Isogenies, Polarisations and Real Multiplication

2015/09/29 — **ICERM** — Providence

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Outline

Isogenies between elliptic curves

Definition

An isogeny is a (non trivial) algebraic map $f: E_1 \to E_2$ between two elliptic curves such that $f(P+Q) = f(P) + f(Q)$ for all geometric points $P, Q \in E_1$.

Theorem

An algebraic map $f: E_1 \to E_2$ is an isogeny if and only if $f(0_{E_1}) = f(0_{E_2})$

Corollary

An algebraic map between two elliptic curves is either

- *trivial (i.e. constant)*
- *or the composition of a translation with an isogeny.*

Remark

Isogenies are surjective (on the geometric points). In particular, if E is ordinary, any curve isogenous to E is also ordinary.

Algorithmic aspect of isogenies

- Given a kernel $K \subset E(\overline{k})$ compute the isogenous elliptic curve E/K);
- Given a kernel $K \subset E(\overline{k})$ and $P \in E(k)$ compute the image of P under the isogeny $E \to E/K$;
- Given a kernel $K \subset E(\overline{k})$ compute the map $E \to E/K$;
- Given an elliptic curve E/k compute all isogenous (of a certain degree d) elliptic curves E';);
- Given two elliptic curves E_1 and E_2 check if they are d-isogenous and if so compute the kernel $K \subset E_1(\overline{k})$.

Algorithmic aspect of isogenies

- Given a kernel $K \subset E(\overline{k})$ compute the isogenous elliptic curve E/K (Vélu's formulae [Vél71]);
- Given a kernel $K \subset E(\overline{k})$ and $P \in E(k)$ compute the image of P under the isogeny $E \to E/K$ (Vélu's formulae [Vél71]);
- Given a kernel $K \subset E(\overline{k})$ compute the map $E \to E/K$ (formal version of Vélu's formulae [Koh96]);
- Given an elliptic curve E/k compute all isogenous (of a certain degree d) elliptic curves E'; (Modular polynomial [Eng09; BLS12]);
- Given two elliptic curves E_1 and E_2 check if they are d-isogenous and if so compute the kernel $K \subset E_1(\overline{k})$ (Elkie's method via a differential equation [Elk92; Bos+08]).

⇒ We have quasi-linear algorithms for all these aspects of isogeny computation over elliptic curves.

Destructive cryptographic applications

- An isogeny $f: E_1 \rightarrow E_2$ transports the DLP problem from E_1 to E_2. This can be used to attack the DLP on E_1 if there is a weak curve on its isogeny class (and an efficient way to compute an isogeny to it).

Example

- extend attacks using Weil descent [GHS02]
- Transfert the DLP from the Jacobian of an hyperelliptic curve of genus 3 to the Jacobian of a quartic curve [Smi09].

Constructive cryptographic applications

- One can recover informations on the elliptic curve E modulo ℓ by working over the ℓ-torsion.
- But by computing isogenies, one can work over a cyclic subgroup of cardinal ℓ instead.
- Since thus a subgroup is of degree ℓ, whereas the full ℓ-torsion is of degree ℓ^2, we can work faster over it.

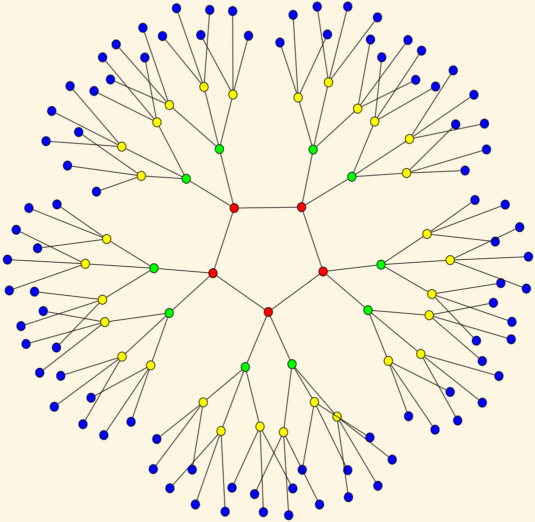
Example

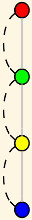
- The SEA point counting algorithm [Sch95; Mor95; Elk97];
- The CRT algorithms to compute class polynomials [Sut11; ES10];
- The CRT algorithms to compute modular polynomials [BLS12].

Further applications of isogenies

- Splitting the multiplication using isogenies can improve the arithmetic [DIK06; Gau07];
- The isogeny graph of a supersingular elliptic curve can be used to construct secure hash functions [CLG09];
- Construct public key cryptosystems by hiding vulnerable curves by an isogeny (the trapdoor) [Tes06], or by encoding informations in the isogeny graph [RS06];
- Take isogenies to reduce the impact of side channel attacks [Sma03];
- Construct a normal basis of a finite field [CL09];
- Improve the discrete logarithm in $\mathbb{F}_q^*$ by finding a smoothness basis invariant by automorphisms [CL08].

A 3-isogeny graph in dimension 1 [Koh96; FM02]





Polarised abelian varieties over $\mathbb{C}$

Definition

A complex abelian variety A of dimension g is isomorphic to a compact Lie group V/Λ with

- A complex vector space V of dimension g;
- A $\mathbb{Z}$-lattice Λ in V (of rank $2g$);

such that there exists an Hermitian form H on V with $E(\Lambda,\Lambda) \subset \mathbb{Z}$ where $E = \operatorname{Im} H$ is symplectic.

- Such an Hermitian form H is called a polarisation on A. Conversely, any symplectic form E on V such that $E(\Lambda,\Lambda) \subset \mathbb{Z}$ and $E(ix,iy) = E(x,y)$ for all $x,y \in V$ gives a polarisation H with $E = \operatorname{Im} H$.
- Over a symplectic basis of Λ, E is of the form.

$$\begin{pmatrix} 0 & D_\delta \\ -D_\delta & 0 \end{pmatrix}$$

  where D_δ is a diagonal positive integer matrix $\delta = (\delta_1, \delta_2, \dots, \delta_g)$, with $\delta_1 \mid \delta_2 | \cdots \mid \delta_g$.
- The product $\prod \delta_i$ is the degree of the polarisation; H is a principal polarisation if this degree is 1.

Isogenies

Let $A = V/\Lambda$ and $B = V'/\Lambda'$.

Definition

An isogeny $f : A \to B$ is a bijective linear map $f : V \to V'$ such that $f(\Lambda) \subset \Lambda'$. The kernel of the isogeny is $f^{-1}(\Lambda')/\Lambda \subset A$ and its degree is the cardinal of the kernel.

- Two abelian varieties over a finite field are isogenous iff they have the same zeta function (Tate);
- A morphism of abelian varieties $f : A \to B$ (seen as varieties) is a group morphism iff $f(0_A) = 0_B$.

The dual abelian variety

Definition

If $A = V/\Lambda$ is an abelian variety, its dual is $\widehat{A} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})/\Lambda^*$. Here $\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ is the space of anti-linear forms and $\Lambda^* = \{f \mid f(\Lambda) \subset \mathbb{Z}\}$ is the orthogonal of Λ.

- If H is a polarisation on A, its dual H^* is a polarisation on $\widehat{A}$. Moreover, there is an isogeny $\Phi_H : A \to \widehat{A}$:
$$x \mapsto H(x, \cdot)$$
of degree $\deg H$. We note $K(H)$ its kernel.
- If $f : A \to B$ is an isogeny, then its dual is an isogeny $\widehat{f} : \widehat{B} \to \widehat{A}$ of the same degree.

