Interpolation: a geometric point of view

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Interpolation at a scheme of fat points

- k a field, $V_{n,d} = \{p \in k[x_1, \ldots, x_n] : \deg p \leqslant d\}$, of dimension $N = \binom{n+d}{n}$.
- $P_1, \ldots, P_s \in \mathbb{A}^n(k)$ pairwise distinct, with multiplicities $m_1, \ldots, m_s \geqslant 1$.
- $Z = m_1 P_1 + \cdots + m_s P_s$ the scheme defined by $I_Z = \bigcap_i \mathfrak{m}_{P_i}^{m_i}$,

$$\mathcal{O}_Z = k[x]/I_Z \simeq \prod_{i=1}^{s} k[x]/\mathfrak{m}_{P_i}^{m_i},$$

of length $\lambda(Z) = \dim_k \mathcal{O}_Z = \sum_i \binom{n+m_i-1}{n}$.

The evaluation map

- $\mathcal{O}_Z = k[x]/I_Z \simeq \prod_{i=1}^{s} k[x]/\mathfrak{m}_{P_i}^{m_i}$.
- The evaluation map

$$\rho\colon V_{n,d} \longrightarrow \mathcal{O}_Z, \qquad p \longmapsto (p \bmod \mathfrak{m}_{P_i}^{m_i})_{1\leqslant i\leqslant s}.$$

- $p \bmod \mathfrak{m}_{P_i}^{m_i}$ is the Taylor exp. of p at P_i up to order $m_i - 1$.
- $m_i = 1$ (Lagrange): $p \mapsto (p(P_i))_i$.
- $m_i = 2$ (Hermite of order 2): $p \mapsto (p(P_i), \nabla p(P_i))_i$.

Maximal rank

- For given points $P_1, \ldots, P_s$, the rank of ρ satisfies

$$\operatorname{rank} \rho \leqslant \min\bigl(\lambda(Z), N\bigr).$$

- ρ has maximal rank if equality holds:
  - $\lambda(Z) \leqslant N$: ρ is surjective, all data can be interpolated;
  - $\lambda(Z) \geqslant N$: ρ is injective, the interpolant is unique.

- The rank depends on the position of the points, not only on n, d and the m_i.

Question. Does ρ have maximal rank?

Dimension one

- $n = 1$: put $Q = \prod_{i=1}^{s} (x - P_i)^{m_i}$, of degree $\lambda(Z)$. By the Chinese remainder theorem $k[x]/(Q) \simeq \mathcal{O}_Z$, so

$$\ker \rho = Q\,k[x] \cap V_{1,d}, \qquad \operatorname{rank} \rho = \min\bigl(\lambda(Z), d+1\bigr).$$

- Square case $\lambda(Z) = d + 1$:

$$\det \rho = \prod_{i<j} (P_j - P_i)^{m_i m_j}.$$

Hence in dim. one, ρ has max. rank: the determinant is nonzero because the points are pairwise distinct.

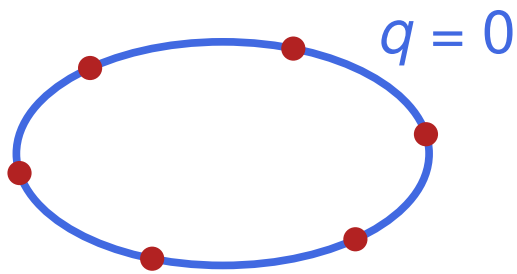
Lagrange interpolation in several variables

- Any n, all $m_i = 1$: $Z = P_1 + \cdots + P_s$, $\lambda(Z) = s$, and

$$\rho : p \longmapsto \left(p(P_i)\right)_i, \qquad \text{matrix } \left(P_i^\alpha\right)_{1 \leqslant i \leqslant s,\ |\alpha| \leqslant d}.$$

- Distinct points do not suffice.

$n = d = 2$, $s = 6 = N$. If the six points lie on a conic $q = 0$, then $0 \neq q \in \ker \rho$, so $\operatorname{rank} \rho \leqslant 5$.



The generic problem

- Fix n, d and $m_1, \ldots, m_s$. Replace the coordinates $P_{i,j}$ of the points by indeterminates.
- The matrix of ρ then has entries in $\mathbb{Z}[P_{i,j}]$. Its rank over $k(P_{i,j})$ is the generic rank.
- For every configuration of points,
  $\operatorname{rank}\rho \leqslant$ generic rank $\leqslant \min(\lambda(Z), N)$, with equality on the left on a nonempty Zariski open subset of

$$\mathcal{C} = (\mathbb{A}^n)^s \setminus \bigcup_{i<i'} \{P_i = P_{i'}\}.$$

- The problem has maximal generic rank if some maximal minor is a nonzero polynomial in the $P_{i,j}$. This depends only on $\operatorname{char} k$.

