_R\Gamma$ this tubular neighborhood of Γ, Π the projection from $Tub_R\Gamma$ on Γ and $\rho(x)$ the distance from $x \in Tub_R\Gamma$ to Γ (we have $\rho(x) = \|x - \Pi(x)\|$, $\forall x \in Tub_R\Gamma$).

- Following A. Miranville (in the canal case [18]) let us introduce the function

$$\phi_\delta(s) = exp(\frac{1}{\delta^4})exp(\frac{R}{\delta^4(s-R)}) \quad \text{if } s \in [0; R[, \quad \phi_\delta(s) = 0 \quad \text{if } s \in [R; +\infty[.$$

This function is C^∞-class on $\mathbb{R}^+$, all its derivatives are bounded and it satisfies

$$\int_0^R |\phi_\delta(s)|^4 ds \le \frac{R}{4}\delta^4. \tag{21}$$

According to the dimension, we are going to proceed differently. In the three-dimensional case, let

$$h_\delta(x) = \phi_\delta(\rho(x))h(\Pi(x)).$$

This vector field is well defined on $\bar{\Omega}$. Moreover, it is obvious that $h_\delta = h$ on Γ and that $h_\delta \in H^2(\Omega)$ (using the fact that the functions ϕ_δ, ρ and Π belong to C^2 and using the regularity of h (see (19))).

- Using the definition of h_δ, we have div $(h_\delta(x)) = \phi_\delta(\rho(x))\,\text{div}\,(h(\Pi(x))) + \phi'_\delta(\rho(x))h(\Pi(x)).\nabla\rho(x)$. The assumption div $(h \circ \Pi) = 0$ allows us to cancel the first term. To cancel the last term, it is necessary to use the hypothesis $h.n = 0$, and show that $\nabla\rho(x) = n_{\Pi(x)}$. This last equality results from two remarks. First of all, the equality is true for a point of the boundary (the boundary is a level line of ρ and so $\nabla\rho$ is orthogonal to the boundary). Then, differentiating the equality $\rho(x)^2 = \|x - \Pi(x)\|^2$ and using $n_{\Pi(x)} = \frac{x-\Pi(x)}{\|x-\Pi(x)\|}$ we deduce the result.

- Since $h \in L^\infty(\bar\Omega)$ and making the change of variable $Tub_R\Gamma \to [0,R]\times\Gamma$, $x \mapsto (\rho(x),\Pi(x))$ whose Jacobian is bounded, we deduce by using (21) that

$$\int_\Omega |h_\delta(x)|^4 dx \le |h|_\infty^4 \int_{\rho(x)\le R} |\phi_\delta(\rho(x))|^4 dx \le C|h|_\infty^4 \int_0^R |\phi_\delta(\rho)|^4 d\rho \le C\delta^4.$$

The last relation is obtained straightforwardly by using the fact that $|u|_4 \le C\|u\|_1$ for $u \in V_1$ (see [18]).

- In the two-dimensional case, O. Ladyzhenskaya [14] proves the existence of a function $\xi : \Omega \to \mathbb{R}$ such that h has a continuation over Ω of kind $\overline{h} = \text{curl } \xi$. Moreover, the construction of ξ ([14], p. 26) shows us that we can chose ξ such that

$$\xi|_\Gamma = 0, \quad \xi(x) = R \text{ for } \rho(x) > R,$$

and the application $x \to (\Pi(x), \xi(x))$ is a diffeomorphism from $Tub_R\Gamma$ to $\Gamma \times [0;R]$.

As in the three-dimensional case, we introduce the function ϕ_δ and we let

$$h_\delta(x) = \phi_\delta(\xi(x))\overline{h}(x).$$

- Using the definition of h_δ, we have $\text{div } (h_\delta(x)) = \phi_\lambda(\xi(x))\,\text{div } (\overline{h}(x)) + \phi'_\lambda(\xi(x))\overline{h}(x).\nabla\xi(x)$. Since $\overline{h} = \text{curl } \xi$ (when $d=2$) we cancel the first term. To cancel the second, we easily remark that $\text{curl } \xi.\nabla\xi = 0$.

The rest of the proof is made exactly in the same way in the three-dimensional case. □

The idea of the proof of theorem 2.1 is to use the previous lemma for a fixed value of δ (which will be clarified after): we let $v = u + h_\delta$ so that the boundary conditions on v are homogeneous. Since $h_\delta - h \in V_1 \cap H^2$ the theorem 2.1 (we have a similary formulation for the theorem 2.2) will be proved if we show that for suitable δ there exists an unique solution (φ, u, σ) of (11) on $[0, T^*[$ such that

$$\begin{aligned} &\varphi \in L^2(0,T^*;\Phi_4) \cap L^\infty(0,T^*;\Phi_2),\\ &u \in L^2(0,T^*;V_1 \cap H^2) \cap L^\infty(0,T^*;V_1),\\ &\sigma \in L^2(0,T^*;\Sigma_2) \cap L^\infty(0,T^*;\Sigma_1). \end{aligned}$$

4.2. **Galerkin approximations.** We use the Fadeo-Galerkin method. Since Φ_2, V_1 and Σ_1 are separable Hilbert spaces, there exist Hilbertian basis $(\psi_i)_{i\ge1}$, $(w_i)_{i\ge1}$ and $(\tau_i)_{i\ge1}$ of Φ_2, V_1 and Σ_1 respectively. Moreover, we can assume that $(\psi_i)_{i\ge1}$ are the eigenfunctions of the operator $-\Delta$ with domain Φ_2, $(w_i)_{i\ge1}$ the eigen functions of the Stokes operator A with domain $V_1\cap H^2$ and $(\tau_i)_{i\ge1}$ the eigen functions of the operator $-\Delta$ with domain Σ_2.

We seek three functions in the form

$$\varphi_n(t) = \sum_{i=1}^n \alpha_i(t)\psi_i, \quad u_n(t) = \sum_{i=1}^n \beta_i(t)w_i, \quad \sigma_n(t) = \sum_{i=1}^n \gamma_i(t)\tau_i,$$

where α_i, β_i and γ_i are functions of C^1 class, $\varphi_n(0)$, $u_n(0)$ and $\sigma_n(0)$ are the respective orthogonal projections in L^2 of φ_0, u_0 and σ_0 on $\Psi_n = Span((\psi_i)_{1\le i\le n})$, $\mathcal{V}_n = Span((w_i)_{1\le i\le n})$ and $\mathcal{T}_n = Span((\tau_i)_{1\le i\le n})$ satisfying the following ordinary differential equations: For any $\psi \in \Psi_n$, any $w \in \mathcal{V}_n$ and any $\tau \in \mathcal{T}_n$,

$$\int_\Omega \partial_t\varphi_n\psi + \int_\Omega B(\varphi_n)\nabla\mu_n.\nabla\psi - \int_\Omega (u_n.\nabla\psi)\varphi_n - \int_\Omega (h_\delta.\nabla\psi)\varphi_n = 0, \qquad (22)$$

$$\begin{aligned}\int_\Omega \partial_t u_n.w + b(u_n+h_\delta,u_n+h_\delta,w) + 2\int_\Omega \eta(\varphi_n)D(u_n):D(w)\\ +2\int_\Omega \eta(\varphi_n)D(h_\delta):D(w) + \int_\Omega \sigma_n:\nabla w = -\int_\Omega (w.\nabla\mu_n)\varphi_n,\end{aligned}\tag{23}$$

$$\begin{aligned}\int_\Omega \partial_t\sigma_n:\tau + \underline{b}(u_n+h_\delta,\sigma_n,\tau) + \int_\Omega g_a(\sigma_n,u_n+h_\delta):\tau + \int_\Omega l(\varphi_n)\sigma_n:\tau\\ +\varepsilon\int_\Omega \nabla\sigma_n \overset{(3)}{:} \nabla\tau = \int_\Omega f(\varphi_n)\eta(\varphi_n)D(u_n+h_\delta):\tau,\end{aligned}\tag{24}$$

$$\mu_n = -\alpha^2\Delta\varphi_n + F'(\varphi_n).\tag{25}$$

This problem is a set of ordinary differential equations in $(\alpha_i)_{1\le i\le n}$, $(\beta_i)_{1\le i\le n}$ and $(\gamma_i)_{1\le i\le n}$. The functions B, η, F', l and f being locally Lipschitz (of C^1-class), the Cauchy theorem ensures that there exists a unique solution (φ_n,u_n,σ_n) on a maximal interval $[0,t_n[$.

