Boundary layers for stress diffusive perturbation in viscoelastic fluids

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Abstract - In this paper, we study a stress diffusive perturbation of the system describing a viscoelastic flow. We analyse the boundary layer which arises near the boundary and we observe in particular that there is no boundary layer on the velocity at the first order.

Key Words - boundary layer, viscoelastic fluid, Oldroyd model, diffusive perturbation

1 Introduction

We study the asymptotic behavior of the solutions of the Oldroyd viscoelastic model when an additive coefficient of stress diffusion goes to zero. The problem models the flow of a viscoelastic incompressible fluid. It is considered on an open and regular domain $\Omega \subset \mathbb{R}^3$ whose boundary is noted Γ. The model we consider contains an additional stress diffusion term which derives from a microscopic dumbbell analysis, see [7]. This perturbation is often present for the determination of shear banding flow, see [12]. For the mathematics study of such a model, the presence of a diffusive term can be interesting, see [3]. More generally, if for theoretical, numerical or physical reasons we need to add such a term, we prove here that such an addition does not basically influence the solution. In order to highlight the dependence in this coefficient of stress diffusion ε, we write the model in the form:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \Delta u^\varepsilon + \nabla p^\varepsilon = \operatorname{div} \sigma^\varepsilon, \quad \operatorname{div} u^\varepsilon = 0, \\ \partial_t \sigma^\varepsilon + u^\varepsilon \cdot \nabla \sigma^\varepsilon + g(\sigma^\varepsilon, \nabla u^\varepsilon) + \sigma^\varepsilon - \varepsilon \Delta \sigma^\varepsilon = D(u^\varepsilon), \\ u^\varepsilon(0) = u_{init}, \quad \sigma^\varepsilon(0) = \sigma_{init}, \end{cases} \tag{1}$$

with Neumann boundary conditions for the stress and Dirichlet for the velocity:

$$\left.\frac{\partial \sigma^\varepsilon}{\partial n}\right|_\Gamma = 0, \quad u^\varepsilon|_\Gamma = 0. \tag{2}$$

Moreover, the bilinear function $g(\sigma, \nabla u)$ is defined by:

$$g(\sigma, \nabla u) = -W(u).\sigma + \sigma.W(u) - a(D(u).\sigma + \sigma.D(u)), \quad a \in [-1, 1],$$

where $D(u)$, $W(u)$ respectively represent the deformation and vorticity tensors.

It is known that such a system admits a solution (see [3, 5, 6]). Our goal is to describe the behavior of this solution $(u^\varepsilon, p^\varepsilon, \sigma^\varepsilon)$ when the viscosity ε goes to zero. We show that the solution converges strongly in L^2 (and in fact in any space which the boundary condition $\partial_n \sigma^\varepsilon\big|_\Gamma = 0$ does not appear) towards (u_0, p_0, σ_0), solution of the system without the stress diffusive term:

$$\begin{cases} \partial_t u_0 + u_0 \cdot \nabla u_0 - \Delta u_0 + \nabla p_0 = \operatorname{div} \sigma_0, \quad \operatorname{div} u_0 = 0, \\ \partial_t \sigma_0 + u_0 \cdot \nabla \sigma_0 + g(\sigma_0, \nabla u_0) + \sigma_0 = D(u_0), \\ u_0(0) = u_{init}, \quad \sigma_0(0) = \sigma_{init}, \quad u_0|_\Gamma = 0. \end{cases} \tag{3}$$

There again, it is already shown that such a system admits a solution (see [4, 8, 11]).

To recover the boundary condition (2) on σ, the solution of (1) oscillates very quickly close to the boundary converging toward (3). Here, we analyse the above-mentioned generated boundary layer. Previous studies have already been undertaken on such phenomena but in different physical cases, see G. Carbou, P. Fabrie and O. Guès [1], O. Guès and E. Grenier [9], O. Guès [10] or D. Sanchez [13]. It is known in particular that if the boundary is characteristic ($u^\varepsilon.n|_\Gamma = 0$) then the size of the generated boundary layer is of order $\sqrt{\varepsilon}$.

