Generalizing Vélu's formulas and some applications

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23/11/2010 (LIX)

Outline

Discrete logarithm

Definition (DLP)

Let $G = \langle g \rangle$ be a cyclic group of prime order. Let $x \in \mathbb{N}$ and $h = g^x$. The discrete logarithm $\log_g(h)$ is x.

- Exponentiation: $O(\log p)$. DLP: $\widetilde{O}(\sqrt{p})$ (in a generic group).

$\Rightarrow$ Usual tools of public key cryptography (and more!)

- $G = \mathbb{F}_p^*$: sub-exponential attacks.

$\Rightarrow$ Find secure groups with efficient law, compact representation.

Abelian varieties

Definition

An Abelian variety is a complete connected group variety over a base field k.

- Abelian variety = points on a projective space (locus of homogeneous polynomials) + an abelian group law given by rational functions.

$\Rightarrow$ Use $G = A(k)$ with $k = \mathbb{F}_q$ for the DLP.

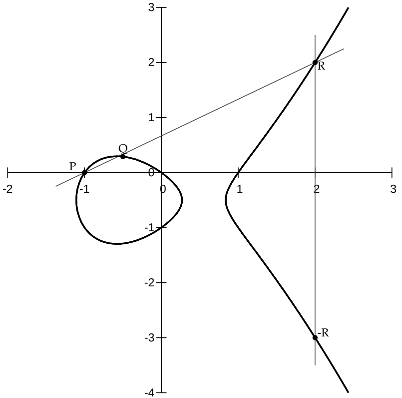
$\Rightarrow$ Pairing-based cryptography with the Weil or Tate pairing. (Only available on abelian varieties.)

Elliptic curves

Definition (car $k \neq 2, 3$)

$E : y^2 = x^3 + ax + b. \quad 4a^3 + 27b^2 \neq 0.$

- An elliptic curve is a plane curve of genus 1.
- Elliptic curves = Abelian varieties of dimension 1.



$$P + Q = -R = (x_R, -y_R)$$

$$\lambda = \frac{y_Q - y_P}{x_Q - x_P}$$

$$x_R = \lambda^2 - x_P - x_Q$$

$$y_R = y_P + \lambda(x_R - x_P)$$

Jacobian of hyperelliptic curves

$C : y^2 = f(x)$, hyperelliptic curve of genus g. $(\deg f = 2g - 1)$

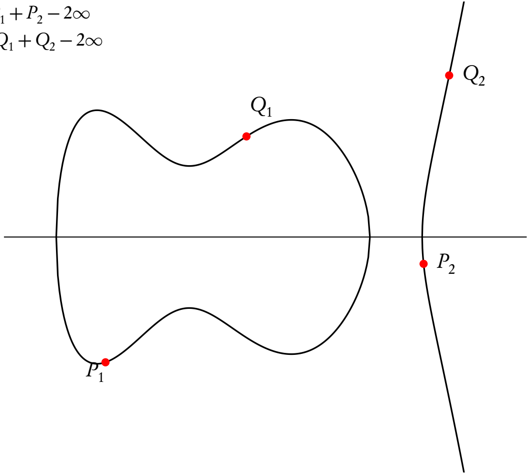
- Divisor: formal sum $D = \sum n_i P_i$, $P_i \in C(\overline{k})$.
  $\deg D = \sum n_i$.
- Principal divisor: $\sum_{P \in C(\overline{k})} v_P(f).P$; $f \in \overline{k}(C)$.
- Jacobian of C = Divisors of degree 0 modulo principal divisors
  = Abelian variety of dimension g.
- Divisor class $D \Rightarrow$ unique representative (Riemann–Roch):

$$D = \sum_{i=1}^{k} (P_i - P_\infty) \qquad k \leqslant g, \quad \text{symmetric } P_i \neq P_j$$

- Mumford coordinates: $D = (u, v) \Rightarrow u = \prod (x - x_i), v(x_i) = y_i$.
- Cantor algorithm: addition law.

Example of the addition law in genus 2

$D = P_1 + P_2 - 2\infty$
$D' = Q_1 + Q_2 - 2\infty$



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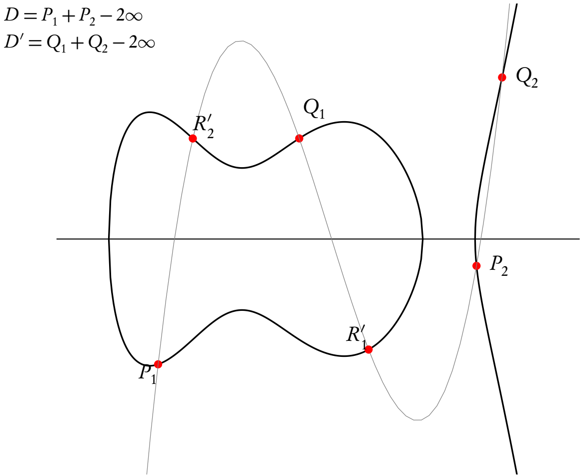

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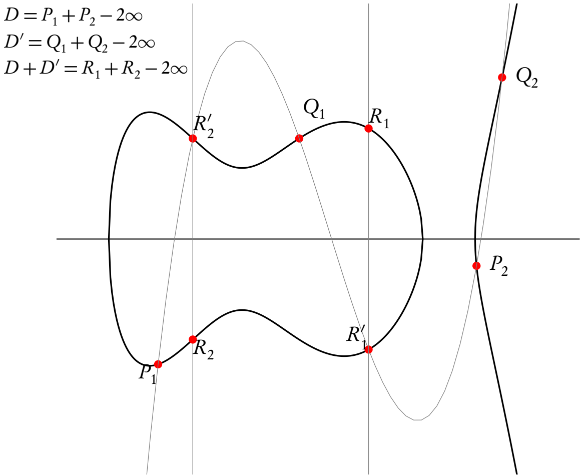

Isogenies

Definition

A (separable) isogeny is a finite surjective (separable) morphism between two Abelian varieties.