4.3. **Estimates.** In this part, we denote φ, μ, u and σ instead of φ_n, μ_n, u_n and σ_n. First of all, we use a result for the Cahn-Hilliard equation which allows us to use the lemma 3.2:

LEMMA 4.2. *For $\varphi\in\Phi_1$ solution of the first equation of* (11)*, with any vector field $v(t)\in V_0$, we have $\partial_t m(\varphi)=0$, and so the average of the order parameter is conserved all along the time (where this solution exists):*

$$m(\varphi(t)) = m(\varphi_0).$$

4.3.1. *H^1-estimate for φ.* We use μ as a test function in (22):

$$\int_\Omega \partial_t\varphi\ \mu + \int_\Omega B(\varphi)\nabla\mu.\nabla\mu - \int_\Omega (u.\nabla\mu)\varphi - \int_\Omega (h_\delta.\nabla\mu)\varphi = 0.$$

From $\frac{\partial\varphi}{\partial n}|_\Gamma = 0$ and (25), we then deduce that

$$\int_\Omega \partial_t\varphi\ \mu = -\alpha^2\int_\Omega \partial_t\varphi\Delta\varphi + \int_\Omega \partial_t\varphi F'(\varphi) = \frac{d}{dt}\left(\frac{\alpha^2|\nabla\varphi|_2^2}{2}\right) + \frac{d}{dt}\int_\Omega F(\varphi).$$

We use the assumption $B_1\le B$ (12) to obtain:

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + B_1|\nabla\mu|_2^2 \le \int_\Omega (u.\nabla\mu)\varphi + \left|\int_\Omega (h_\delta.\nabla\mu)\varphi\right|.$$

We remark that, thanks to the boundary conditions

$$\int_\Omega h_\delta.\nabla\mu = \int_\Omega \text{div}\ (\mu h_\delta) = \int_{\partial\Omega} \mu(h_\delta.n) = 0,$$

then we can write (using successively the lemma 4.2, the Young inequality, the lemma 3.1 and the lemma 4.1)

$$\left|\int_\Omega (h_\delta.\nabla\mu)\varphi\right| = \left|\int_\Omega (h_\delta.\nabla\mu)(\varphi - m(\varphi))\right| \le \frac{B_1}{2}|\nabla\mu|_2^2 + C_1(\Omega)^2\delta^2|\nabla\varphi|_2^2.$$

Following F. Boyer [2], under the assumptions (18) and (15), we have

$$|\nabla\mu|_2^2 \ge \alpha^2|\Delta\varphi|_2^2 + C_1|\nabla\varphi|_2^2 + C_1F_3(m(\varphi_0))\int_\Omega F(\varphi) - 2C_2F_4(m(\varphi_0)),$$

where C_1 and C_2 are two positive constants which only depend on α, $|\Omega|$ and F_5.

In the sequel, we choose δ such that $2C_1\delta \le \tilde{C}\alpha^2$. The new estimate is writen

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + C\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \frac{B_1}{2}|\nabla\mu|_2^2 \\ + \alpha^2 B_1|\Delta\varphi|_2^2 \le \int_\Omega (u.\nabla\mu)\varphi + C. \tag{26}$$

4.3.2. $\mathbb{L}^2$*-estimate for* u. Now, we use u as a test function in (23):

$$\int_\Omega \partial_t u.u + b(u,u,u) + b(u,h_\delta,u) + b(h_\delta,u,u) + b(h_\delta,h_\delta,u) \\ + 2\int_\Omega \eta(\varphi)D(u):D(u) + 2\int_\Omega \eta(\varphi)D(h_\delta):D(u) + \int_\Omega \sigma:\nabla u = -\int_\Omega (u.\nabla\mu)\varphi.$$

Here, we use the fact that u and h_δ are a divergence-free vector fields and therefore $b(u,u,u) = b(h_\delta,u,u) = 0$ (see (20)). From the assumption (13): $\eta_1 \le \eta$ and the Korn inequality, we deduce

$$2\int_\Omega \eta(\varphi)D(u):D(u) \ge \eta_1|\nabla u|_2^2.$$

We use the lemma 4.1 (assuming $\delta < \eta_1/(8C(\Omega)C_1)$) and the Poincaré inequality:

$$|b(u,h_\delta,u)| \le C_1(\Omega)\delta\|u\|_1^2 \le \frac{\eta_1}{8}|\nabla u|_2^2.$$

Moreover, we have the classical bounds

$$|b(h_\delta,h_\delta,u)| \le |h_\delta|_4|\nabla h_\delta|_2|u|_4 \le C|\nabla u|_2 \le C + \frac{\eta_1}{8}|\nabla u|_2^2,$$
$$\left|\int_\Omega \eta(\varphi)D(h_\delta):D(u)\right| \le \eta_2|\nabla h_\delta|_2|\nabla u|_2 \le C + \frac{\eta_1}{8}|\nabla u|_2^2,$$
$$\left|\int_\Omega \sigma:\nabla u\right| \le \frac{\eta_1}{8}|\nabla u|_2^2 + C|\sigma|_2^2,$$

and so, finally

$$\frac{d}{dt}\left(\frac{|u|_2^2}{2}\right) + \frac{\eta_1}{2}|\nabla u|_2^2 \le -\int_\Omega (u.\nabla\mu)\varphi + C|\sigma|_2^2 + C. \tag{27}$$

4.3.3. $\mathbb{L}^2$*-estimate for* σ. We use σ as a test function in (24). Since u and h_δ are divergence-free and $h_\delta.n = u.n = 0$ on Γ, we can use (20):

$$\frac{d}{dt}\left(\frac{|\sigma|_2^2}{2}\right) + l_1|\sigma|_2^2 + \varepsilon|\nabla\sigma|_2^2 \le \left|\int_\Omega f(\varphi)\eta(\varphi)D(u):\sigma\right| \\ + \left|\int_\Omega g_a(\sigma,u):\sigma\right| + \left|\int_\Omega g_a(\sigma,h_\delta):\sigma\right| + \left|\int_\Omega f(\varphi)\eta(\varphi)D(h_\delta):\sigma\right|.$$

The first term of the right member is estimated by

$$\left|\int_\Omega f(\varphi)\eta(\varphi)D(u):\sigma\right| \le f_2\eta_2|\nabla u|_2|\sigma|_2 \le \frac{\eta_1}{8}|\nabla u|_2^2 + C|\sigma|_2^2.$$

Moreover, by lemma 3.3, the Cauchy-Schwarz inequality, the Hölder inequality and Sobolev embeddings:

$$\left|\int_\Omega g_a(\sigma,u):\sigma\right| \le 2|a||(\sigma.\sigma,\nabla u)| \le 2|a||\sigma|_3|\sigma|_6|\nabla u|_2 \le \frac{\eta_1}{8}|\nabla u|_2^2 + a^2C\|\sigma\|_1^4,$$

where C does not depend on a (we obtain the same results when h_δ takes the place u). Finally, we have

$$\frac{d}{dt}\left(\frac{|\sigma|_2^2}{2}\right) + \varepsilon|\nabla\sigma|_2^2 \leq \frac{\eta_1}{4}|\nabla u|_2^2 + C|\sigma|_2^2 + a^2C\|\sigma\|_1^4 + C. \tag{28}$$

It is important to notice that the constants C here do not depend on the rheologic coefficient a. It is the reason for which the particular case $a = 0$ will be interessant to study: all the powers of $|\sigma|_2$ in the right member will be lower than 2 and we can apply the Gronwall lemma (see the section 6).