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2 Statements of the results

The main result is the following

Theorem 2.1 *Assume $u_{init} \in H^4(\Omega)$ verifies* div $(u_{init}) = 0$ *and $u_{init}.n|_\Gamma = 0$, and $\sigma_{init} \in H^4(\Omega)$ then there exists $T > 0$ and two functions $P \in L^\infty(0,T; H^1(\Omega \times \mathbb{R}^+)) \cap L^2(0,T; H^2(\Omega \times \mathbb{R}^+))$ and $\Sigma \in L^\infty(0,T; H^2(\Omega \times \mathbb{R}^+))$ such that, on $[0,T] \times \Omega$, we have*

$$\begin{cases} u^\varepsilon(t,x) = u_0(t,x) + \sqrt{\varepsilon}\ w(t,x), \\ p^\varepsilon(t,x) = p_0(t,x) + \sqrt{\varepsilon}\ P\left(t, x, \dfrac{d(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon} q(t,x), \\ \sigma^\varepsilon(t,x) = \sigma_0(t,x) + \sqrt{\varepsilon}\ \Sigma\left(t, x, \dfrac{d(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\ \tau(t,x). \end{cases}$$

where $d(x)$ represents the distance from $x \in \Omega$ to the boundary Γ. The functions w, q and τ verify :

$$\begin{aligned} & w \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ & q \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \\ & \tau \in L^\infty(0,T; H^1(\Omega)) \quad \text{and} \quad \sqrt{\varepsilon}\tau \in L^2(0,T; H^2(\Omega)). \end{aligned}$$

Remark 2.1

- *The functions P and Σ introduced above are profiles of boundary layer which satisfy*

$$\lim_{z \to +\infty} P(t,x,z) = 0 \quad \text{and} \quad \lim_{z \to +\infty} \Sigma(t,x,z) = 0.$$

- *It is interesting to note there is no boundary layer for the velocity term. In fact, we will see during the proof of this theorem that the free divergence condition implies the lack of the boundary layer term in the normal velocity component. However, this term naturally appears in the higher order term of the tangential velocity component.*

The proof is organised in three steps. The first consists in building an approximate solution: we carry out a formal asymptotic extension of the solution. In the second step, we solve two profile equations: the first one corresponding to the initial equations (1) without the term $\varepsilon\Delta\sigma$, the second one to a hyperbolic-parabolic type in which it is necessary to control the decay in the fast variable. The third step consists in showing that the remainder of the extension is bounded in an adequate space.

3 Boundary layer profiles

According to O. Guès [10], we seek an asymptotic extension of u^ε, p^ε and σ^ε in the form:

$$f^\varepsilon(t,x) = f_0\left(t, x, \frac{d(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\ f_1\left(t, x, \frac{d(x)}{\sqrt{\varepsilon}}\right) + \ldots$$

For such a method, it is convenient to introduce the following notations. For all $i \in \mathbb{N}$, we write:

$$f_i(t,x,z) = \overline{f_i}(t,x) + \widetilde{f}_i(t,x,z) \quad \text{with} \quad \overline{f_i}(t,x) = \lim_{z \to +\infty} f_i(t,x,z) \quad \text{and} \quad \widetilde{f} \text{ with fast decay in } z.$$

We then replace formally $(u^\varepsilon, p^\varepsilon, \sigma^\varepsilon)$ by its asymptotic extension in the equations (1). We then seek to determine the profiles (u_i, p_i, σ_i) by identifying all terms of the same order in ε.

Order −1 The only contribution $-\partial_z^2 u_0$ is zero. Since $\overline{u_0}$ does not depend on z, this condition can be written as $-\partial_z^2 \widetilde{u_0} = 0$. Moreover, the decay condition of all the derivatives of $\widetilde{u_0}$ necessarily implies $\widetilde{u_0} = 0$.

Order $-\frac{1}{2}$ Knowing that u_0 does not depend on z, one can conclude that the only terms of order $-\frac{1}{2}$ resulting from the Navier-Stokes equation are:

$$-\partial_z^2\widetilde{u_1} + \nabla d\ \partial_z\widetilde{p_0} = \nabla d.\partial_z\widetilde{\sigma_0}. \tag{4}$$

Let us note that the term $\frac{1}{\sqrt{\varepsilon}}\ (u_0\cdot\nabla d)\ \partial_z\widetilde{\sigma_0}$ is of order 0. Indeed, using the condition $(u_0\cdot\nabla d)|_\Gamma = 0$, it is written

$$\frac{1}{\sqrt{\varepsilon}}\ (u_0\cdot\nabla d)\ \partial_z\widetilde{\sigma_0} = ((u_0\cdot\nabla d)^\sharp z\partial_z\widetilde{\sigma_0})$$

where $(u_0\cdot\nabla d)^\sharp$ is a regular and bounded function.