- Isogenies = Rational map + group morphism + finite kernel.
- Isogenies $\Leftrightarrow$ Finite subgroups.

$$(f : A \to B) \mapsto \operatorname{Ker} f$$
$$(A \to A/H) \leftarrow H$$

- *Example:* Multiplication by ℓ ($\Rightarrow \ell$-torsion), Frobenius (non separable).

Cryptographic usage of isogenies

- Transfer the DLP from one Abelian variety to another.
- Point counting algorithms (ℓ-adic or p-adic) $\Rightarrow$ Verify a curve is secure.
- Compute the class field polynomials (CM-method) $\Rightarrow$ Construct a secure curve.
- Compute the modular polynomials $\Rightarrow$ Compute isogenies.
- Determine $\mathrm{End}(A)$ $\Rightarrow$ CRT method for class field polynomials.

Explicit isogeny computation

- Given an isotropic subgroup $K \subset A(\overline{k})$ compute the isogeny $A \mapsto A/K$. (Vélu's formula.)
- Given an abelian variety compute all the isogeneous varieties. (Modular polynomials.)
- Given two isogeneous abelian variety A and B find the isogeny $A \mapsto B$. (Clever use of Vélu's formula $\Rightarrow$ SEA algorithm).

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Vélu's formula

Theorem

Let $E : y^2 = f(x)$ be an elliptic curve and $G \subset E(k)$ a finite subgroup. Then E/G is given by $Y^2 = g(X)$ where

$$X(P) = x(P) + \sum_{Q \in G \setminus \{0_E\}} \left(x(P+Q) - x(Q)\right)$$

$$Y(P) = y(P) + \sum_{Q \in G \setminus \{0_E\}} \left(y(P+Q) - y(Q)\right).$$

- Uses the fact that x and y are characterised in $k(E)$ by

$$\begin{aligned} v_{0_E}(x) &= -2 & v_P(x) &\geqslant 0 \quad \text{if } P \neq 0_E \\ v_{0_E}(y) &= -3 & v_P(y) &\geqslant 0 \quad \text{if } P \neq 0_E \\ y^2/x^3(0_E) &= 1 \end{aligned}$$

- No such characterisation in genus $g \geqslant 2$ for Mumford coordinates.

Complex abelian varieties and theta functions of level n

- $(\vartheta_i)_{i\in Z(\overline{n})}$: basis of the theta functions of level n. $(Z(\overline{n}) := \mathbb{Z}^g/n\mathbb{Z}^g)$
  $\Leftrightarrow A[n] = A_1[n] \oplus A_2[n]$: symplectic decomposition.
- $(\vartheta_i)_{i\in Z(\overline{n})} = \begin{cases} \text{coordinates system} & n \geqslant 3 \\ \text{coordinates on the Kummer variety } A/\pm 1 & n = 2 \end{cases}$
- Theta null point: $\vartheta_i(0)_{i\in Z(\overline{n})}$ = modular invariant.

Example ($k = \mathbb{C}$)

Abelian variety over $\mathbb{C}$: $A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$; $\Omega \in \mathcal{H}_g(\mathbb{C})$ the Siegel upper half space (Ω symmetric, $\operatorname{Im}\Omega$ positive definite).

$$\vartheta_i := \Theta\left[\begin{smallmatrix} 0 \\ i/n \end{smallmatrix}\right](z, \Omega/n).$$

Changing level

Theorem (Koizumi–Kempf)

Let F be a matrix of rank r such that ${}^tFF = \ell\,\mathrm{Id}_r$. Let $X \in (\mathbb{C}^g)^r$ and $Y = F(X) \in (\mathbb{C}^g)^r$. Let $j \in (\mathbb{Q}^g)^r$ and $i = F(j)$. Then we have

$$\vartheta\left[\begin{smallmatrix}0\\ i_1\end{smallmatrix}\right]\left(Y_1, \frac{\Omega}{n}\right)\ldots\vartheta\left[\begin{smallmatrix}0\\ i_r\end{smallmatrix}\right]\left(Y_r, \frac{\Omega}{n}\right) = \sum_{\substack{t_1,\ldots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta\left[\begin{smallmatrix}0\\ j_1\end{smallmatrix}\right]\left(X_1+t_1, \frac{\Omega}{\ell n}\right)\ldots\vartheta\left[\begin{smallmatrix}0\\ j_r\end{smallmatrix}\right]\left(X_r+t_r, \frac{\Omega}{\ell n}\right),$$

- If $\ell = a^2 + b^2$, we take $F = \left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right)$, so $r = 2$.
- In general, $\ell = a^2 + b^2 + c^2 + d^2$, we take F to be the matrix of multiplication by $a + bi + cj + dk$ in the quaternions, so $r = 4$.

Changing level and isogenies

Corollary

Let $A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ and $B = \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g)$. We can express the isogeny $A \to B, z \mapsto \ell z$ of kernel $K = \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g$ in term of the theta functions of level n on A and B:

$$\vartheta\left[\begin{smallmatrix}0\\ i_1\end{smallmatrix}\right](\ell z, \ell\frac{\Omega}{n})\vartheta\left[\begin{smallmatrix}0\\ i_2\end{smallmatrix}\right](0, \ell\frac{\Omega}{n})\ldots\vartheta\left[\begin{smallmatrix}0\\ i_r\end{smallmatrix}\right](0, \ell\frac{\Omega}{n}) = \sum_{\substack{t_1,\ldots,t_r\in K\\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta\left[\begin{smallmatrix}0\\ j_1\end{smallmatrix}\right](X_1 + t_1, \frac{\Omega}{n})\ldots\vartheta\left[\begin{smallmatrix}0\\ j_r\end{smallmatrix}\right]^{\mathcal{L}}(X_r + t_r, \frac{\Omega}{n}),$$

where $X = F^{-1}(\ell z, 0, \ldots, 0)$.