4.3.4. *H^2-estimate for φ.* Substituting $\psi = \Delta^2\varphi$ into (22):

$$\int_\Omega \partial_t\varphi\ \Delta^2\varphi - b(u,\varphi,\Delta^2\varphi) - b(h_\delta,\varphi,\Delta^2\varphi) - \int_\Omega \text{div}\ (B(\varphi)\nabla\mu)\Delta^2\varphi = 0.$$

Then, integrating by parts the first term, writing $\Delta\mu = -\alpha^2\Delta^2\varphi + \Delta(F'(\varphi))$ for the last one and using Young inequality, we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\Delta\varphi|_2^2}{2}\right) + \alpha^2B_1|\Delta^2\varphi|_2^2 \leq\ &\frac{\alpha^2B_1}{6}|\Delta^2\varphi|_2^2\\ &+ C(|u|_2^2|\nabla\varphi|_\infty^2 + |h_\delta|_\infty^2|\nabla\varphi|_2^2 + |\Delta(F'(\varphi))|_2^2 + |\nabla\varphi\nabla\mu|_2^2).\end{aligned}$$

We have (using the lemma 3.2):

$$|u|_2^2|\nabla\varphi|_\infty^2 \leq |u|_2^2|\nabla\varphi|_2|\Delta^2\varphi|_2 \leq \frac{\alpha^2B_1}{6}|\Delta^2\varphi|_2^2 + C|u|_2^4|\nabla\varphi|_2^2.$$

Following [24] and [2], the inequality $|\nabla\varphi|_3^2|\nabla\varphi|_6^2 \leq C\|\nabla\varphi\|_1^4$ (as we did for σ) allows us to estimate the term $|\Delta(F'(\varphi))|_2$ under the assumptions (16):

$$\begin{aligned}|\Delta(F'(\varphi))|_2^2 &\leq |F'''(\varphi)|_\infty|\nabla\varphi|_6^2|\nabla\varphi|_3^2 + |F''(\varphi)|_\infty^2|\Delta\varphi|_2^2,\\ &\leq C(1 + |\varphi - m(\varphi)|_\infty^{2q})\|\nabla\varphi\|_1^4 + C(1 + |\varphi - m(\varphi)|_\infty^{2q+2})|\Delta\varphi|_2^2,\\ &\leq C(1 + |\Delta\varphi|_2^{2q})\|\nabla\varphi\|_1^4 + C_3|\Delta\varphi|_2^2.\end{aligned}$$

By definition of the potential μ, we have

$$|\nabla\varphi\nabla\mu|_2^2 \leq \alpha^2|\nabla\varphi|_\infty^2|\nabla\Delta\varphi|_2^2 + |\nabla\varphi|_6^2|\nabla\varphi|_3^2|F''(\varphi)|_\infty^2.$$

For the first term, we use the Agmon's inequality and the lemma 3.1

$$|\nabla\varphi|_\infty^2 = |\nabla(\varphi - m(\varphi))|_\infty^2 \leq \|\nabla(\varphi - m(\varphi))\|_1\|\nabla(\varphi - m(\varphi))\|_2 \leq |\Delta\varphi|_2|\nabla\Delta\varphi|_2.$$

Moreover, integration by parts yields $|\nabla\Delta\varphi|_2^2 \leq |\Delta\varphi|_2|\Delta^2\varphi|_2$ and so,

$$|\nabla\varphi|_\infty^2|\nabla\Delta\varphi|_2^2 \leq |\Delta\varphi|_2^{5/2}|\Delta^2\varphi|_2^{3/2} \leq \frac{\alpha^2B_1}{6}|\Delta^2\varphi|_2^2 + C|\Delta\varphi|_2^{10}.$$

For the second, the assumption (16) with the embedding $H^2 \subset L^\infty$ show us that:

$$|\nabla\varphi|_3^2|\nabla\varphi|_6^2|F''(\varphi)|_\infty^2 \leq C(1 + |\Delta\varphi|_2^{q+1})\|\nabla\varphi\|_1^4.$$

We deduce that there exists a polynomial function P such that

$$\frac{d}{dt}\left(\frac{|\Delta\varphi|_2^2}{2}\right) + \frac{\alpha^2B_1}{2}|\Delta^2\varphi|_2^2 \leq (\|\nabla\varphi\|_1^2 + |u|_2^2)^2P(\|\nabla\varphi\|_1^2) + C_3|\Delta\varphi|_2^2 + C. \tag{29}$$

4.3.5. $\mathbb{H}^1$*-estimate for* u. For this estimate, we use Au as a test function in (23):

$$\int_\Omega \partial_t u.Au + b(u+h_\delta, u+h_\delta, Au) - 2\int_\Omega \text{div } (\eta(\varphi)D(u)).Au - 2\int_\Omega \text{div } (\eta(\varphi)D(h_\delta)).Au - \int_\Omega \text{div } \sigma : Au = -\int_\Omega \mu\nabla\varphi.Au.$$

For the second term, we write:

$$|b(u,u,Au)| \leq C|u|_6|\nabla u|_3|Au|_2 \leq C(|u|_2 + |\nabla u|_2)(|\nabla u|_2 + |\nabla u|_2^{1/2}|\Delta u|_2^{1/2})|Au|_2,$$
$$\leq C\|u\|_1^2|Au|_2 + C\|u\|_1^{3/2}|Au|_2^{3/2} \leq \frac{\eta_1}{18}|Au|_2^2 + C(\|u\|_1^4 + \|u\|_1^6).$$

With Agmon's inequality, it follows

$$|b(u,h_\delta,Au)| \leq |u|_\infty|\nabla h_\delta|_2|Au|_2 \leq C|\nabla u|_2^{1/2}|Au|_2^{3/2} \leq \frac{\eta_1}{18}|Au|_2^2 + C|\nabla u|_2^2,$$
$$|b(h_\delta,u,Au)| \leq \frac{\eta_1}{18}|Au|_2^2 + C|h_\delta|_\infty^2|\nabla u|_2^2, \quad \text{and}$$
$$|b(h_\delta,h_\delta,Au)| \leq \frac{\eta_1}{18}|Au|_2^2 + C|h_\delta|_\infty^2|\nabla h_\delta|_2^2.$$

The next term splits into three parts:

$$\int_\Omega \text{div } (\eta(\varphi)D(u)).Au = \int_\Omega \eta'(\varphi)\nabla\varphi D(u).Au - \int_\Omega \eta(\varphi)Au.Au + \int_\Omega \eta(\varphi)\nabla\pi.Au.$$

The assumption (13) on η implies $\int_\Omega \eta(\varphi)Au.Au \geq \eta_1|Au|_2^2$, so we use successively lemma 3.2 and the regularity of the Stokes operator [23] to show that we have an estimate where all the powers of $|Au|_2$ and $|\Delta^2\varphi|_2$ are small enough to apply the Young inequality and obtain as follows an estimate of kind (30).

On the one hand, after integration by parts (remember that div $u = 0$ and $u = u_n$ is in $\mathcal{V}_n$, subset of the set of eigen functions of A)

$$\left|\int_\Omega \eta(\varphi)\nabla\pi.Au\right| = \left|\int_\Omega \eta'(\varphi)\nabla\varphi\ \pi.Au\right| \leq |\eta'|_\infty|\nabla\varphi|_\infty|\pi|_2|Au|_2,$$
$$\leq C|\eta'|_\infty|\nabla\varphi|_2^{1/2}|\Delta^2\varphi|_2^{1/2}\|u\|_1|Au|_2 \leq \frac{\eta_1}{18}|Au|_2^2 + \frac{\alpha^2B_1}{48}|\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2\|u\|_1^4.$$

On the other hand, using the lemma 3.2 again,

$$\left|\int_\Omega \eta'(\varphi)\nabla\varphi D(u).Au\right| \leq \frac{\eta_1}{18}|Au|_2^2 + \frac{\alpha^2B_1}{48}|\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2|\nabla u|_2^4.$$

We write the same term with h_δ as follows:

$$\left|\int_\Omega \text{div } (\eta(\varphi)D(h_\delta)).Au\right| = \left|\int_\Omega \eta'(\varphi)\nabla\varphi D(h_\delta).Au + \int_\Omega \eta(\varphi)\Delta h_\delta.Au\right|,$$
$$\leq \frac{\eta_1}{18}|Au|_2^2 + \frac{\alpha^2B_1}{48}|\Delta^2\varphi|_2^2 + C(\|\nabla h_\delta\|_1^2 + |\nabla\varphi|_2^2).$$