Order 0 The free divergence condition writes $\operatorname{div} u_0 + \nabla d.\partial_z\widetilde{u_1} = 0$. We let $z\to+\infty$. We deduce:

$$\operatorname{div} u_0 = 0. \tag{5}$$

$$\partial_z(\widetilde{u_1}\cdot\nabla d) = 0. \tag{6}$$

In the same way, the terms of order 0 in the equation in σ are written:

$$\partial_t\overline{\sigma_0} + u_0\cdot\nabla\overline{\sigma_0} + g(\overline{\sigma_0},\nabla u_0) + \overline{\sigma_0} = D(u_0), \tag{7}$$

$$\partial_t\widetilde{\sigma_0} + u_0\cdot\nabla\widetilde{\sigma_0} + (u_0\cdot\nabla d)^\sharp z\partial_z\widetilde{\sigma_0} + g(\widetilde{\sigma_0},\nabla u_0) + \widetilde{\sigma_0} = 0. \tag{8}$$

We can then determine the profiles. Indeed, $\widetilde{\sigma_0} = 0$ is the unique solution of (8), zero at $t = 0$ and at the boundary. Then taking the scalar product of (4) with ∇d, and using the equation (6) we deduce that $\widetilde{p_0} = 0$. The equation (4) thus rewrites $\partial_z^2\widetilde{u_1} = 0$. We thus obtain:

$$\widetilde{u_1} = 0,\quad \widetilde{p_0} = 0,\quad \widetilde{\sigma_0} = 0.$$

This result makes possible to write more easily the equation resulting from Navier-Stokes at zero-order (by separating the slow part and the oscillating part):

$$\partial_t u_0 + u_0\cdot\nabla u_0 - \Delta u_0 + \nabla p_0 = \operatorname{div}\sigma_0, \tag{9}$$

$$-\partial_z^2\widetilde{u_2} + \nabla d\ \partial_z\widetilde{p_1} = \nabla d.\partial_z\widetilde{\sigma_1}. \tag{10}$$

We defer these results in the equations concerning the nonoscillating parts (5), (7), (9) and we deduce from it that u_0, p_0 and σ_0 are solutions of the problem (3).

Order $\frac{1}{2}$ We start again computations: the free divergence condition is written:

$$\operatorname{div} u_1 = 0,\quad \partial_z(\widetilde{u_2}\cdot\nabla d) = 0.$$

Taking the scalar product of (10) with ∇d, we deduce the first oscillating profile in pressure: $\widetilde{p_1} = \nabla d.(\nabla d.\widetilde{\sigma_1})$. Concerning the stress equation, we find:

$$\begin{cases} \partial_t\widetilde{\sigma_1} + u_0\cdot\nabla\widetilde{\sigma_1} + (u_0\cdot\nabla d)^\sharp z\partial_z\widetilde{\sigma_1} - \partial_z^2\widetilde{\sigma_1} = L(\widetilde{\sigma_1}), \\ \partial_z\widetilde{\sigma_1}\big|_{z=0,x\in\Gamma} = -\left.\dfrac{\partial\,\sigma_0}{\partial n}\right|_\Gamma, \\ \widetilde{\sigma_1}(0,x,z) = 0, \\ \widetilde{\sigma_1} \text{ with fast decay in } z. \end{cases} \tag{11}$$

where L is a linear application, whose coefficients depend only on the slow variables t and x (see [5, 6]):

$$L(\tau) = (\nabla d\ \nabla d\ \nabla d.(\nabla d.\tau) - \nabla d\ \nabla d.\tau) - \tau - g(\tau,\nabla u_0) - g(\sigma_0,(\nabla d\ \nabla d\ \nabla d.(\nabla d.\tau) - \nabla d\ \nabla d.\tau)).$$

4 Asymptotic extension

4.1 First term of the extension

In the study which we have just undertaken, we obtained the principal terms of extension u_0, p_0 and σ_0 as solutions of the system (3). This system was studied previously (see [4, 8, 11]): the existence results are obtained by fixed point methods. Using a similar method, we obtained in [6]:

Theorem 4.1 *Assume $u_{init} \in H^4$ satisfies $\operatorname{div}(u_{init}) = 0$ and $u_{init}.n|_\Gamma = 0$, and $\sigma_{init} \in H^4$ then there exists $T > 0$ such that (3) admits a solution (u_0, σ_0) verifying*

$$u_0 \in L^\infty(0,T;H^4) \cap L^2(0,T;H^5) \quad \textit{and} \quad \sigma_0 \in L^\infty(0,T;H^4).$$