Remark

We need a way to compute the coordinates $\vartheta\left[\begin{smallmatrix}0\\ j_i\end{smallmatrix}\right](X_i + t_i, \frac{\Omega}{n})$ not in A but in $\mathbb{C}^g$.

The differential addition law ($k = \mathbb{C}$)

Applying twice the level formulas to $F = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ ($l = 2$) yields:

$$\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{i+t}(x+y)\vartheta_{j+t}(x-y)\Big).\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{k+t}(0)\vartheta_{l+t}(0)\Big) = \Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{-i'+t}(y)\vartheta_{j'+t}(y)\Big).\Big(\sum_{t \in Z(\overline{2})} \chi(t)\vartheta_{k'+t}(x)\vartheta_{l'+t}(x)\Big).$$

where $\chi \in \hat{Z}(\overline{2}), i, j, k, l \in Z(\overline{n})$

$$(i', j', k', l') = A(i, j, k, l)$$

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

An example with $g = 1$, $n = 2$, $\ell = 3$

$$\begin{tikzcd}
z \in \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g) \arrow[rr, "{[\ell]}"] \arrow[dr, "\pi"'] & & \ell z \in \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g) \\
& z \in \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g) \arrow[ur, "\widehat{\pi}"'] &
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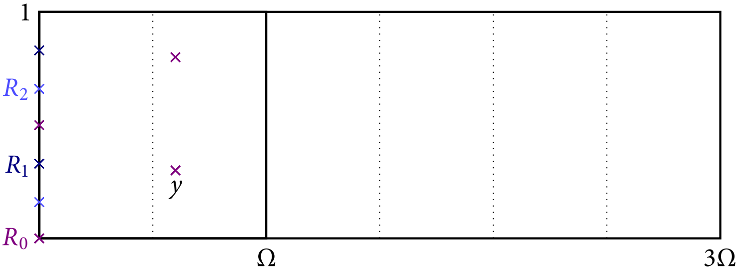

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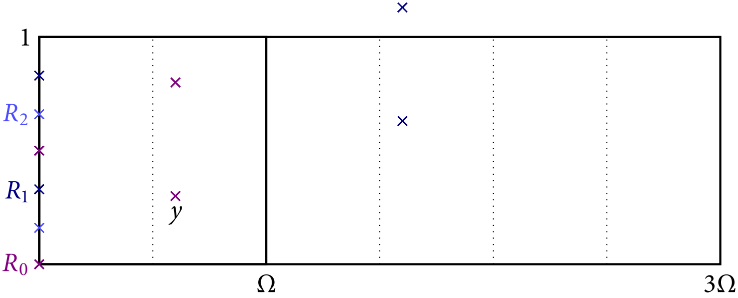

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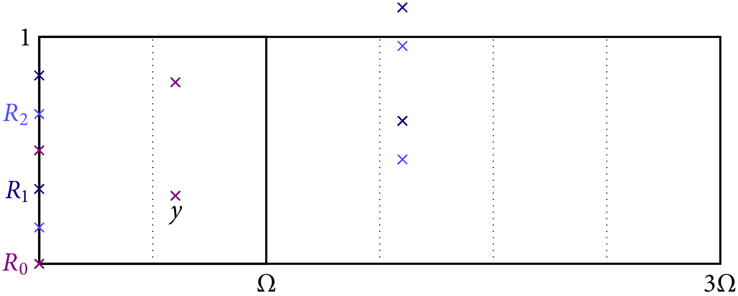

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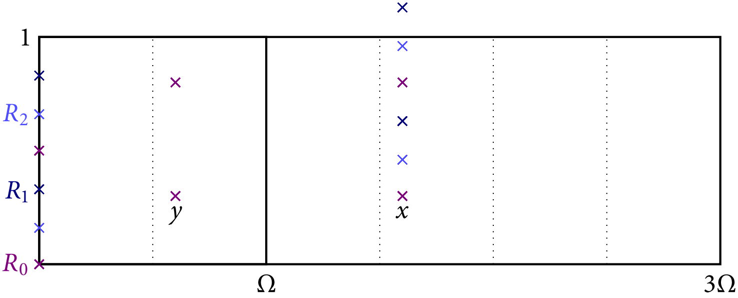

Computing isogenies [Cosset, Lubicz, R.]

- Let A/k be an abelian variety of dimension g over k given in theta coordinates. Let $K \subset A$ be a maximal isotropic subgroup of $A[\ell]$ (ℓ prime to 2 and the characteristic). Then we have an algorithm to compute the isogeny $A \mapsto A/K$.
- Need $O(\#K)$ differential additions in A
  $+ O(\ell^g)$ or $O(\ell^{2g})$ multiplications $\Rightarrow$ fast.
- The formulas are rational if the kernel K is rational.

- Blocking part: compute $K \Rightarrow$ compute all the ℓ-torsion on B.
  $g = 2$: ℓ-torsion, $\widetilde{O}(\ell^6)$ vs $O(\ell^2)$ or $O(\ell^4)$ for the isogeny.
- Theta coordinates are not rationnal.

$\Rightarrow$ Work in level 2.

$\Rightarrow$ Convert back and forth to Mumford coordinates:

$$\begin{tikzcd} A \arrow[r, "\widehat{\pi}"] \arrow[d, equal] & B \arrow[d, equal] \\ \mathrm{Jac}(C_1) \arrow[r, dotted] & \mathrm{Jac}(C_2) \end{tikzcd}$$

Avisogenies

- Avisogenies: Magma code written by Bisson, Cosset and R.
- Released under LGPL 2+.
- Implement isogeny computation (and applications thereof) for abelian varieties using theta functions.
- Current release 0.1: isogenies in genus 2.