Remark

The canonical pairing $A \times \widehat{A} \to \mathbb{C}, (x, f) \mapsto f(x)$ induces a canonical principal polarisation on $A \times \widehat{A}$ (the Poincaré bundle):

$$E_P((x_1, f_1), (x_2, f_2)) = f_1(x_2) - f_2(x_1).$$

The pullback $(\mathrm{Id}, \varphi_H)^* E_P = 2E$.

Isogenies and polarisations

Definition

- An isogeny $f:(A,H_1)\to(B,H_2)$ between polarised abelian varieties is an isogeny such that
$$f^*H_2 := H_2(f(\cdot),f(\cdot)) = H_1.$$
- f is an ℓ-isogeny between principally polarised abelian varieties if H_1 and H_2 are principal and $f^*H_2 = \ell H_1$.

An isogeny $f:(A,H_1)\to(B,H_2)$ respect the polarisations iff the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\widehat{f}} & B \\ \downarrow{\scriptstyle \Phi_{H_1}} & & \downarrow{\scriptstyle \Phi_{H_2}} \\ \widehat{A} & \xleftarrow[f]{} & \widehat{B} \end{array}$$

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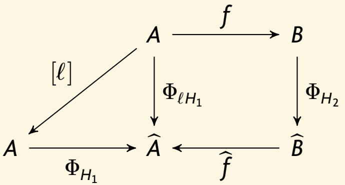

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- f is an ℓ-isogeny between principally polarised abelian varieties if H_1 and H_2 are principal and $f^*H_2 = \ell H_1$.

Proposition

If $K\subset A(\overline{k})$, H_1 descends to a polarisation H_2 on A/K (ie $f^*H_2 = H_1$) if and only if $\operatorname{Im} H_1(K+\Lambda_1, K+\Lambda_1)\subset \mathbb{Z}$. The degree of H_2 is then $\deg H_1/\deg f^2$.

Example

Let $\Lambda_1 = \Omega_1\mathbb{Z}^g + \mathbb{Z}^g$, $H_1 = \ell(\operatorname{Im}\Omega_1)^{-1}$, then A/K is principally polarised ($A/K = \mathbb{C}^g/(\Omega_2\mathbb{Z}^g+\mathbb{Z}^g)$) if $K = \frac{1}{\ell}\mathbb{Z}^g$ or $K = \frac{1}{\ell}\Omega\mathbb{Z}^g$.

Theta functions

- Let (A, H_0) be a principally polarised abelian variety over $\mathbb{C}$;
- $A = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ with $\Omega \in \mathfrak{H}_g$ and $H_0 = (\Im\Omega)^{-1}$.
- All automorphic forms corresponding to a multiple of H_0 come from the theta functions with characteristics:

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^t(n+a)\Omega(n+a) + 2\pi i\, {}^t(n+a)(z+b)} \qquad a,b \in \mathbb{Q}^g$$

- Automorphic property:

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z + m_1\Omega + m_2, \Omega) = e^{2\pi i({}^t a \cdot m_2 - {}^t b \cdot m_1) - \pi i\, {}^t m_1 \Omega m_1 - 2\pi i\, {}^t m_1 \cdot z} \vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\Omega).$$

- Define $\vartheta_i = \vartheta\left[\begin{smallmatrix} 0 \\ \frac{i}{n} \end{smallmatrix}\right](., \frac{\Omega}{n})$ for $i \in Z(\overline{n}) = \mathbb{Z}^g/n\mathbb{Z}^g$
- $(\vartheta_i)_{i\in Z(\overline{n})} = \begin{cases} \text{coordinates system} & n \geqslant 3 \\ \text{coordinates on the Kummer variety } A/\pm 1 & n = 2 \end{cases}$

Theta group ($k = \bar{k}$)

- Let $(A, \mathscr{L})$ be a polarised abelian variety with $\mathscr{L}$ an ample line bundle of degree prime to $\operatorname{char} k$;
- The Theta group $G(\mathscr{L})$ is the group $\{(x, \psi_x)\}$ where $x \in K(\mathscr{L})$ and ψ_x is an isomorphism
$$\psi_x : \mathscr{L} \to \tau_x^* \mathscr{L}$$
The composition is given by $(y, \psi_y).(x, \psi_x) = (y + x, \tau_x^* \psi_y \circ \psi_x)$.
- $G(\mathscr{L})$ is an Heisenberg group:
$$0 \longrightarrow k^* \longrightarrow G(\mathscr{L}) \longrightarrow K(\mathscr{L}) \longrightarrow 0$$
where $K(\mathscr{L})$ is the kernel of the polarisation
$$\begin{array}{rcl} \Phi_{\mathscr{L}} : A & \longrightarrow & \widehat{A} = \mathrm{Pic}^0(A) \\ x & \longmapsto & t_x^* \mathscr{L} \otimes \mathscr{L}^{-1} \end{array}.$$

Remark

The polarisation $\Phi_{\mathscr{L}}$ only depend on the algebraic equivalent class of $\mathscr{L}$ in the Néron-Severi group $NS(A)$. When $\mathscr{L}$ is ample, $\mathscr{L}'$ is algebraically equivalent to $\mathscr{L}$ if $\mathscr{L}' = t_x^* \mathscr{L}$ for a $x \in A(\bar{k})$.

Theta group ($k = \bar{k}$)

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Definition (Pairings)

- Let $g_P = (P, \psi_P) \in G(\mathscr{L})$ and $g_Q = (Q, \psi_Q) \in G(\mathscr{L})$,

$$e_{\mathscr{L}}(P,Q) = g_P g_Q g_P^{-1} g_Q^{-1};$$

- If $\psi : K(\mathscr{L}) \times K(\mathscr{L}) \to k^*$ is the 2-cocycle associated to $G(\mathscr{L})$, we also have

$$e_{\mathscr{L}}(P,Q) = \frac{\psi(P,Q)}{\psi(Q,P)}.$$

- The $e_{\mathscr{L}^n}$ glue together to give a pairing on the Tate modules $T_\ell A$.

Descent

- Let $(A, \mathscr{L})$ be a polarised abelian variety as above;
- Let $K \subset A(\overline{k})$ and $f : A \to B = A/K$.

Theorem ([Mum66])

$\mathscr{L}$ descends to a polarisation $\mathscr{M}$ on B (ie $f^* \mathscr{M} \simeq \mathscr{L}$) if and only if either

- *K has a level subgroup $\widetilde{K} \subset G(\mathscr{L})$;*
- *K is isotropic for $e_{\mathscr{L}}$.*

$\mathscr{L}$ descends to a principal polarisation $\mathscr{M}$ if and only if K is maximal isotropic.

Theorem ([Mil86] ($\operatorname{char} k \neq 2$))

A morphism $\lambda : A \to \widehat{A}$ is induced by a line bundle $\mathscr{L}$ if and only if the induced pairing $e_{\lambda,\ell}$ on the Tate module $T_\ell(A)$ (for a $\ell > 2$) is skew-symmetric.