We have clearly,

$$\left|\int_\Omega \text{div } (\sigma) : Au\right| \leq \frac{\eta_1}{18}|Au|_2^2 + C|\nabla\sigma|_2^2.$$

We can repeat the same argument with the last term $\int_\Omega \mu\nabla\varphi.Au$:

$$\left|\int_\Omega \mu\nabla\varphi.Au\right| \leq \alpha^2\left|\int_\Omega \Delta\varphi\nabla\varphi.Au\right| + \left|\int_\Omega \nabla(F(\varphi)).Au\right|,$$
$$\leq \frac{\eta_1}{18}|Au|_2^2 + \frac{\alpha^2B_1}{48}|\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2|\Delta\varphi|_2^4.$$

The term $\left|\int_\Omega \nabla(F(\varphi)).Au\right|$ vanishes as $u = u_n$ is a sum of eigenvectors for A with div $u = 0$. Finally, from the Sobolev embeddings, we have

$$\frac{d}{dt}\left(\frac{|\nabla u|_2^2}{2}\right) + \frac{\eta_1}{2}|Au|_2^2 \le \frac{\alpha^2 B_1}{8}|\Delta^2\varphi|_2^2 + C(\|\nabla\varphi\|_1^2 + \|u\|_1^2)^3 + C + |\nabla\sigma|_2^2. \quad (30)$$

4.3.6. $\mathbb{H}^1$*-estimate for* σ. Substituting $\tau = -\Delta\sigma$ into (24). After an integration by parts and using the result of the lemma 3.3, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla\sigma|_2^2}{2}\right) &+ l_1|\nabla\sigma|_2^2 + \varepsilon|\Delta\sigma|_2^2 \le f_2\eta_2|\nabla u|_2|\Delta\sigma|_2 + f_2\eta_2|\nabla h_\delta|_2|\Delta\sigma|_2 \\ &+ |u|_6|\nabla\sigma|_3|\Delta\sigma|_2 + |h_\delta|_4|\nabla\sigma|_4|\Delta\sigma|_2 + (a+1)|(\nabla u.\sigma,\Delta\sigma)| + (1-a)|(\sigma.\nabla u,\Delta\sigma)| \\ &+ (a+1)|(\nabla h_\delta.\sigma,\Delta\sigma)| + (1-a)|(\sigma.\nabla h_\delta,\Delta\sigma)| + |l'|_\infty|\nabla\varphi|_\infty|\sigma|_2|\nabla\sigma|_2\end{aligned}$$

Remember that $|(\nabla u.\sigma,\Delta\sigma)|$ and $|(\sigma.\nabla u,\Delta\sigma)|$ can be controlled by $|\nabla u|_3|\sigma|_6|\Delta\sigma|_2$ (use the same estimate for the similar term with h_δ). A Young inequality gives

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla\sigma|_2^2}{2}\right) + l_1|\nabla\sigma|_2^2 + \varepsilon|\Delta\sigma|_2^2 &\le \frac{\varepsilon}{4}|\Delta\sigma|_2^2 + C|\nabla u|_2^2 + C|\nabla h_\delta|_2^2 + C|u|_6^2|\nabla\sigma|_3^2 \\ &+ C|h_\delta|_4^2|\nabla\sigma|_4^2 + C|\nabla u|_3^2|\sigma|_6^2 + |l'|_\infty|\nabla\varphi|_\infty|\sigma|_2|\nabla\sigma|_2 + C.\end{aligned}$$

Finally, we use the Sobolev embeddings again and the Young inequality to write

$$\begin{aligned}|u|_6^2|\nabla\sigma|_3^2 &\le \frac{\varepsilon}{4}|\Delta\sigma|_2^2 + C(|u|_2^2 + |\nabla u|_2^2 + |\nabla\sigma|_2^2)^3,\\ |\sigma|_6^2|\nabla u|_3^2 &\le \frac{\eta_1}{4}|Au|_2^2 + C(|\nabla u|_2^2 + |\sigma|_2^2 + |\nabla\sigma|_2^2)^3.\end{aligned}$$

And the last term gives,

$$|\nabla\varphi|_\infty|\sigma|_2|\nabla\sigma|_2 \le \frac{\alpha^2 B_1}{8}|\Delta^2\varphi|_2^2 + C(|\nabla\varphi|_2^2 + |\sigma|_2^2|\nabla\sigma|_2^2).$$

These estimates give

$$\frac{d}{dt}\left(\frac{|\nabla\sigma|_2^2}{2}\right) + \frac{\varepsilon}{2}|\Delta\sigma|_2^2 \le \frac{\eta_1}{4}|Au|_2^2 + \frac{\alpha^2 B_1}{8}|\Delta^2\varphi|_2^2 + C(\|u\|_1^2 + \|\sigma\|_1^2 + \|\nabla\varphi\|_1^2)^3 + C. \quad (31)$$

4.4. **Estimates for the derivatives.** Equations (22), (23), (24) and (25) can also be written:

$$\begin{cases}\dfrac{d\varphi_n}{dt} = -P^*_{\Psi_n}\Big(\text{div }(B(\varphi_n)\nabla\mu_n) + \text{ div }(\varphi_n v_n)\Big),\\ \dfrac{dv_n}{dt} = -P^*_{V_n}\Big(B(v_n,v_n) + E(\varphi_n,v_n) - \text{ div }(\sigma_n) + \varphi_n\nabla\mu_n\Big),\\ \dfrac{d\sigma_n}{dt} = -P^*_{\mathcal{T}_n}\Big(\text{div }(\sigma_n v_n) + g_a(\sigma_n,v_n) + l(\varphi_n)\sigma_n - \varepsilon\Delta\sigma_n - f\eta(\varphi_n)D(v_n)\Big),\end{cases}$$

with $E(\varphi,v).w = 2\int_\Omega \eta(\varphi)D(v) : D(w)$.

Remark 5. *The fact that all the projectors are orthogonal in* L^2 *implies* $\|P\| \le 1$ *and* $\|P^*\| \le 1$ *(as linear operator). More precisely,* P_{Ψ_n}, $P_{\mathcal{V}_n}$ *and* $P_{\mathcal{T}_n}$ *are projectors respectively in,* Φ_1, V_s *and* Σ_s *for any* $s \ge 0$*. So, we have:*

$$\begin{aligned}&\|P_{\Psi_n}\|_{L(\Phi_1,\Phi_1)} \le 1, \quad \|P_{\mathcal{V}_n}\|_{L(V_s,V_s)} \le 1, \quad \|P_{\mathcal{T}_n}\|_{L(\Sigma_s,\Sigma_s)} \le 1,\\ &\|P^*_{\Psi_n}\|_{L(\Phi_1',\Phi_1')} \le 1, \quad \|P^*_{\mathcal{V}_n}\|_{L(V_s',V_s')} \le 1, \quad \|P^*_{\mathcal{T}_n}\|_{L(\Sigma_s',\Sigma_s')} \le 1.\end{aligned}$$

4.4.1. *L^2-estimate for $\partial_t\varphi_n$.* We first show that if φ_n and v_n are regular enough, for example $\{\varphi_n\}$ bounded in $L^2(0,t;\Phi_4)\cap L^\infty(0,t;\Phi_2)$ and $\{v_n\}$ bounded in $L^2(0,t;V_1\cap H^2)$, then we have an estimate

$$\{\partial_t\varphi_n\} \text{ bounded in } L^2(0,t;\Phi_0). \tag{32}$$

Indeed, $\partial_t\varphi_n$ has the same regularity than $\operatorname{div}(B(\varphi_n)\nabla\Delta\varphi_n)$ (the other terms are clearly more regular) and since

$$\operatorname{div}(B(\varphi_n)\nabla\Delta\varphi_n) = B'(\varphi_n)\nabla\varphi_n\nabla\Delta\varphi_n + \Delta^2\varphi_n,$$

we obtain a bound on this term using interpolation results: $\{\varphi_n\}$ is bounded in $L^4(0,t;\Phi_3)$ which implies in particular $\{\nabla\varphi_n\nabla\Delta\varphi_n\}$ bounded in $L^2(0,t;\Phi_0)$.