Implementation

H hyperelliptic curve of genus 2 over $k = \mathbb{F}_q$, $J = \mathrm{Jac}(H)$, ℓ odd prime, $2\ell \wedge \operatorname{car} k = 1$. Compute all rational (ℓ, ℓ)-isogenies $J \mapsto \mathrm{Jac}(H')$ (we suppose the zeta function known):

1. Compute the extension $\mathbb{F}_{q^n}$ where the geometric points of the maximal isotropic kernel of $J[\ell]$ lives.
2. Compute a "symplectic" basis of $J[\ell](\mathbb{F}_{q^n})$.
3. Find the rational maximal isotropic kernels K.
4. For each kernel K, convert its basis from Mumford to theta coordinates of level 2. (Rosenhain then Thomae).
5. Compute the other points in K in theta coordinates using differential additions.
6. Apply the change level formula to recover the theta null point of J/K.
7. Compute the Igusa invariants of J/K ("Inverse Thomae").
8. Distinguish between the isogeneous curve and its twist.

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Computing the right extension

- $J = \mathrm{Jac}(H)$ abelian variety of dimension 2. $\chi(X)$ the corresponding zeta function.
- Degree of a point of ℓ-torsion | the order of X in $\mathbb{F}_\ell[X]/\chi(X)$.
- If K rational, $K(\overline{k}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, the degree of a point in K | the LCM of orders of X in $\mathbb{F}_\ell[X]/P(X)$ for $P \mid \chi$ of degree two.
- Since we are looking to K maximal isotropic, $J[\ell] \simeq K \oplus K'$ and we know that $P \mid \chi$ is such that $\chi(X) \equiv P(X)P(\overline{X}) \mod \ell$ where $\overline{X} = q/X$ represents the Verschiebung.

Remark

The degree n is $\leqslant \ell^2 - 1$. If ℓ is totally split in $\mathbb{Z}[\pi, \overline{\pi}]$ then $n \mid \ell - 1$.

Computing the ℓ-torsion

- We want to compute $J(\mathbb{F}_{q^n})[\ell]$.
- From the zeta function $\chi(X)$ we can compute random points in $J(\mathbb{F}_{q^n})[\ell^\infty]$ uniformly.
- If P is in $J(\mathbb{F}_{q^n})[\ell^\infty]$, $\ell^m P \in J(\mathbb{F}_{q^n})[\ell]$ for a suitable m. This does not give uniform points of ℓ-torsion but we can correct the points obtained.

Example

- Suppose $J(\mathbb{F}_{q^n})[\ell^\infty] = < P_1, P_2 >$ with P_1 of order ℓ^2 and P_2 of order ℓ.
- First random point $Q_1 = P_1 \Rightarrow$ we recover the point of ℓ-torsion: $\ell.P_1$.
- Second random point $Q_2 = \alpha P_1 + \beta P_2$. If $\alpha \neq 0$ we recover the point of ℓ-torsion $\alpha \ell P_1$ which is not a new generator.
- We correct the original point: $Q_2' = Q_2 - \alpha Q_1 = \beta P_2$.

Weil pairing

- Used to decompose a point $P \in J[\ell]$ in term of a basis of the ℓ-torsion (and to construct a symplectic basis).
- The magma implementation is **extremely** slow in genus 2 for non degenerate divisors.
- But since we convert the points in theta coordinates we can use the pairing in theta coordinates [LR10].

Timings for isogenies computations ($\ell = 7$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = t^254*x^6 + t^223*x^5 +
  t^255*x^4 + t^318*x^3 + t^668*x^2 + t^543*x + t^538 over GF(3^6)
> time RationallyIsogenousCurvesG2(J,7);
** Computing  7 -rationnal isotropic subgroups
  -- Computing the 7 -torsion over extension of deg 4
  !! Basis: 2 points in Finite field of size 3^24
  -- Listing subgroups
  1 subgroups over Finite field of size 3^24
  -- Convert the subgroups to theta coordinates
  Time: 0.060
Computing the 1 7 -isogenies
  ** Precomputations for l= 7 Time: 0.180
  ** Computing the 7 -isogeny
    Computing the l-torsion Time: 0.030
    Changing level Time: 0.210
  Time: 0.430
Time: 0.490
[ <[ t^620, t^691, t^477 ], Jacobian of Hyperelliptic Curve defined by
y^2 = t^615*x^6 + t^224*x^5 + t^37*x^4 + t^303*x^3 + t^715*x^2 + t^128*x
```

Timings for isogenies computations ($\ell = 5$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = 39*x^6 + 4*x^5 + 82*x^4
  + 10*x^3 + 31*x^2 + 39*x + 2 over GF(83)
> time RationallyIsogenousCurvesG2(J,5);
** Computing  5 -rationnal isotropic subgroups
  -- Computing the 5 -torsion over extension of deg 24
  Time: 0.940
  !! Basis: 4 points in Finite field of size 83^24
  -- Listing subgroups
  Time: 1.170
  6 subgroups over Finite field of size 83^24
  -- Convert the subgroups to theta coordinates
  Time: 0.360
Time: 2.630
Computing the 6 5 -isogenies
Time: 0.820
Time: 3.460
 [ <[ 36, 69, 38 ], Jacobian of Hyperelliptic Curve defined by
 y^2 = 27*x^6 + 63*x^5 + 5*x^4 + 24*x^3 + 34*x^2 + 6*x + 76 over GF(83)>,
   ...]
```

Timings for isogeny graphs $(\ell = 3)$

```
Jacobian of Hyperelliptic Curve defined by y^2 = 41*x^6 + 131*x^5 +
  55*x^4 + 57*x^3 + 233*x^2 + 225*x + 51 over GF(271)
time isograph,jacobians:=IsoGraphG2(J,{3}: save_mem:=-1);
Computed 540 isogenies and found 135 curves.
Time: 14.410
```

- Core 2 with 4BG of RAM.
- Computing kernels: $\approx 5s$.
- Computing isogenies: $\approx 7s$ (Torsion: $\approx 2s$, Changing level: $\approx 3.5s$.)