Algebraic theta functions

- Let $H(\delta) = \overline{k}^* \times Z(\delta) \times \hat{Z}(\delta)$ be the canonical Heisenberg group of level δ (with $Z(\delta) = \mathbb{Z}/\delta_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\delta_g\mathbb{Z}$ and $\hat{Z}(\delta) = \widehat{\mathbb{Z}}/\delta_1\widehat{\mathbb{Z}} \times \cdots \times \widehat{\mathbb{Z}}/\delta_g\widehat{\mathbb{Z}}$);
- It admits a unique irreducible (projective) representation:

$$(\alpha, i, j).\delta_k = \langle i+k, -j \rangle \delta_{i+k}.$$

- $G(\mathscr{L})$ acts (projectively) on $\Gamma(\mathscr{L})$. If $\mathscr{L}$ is ample this action is irreducible;
- If $\mathscr{L}$ has level δ, fixing an isomorphism $H(\delta) \simeq G(\mathscr{L})$ fixes a basis of section uniquely (up to a multiplication by a constant): the theta functions;
- If $\mathscr{L} = \mathscr{L}_0^3$ then $\mathscr{L}$ is very ample:

$$z \longmapsto (\vartheta_i(z))_{i \in Z(\delta)}$$

  is a projective embedding $A \rightarrow \mathbb{P}_k^{\prod \delta_i - 1}$.
- Technical details: we work with totally symmetric line bundles which are unique in their algebraic equivalence class and so are canonically defined from the induced polarization.

Computing isogenies in dimension 2

- Richelot formluae [Ric36; Ric37] allows to compute 2-isogenies between Jacobians of hyperelliptic curves of genus 2 (ie maximal isotropic kernels in $A[2]$);
- The duplication formulae for theta functions

$$\vartheta\left[\begin{smallmatrix}\chi\\0\end{smallmatrix}\right](0, 2\frac{\Omega}{n})^2 = \frac{1}{2^g} \sum_{t \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} e^{-2i\pi 2\, {}^t\chi \cdot t} \vartheta\left[\begin{smallmatrix}0\\t\end{smallmatrix}\right](0, \frac{\Omega}{n})^2$$

$$\vartheta\left[\begin{smallmatrix}0\\i/2\end{smallmatrix}\right](0, 2\Omega)^2 = \frac{1}{2^g} \sum_{i_1 + i_2 = 0 \pmod 2} \vartheta\left[\begin{smallmatrix}0\\i_1/2\end{smallmatrix}\right](0, \Omega)\vartheta\left[\begin{smallmatrix}0\\i_2/2\end{smallmatrix}\right](0, \Omega) \quad \text{(for all } \chi \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g\text{);}$$

  allows to generalize Richelot formulae to any dimension;
- Dupont compute modular polynomials of level 2 in [Dup06] and started the computation of modular polynomials of level 3.
- Low degree formulae [DL08] effective for $\ell = 3$ and made explicit in [Smi12].

The isogeny theorem

Theorem ([Mum66])

- *Let* $\varphi : Z(\overline{n}) \to Z(\overline{\ell n}), x \mapsto \ell.x$ *be the canonical embedding.*
  Let $K = A_2[\ell] \subset A_2[\ell n]$.
- *Let* $(\vartheta_i^A)_{i \in Z(\overline{\ell n})}$ *be the theta functions of level* ℓn *on* $A = \mathbb{C}^g / (\mathbb{Z}^g + \ell \Omega \mathbb{Z}^g)$.
- *Let* $(\vartheta_i^B)_{i \in Z(\overline{n})}$ *be the theta functions of level* n *of* $B = A/K = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$.
- *We have:*

$$(\vartheta_i^B(x))_{i \in Z(\overline{n})} = (\vartheta_{\varphi(i)}^A(x))_{i \in Z(\overline{n})}$$

Example

$f : (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}) \mapsto (x_0, x_3, x_6, x_9)$ is a 3-isogeny between elliptic curves.

Changing level

Theorem (Koizumi–Kempf)

Let F be a matrix of rank r such that ${}^tFF = \ell \operatorname{Id}_r$. Let $X \in (\mathbb{C}^g)^r$ and $Y = F(X) \in (\mathbb{C}^g)^r$. Let $j \in (\mathbb{Q}^g)^r$ and $i = F(j)$. Then we have

$$\vartheta\left[\begin{smallmatrix}0\\ i_1\end{smallmatrix}\right]\left(Y_1, \frac{\Omega}{n}\right)\dots\vartheta\left[\begin{smallmatrix}0\\ i_r\end{smallmatrix}\right]\left(Y_r, \frac{\Omega}{n}\right) = \sum_{\substack{t_1,\dots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ F(t_1,\dots,t_r)=(0,\dots,0)}} \vartheta\left[\begin{smallmatrix}0\\ j_1\end{smallmatrix}\right]\left(X_1 + t_1, \frac{\Omega}{\ell n}\right)\dots\vartheta\left[\begin{smallmatrix}0\\ j_r\end{smallmatrix}\right]\left(X_r + t_r, \frac{\Omega}{\ell n}\right),$$

(This is the isogeny theorem applied to $F_A : A^r \to A^r$.)

- If $\ell = a^2 + b^2$, we take $F = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so $r = 2$.
- In general, $\ell = a^2 + b^2 + c^2 + d^2$, we take F to be the matrix of multiplication by $a + bi + cj + dk$ in the quaternions, so $r = 4$.

The isogeny formula [Cosset, R.]

$$\ell \wedge n = 1, \quad B = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g), \quad A = \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g)$$
$$\vartheta_b^B := \vartheta\left[\begin{smallmatrix} 0 \\ \frac{b}{n} \end{smallmatrix}\right]\left(\cdot, \frac{\Omega}{n}\right), \quad \vartheta_b^A := \vartheta\left[\begin{smallmatrix} 0 \\ \frac{b}{n} \end{smallmatrix}\right]\left(\cdot, \frac{\ell\Omega}{n}\right)$$

Proposition

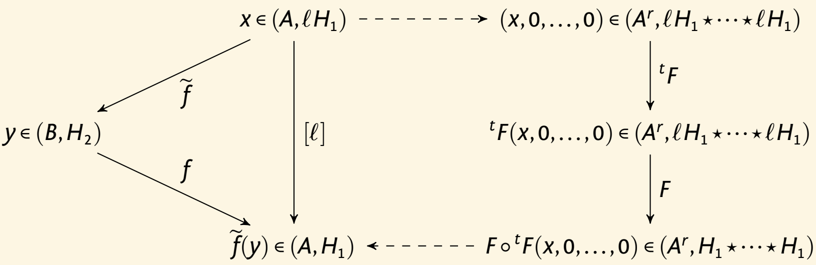
Let F be a matrix of rank r such that ${}^tFF = \ell\,\mathrm{Id}_r$. Let $Y = (\ell x, 0, \dots, 0)$ in $(\mathbb{C}^g)^r$ and $X = YF^{-1} = (x, 0, \dots, 0)t_F \in (\mathbb{C}^g)^r$. Let $i \in (Z(\overline{n}))^r$ and $j = iF^{-1}$. Then we have

$$\vartheta_{i_1}^A(\ell z)\dots\vartheta_{i_r}^A(0) = \sum_{\substack{t_1,\dots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ F(t_1,\dots,t_r)=(0,\dots,0)}} \vartheta_{j_1}^B(X_1 + t_1)\dots\vartheta_{j_r}^B(X_r + t_r),$$

Corollary

$$\vartheta_k^A(0)\vartheta_0^A(0)\dots\vartheta_0^A(0) = \sum_{\substack{t_1,\dots,t_r \in K \\ (t_1,\dots,t_r)F=(0,\dots,0)}} \vartheta_{j_1}^B(t_1)\dots\vartheta_{j_r}^B(t_r), \quad (j = (k, 0, \dots, 0)F^{-1} \in Z(\overline{n}))$$

The Algorithm [Cosset, R.]



Theorem ([Lubicz, R.])