4.4.2. *L^2-estimate for $\partial_t v_n$.* To obtain this estimate, we assume first that we have $\{\varphi_n\}$, $\{v_n\}$ and $\{\sigma_n\}$ bounded in $L^2(0,t;\Phi_4)\cap L^\infty(0,t;\Phi_2)$, $L^2(0,t;V_1\cap H^2)\cap L^\infty(0,t;V_1)$ and $L^2(0,t;\Sigma_1)$ respectively. Using the products of Lebesgue spaces (for example if $d<4$, then $H^1\times H^1\subset L^2$), we obtain directly a bound for $\{v_n.\nabla v_n\}$, $\{\mu\nabla\varphi_n\}$, $\{\nabla\varphi_n\nabla v_n\}$ and $\{Av_n\}$ in $L^2(0,t;L^2)$, and so

$$\{\operatorname{div}(\eta(\varphi_n)D(v_n))\} = \{\eta'(\varphi_n)\nabla\varphi_n\nabla v_n + \eta(\varphi_n)Av_n\} \text{ is bounded in } L^2(0,t;L^2)$$

which proves that

$$\{\partial_t v_n\} \text{ is bounded in } L^2(0,t;V_0). \tag{33}$$

4.4.3. *L^2-estimate for $\partial_t\sigma$.* Of course, when we have $\{v_n\}$ is bounded in $L^2(0,t;V_1\cap H^2)\cap L^\infty(0,t;V_1)$ and $\{\sigma_n\}$ is bounded in $L^2(0,t;\Sigma_2)\cap L^\infty(0,t;\Sigma_1)$ then with simple arguments like the above we deduce that:

$$L^2(0,t;H^2)\times L^\infty(0,t;H^1)\subset L^2(0,t;H^1) \text{ then } \{v_n\sigma_n\} \text{ is bounded in } L^2(0,t;H^1),$$

$$L^2(0,t;H^1)\times L^\infty(0,t;H^1)\subset L^2(0,t;L^2) \text{ then } \{Dv_n.\sigma_n\} \text{ is bounded in } L^2(0,t;L^2),$$

and more generally, $\{g_a(\sigma_n,v_n)\}$ is bounded in $L^2(0,t;L^2)$. The other term are clearly bounded in $L^2(0,t;L^2)$, we deduce

$$\{\partial_t\sigma_n\} \text{ is bounded in } L^2(0,t;\Sigma_0). \tag{34}$$

4.5. **Passing to the limit.** We are going to show now the result in the case $a\neq 0$: if we define the total energy of the system by

$$z(t) = \frac{\alpha^2}{2}\|\nabla\varphi_n\|_1^2 + \int_\Omega F(\varphi_n) + \frac{1}{2}\|v_n\|_1^2 + \frac{1}{2}\|\sigma_n\|_1^2,$$

we see that the sum of inequalities (26), (27), (28), (29), (30) and (31) has the following form ($\mathcal{P}$ is a polynomial function):

$$z' + \frac{\alpha^2 B_1}{2}|\Delta^2\varphi_n|_2^2 + \frac{\eta_1}{2}|\Delta v_n|_2^2 + \frac{\varepsilon}{2}|\Delta\sigma_n|_2^2 \le \mathcal{P}(z).$$

This estimate shows us that there exists a $T^*>0$ such that

$$\begin{aligned}
&\{\varphi_n\} \text{ is bounded in } L^2(0,T^*;\Phi_4)\cap L^\infty(0,T^*;\Phi_2),\\
&\{v_n\} \text{ is bounded in } L^2(0,T^*;V_1\cap H^2)\cap L^\infty(0,T^*;V_1),\\
&\{\sigma_n\} \text{ is bounded in } L^2(0,T^*;\Sigma_2)\cap L^\infty(0,T^*;\Sigma_1).
\end{aligned}$$

Using the inequalities (32), (33), (34), it follows that $\{\partial_t\varphi_n\}$, $\{\partial_t v_n\}$, and $\{\partial_t\sigma_n\}$ are bounded in $L^2(0,T^*;\Phi_0)$, $L^2(0,T^*;V_0)$ and $L^2(0,T^*;\Sigma_0)$ respectively. We apply the classical results (see for instance [22]) to extract subsequences of $\{\varphi_n\}$, $\{v_n\}$ and $\{\sigma_n\}$, still denoted by $\{\varphi_n\}$, $\{v_n\}$ and $\{\sigma_n\}$ such that

$$\begin{aligned}&\varphi_n \rightharpoonup \varphi \text{ in } L^\infty(0,T^*;\Phi_2) \text{ weak-}*, \quad \varphi_n \rightharpoonup \varphi \text{ in } L^2(0,T^*;\Phi_4) \text{ weakly},\\ &\partial_t\varphi_n \rightharpoonup \partial_t\varphi \text{ in } L^2(0,T^*;\Phi_0) \text{ weakly}, \quad \varphi_n \longrightarrow \varphi \text{ in } L^2(0,T^*;\Phi_0) \text{ strong a.e.},\end{aligned}$$

$$\begin{aligned}&v_n \rightharpoonup v \text{ in } L^\infty(0,T^*;V_1) \text{ weak-}*, \quad v_n \rightharpoonup v \text{ in } L^2(0,T^*;V_1\cap H^2) \text{ weakly},\\ &\partial_t v_n \rightharpoonup \partial_t v \text{ in } L^2(0,T^*;V_0) \text{ weakly}, \quad v_n \longrightarrow v \text{ in } L^2(0,T^*;V_0) \text{ strong a.e.},\end{aligned}$$

$$\begin{aligned}&\sigma_n \rightharpoonup \sigma \text{ in } L^\infty(0,T^*;\Sigma_1) \text{ weak-}*, \quad \sigma_n \rightharpoonup \sigma \text{ in } L^2(0,T^*;\Sigma_2) \text{ weakly},\\ &\partial_t\sigma_n \rightharpoonup \partial_t\sigma \text{ in } L^2(0,T^*;\Sigma_0) \text{ weakly}, \quad \sigma_n \longrightarrow \sigma \text{ in } L^2(0,T^*;\Sigma_0) \text{ strong a.e..}\end{aligned}$$

Moreover, a compacity result (see [22]) ensures that $\varphi_n \rightharpoonup \varphi \in C^0([0,T^*[;\Phi_2)$ weakly, $v_n \rightharpoonup v \in C^0([0,T^*[;V_1)$ weakly and $\sigma_n \rightharpoonup \sigma \in C^0([0,T^*[;\Sigma_1)$ weakly.

Consequently, the limit functions φ, v and σ verify the initial conditions. Indeed, $\varphi_n(0) = P_{\Phi_n}(\varphi_0)$ converges weakly to $\varphi(0)$ in Φ_1, and since P_{Φ_n} converges strongly to the identity, $\varphi(0) = \varphi_0$. From the strong convergence of v_n, we can pass to the limit in equation (11), in particular in the quadratic terms like $\sigma_n.\nabla v_n$, $v_n.\nabla v_n$. This proves the result.