Going further $(\ell = 53)$

```
Jacobian of Hyperelliptic Curve defined by y^2 = 97*x^6 + 77*x^5 +
  62*x^4 + 14*x^3 + 33*x^2 + 18*x + 40 over GF(113)
> time RationallyIsogenousCurvesG2(J,53);
** Computing  53 -rationnal isotropic subgroups
  -- Computing the 53 -torsion over extension of deg 52 Time: 8.610
  !! Basis: 3 points in Finite field of size 113^52
  -- Listing subgroups Time: 1.210
  2 subgroups over Finite field of size 113^52
  -- Convert the subgroups to theta coordinates Time: 0.100
  Time: 9.980
Computing the 2 53 -isogenies
  ** Precomputations for l= 53 Time: 0.240
  ** Computing the 53 -isogeny
    Computing the l-torsion Time: 7.570
    Changing level Time: 1.170
  Time: 8.840
  ** Computing the 53 -isogeny
  Time: 8.850
Time: 27.950
```

Going further $(\ell = 19)$

```
  Jacobian of Hyperelliptic Curve defined by y^2 = 194*x^6 + 554*x^5 +
  606*x^4 + 523*x^3 + 642*x^2 + 566*x + 112 over GF(859)
  > time RationallyIsogenousCurvesG2(J,19);
  ** Computing  19 -rationnal isotropic subgroups (extension degree 18)
  Time: 0.760
Computing the 2 19 -isogenies
  ** Precomputations for l= 19 Time: 11.160
  ** Computing the 19 -isogeny
    Computing the l-torsion Time: 0.250
    Changing level Time: 18.590
  Time: 18.850
  ** Computing the 19 -isogeny
    Computing the l-torsion Time: 0.250
    Changing level Time: 18.640
  Time: 18.900
Time: 51.060
[ <[ 341, 740, 389 ], Jacobian of Hyperelliptic Curve defined by y^2 = 72
    680*x^5 + 538*x^4 + 613*x^3 + 557*x^2 + 856*x + 628 over GF(859)>,
  ... ]
```

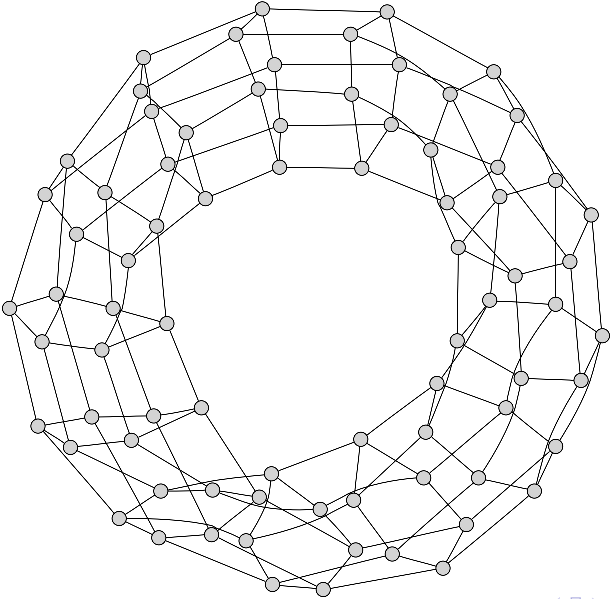
A record isogeny computation! ($\ell = 1321$)

- J Jacobian of $y^2 = x^5 + 41691x^4 + 24583x^3 + 2509x^2 + 15574x$ over $\mathbb{F}_{42179}$.
- $\#J = 2^{10}1321^2$.

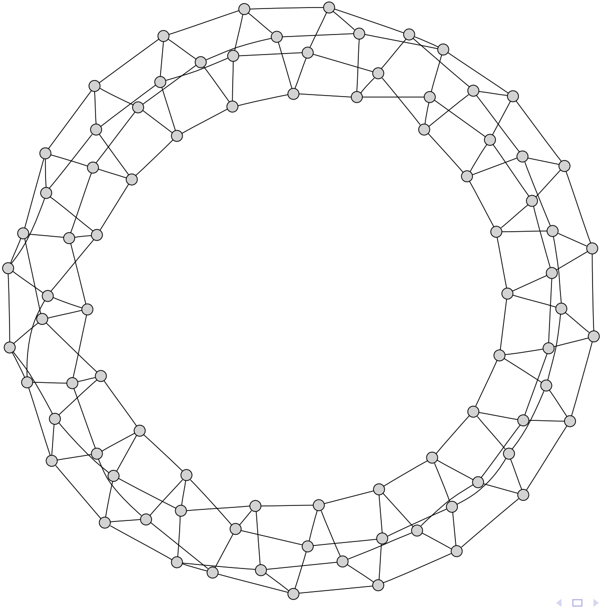
```
> time RationallyIsogenousCurvesG2(J,1321:ext_degree:=1);
** Computing  1321 -rationnal isotropic subgroups
Time: 0.350
Computing the 1 1321 -isogenies
  ** Precomputations for l= 1321
  Time: 1276.950
  ** Computing the 1321 -isogeny
    Computing the l-torsion
    Time: 1200.270
    Changing level
    Time: 1398.780
  Time: 5727.250
Time: 7004.240
Time: 7332.650
[ <[ 9448, 15263, 31602 ], Jacobian of Hyperelliptic Curve defined by
  y^2 = 33266*x^6 + 20155*x^5 + 31203*x^4 + 9732*x^3 +
  4204*x^2 + 18026*x + 29732 over GF(42179)> ]
```