We can compute the isogeny directly given the equations (in a suitable form) of the kernel K of the isogeny. When K is rational, this gives a complexity of $\widetilde{O}(\ell^g)$ or $\widetilde{O}(\ell^{2g})$ operations in $\mathbb{F}_q$ according to whether $\ell \cong 1$ or 3 modulo 4.

- "Record" isogeny computation: $\ell = 1321$.

The case $\ell \equiv 1 \pmod 4$

- The isogeny formula assumes that the points are in affine coordinates. But $A/\mathbb{F}_q$ is given by projective coordinates $\Rightarrow$ normalize the coordinates using the 2-cocycle defining the theta group;
- Suppose that we have (projective) equations of K in diagonal form over the base field k:

$$P_1(X_0, X_1) = 0$$
$$\ldots$$
$$X_n X_0^d = P_n(X_0, X_1)$$

- By setting $X_0 = 1$ we can work with affine coordinates. The projective solutions can be written $(x_0, x_0 x_1, \ldots, x_0 x_n)$ so X_0 can be seen as the normalization factor.
- We work in the algebra $\mathfrak{A} = k[X_1]/(P_1(X_1))$; each operation takes $\widetilde{O}(\ell^g)$ operations in k
- Let $F = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $\ell = a^2 + b^2$. Let $c = -a/b \pmod \ell$. The couples in the kernel of F are of the form (x, cx) for each $x \in K$.
- So we normalize the generic point η, compute $c.\eta$ and then $R := \vartheta_{j_1}^A(\eta)\vartheta_{j_2}^A(c.\eta) \in \mathfrak{A}$.
- We compute $\sum_{x \in K} R(x_1) = Q(0) \in k$ where Q comes from the euclidean division $XRP_1' = PQ + S$.

Birational invariants for $\mathfrak{H}_g/\mathrm{Sp}_4(\mathbb{Z})$

Definition

- The Igusa invariants are Siegel modular functions j_1, j_2, j_3 for $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$ defined by
$$j_1 := \frac{h_{12}^5}{h_{10}^6}, \quad j_2 := \frac{h_4 h_{12}^3}{h_{10}^4}, \quad j_3 := \frac{h_{16} h_{12}^2}{h_{10}^4}$$
where the h_i are modular forms of weight i given by explicit polynomials in terms of theta constants.
- Invariants derived by Streng are better suited for computations:
$$i_1 := \frac{h_4 h_6}{h_{10}}, \quad i_2 := \frac{h_4^2 h_{12}}{h_{10}^2}, \quad i_3 := \frac{h_4^5}{h_{10}^2}.$$

- The three invariants $j_{i,\ell}(\Omega) = j_i(\ell\Omega)$ encode a principally polarised abelian surface ℓ-isogeneous to $A = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$;
- All others ppav ℓ-isogenous to A comes from the action of $\Gamma/\Gamma_0(\ell)$ on Ω. The index is $\ell^3 + \ell^2 + \ell + 1$.

Modular polynomials in dimension 2

Definition

$$\Phi_{1,\ell}(X, j_1, j_2, j_3) = \prod_{\gamma \in \Gamma/\Gamma_0(\ell)} (X - j_{1,\ell}^{\gamma})$$

$$\Psi_{i,\ell}(X, j_1, j_2, j_3) = \sum_{\gamma \in \Gamma/\Gamma_0(\ell)} j_{i,\ell}^{\gamma} \prod_{\gamma' \in \Gamma/\Gamma_0(\ell)\setminus\{\gamma\}} (X - j_{1,\ell}^{\gamma'}) \quad (i = 2, 3)$$

$$\Phi_{1,\ell}, \Psi_{2,\ell}, \Psi_{3,\ell} \in \mathbb{Q}(j_1, j_2, j_3)[X].$$

- Computed via an evaluation-interpolation approach;
- Evaluation requires evaluating the modular invariants on Ω at high precision;
- Interpolation requires finding Ω from the value of the modular invariants;

⇒ Uses a generalized version of the AGM to compute theta functions in quasi-linear time in the precision [Dup06];

⇒ Need to interpolate rational functions;

- Denominator describes Humbert surface of discriminant ℓ^2 [BL09; Gru10];
- Quasi-linear algorithm [Dup06; Mil14];
- Can be generalized to smaller modular invariants [Mil14].

Example of modular polynomials in dimension 2 [Mil14]

Invariant	ℓ	Size
Igusa	2	57 MB
Streng	2	2.1 MB
Streng	3	890 MB
Theta	3	175 KB
Theta	5	200 MB
Theta	7	29 GB

Example

The denominator of $\Phi_{1,3}$ for modular functions b_1, b_2, b_3 derived from theta constant of level 2 is:

$$1024b_3^6b_2^6b_1^{10} - ((768b_3^8 + 1536b_3^4 - 256)b_3^8 + 1536b_3^8b_3^4 - 256b_3^8)b_1^8 + (1024b_3^6b_2^{10} + (1024b_3^{10} + 2560b_3^6 - 512b_3^2)b_2^6 - (512b_3^6 - 64b_3^2)b_2^2)b_1^6 - (1536b_3^8b_2^8 + (-416b_3^4 + 32)b_2^4 + 32b_3^4)b_1^4 - ((512b_3^6 - 64b_3^2)b_2^6 - 64b_3^6b_2^2)b_1^2 + 256b_3^8b_2^8 - 32b_3^4b_2^4 + 1.$$

Non principal polarisations

- Let $f:(A,H_1)\rightarrow(B,H_2)$ be an isogeny between principally polarised abelian varieties;
- When $\operatorname{Ker} f$ is not maximal isotropic in $A[\ell]$ then f^*H_2 is not of the form ℓH_1;
- How can we go from the principal polarisation H_1 to f^*H_1?

Non principal polarisations

Theorem (Birkenhake-Lange, Th. 5.2.4)

Let A be an abelian variety with a principal polarisation H_1;

- *Let $O_0 = \mathrm{End}(A)^s$ be the real algebra of endomorphisms symmetric under the Rosati involution;*
- *Let $\mathrm{NS}(A)$ be the Néron-Severi group of line bundles modulo algebraic equivalence.*

Then

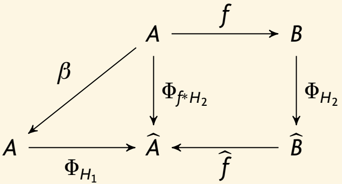
- *$\mathrm{NS}(A)$ is isomorphic to O_0 via*

$$\beta \in O_0 \mapsto H_\beta = \beta H_1 = H_1(\beta \cdot, \cdot);$$

- *This induces a bijection between polarisations of degree d in $\mathrm{NS}(A)$ and totally positive symmetric endomorphisms of norm d in O_0^{++};*
- *The isomorphic class of a polarisation $\mathscr{L}_\beta \in \mathrm{NS}(A)$ for $f \in O_0^{++}$ correspond to the action $\varphi \mapsto \varphi^* \beta \varphi$ of the automorphisms of A.*

Cyclic isogeny

- Let $f:(A,H_1)\rightarrow(B,H_2)$ be an isogeny between principally polarised abelian varieties with cyclic kernel of degree ℓ;
- There exists β such that the following diagram commutes:



- β is an $(\ell,0,\ldots,\ell,0,\ldots)$-isogeny whose kernel is not isotropic for the H_1-Weil pairing on $A[\ell]$!
- β commutes with the Rosatti involution so is a real endomorphism (β is H_1-symmetric). Since H_1 is Hermitian, β is totally positive.
- $\operatorname{Ker} f$ is maximal isotropic for βH_1; conversely if K is a maximal isotropic kernel in $A[\beta]$ then $f:A\rightarrow A/K$ fits in the diagram above.