Moreover, the solution verify the following estimates:

$$\begin{aligned} &\partial_t u_0 \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \qquad && \partial_t \sigma_0 \in L^\infty(0,T;H^3),\\ &\partial_t^2 u_0 \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), && \partial_t^2 \sigma_0 \in L^\infty(0,T;H^1). \end{aligned}$$

Remark 4.1 *More precisely, the data have to fulfil compatibility conditions (see [6]). If the data are not appropriately prepared, we expect to observe a boundary layer with time. J.Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [2] proved (in a different physical framework) that the solution also depends on variables $\frac{t}{\varepsilon^k}$, for $k \in \mathbb{N}$. For a rather long time, even for no appropriately prepared data, the profile which we found approaches the exact profile.*

4.2 First boundary layer terms

The only equation of profile which we have to solve is that relating to $\widetilde{\sigma_1}$. We seek $\widetilde{\sigma_1}(t,x,z)$ solution of the linear equation (11). We then prove an existence theorem of regular solutions:

Theorem 4.2 *If u_0 verifies $u_0 \in L^\infty(0,T;H^3(\Omega))$, $\operatorname{div}(u_0) = 0$ and $u_0.n|_\Gamma = 0$ and if σ_0 verifies $\sigma_0 \in L^2(0,T;H^4(\Omega))$ and $\partial_t\sigma_0 \in L^2(0,T;H^3(\Omega))$ then the equation (11) admits a solution $\widetilde{\sigma_1}$ such that*

$$\begin{aligned} &\widetilde{\sigma_1} \in L^\infty(0,T;H^1(\Omega)\otimes H^2(\mathbb{R}^+)) \cap L^2(0,T;H^1(\Omega)\otimes H^3(\mathbb{R}^+)),\\ &\widetilde{\sigma_1} \in L^\infty(0,T;H^2(\Omega)\otimes L^2(\mathbb{R}^+)) \cap L^2(0,T;H^2(\Omega)\otimes H^1(\mathbb{R}^+)). \end{aligned}$$

Remark 4.2 *The regularity we prove in this theorem implies in particular:*

$$\widetilde{\sigma_1} \in L^\infty(0,T;H^2(\Omega\times\mathbb{R}^+)), \quad \partial_z\widetilde{\sigma_1} \in L^\infty(0,T;H^1(\Omega\times\mathbb{R}^+)) \cap L^2(0,T;H^2(\Omega\times\mathbb{R}^+)).$$

The proof is articulated around three points. We carry out initially an extension of the condition at the boundary in order to obtain an equivalent system with homogeneous boundary conditions. We then show by a Galerkin method that this new system has weak solutions. In a next step, we set up a method allowing to increase the regularity of the solution until obtaining the desired regularity.

Proof of the theorem 4.2

Step 1 : extension. The first step consists in being brought back to a homogeneous problem at the boundary of the domain $\Omega \times [0, +\infty[$. For this, we carry out an extension of the Neumann condition. Precisly, we define the function $\zeta \in C^\infty(\mathbb{R}^+, \mathbb{R})$ such as $\zeta'(0) = 1$ and $\operatorname{supp}(\zeta) \subset [0,1]$. We note

$$\theta(t,x,z) = \zeta(z)\nabla d(x)\cdot\nabla\sigma_0(t,x), \text{ we have } \theta \in L^2(0,T;H^3(\Omega)\otimes H_0^\infty(\mathbb{R}^+))$$

and let $\widetilde{\sigma_1} = s - \theta$. The tensor s is the solution of

$$\begin{cases} \partial_t s + u_0\cdot\nabla s + (u_0\cdot\nabla d)^\sharp z\partial_z s - \partial_z^2 s = L(s) + K,\\ \partial_z s\big|_{z=0} = 0,\\ s(0,x,z) = 0,\\ s \text{ with fast decay in } z \end{cases} \tag{12}$$

where $K = \partial_t\theta + u_0 \cdot \nabla\theta + (u_0 \cdot \nabla d)^\sharp z\partial_z\theta - \partial_z^2\theta - L(\theta)$ and $K \in L^2(0,T;H^2(\Omega)\otimes H_0^\infty(\mathbb{R}^+))$.

Step 2 : Existence of solution. Let $M > 0$. We start solving the problem for $z \in]0, M[$ with the following boundary conditions: $\partial_z s|_{z=0} = s|_{z=M} = 0$. We use Galerkin approximations choosing s as a test function (see [5, 6]):

$$\frac{d}{dt}|s|^2_{xz,M} + |\partial_z s|^2_{xz,M} \le C(|s|^2_{xz,M} + |K|^2_{xz,\infty}). \tag{13}$$

where the norm $|\cdot|_{xz,M}$ on the space $L^2(\Omega)\otimes L^2(0,M)$ is defined by $|f|_{xz,M} = \displaystyle\int_\Omega\int_0^M |f|^2$.