- Core 2 with 32GB of RAM

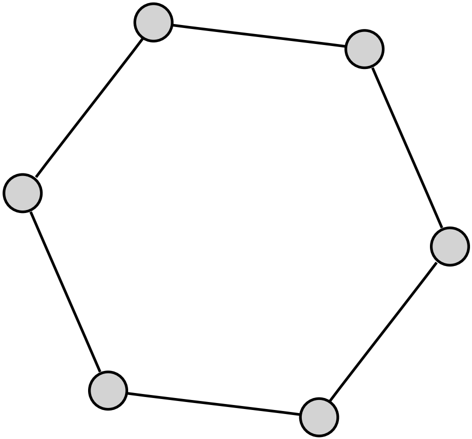
Isogeny graphs: $\ell = q_1 q_2 = \mathfrak{Q}_1 \overline{\mathfrak{Q}_1} \mathfrak{Q}_2 \overline{\mathfrak{Q}_2}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



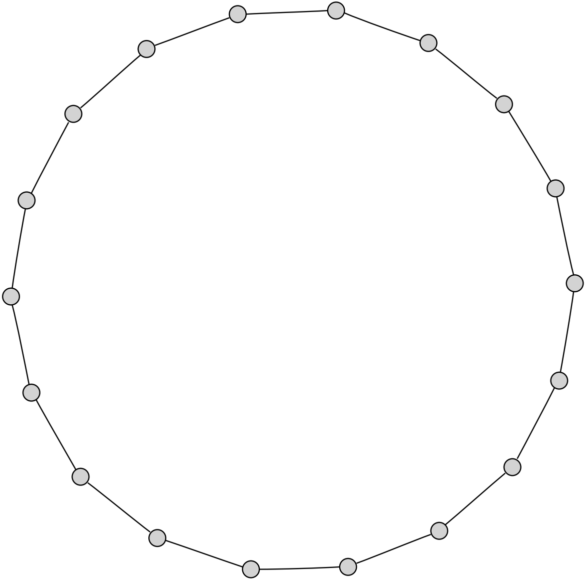
Isogeny graphs: $\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2 \overline{Q_2}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



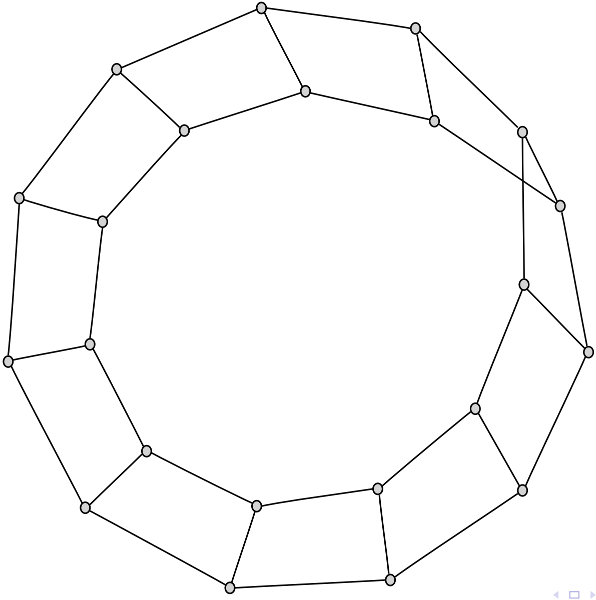
Isogeny graphs: $\ell = q = \mathfrak{Q}\overline{\mathfrak{Q}}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



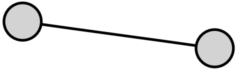
Isogeny graphs: $\ell = q_1 q_2 = \mathfrak{Q}_1 \overline{\mathfrak{Q}_1} \mathfrak{Q}_2^2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



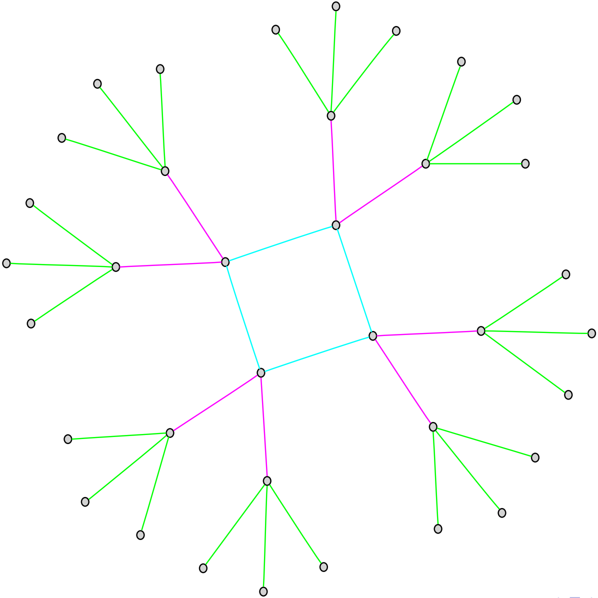
Isogeny graphs: $\ell = q^2 = Q^2\overline{Q}^2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