β-isogenies

Lemma ([Dudeanu, Jetchev, R.])

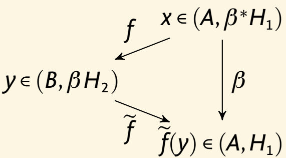
- *Let $(A, \mathscr{L})$ be a ppav and $\beta \in \mathrm{End}(A)^{++}$ be a totally positive real element of degree ℓ. Let $K \subset \mathrm{Ker}\,\beta$ be cyclic of degree ℓ (note that it is automatically isotropic). Then A/K is principally polarised.*
- *Conversely if there is a cyclic isogeny $f: A \to B$ of degree ℓ between ppav then there exists $\beta \in \mathrm{End}(A)^{++}$ such that $\mathrm{Ker} f \subset \mathrm{Ker}\,\beta$.*

Corollary

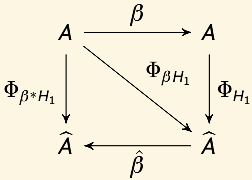
- *If $NS(A) = \mathbb{Z}$ there are no cyclic isogenies to a ppav;*
- *For an ordinary abelian surface, if there is a cyclic isogeny of degree ℓ then ℓ splits into totally positive principal ideals in the real quadratic order which is locally maximal at ℓ. A cyclic isogeny does not change the real multiplication.*

β-change of level

- β-contragredient isogeny $\tilde{f}$:



- Use the isogeny theorem to compute f from $(A, \beta H_1)$ down to (B, H_2) or $\tilde{f}$ from (B, H_2) up to $(A, \beta H_1)$ as before;
- What about changing level between $(A, \beta H_1)$ and (A, H_1)?
- βH_1 fits in the following diagram:



- Applying the isogeny theorem on β allows to find relations between $\beta^* H_1$ and H_1 but we want βH_1.

β-change of level

- β is a totally positive element of a totally positive order O_0;
- A theorem of Siegel show that β is a sum of m squares in $K_0 = O_0 \otimes \mathbb{Q}$;
- Clifford's algebras give a matrix $F \in \mathrm{Mat}_r(K_0)$ such that $\mathrm{diag}(\beta) = F^* F$;
- Use this matrix F to change level as before: If $X \in (\mathbb{C}^g)^r$ and $Y = F(X) \in (\mathbb{C}^g)^r$, $j \in (\mathbb{Q}^g)^r$ and $i = F(j)$, then (up to a modular automorphism)

$$\vartheta\left[\begin{smallmatrix}0\\ i_1\end{smallmatrix}\right](Y_1, \frac{\Omega}{n}) \ldots \vartheta\left[\begin{smallmatrix}0\\ i_r\end{smallmatrix}\right](Y_r, \frac{\Omega}{n}) = \sum_{\substack{t_1,\ldots,t_r \in K(\beta H_1)\\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta\left[\begin{smallmatrix}0\\ j_1\end{smallmatrix}\right](X_1 + t_1, \frac{\beta^{-1}\Omega}{n}) \ldots \vartheta\left[\begin{smallmatrix}0\\ j_r\end{smallmatrix}\right](X_r + t_r, \frac{\beta^{-1}\Omega}{n}),$$

Remark

- In general r can be larger than m;
- The matrix F acts by real endomorphisms rather than by integer multiplication;
- There may be denominators in the coefficients of F.

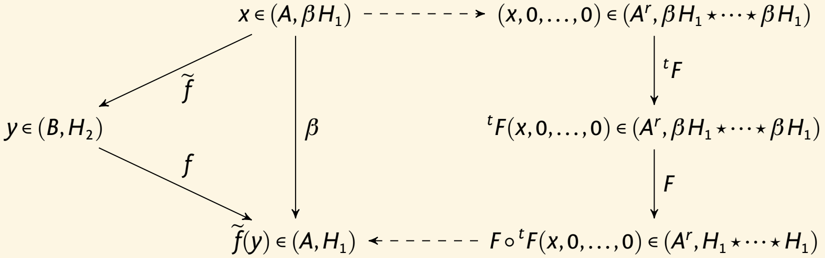
The Algorithm for cyclic isogenies [Dudeanu, Jetchev, R.]

$$B=\mathbb{C}^g/(\mathbb{Z}^g+\Omega\mathbb{Z}^g),\quad A=\mathbb{C}^g/(\mathbb{Z}^g+\beta\Omega\mathbb{Z}^n),\quad \vartheta_b^B:=\vartheta\left[\begin{smallmatrix}0\\ \frac{b}{n}\end{smallmatrix}\right]\left(\cdot,\frac{\Omega}{n}\right),\quad \vartheta_b^A:=\vartheta\left[\begin{smallmatrix}0\\ \frac{b}{n}\end{smallmatrix}\right]\left(\cdot,\frac{\beta\Omega}{n}\right)$$

Theorem

Let Y in $(\mathbb{C}^g)^r$ and $X=YF^{-1}\in(\mathbb{C}^g)^r$. Let $i\in(Z(\overline{n}))^r$ and $j=iF^{-1}$. Up to a modular automorphism:

$$\vartheta_{i_1}^A(Y_1)\dots\vartheta_{i_r}^A(Y_r)=\sum_{\substack{t_1,\dots,t_r\in K(\beta H_2)\\ (t_1,\dots,t_r)F=(0,\dots,0)}}\vartheta_{j_1}^B(X_1+t_1)\dots\vartheta_{j_r}^B(X_r+t_r),$$



Hidden details

- Normalize the coordinates by using multi-way additions;
- The real endomorphisms are codiagonalisables (in the ordinary case), this is important to apply the isogeny theorem;
- If $g = 2$, $K_0 = \mathbb{Q}(\sqrt{d})$, the action of $\sqrt{d}$ is given by a standard (d,d)-isogeny, so we can compute it using the previous algorithm for d-isogenies!
- The important point is that this algorithm is such that we can keep track of the projective factors when computing the action of $\sqrt{d}$.
- Unlike the case of maximal isotropic kernels for the Weil pairing, for cyclic isogenies the Koizumi formula does not yield a product theta structure. We compute the action of the modular automorphism coming from F that gives a product theta structure.

Remark

Computing the action of $\sqrt{d}$ directly may be expensive if d is big. If possible we replace it with Frobeniuses.

Cyclic modular polynomials in dimension 2 [Milio-R.]

- Given $\beta \in O_{K_0}$ one can define the β-modular polynomial in terms of symmetric invariants of the Hilbert space $\mathfrak{H}_1^g / \mathrm{Sl}_2(O_{K_0})$;
- If $D = 2$ or $D = 5$ the symmetric Hilbert moduli space is rational and parametrized by two invariants: the Gundlach invariants;
- Use an evaluation-interpolation approach via the action of $\mathrm{Sl}_2(O_{K_0})/\Gamma_0(\beta_i)$ (by symmetry, to get a rational polynomial we may need to take the product of the polynomial computed via the action of β_1 and the one obtained via the action of β_2);
- Evaluation and interpolation done by computing the explicit maps back to Siegel;
- For general D the Hilbert space is not unirational $\Rightarrow$ we need to interpolate three invariants (the pull back of the Igusa invariants or the level 2 theta constant);
- There is now a relation between the invariants we interpolate, so we need to fix a Gröbner basis for unicity;
- The modular polynomials are much smaller: the total degree is $\ell + 1$ or $2(\ell + 1)$ once the invariants are plugged in;
- Need a precomputation for each K_0 (the equation of the Humbert surface [Gru10]).