The Gronwall lemma provides a L^∞ bound for $|s|_{xz,m}$ then L^2 for $|\partial_z s|_{xz,M}$. These bounds being independent of M and the equation being linear, we deduce:

$$s \in L^\infty(0,T;L^2(\Omega)\otimes L^2(\mathbb{R}^+)) \cap L^2(0,T;L^2(\Omega)\otimes H^1(\mathbb{R}^+)).$$

Step 3 : Regularity. We apply a differential operator $\mathcal{Z}$ to the equation (12). The function $\mathcal{Z}s$ satisfies an equation of the same type as (12) where the second member contains not only $\mathcal{Z}F$ but also all the commutators $[\partial_t, \mathcal{Z}]$, $[u_0\cdot\nabla, \mathcal{Z}]$, etc. Choosing various operators $\mathcal{Z}$ in a precise order, we should be able, by each step, control the second member and show that $\mathcal{Z}s$ is as regular as s.

The choice of the operator $\mathcal{Z}$ is limited to the fact that $\mathcal{Z}s$ must also satisfies the boundary conditions of the problem (12). We distinguish the conormal derivatives (by the order $z\partial_z$, ∇, $(z\partial_z)^2$, $\nabla Z\partial_z$ and ∇^2) for which we can apply this method, and the normal derivative (∂_z) which will be handled at last using the regularisation of the operator $\partial_t - \partial_z^2$. We deduce

$$s \in L^\infty(0,T;H^1(\Omega)\otimes H^2(\mathbb{R}^+)) \cap L^2(0,T;H^1(\Omega)\otimes H^3(\mathbb{R}^+)).$$

5 Convergence of the extension

The computations carried out during the part 3 are valid only in the neighbourhood of the boundary Γ (it is necessary to take a well defined and regular d function). To justify the calculations not only near the boundary but also on Ω, we introduce a function $\psi \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ whose support is contained in a neighbourhood of the boundary. ψ is equal 1 in a more restricted neighbourhood of this boundary. To extend the condition on the boundary $\partial_n\widetilde{\sigma_1}(t,x,0)$, we introduce $\theta \in L^\infty(\mathbb{R}^+, H^2)$ such that $\partial_t\theta \in L^2(\mathbb{R}^+, H^1)$ and $\partial_n\theta(t,x) = \partial_n\widetilde{\sigma_1}(t,x,0)$ on $\mathbb{R}^+\times\Gamma$. We write finally $(u^\varepsilon, p^\varepsilon, \sigma^\varepsilon)$ in the form

$$\begin{cases} u^\varepsilon(t,x) = u_0(t,x) + \sqrt{\varepsilon}\, w(t,x), \\ p^\varepsilon(t,x) = p_0(t,x) + \sqrt{\varepsilon}\, \psi(x)\widetilde{p_1}\left(t,x,\dfrac{d(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon} q(t,x), \\ \sigma^\varepsilon(t,x) = \sigma_0(t,x) + \sqrt{\varepsilon}\, \psi(x)\widetilde{\sigma_1}\left(t,x,\dfrac{d(x)}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\, \theta(t,x) + \sqrt{\varepsilon}\, \tau(t,x). \end{cases}$$

Then, these profiles are introduced in the equations (1). Using the equations satisfying by u_0, p_0, σ_0, $\widetilde{p_1}$ and $\widetilde{\sigma_1}$, a Navier-Stokes like equation for (w,q) and another equation for the stress τ, are deduced.

Formally multiplying the equation of evolution for w by Aw (A being the Stokes operator) and the equation of evolution for the stress (τ) by τ then by $-\Delta\tau$, and gathering the two estimates, we find an estimate of the form:

$$y'(t) + z(t) \le P(y(t)),$$
$$\text{where} \quad y = \|\tau\|^2_{H^1} + \|w\|^2_{H^1}, \quad \text{and} \quad z = \|\tau\|^2_{H^1} + \varepsilon\|\tau\|^2_{H^2} + \|w\|^2_{H^2},$$

where the polynomial P is independent of ε. In a traditional way, it is deduce that there exists a time T independent of ε such that w and τ are bounded in $L^\infty(0,T;H^1)$, w and $\sqrt{\varepsilon}\tau$ being bounded in $L^2(0,T;H^2)$. Concenring the regularity of the remainder in pressure q, we use the regularity of the Stokes problem.

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