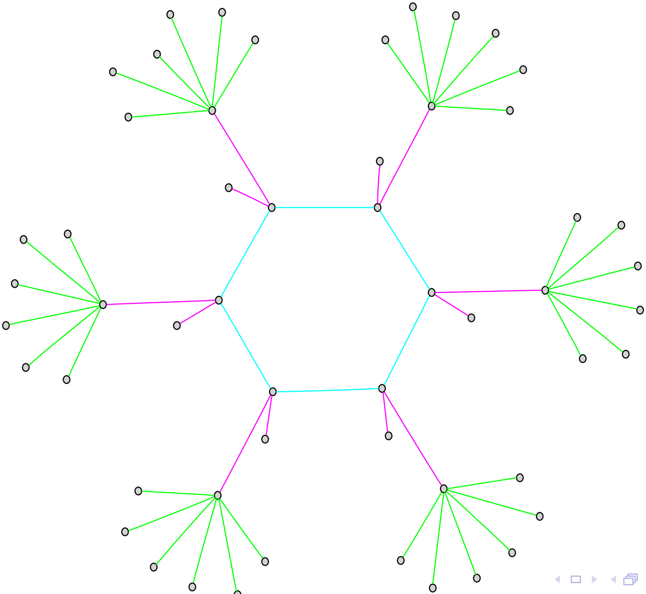
Isogeny graphs: $\ell = q^2 = Q^4$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$

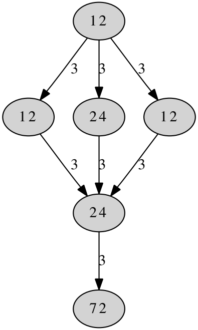


Non maximal isogeny graphs ($\ell = q = Q\overline{Q}$)

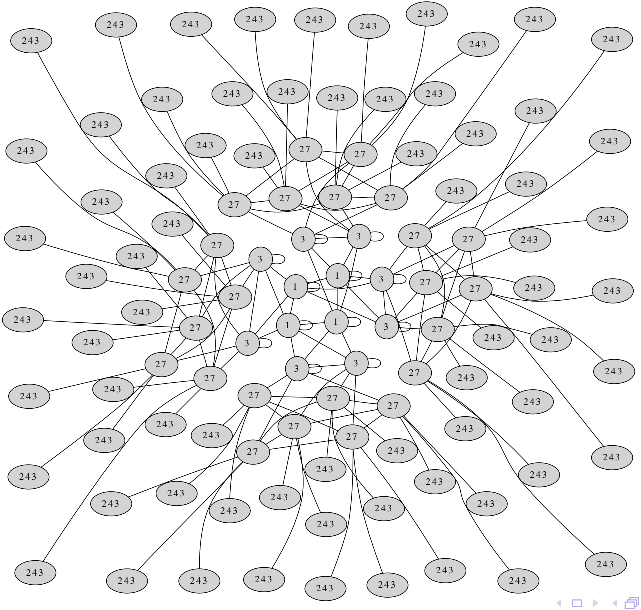


Non maximal isogeny graphs ($\ell = q = Q\overline{Q}$)

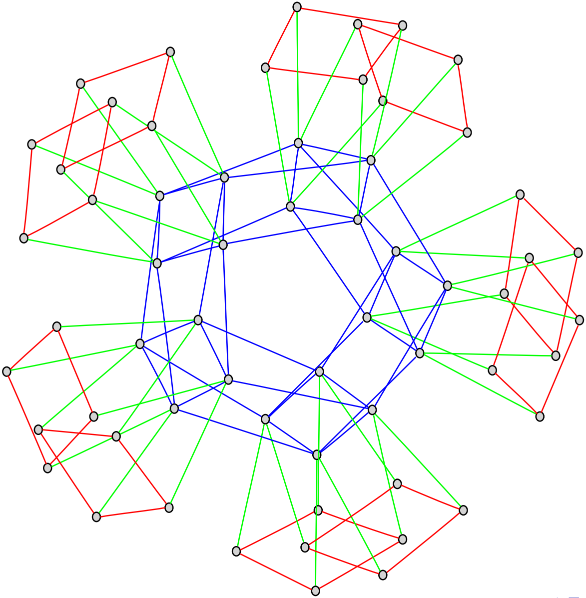




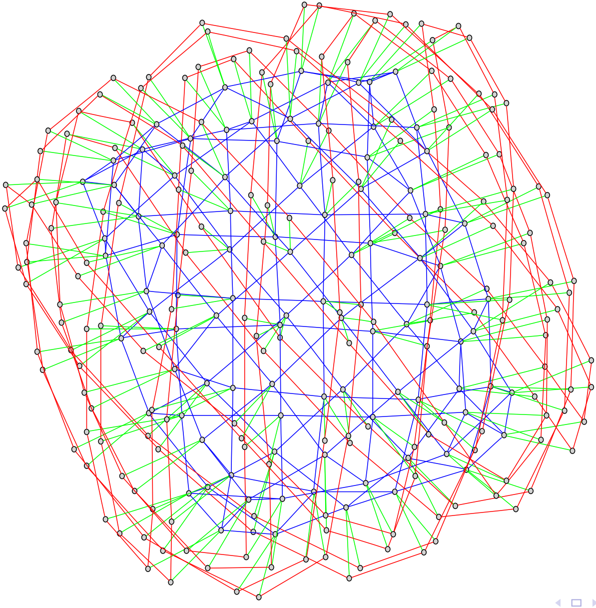
Non maximal isogeny graphs ($\ell = q = Q\overline{Q}$)



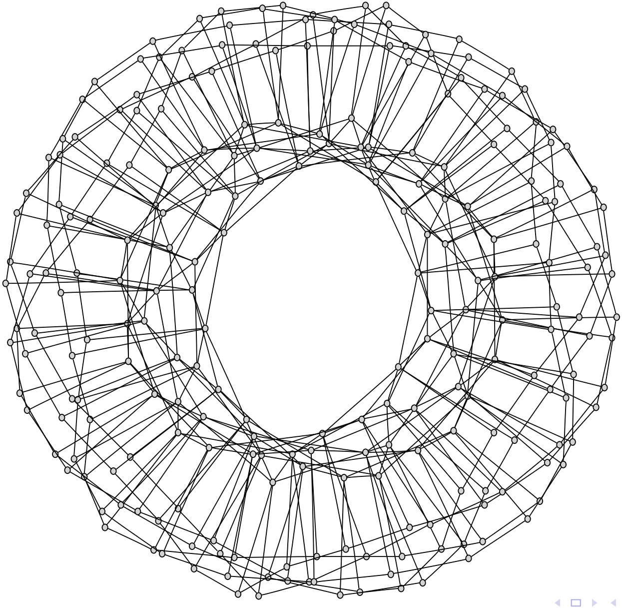
Non maximal isogeny graphs ($\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2 \overline{Q_2}$)