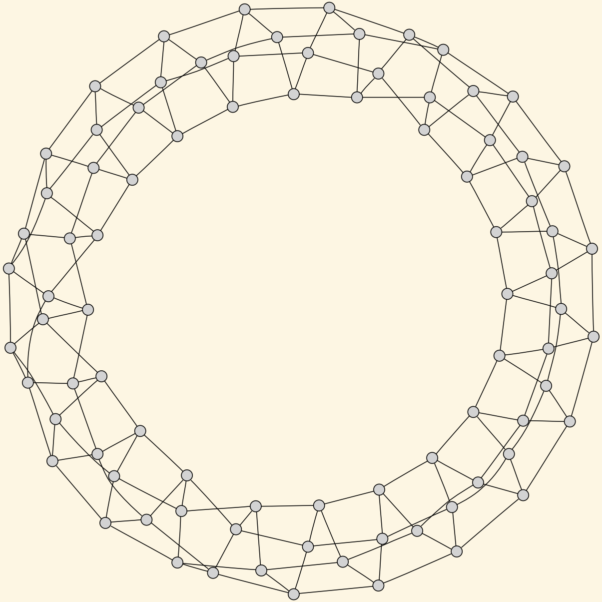
Example of cyclic modular polynomials in dimension 2 [Milio-R.]

ℓ ($D=2$)	Size (Gundlach)	Theta	ℓ ($D=5$)	Size (Gundlach)	Theta
2	8.5KB		5	22KB	45KB
7	172KB		11	3.5MB	308KB
17	5.8MB	221KB	19	33MB	3.6MB
23	21 MB		29	188MB	
31	70 MB		31	248 MB	
41	225 MB	7.2MB			

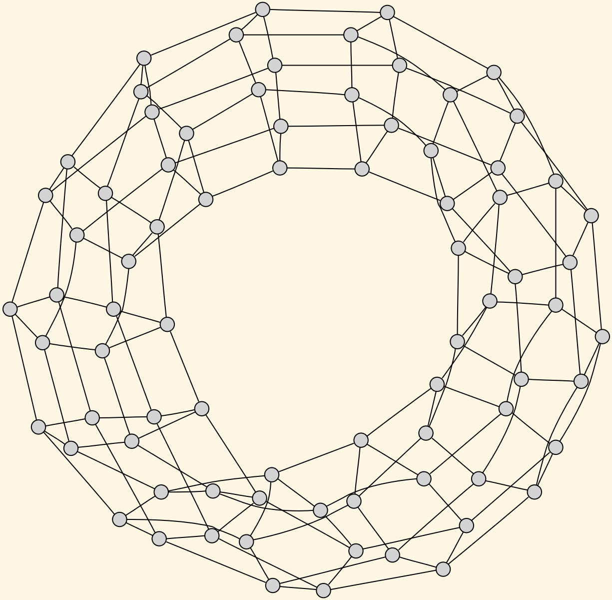
Example

For $D=2$, $\beta=5+2\sqrt{2} \mid 17$, using b_1, b_2, b_3 pullback of level 2 theta functions on the Hilbert space, the denominator of $\Phi_{1,\beta}$ is $b_3^6 b_2^{18} + (6b_3^8 6b_3^4 + 1)b_2^{16} + (15b_3^{10} 24b_3^6 + 7b_3^2)b_2^{14} + (20b_3^{12} 42b_3^8 + 9b_3^4 + 2)b_2^{12} + (15b_3^{14} 48b_3^{10} + 37b_3^6 + 4b_3^2)b_2^{10} + (6b_3^{16} 42b_3^{12} + 68b_3^8 26b_3^4 + 3)b_2^8 + (b_3^{18} 24b_3^{14} + 37b_3^{10} + 8b_3^6 b_3^2)b_2^6 + (6b_3^{16} + 9b_3^{12} 26b_3^8 24b_3^4 + 2)b_2^4 + (7b_3^{14} + 4b_3^{10} b_3^6)b_2^2 + (b_3^{16} + 2b_3^{12} + 3b_3^8 + 2b_3^4 + 1)$.

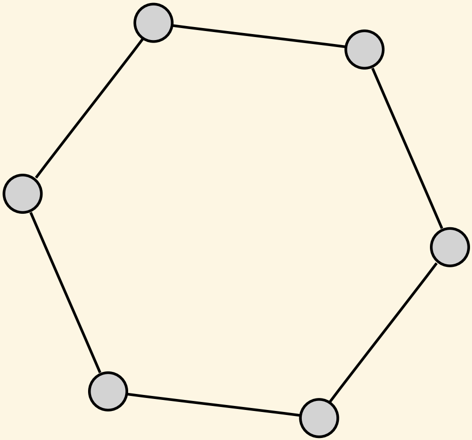
Horizontal isogeny graphs: $\ell = q_1 q_2 = Q_1\overline{Q_1}Q_2\overline{Q_2}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



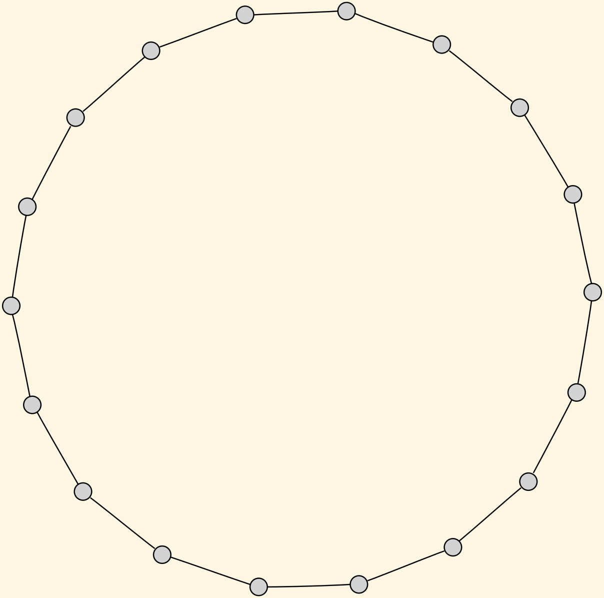
Horizontal isogeny graphs: $\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2 \overline{Q_2}$ ($\mathbb{Q} \mapsto K_0 \mapsto K$)



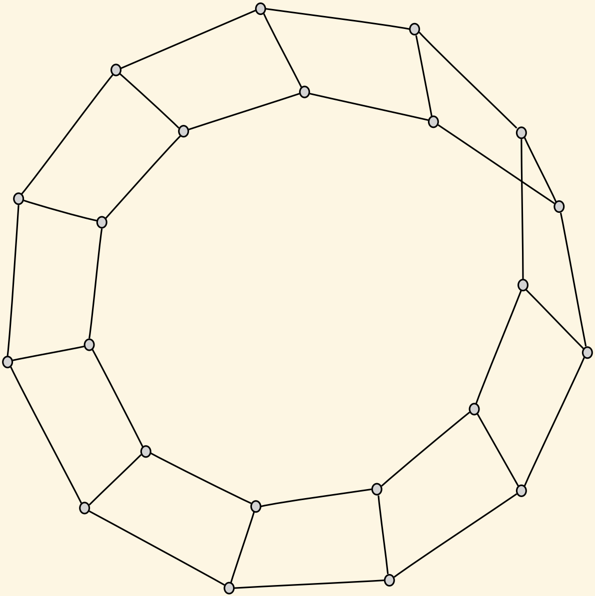
Horizontal isogeny graphs: $\ell = q = Q\overline{Q}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



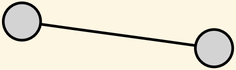
Horizontal isogeny graphs: $\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2^2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