Generic Lagrange interpolation

Theorem. Lagrange interpolation has maximal generic rank, in every characteristic: the generic rank of $\left(P_i^\alpha\right)_{i,\,|\alpha|\leqslant d}$ is $\min(s, N)$.

- Let $t = \min(s, N)$ and choose distinct exponents $\alpha_1, \ldots, \alpha_t$. Consider the minor

$$D = \det\left(P_i^{\alpha_j}\right)_{1\leqslant i,j\leqslant t} = \sum_{\sigma\in\mathfrak{S}_t} \operatorname{sgn}(\sigma) \prod_{i=1}^{t} P_i^{\alpha_{\sigma(i)}}.$$

- The σ-term determines $\alpha_{\sigma(i)}$ for each i, hence σ.

- So the $t!$ terms are distinct monomials with coefficients ± 1: nothing cancels and $D \neq 0$.

Hermite interpolation of order two

- Any n, all $m_i = 2$: $Z = 2P_1 + \cdots + 2P_s$, $\lambda(Z) = s(n+1)$, and

$$\rho : p \longmapsto \big(p(P_i), \nabla p(P_i)\big)_i.$$

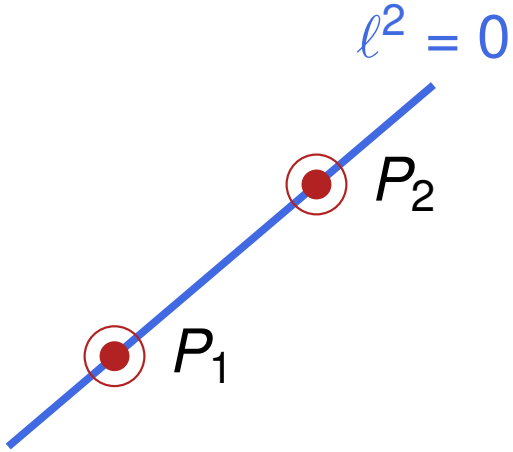
- The problem may never have max. rank.

$n = d = 2$, $s = 2$, so $\lambda(Z) = 6 = N$.

$\ell = 0$ the line through P_1 and P_2.

$\nabla(\ell^2) = 2\ell\,\nabla\ell$ vanishes on the line.

So $0 \neq \ell^2 \in \ker\rho$ for *all* $P_1 \neq P_2$.



The generic rank may not be maximal

- Unluckily, even the generic rank may not be maximal.
- Already seen: Hermite of order two with $n = d = 2$, $s = 2$.
- Another example: plane quartics through five double points.

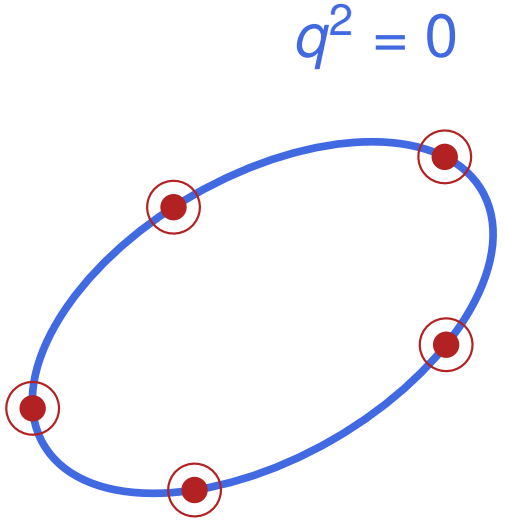
$n = 2$, $d = 4$, $s = 5$, all $m_i = 2$.

So $\lambda(Z) = 5 \cdot 3 = 15 = N$.

A conic $q = 0$ passes through any five points.

$\nabla(q^2) = 2q\nabla q$ vanishes on the conic.

So $0 \neq q^2 \in \ker \rho$ for *all* $P_1, \ldots, P_5$.