5. **Uniqueness of the solution.** In this section, we show that the solution obtained in the theorem 2.1 is unique. Let $(\varphi_1, v_1, \sigma_1)$ and $(\varphi_2, v_2, \sigma_2)$ be two solutions of equation (11) such that

$$\begin{cases}\varphi_1,\ \varphi_2 \in L^2(0,T;\Phi_4)\cap L^\infty(0,T;\Phi_2),\\ v_1,\ v_2 \in L^2(0,T;V_1\cap H^2)\cap L^\infty(0,T;V_1),\\ \sigma_1,\ \sigma_2 \in L^2(0,T;\Sigma_2)\cap L^\infty(0,T;\Sigma_1),\end{cases}$$

with same initial values $(\varphi_0, v_0, \sigma_0)$. The scalar function $\varphi = \varphi_1 - \varphi_2$, the vector function $v = v_1 - v_2$ and the tensor function $\sigma = \sigma_1 - \sigma_2$ satisfy the following relations: For any $\psi \in \Phi_3$, any $w \in V_1$ and any $\tau \in \Sigma_1$,

$$\begin{aligned}\int_\Omega &\partial_t\varphi\ \psi + b(v,\varphi_1,\psi) + b(v_2,\varphi,\psi)\\ &- \alpha^2\int_\Omega (B(\varphi_1)-B(\varphi_2))\nabla\Delta\varphi_1.\nabla\psi - \alpha^2\int_\Omega B(\varphi_2)\nabla\Delta\varphi.\nabla\psi\\ &+ \int_\Omega (BF''(\varphi_1)-BF''(\varphi_2))\nabla\varphi_1.\nabla\psi + \int_\Omega BF''(\varphi_2)\nabla\varphi.\nabla\psi = 0,\end{aligned} \tag{35}$$

$$\begin{aligned}\int_\Omega &\partial_t v.w + b(v,v_1,w) + b(v_2,v,w) - \int_\Omega \operatorname{div}\sigma.w\\ &+ 2\int_\Omega (\eta(\varphi_1)-\eta(\varphi_2))D(v_1):D(w) + 2\int_\Omega \eta(\varphi_2)D(v):D(w)\\ &= -\alpha^2\int_\Omega (w.\nabla\varphi_1)\Delta\varphi - \alpha^2\int_\Omega (w.\nabla\varphi)\Delta\varphi_2,\end{aligned} \tag{36}$$

$$\begin{aligned}\int_\Omega &\partial_t\sigma:\tau + \underline{b}(v,\sigma_1,\tau) + \underline{b}(v_2,\sigma,\tau) + \int_\Omega l(\varphi_2)\sigma:\tau + \int_\Omega (l(\varphi_1)-l(\varphi_2))\sigma_1:\tau\\ &+ \varepsilon\int_\Omega \nabla\sigma:\nabla\tau + \int_\Omega (g_a(\sigma,v_1)+g_a(\sigma_2,v)):\tau\\ &= \int_\Omega (f\eta(\varphi_1)-f\eta(\varphi_2))D(v_1):\tau + \int_\Omega f\eta(\varphi_2)D(v):\tau,\end{aligned} \tag{37}$$

together with zero initial conditions, $\varphi(0) = 0$, $v(0) = 0$, $\sigma(0) = 0$. For $i = 1$ or 2, we have $\varphi_i \in C([0,T] \times \bar{\Omega})$ and so there exists $R > 0$ such that

$$|\varphi_i(t,x)|_\infty \leq R, \quad \|\varphi_i(t)\|_2 \leq R, \quad \|v_i(t)\|_1 \leq R, \quad \|\sigma_i(t)\|_1 \leq R.$$

5.1. **Estimate on $\varphi_1 - \varphi_2$.** We take $-\alpha^2\Delta\varphi$ as a test function in (35), we use the assumption (12) on B: $B_1 \leq B$ to deduce:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2\right) + \alpha^4 B_1|\nabla\Delta\varphi|_2^2 \leq\ & \alpha^2|v|_2|\nabla\varphi_1|_6|\Delta\varphi|_3\\ &+ \alpha^2|v_2|_6|\nabla\varphi|_2|\Delta\varphi|_3 + \alpha^4|B'|_\infty|\varphi|_\infty|\nabla\Delta\varphi_1|_2|\nabla\Delta\varphi|_2\\ &+ \alpha^2|(BF'')'|_\infty|\varphi|_3|\nabla\varphi_1|_6|\nabla\Delta\varphi|_2 + \alpha^2|BF''|_\infty|\nabla\varphi|_2|\nabla\Delta\varphi|_2.\end{aligned}$$

In the two first terms of the right member, using $|\Delta\varphi|_2^2 \leq |\nabla\varphi|_2|\nabla\Delta\varphi|_2$ we have

$$|\Delta\varphi|_3 \leq C(|\Delta\varphi|_2 + |\Delta\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2}) \leq C(|\nabla\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2} + |\nabla\varphi|_2^{1/4}|\nabla\Delta\varphi|_2^{3/4}).$$

In the third term, a straightforward integration by parts gives

$$|\nabla\Delta\varphi_1|_2 \leq |\Delta\varphi_1|_2^{1/2}|\Delta^2\varphi_1|_2^{1/2} \leq \sqrt{R}\ |\Delta^2\varphi_1|_2^{1/2},$$

and successively, the Agmon's inequality $|\varphi|_\infty \leq C\|\varphi\|_1^{1/2}\|\varphi\|_2^{1/2}$, and the lemma 3.1 (with $m(\varphi) = m(\varphi_1) - m(\varphi_2) = 0$) give:

$$|\varphi|_\infty \leq C|\nabla\varphi|_2^{3/4}|\nabla\Delta\varphi|_2^{1/4}.$$

In the same way, we have

$$|\varphi|_3 \leq \|\varphi\|_1 \leq \|\varphi - m(\varphi)\|_1 \leq C|\nabla\varphi|_2.$$

The above estimates, the bound R on the solutions and a straightforward application of the Young inequality give the following energy estimate

$$\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2\right) + \frac{\alpha^4 B_1}{2}|\nabla\Delta\varphi|_2^2 \leq C(|v|_2^2 + |\nabla\varphi|_2^2) + C|\Delta^2\varphi_1|_2^{4/3}|\nabla\varphi|_2^2. \tag{38}$$

5.2. **Estimate of $v_1 - v_2$.** In the same way, we take $w = v$ as a test function in (36), using the assumption on η, the fact that $b(v_2, v, v) = 0$ (20) and the Korn inequality we obtain directly

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}|v|_2^2\right) + \eta_1|\nabla v|_2^2 \leq |b(v,v_1,v)| + \int_\Omega|\sigma||\nabla v| + 2|\eta'|_\infty\int_\Omega|\varphi||D(v_1)||D(v)|\\ + \alpha^2|b(v,\varphi_1,\Delta\varphi)| + \alpha^2|b(v,\varphi,\Delta\varphi_2)|.\end{aligned}$$

Using the Hölder inequality again, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}|v|_2^2\right) + \eta_1|\nabla v|_2^2 \leq |v|_6|\nabla v_1|_2|v|_3 + |\sigma|_2|\nabla v|_2 + 2|\eta'|_\infty|\nabla v_1|_2|\varphi|_\infty|\nabla v|_2\\ + \alpha^2|\nabla\varphi_1|_6|v|_2|\Delta\varphi|_3 + \alpha^2|\Delta\varphi_2|_2|v|_2|\nabla\varphi|_\infty.\end{aligned}$$

As in the last part, the Sobolev inequalities give

$$|v|_6|v|_3 \leq C(|v|_2^2 + |v|_2|\nabla v|_2 + |v|_2^{3/2}|\nabla v|_2^{1/2} + |v|_2^{1/2}|\nabla v|_2^{3/2}).$$

The lemma 3.1 with $m(\varphi) = 0$ gives $|\nabla\varphi|_\infty \leq C|\nabla\Delta\varphi|_2$, then following the technics introduced in the estimate of φ, we can deduce that

$$\frac{d}{dt}\left(|v|_2^2\right) + \eta_1|\nabla v|_2^2 \leq C(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2) + \frac{\alpha^4 B_1}{2}|\nabla\Delta\varphi|_2^2. \tag{39}$$

5.3. **Estimate of $\sigma_1 - \sigma_2$.** Finally, substituting $\tau = \sigma$ in (37). As in the proof of lemma 3.3, we have

$$\int_\Omega g_a(\sigma, v_1) : \sigma = 2a \int_\Omega Tr(\nabla v_1.\sigma.\sigma),$$
$$\int_\Omega g_a(\sigma_2, v) : \sigma = (a+1) \int_\Omega Tr(\nabla v.\sigma_2.\sigma) + (a-1) \int_\Omega Tr(\nabla v.\sigma.\sigma_2).$$