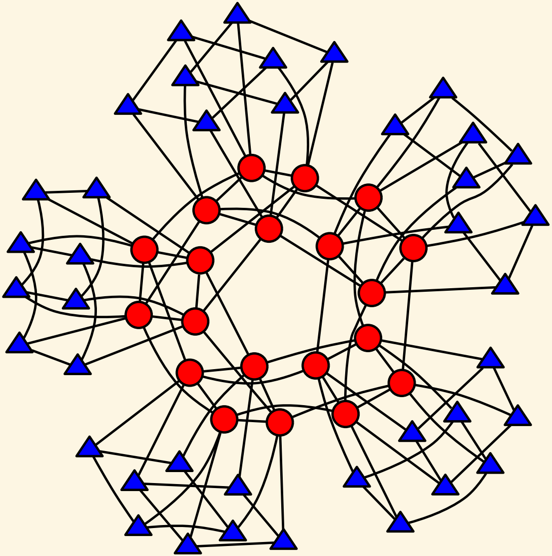
Horizontal isogeny graphs: $\ell = q^2 = Q^2\overline{Q}^2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$

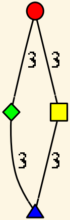


Horizontal isogeny graphs: $\ell = q^2 = Q^4$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$

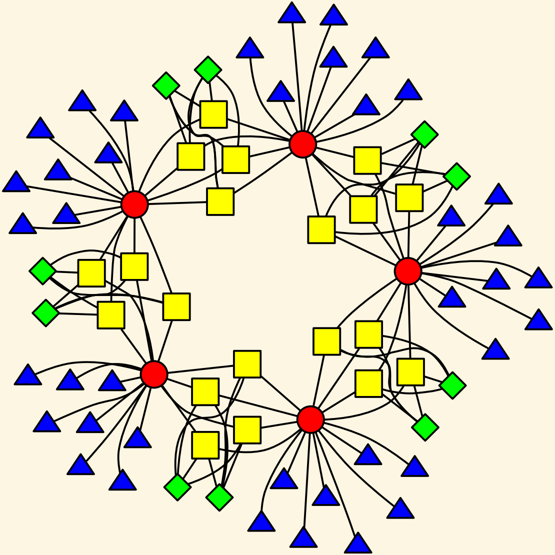


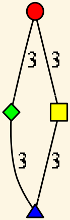
Isogeny graphs in dimension 2 ($\ell = q_1 q_2 = Q_1\overline{Q_1}Q_2\overline{Q_2}$)



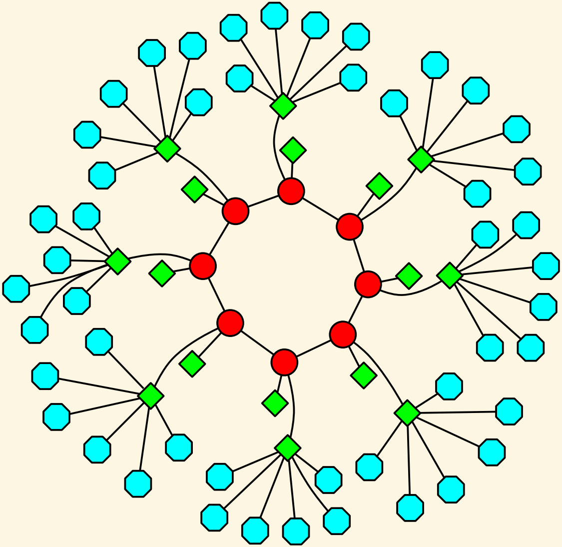


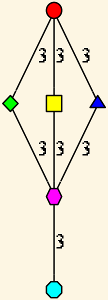
Isogeny graphs in dimension 2 ($\ell = q = Q\overline{Q}$)



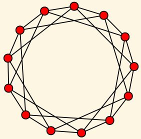


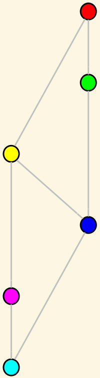
Isogeny graphs in dimension 2 ($\ell = q = Q\overline{Q}$)





Isogeny graphs and lattice of orders [Bisson, Cosset, R.]





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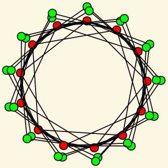

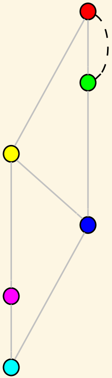

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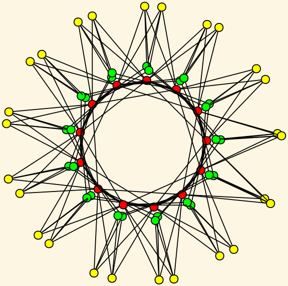

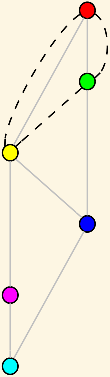

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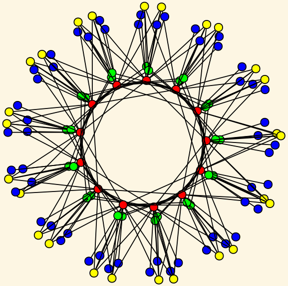

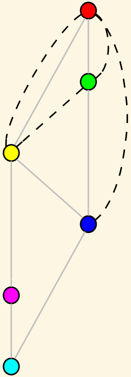

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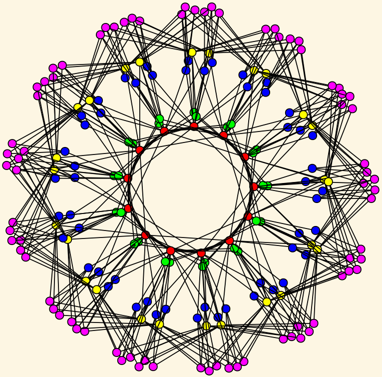

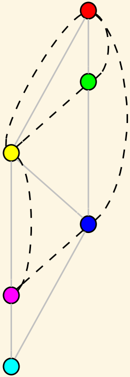

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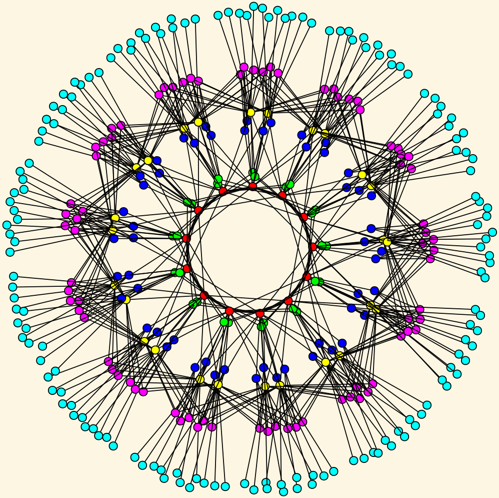
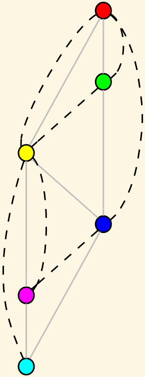