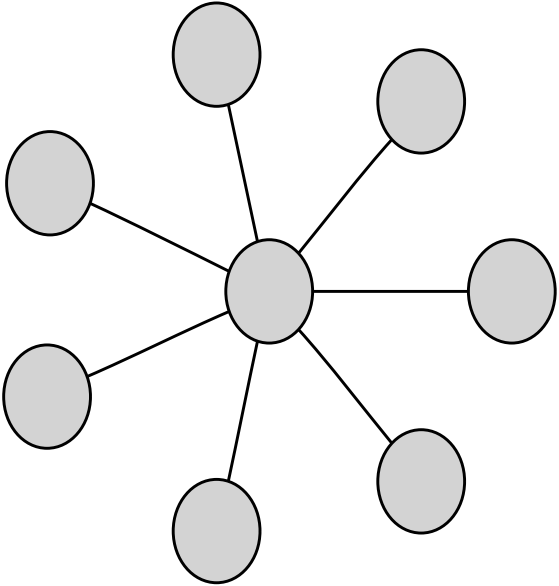
Non maximal isogeny graphs ($\ell = q_1 q_2 = Q_1\overline{Q_1}Q_2\overline{Q_2}$)



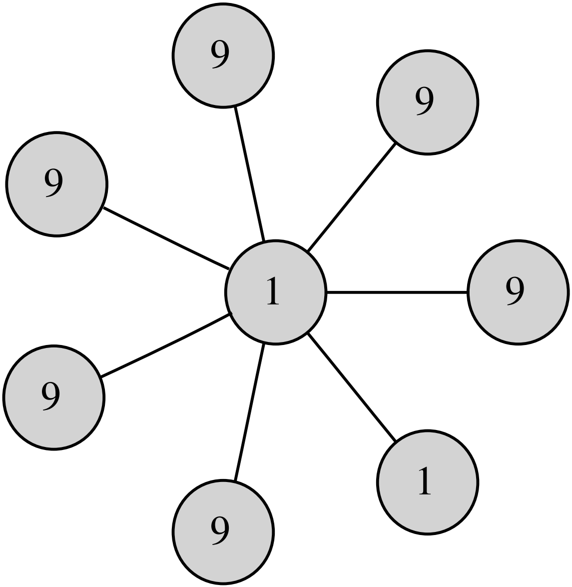
Non maximal isogeny graphs ($\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2 \overline{Q_2}$)



Non maximal isogeny graphs ($\ell = q = Q^2$)



Non maximal isogeny graphs ($\ell = q = Q^2$)

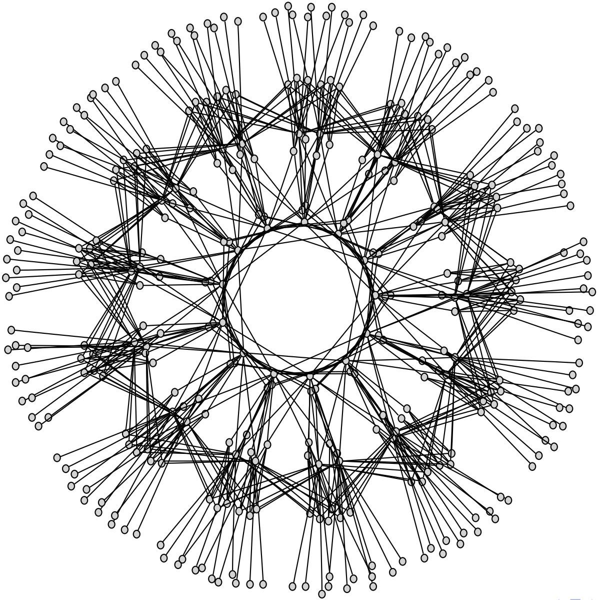


Applications and perspectives

- Computing endomorphism ring. Generalize [BS09] to higher genus, work by Bisson.
- Class polynomials in genus 2 using the CRT. If K is a CM field and $J/\mathbb{F}_p$ is such that $\operatorname{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q} = K$, use isogenies to find the Jacobians whose endomorphism ring is O_K. Work by Lauter+R.

- Modular polynomials in genus 2 using theta null points: computed by Gruenewald using analytic methods for $\ell = 3$.
- Isogenies using rational coordinates? Work by Smith using the geometry of Kummer surfaces for $\ell = 3$ ($g = 2$). Cassels and Flynn: modification of theta coordinates to have rational coordinates on hyperelliptic curves of genus 2.
- How to compute $(\ell, 1)$-isogenies in genus 2?
- Look at $g = 3$ (associate theta coordinates to the Jacobian of a non hyperelliptic curve).

Thank you for your attention!



Bibliography

[BS09] G. Bisson and A.V. Sutherland. "Computing the endomorphism ring of an ordinary elliptic curve over a finite field". In: *Journal of Number Theory* (2009). (Cit. on p. 62).

[LR10] David Lubicz and Damien Robert. *Efficient pairing computation with theta functions*. Ed. by Guillaume Hanrot, François Morain, and Emmanuel Thomé. 9th International Symposium, Nancy, France, ANTS-IX, July 19-23, 2010, Proceedings. Jan. 2010. URL: http://www.normalesup.org/~robert/pro/publications/articles/pairings.pdf. (Cit. on p. 41).