Therefore, by Hölder inequality we deduce the following estimate

$$\frac{d}{dt}\left(\frac{1}{2}|\sigma|_2^2\right) + \varepsilon|\nabla\sigma|_2^2 \leq\ |v|_6|\nabla\sigma_1|_2|\sigma|_3 + 2|a||\nabla v_1|_2|\sigma|_6|\sigma|_3 + 2|\nabla v|_2|\sigma_2|_6|\sigma|_3$$
$$+ |(f\eta)'|_\infty|\varphi|_3|\nabla v_1|_2|\sigma|_6 + f_2\eta_2|\nabla v|_2|\sigma|_2 + |l'|_\infty|\varphi|_3|\sigma_1|_2|\sigma|_6.$$

Finally, by Sobolev lemmas, using the bound R on the solutions and the Young inequality we obtain

$$\frac{d}{dt}\left(|\sigma|_2^2\right) + \varepsilon|\nabla\sigma|_2^2 \leq \frac{C}{\varepsilon}(|v|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2) + \frac{\eta_1}{2}|\nabla v|_2^2. \tag{40}$$

5.4. **Conclusion.** Let us add the inequalities (38), (39) and (40) to obtain

$$y'(t) \leq h(t)y(t),$$

where $y(t) = |v|_2^2 + \frac{\alpha^2}{2}|\nabla\varphi|_2^2 + |\sigma|_2^2$ and $h(t) = C + C|\Delta^2\varphi_1|_2^{4/3} \in L^1(0,T)$. It 's clear that the uniqueness of the solution follows from Gronwall's lemma.

6. **The particular case $a = 0$.** In this part, we study the case where the rheological coefficient a is zero. The result (theorem 2.2) shows us that we need less regularity on the initial conditions to obtain an existence result. This model for the evolution of the stress tensor was studied by N. Masmoudi and P.L. Lions [16]. The explanation comes from the fact that the quadratic terms in the constitutive equation of σ nullify (see lemma 3.3).

6.1. **Energy-type bounds.** We have the same estimates than (26), (27) excepted that (28) becomes now:

$$\frac{d}{dt}\left(\frac{|\sigma|_2^2}{2}\right) + \varepsilon|\nabla\sigma|_2^2 \leq \frac{\eta_1}{4}|\nabla u|_2^2 + C|\sigma|_2^2 + C. \tag{41}$$

Then, we add the results (26), (27), (41) to deduce an estimate of kind

$$\frac{d}{dt}\left(|\nabla\varphi|_2^2 + \int_\Omega F(\varphi) + |u|_2^2 + |\sigma|_2^2\right) + \left(|\nabla\mu|_2^2 + |\nabla u|_2^2 + \varepsilon|\nabla\sigma|_2^2\right) \leq\ C|\sigma|_2^2 + C.$$

In this case, it's clear that we can show that the sequences $\{\varphi_n\}$, $\{v_n\}$ and $\{\sigma_n\}$ are bounded, for any $T > 0$, respectively in

$$L^\infty(0,T;\Phi_1) \text{ for } \{\varphi_n\}, \quad L^\infty(0,T;V_0) \cap L^2(0,T;V_1) \text{ for } \{v_n\},$$
$$L^2(0,T;\Phi_1) \text{ for } \{\mu_n\}, \quad \text{and} \quad L^\infty(0,T;\Sigma_0) \cap L^2(0,T;\Sigma_1) \text{ for } \{\sigma_n\}.$$

To have good estimates on the derivatives, we shall need that φ is bounded in $L^2(0,T;\Phi_3)$. This result can be obtained only by using the supplementary hypothesis (17) and the equation (25):

$$\alpha^4|\nabla\Delta\varphi|_2^2 \leq 2|\nabla\mu|_2^2 + 2|\nabla F'(\varphi)|_2^2$$

We have just seen that the term $|\nabla\mu|_2^2$ is bounded. For the last term, since q satisfies (17) the embeddings $H^1 \subset L^6$ and $H^1 \subset L^{3q-3}$ hold in the three-dimensional case and the two dimensional case. We deduce

$$|\nabla F'(\varphi)|_2^2 \leq C \int_\Omega (1+|\varphi|^{2q-2})|\nabla\varphi|^2 \leq C|\nabla\varphi|_2^2 + C|\nabla\varphi|_2^{2q-2}|\Delta\varphi|_2^2,$$

and we obtain that $\{\varphi_n\}$ is bounded in $L^2(0,T;\Phi_3)$.

6.2. **Bounds on the derivatives.** We can show that, if φ_n and v_n are less regular than in the part 4.4 then we have an estimate of $\partial_t\varphi_n$ in $L^2(0,t;\Phi_{-1})$: assume that $\{\varphi_n\}$ and $\{v_n\}$ are bounded in $L^2(0,t;\Phi_3)\cap L^\infty(0,t;\Phi_1)$ and $L^2(0,t;V_1)$ respectively then we can write

$$\|\operatorname{div}(\varphi_n v_n)\|_{-1} \leq C\|\varphi_n\|_1\|v_n\|_1, \quad \|\operatorname{div}(B(\varphi_n)\nabla\mu_n)\|_{-1} \leq B_2\|\varphi_n\|_3,$$

hence, we get $\|\partial_t\varphi_n\|_{-1} \leq C\|\varphi_n\|_1\|v_n\|_1 + B_2\|\varphi_n\|_3$. Time integrating gives

$$\|\partial_t\varphi_n\|^2_{L^2(0,t;\Phi_{-1})} \leq 2C^2\|\varphi_n\|^2_{L^\infty(0,t;\Phi_1)}\|v_n\|^2_{L^2(0,t;V_1)} + 2B_2^2\|\varphi_n\|^2_{L^2(0,t;\Phi_3)}. \qquad (42)$$

As for the estimate of $\partial_t\varphi_n$, we show here a similar result than (33) in which we have less regularity. We only suppose that $\{\varphi_n\}$, $\{v_n\}$ and $\{\sigma_n\}$ are bounded in $L^2(0,t;\Phi_3)\cap L^\infty(0,t;\Phi_1)$, $L^2(0,t;V_1)\cap L^\infty(0,t;V_0)$ and $L^2(0,t;\Sigma_0)$ respectively and we clearly obtain

$$\|E(\varphi_n,v_n)\|_{-1} \leq 2\eta_2\|v_n\|_1, \quad \|\operatorname{div}(\sigma_n)\|_{-1} \leq |\sigma_n|_2, \quad \|\varphi_n\nabla\mu_n\|_{-1} \leq \|\varphi_n\|_1\|\varphi_n\|_3.$$

We know (see lemma 3.4) that for $w \in V_{d/2}$, we have

$$|b(v_n,v_n,w)|_2 \leq C|v_n|_2\|v_n\|_1\|w\|_{d/2}, \quad \|B(v_n,v_n)\|_{-d/2} \leq C|v_n|_2\|v_n\|_1.$$

We use here the embedding $H^{-1} \subset H^{-d/2}$, to get

$$\|\partial_t v_n\|_{-d/2} \leq 2\eta_2\|v_n\|_1 + C|\sigma_n|_2 + C\|\varphi_n\|_1\|\varphi_n\|_3 + C|v_n|_2\|v_n\|_1. \qquad (43)$$

Finally, for $\partial_t\sigma$ we have a similar result: suppose that $\{v_n\}$ and $\{\sigma_n\}$ are bounded in $L^2(0,t;V_1)$ and $L^2(0,t;\Sigma_1)$ respectively, then we clearly have

$$\|l(\varphi_n)\sigma_n\|_{-1} \leq C|\sigma_n|_2, \quad \|\Delta\sigma_n\|_{-1} \leq \|\sigma_n\|_1, \quad \|f(\varphi_n)\eta(\varphi_n)D(v_n)\|_{-1} \leq C\|v_n\|_1.$$