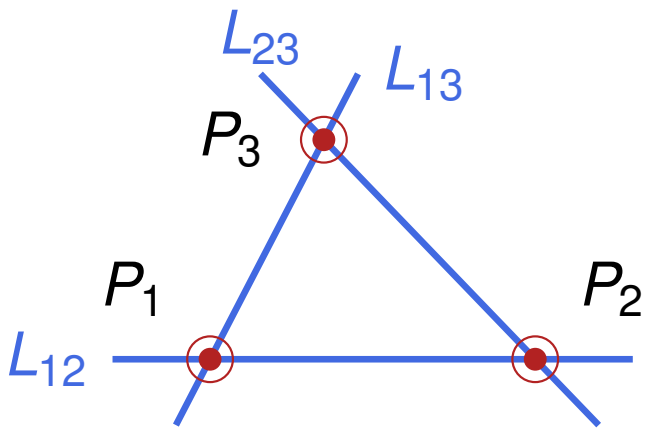
Plane cubics through three double points

- $n = 2$, $d = 3$, $s = 3$, all $m_i = 2$: $\lambda(Z) = 9 < N = 10$.

  Proposition. For any three non-collinear points, $\operatorname{rank} \rho = 9$ is maximal. The kernel is spanned by $\ell_{12}\ell_{13}\ell_{23}$.

Let $F \in \ker \rho$. Restricted to L_{ij}, it has degree $\leqslant 3$ and two double roots, so it vanishes.

Hence $\ell_{12}\ell_{13}\ell_{23}$ divides F, so $F = c\,\ell_{12}\ell_{13}\ell_{23}$.



The Alexander–Hirschowitz theorem

Theorem (Alexander–Hirschowitz, 1995). Let k be any field. Hermite interpolation of order two at s points of $\mathbb{A}^n$ in degree d has maximal generic rank $\min\big(s(n+1), N\big)$, except in the following cases.

n	d	s	in the kernel
any	2	$2 \leqslant s \leqslant n$	quadrics singular on $\langle P_1, \ldots, P_s \rangle$
2	4	5	q^2, q conic through the P_i
3	4	9	q^2, q quadric through the P_i
4	4	14	q^2, q quadric through the P_i
4	3	7	$\sigma_2(C)$, C rat. normal quartic through the P_i

Waring rank

- k algebraically closed of characteristic 0. $S_{n,d}$ the forms of degree d in $x_0, \ldots, x_n$, of dimension N.

- Every $F \in S_{n,d}$ is a sum of d-th powers of linear forms:

$$F = \ell_1^d + \cdots + \ell_s^d.$$

- The generic Waring rank $r(n, d)$ is the least s that suffices for a generic F. Counting parameters: $s(n+1) \geqslant N$.

  Theorem (consequence of AH).

$$r(n, d) = \left\lceil \frac{N}{n+1} \right\rceil,$$

  except for quadrics, where $r(n, 2) = n + 1$, and for $(n, d) = (2, 4), (3, 4), (4, 4), (4, 3)$, where it is one more.

From Waring to Hermite: Terracini's lemma

- Consider

$$\varphi : S_{n,1}^{s} \longrightarrow S_{n,d}, \qquad (\ell_1, \ldots, \ell_s) \longmapsto \ell_1^d + \cdots + \ell_s^d.$$

  Then $r(n, d)$ is the least s such that $\dim \operatorname{Im} \varphi = N$.

- In characteristic 0, $\dim \operatorname{Im} \varphi$ is the rank of the differential at a generic point:

$$D\varphi(\mu_1, \ldots, \mu_s) = d\left(\ell_1^{d-1}\mu_1 + \cdots + \ell_s^{d-1}\mu_s\right).$$

- Write $\ell_i = \sum_j P_{i,j} x_j$. There is a perfect pairing on $S_{n,d}$ such that the orthogonal of $\operatorname{Im} D\varphi$ is the space of forms singular at $P_1, \ldots, P_s$.

- So $\dim \operatorname{Im} \varphi = \operatorname{rank} \rho$ for Hermite interpolation of order two at $P_1, \ldots, P_s$, and AH gives $r(n, d)$.

The role of the characteristic

- The generic rank depends only on $\mathrm{char}\, k$.
- Any field: Lagrange and Alexander–Hirschowitz.
- Characteristic 0: Terracini and Waring. If $\mathrm{char}\, k = d = p$,

$$\left(\sum_j a_j x_j\right)^p = \sum_j a_j^p\, x_j^p,$$

so most forms are not sums of p-th powers.

An open question

- Lagrange: explicit unisolvent sets (principal lattice) give a one-line proof.

- Hermite: find explicit configurations of double points, over a finite field or over $k((t))$, whose rank can be computed directly. This would give an alternative proof of AH.

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