Abelian varieties with real and complex multiplication

- Let K be a CM field (a totally imaginary quadratic extension of a totally real field K_0 of dimension g);
- An abelian variety with RM by K_0 is of the form $\mathbb{C}^g/(\Lambda_1 \oplus \Lambda_2 \tau)$ where Λ_i is a lattice in K_0, K_0 is embedded into $\mathbb{C}^g$ via $K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^g \subset \mathbb{C}^g$, and $\tau \in \mathfrak{H}_1^g$;
- Furthermore the polarisations are of the form

$$H(z_1, z_2) = \sum_{\varphi_i : K \to \mathbb{C}} \varphi_i(\lambda z_1 \overline{z_2}) / \Im \tau_i$$

  for a totally positive element $\lambda \in K_0^{++}$. In other words if $x_i, y_i \in K_0$, then $E(x_1 + y_1\tau, x_2 + y_2\tau) = \mathrm{Tr}_{K_0/\mathbb{Q}}(\lambda(x_2 y_1 - x_1 y_2))$.
- An abelian variety with CM by K is of the form $\mathbb{C}^g/\Phi(\Lambda)$ where Λ is a lattice in K and Φ is a CM-type.
- Furthermore, the polarisations are of the form

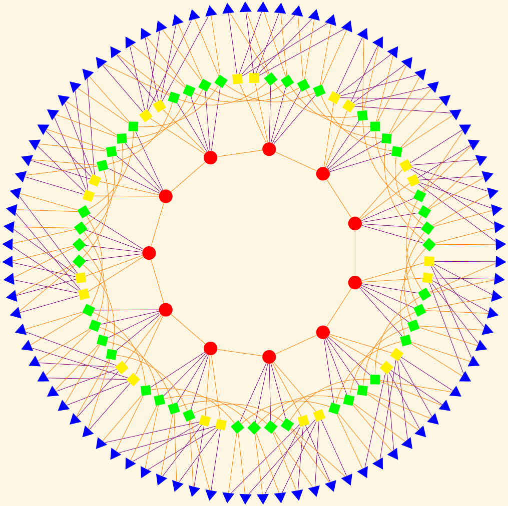
$$E(z_1, z_2) = \mathrm{Tr}_{K/Q}(\xi z_1 \overline{z_2})$$

  for a totally imaginary element $\xi \in K$. The polarisation is principal iff $\xi \overline{\Lambda} = \Lambda^\star$ where $\Lambda^\star$ is the dual of Λ for the trace.

Cyclic isogeny graph in dimension 2 [IT14]

- Let A be a principally polarised abelian surface over $\mathbb{F}_q$ with CM by $O \subset O_K$ and RM by $O_0 \subset O_{K_0}$;
- Assume that O_0 is maximal (locally at ℓ) and that we are in the split case: $(\ell) = (\beta_1)(\beta_2)$ in O_0 (where β_i is totally positive). Then $A[\ell] = A[\beta_1] \oplus A[\beta_2]$.
- There are two kind of cyclic isogenies: β_1-isogenies ($K \subset A[\beta_1]$) and β_2-isogenies.
- Looking at β_1 isogenies, we recover the structure of a volcano: $O = O_0 + \mathfrak{f} O_K$ for a certain O_0-ideal $\mathfrak{f}$ such that the conductor of O is $\mathfrak{f} O_K$.
  - If $\mathfrak{f}$ is prime to β_1, there are 2, 1, or 0 horizontal-isogenies according to whether β_1 splits, is ramified or is inert in O, and the rest are descending to $O_0 + \mathfrak{f}\beta_1 O_K$;
  - If $\mathfrak{f}$ is not prime to β_1 there is one ascending isogeny (to $O_0 + \mathfrak{f}/\beta_1 O_K$) and ℓ descending ones;
  - We are at the bottom when the β_1-valuation of $\mathfrak{f}$ is equal to the valuation of the conductor of $\mathbb{Z}[\pi, \overline{\pi}]$.
- ℓ-isogenies preserving O_0 are a composition of a β_1-isogeny with a β_2-isogeny.

Cyclic isogeny graph in dimension 2 [IT14]



$[A, B] = [81, 1181]$, $p = 211$, $\ell = 3$

Changing the real multiplication: moving between pancakes

Cyclic isogenies (that preserve principal polarisations) preserve real multiplication; so we need to look at ℓ-isogenies.

Proposition

- *Let O_ℓ be the order of conductor ℓ inside O_{K_0}. ℓ-isogenies going from O_ℓ to O_{K_0} are of the form*

$$\mathbb{C}^g/(O_\ell \oplus O_\ell^\vee \tau) \to \mathbb{C}^g/(O_{K_0} \oplus O_{K_0}^\vee \tau).$$

- $\mathrm{Sl}_2(O_{K_0} \oplus O_{K_0}^\vee)/\mathrm{Sl}_2(O_\ell \oplus O_\ell^\vee)$ *acts on such isogenies;*
- *When ℓ splits in O_{K_0},* $\mathrm{Sl}_2(O_{K_0} \oplus O_{K_0}^\vee)/\mathrm{Sl}_2(O_\ell \oplus O_\ell^\vee) \simeq \mathrm{Sl}_2(O_{K_0}/\ell O_{K_0})/\mathrm{Sl}_2(O_\ell/\ell O_\ell) \simeq \mathrm{SL}_2(\mathbb{F}_l^2)/\mathrm{Sl}_2(\mathbb{F}_l) \simeq \mathrm{Sl}_2(\mathbb{F}_l)$*, so we find $\ell^3 - \ell$ ℓ-isogenies changing the real multiplication.*
- *On the other end there is $(\ell+1)^2$ ℓ-isogenies preserving the real multiplication*
- *In total we find all $\ell^3 + \ell^2 + \ell + 1$ ℓ-isogenies.*

Changing the real multiplication: moving between pancakes

Corollary ([Ionica, Martindale, R., Streng])

If O is maximal at ℓ,

- *If ℓ is split there are $\ell^2+2\ell+1$ RM-horizontal ℓ-isogenies and $\ell^3-\ell$ RM-descending ℓ-isogenies;*
- *If ℓ is inert there are ℓ^2+1 RM-horizontal ℓ-isogenies and $\ell^3+\ell$ RM-descending ℓ-isogenies;*
- *If ℓ is ramified there are $\ell^2+\ell+1$ RM-horizontal ℓ-isogenies and ℓ^3 RM-descending ℓ-isogenies;*

If O is not maximal at ℓ, there are 1 RM-ascending ℓ-isogeny, $\ell^2+\ell$ RM-horizontal ℓ-isogenies and ℓ^3 RM-descending ℓ-isogenies.

AVIsogenies [Bisson, Cosset, R.]

- AVIsogenies: Magma code written by Bisson, Cosset and R.
  http://avisogenies.gforge.inria.fr
- Released under LGPL 2+.
- Implement isogeny computation (and applications thereof) for abelian varieties using theta functions.
- Current release 0.6.
- Cyclic isogenies coming "soon"!

Higher dimension

- Abelian surfaces with maximal real multiplication are very similar to elliptic curves;
- But their moduli space is two compared to one, more choice of parameters;
- ☺ Explicit isogeny computations in term of theta functions work for any dimension;
- ☹ But the number of coordinates is exponential in g;
- For a Jacobian need to convert between the divisors on the curve and the theta functions;
- For modular polynomials no good modular invariants for $g \geqslant 3$ (lot of secondary invariants: 36 even theta functions for a space of dimension 6);
- In dimension 2 the real orders are Gorenstein rings, this simplify the description of the isogeny graph.

Non principally polarised abelian varieties

- Why focus on principally polarised abelian varieties?
- In dimension 2 and 3 to recover the underlying curve;
- In general starting from a ppav A given by level n theta functions and a cyclic kernel K of order ℓ, we could compute theta functions of level $(n, n, \ldots, n\ell)$ on A/K.
- We could iterate and follow an isogeny trail and get polarisations of level $(n, n, \ldots, n\ell^m)$;
- But without adequate real multiplication, there is no way to descend the level of the polarisation.

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