For the two other terms, we remark that, with Sobolev inequalities for $\tau \in \Sigma_s$, $s > \frac{d}{2}+1$, we can write $\nabla\tau \in \Sigma_{s-1} \subset \mathbb{L}^\infty$. We obtain, on the one hand,

$$(\operatorname{div}(\sigma_n v_n),\tau) \leq |\sigma_n|_2|v_n|_2|\nabla\tau|_\infty, \quad \|\operatorname{div}(\sigma_n v_n)\|_{-s} \leq |\sigma_n|_2|v_n|_2,$$

and on the other hand,

$$(g_a(\sigma_n,v_n),\tau) \leq C|\nabla v_n|_2|\sigma_n|_2\|\tau\|_s, \quad \|g_a(\sigma_n,v_n)\|_{-s} \leq C\|v_n\|_1|\sigma_n|_2,$$

which implies ($\mathbb{H}^{-1} \subset \mathbb{H}^{-s}$ since $s \geq 1$):

$$\|\partial_t\sigma_n\|_{-s} \leq C(|\sigma_n|_2|v_n|_2 + \|v_n\|_1|\sigma_n|_2 + |\sigma_n|_2 + \varepsilon\|\sigma_n\|_1 + \|v_n\|_1),$$

$$\begin{aligned}\|\partial_t\sigma_n\|^2_{L^2(0,t;\Sigma_{-s})} \leq C\Big(&|\sigma_n|^2_{L^\infty(0,t;\Sigma_0)}|v_n|^2_{L^2(0,t;V_0)} + \|v_n\|^2_{L^2(0,t;V_1)}|\sigma_n|^2_{L^\infty(0,t;\Sigma_0)}\\ &+ |\sigma_n|^2_{L^2(0,t;\Sigma_0)} + \|\sigma_n\|^2_{L^2(0,t;\Sigma_1)} + \|v_n\|^2_{L^2(0,t;V_1)}\Big).\end{aligned} \qquad (44)$$

6.3. **Passing to the limit.** We can also use the inequalities (42), (43), (44): $\{\partial_t\varphi_n\}$, $\{\partial_t v_n\}$ and $\{\partial_t\sigma_n\}$ are bounded in $L^2(0,T;\Phi_{-1})$, $L^2(0,T;V_{-d/2})$ and $L^2(0,T;\Sigma_{-s})$ respectively. Just like above (section 4.5), the compacity results [22] allows us to conclude the proof in the case $a=0$. For instance, a classical compacity result [22] ensures that

$$\begin{aligned} &\varphi_n \rightharpoonup \varphi \text{ in } C^0([0,T[;\Phi_1) \text{ weakly},\\ &v_n \rightharpoonup v \text{ in } C^0([0,T[;V_{(2-d)/4}) \text{ weakly},\\ &\sigma_n \rightharpoonup \sigma \text{ in } C^0([0,T[;\Sigma_{(1-s)/2}) \text{ weakly}.\end{aligned}$$

REMARK 6. *In the case where φ is constant, we can obtain an uniform bound in time. The term $\int_\Omega \sigma : \nabla u$ of the $\mathbb{L}^2$-estimate for u canceling with the term $\int_\Omega \eta f\sigma : \nabla u$ of the $\mathbb{L}^2$-estimate for σ. We find the results presented in* [16].

6.4. **Proof of the corollary.** We resume the estimations realized in the proof of the theorem 2.1 by using the fact that we have a global weak solution. Here, the assumption $d=2$ allows us to obtain better estimates. First of all, in the H^2-estimate for φ (see section 4.3.4) we can use two results due to F. Boyer based on the fact that $H^{1+\delta}\subset L^\infty$ for all $\delta>0$ in the two-dimensional case (see [2]):

$$|\Delta(F'(\varphi))|_2|\Delta^2\varphi|_2 \le \frac{\alpha^2 B_1}{4}|\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^{4/3}(1+|\nabla\varphi|_2^{\alpha_1}),\quad \alpha_1>0,$$

$$|\nabla\varphi\nabla\mu|_2|\Delta^2\varphi|_2 \le \frac{\alpha^2 B_1}{4}|\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2|\Delta\varphi|_2^2|\nabla\mu|_2^2 + C|\nabla\varphi|_2^2 + C|\Delta\varphi|_2^2|\nabla\mu|_2^2.$$

The estimate (29) becomes

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\Delta\varphi|_2^2}{2}\right) + \frac{\alpha^2 B_1}{2}|\Delta^2\varphi|_2^2 \le{}& C|\nabla\varphi|_2^{4/3}(1+|\nabla\varphi|_2^{\alpha_1}) + C|\nabla\varphi|_2^2|\Delta\varphi|_2^2|\nabla\mu|_2^2\\ &+ C|\nabla\varphi|_2^2 + C|\Delta\varphi|_2^2|\nabla\mu|_2^2 + C|\nabla\varphi|_2^2|u|_2^4 + C.\end{aligned}\tag{45}$$

For the $\mathbb{H}^1$-estimate for u (see section 4.3.5), we juste change the estimate of the term $b(u,u,Au)$ by

$$\begin{aligned}|b(u,u,Au)| &\le C|u|_4|\nabla u|_4|Au|_2,\\ &\le \frac{\eta_1}{18}|Au|_2^2 + C|u|_2^2|\nabla u|_2^2 + C|u|_2|\nabla u|_2^3 + C|u|_2^4|\nabla u|_2^2 + C|u|_2^2|\nabla u|_2^4.\end{aligned}$$

So, the estimate (30) becomes in the two-dimensional case

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla u|_2^2}{2}\right) + \frac{\eta_1}{2}|Au|_2^2 \le \frac{\alpha^2 B_1}{8}|\Delta^2\varphi|_2^2 + C|u|_2^2|\nabla u|_2^2 + C|u|_2|\nabla u|_2^3 + C|u|_2^4|\nabla u|_2^2\\ + C|u|_2^2|\nabla u|_2^4 + C|\nabla\varphi|_2^2 + C|\nabla u|_2^2 + C|\nabla\varphi|_2^2|\nabla u|_2^4 + C|\nabla\sigma|_2^2 + C|\nabla\varphi|_2^2|\Delta\varphi|_2^4 + C.\end{aligned}\tag{46}$$

And for the $\mathbb{H}^1$-estimate for σ (see section 4.3.6), we deduct easily

$$\begin{aligned}\frac{d}{dt}\left(\frac{|\nabla\sigma|_2^2}{2}\right) + \frac{\varepsilon}{2}|\Delta\sigma|_2^2 \le \frac{\alpha^2 B_1}{8}|\Delta^2\varphi|_2^2 + \frac{\eta_1}{4}|Au|_2^2 + C(1 + |\nabla u|_2^2 + |u|_2^2|\nabla\sigma|_2^2\\ + |u|_2|\nabla u|_2|\nabla\sigma|_2^2 + |u|_2^4|\nabla\sigma|_2^2 + |u|_2^2|\nabla u|_2^2|\nabla\sigma|_2^2 + |\nabla\sigma|_2^2 + |\nabla u|_2^2|\sigma|_2^2\\ + |\nabla u|_2^2|\sigma|_2|\nabla\sigma|_2 + |\sigma|_2^4|\nabla u|_2^2 + |\nabla u|_2^2|\sigma|_2^2|\nabla\sigma|_2^2 + |\sigma|_2^2 + |\nabla\varphi|_2^2 + |\sigma|_2^2|\nabla\sigma|_2^2).\end{aligned}\tag{47}$$

Finally, adding the equations (45), (46) and (47) we obtain an estimate of kind

$$z'(t) + \alpha^2 B_1|\Delta^2\varphi|_2^2 + \eta_1|\Delta v|_2^2 + \varepsilon|\Delta\sigma|_2^2 \le g(t) + h(t)z(t),$$

where $z(t) = |\Delta\varphi|_2^2 + |\nabla u|_2^2 + |\nabla\sigma|_2^2$, and where, using the estimates on the weak solutions (see theorem 2.2) we have g and h in $\mathbb{L}^1_{loc}(0,T)$. With the Gronwall lemma, it's now straightforward to deduce the global existence of strong solution in the two-dimensional case.

Acknowledgments: The author wishes to thank Professor P. Fabrie for directing this work and giving much advice, and F. Boyer for helpful discussions. I also thank the referee for the care she/he brought to the reading of this paper.

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Received June 2001; first revision February 2002; final version August 2002.

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