Isogenies, Polarisations and Real Multiplication

2015/10/06 — **Journées C2** — La Londe-Les-Maures

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Outline

Isogenies between elliptic curves

Definition

An isogeny is a (non trivial) algebraic map $f: E_1 \to E_2$ between two elliptic curves such that $f(P+Q) = f(P) + f(Q)$ for all geometric points $P, Q \in E_1$.

Theorem

An algebraic map $f: E_1 \to E_2$ is an isogeny if and only if $f(0_{E_1}) = 0_{E_2}$

Corollary

An algebraic map between two elliptic curves is either

- *trivial (i.e. constant)*
- *or the composition of a translation with an isogeny.*

Remark

- Isogenies are surjective (on the geometric points). In particular, if E is ordinary, any curve isogenous to E is also ordinary.
- Two elliptic curves over $\mathbb{F}_q$ are isogenous if and only if they have the same number of points (Tate).

Algorithmic aspect of isogenies

- Given a kernel $K \subset E(\overline{k})$ compute the isogenous elliptic curve E/K ;
- Given a kernel $K \subset E(\overline{k})$ and $P \in E(k)$ compute the image of P under the isogeny $E \to E/K$;
- Given a kernel $K \subset E(\overline{k})$ compute the map $E \to E/K$;
- Given an elliptic curve E/k compute all isogenous (of a certain degree d) elliptic curves E' ;
- Given two elliptic curves E_1 and E_2 check if they are d-isogenous and if so compute the kernel $K \subset E_1(\overline{k})$.

Algorithmic aspect of isogenies

- Given a kernel $K \subset E(\overline{k})$ compute the isogenous elliptic curve E/K (Vélu's formulae [Vél71]);
- Given a kernel $K \subset E(\overline{k})$ and $P \in E(k)$ compute the image of P under the isogeny $E \to E/K$ (Vélu's formulae [Vél71]);
- Given a kernel $K \subset E(\overline{k})$ compute the map $E \to E/K$ (formal version of Vélu's formulae [Koh96]);
- Given an elliptic curve E/k compute all isogenous (of a certain degree d) elliptic curves E' (Modular polynomial [Eng09; BLS12]);
- Given two elliptic curves E_1 and E_2 check if they are d-isogenous and if so compute the kernel $K \subset E_1(\overline{k})$ (Elkie's method via a differential equation [Elk92; Bos+08]).

$\Rightarrow$ We have quasi-linear algorithms for all these aspects of isogeny computation over elliptic curves.

Destructive cryptographic applications

- An isogeny $f : E_1 \to E_2$ transports the DLP from E_1 to E_2. This can be used to attack the DLP on E_1 if there is a weak curve on its isogeny class (and an efficient way to compute an isogeny to it).

Example

- Extend attacks using Weil descent [GHS02]
- Transfert the DLP from the Jacobian of an hyperelliptic curve of genus 3 to the Jacobian of a quartic curve [Smi09].

Constructive cryptographic applications

- One can recover informations on the elliptic curve E modulo ℓ by working over the ℓ-torsion.
- But by computing isogenies, one can work over a cyclic subgroup of cardinal ℓ instead.
- Since thus a subgroup is of degree ℓ, whereas the full ℓ-torsion is of degree ℓ^2, we can work faster over it.

Example

- The SEA point counting algorithm [Sch95; Mor95; Elk97];
- The CRT algorithms to compute class polynomials [Sut11; ES10];
- The CRT algorithms to compute modular polynomials [BLS12].

Further applications of isogenies

- Splitting the multiplication using isogenies can improve the arithmetic [DIK06; Gau07];
- The isogeny graph of a supersingular elliptic curve can be used to construct secure hash functions [CLG09];
- Construct public key cryptosystems by hiding vulnerable curves by an isogeny (the trapdoor) [Tes06], or by encoding informations in the isogeny graph [RS06];
- Take isogenies to reduce the impact of side channel attacks [Sma03];
- Construct a normal basis of a finite field [CL09];
- Improve the discrete logarithm in $\mathbb{F}_q^*$ by finding a smoothness basis invariant by automorphisms [CL08].

Computing explicit isogenies

- If E_1 and E_2 are two elliptic curves given by short Weierstrass equations $y^2 = x^3 + a_i x + b_i$ an isogeny $f : E_1 \to E_2$ is of the form

$$f(x,y) = (R_1(x), yR_2(x))$$

where R_1 and R_2 are rational functions. (Exercice: $f(0_{E_1}) = 0_{E_2}$; what does this implies on the degrees of R_1 and R_2?)

- Let $w_E = dx/2y$ be the canonical differential. Then $f^* w_{E'} = c w_E$, with c in k so

$$f(x,y) = \left(\frac{g(x)}{h(x)}, cy \left(\frac{g(x)}{h(x)} \right)' \right),$$

where $h(x) = \prod_{P \in \operatorname{Ker} f \setminus \{0_E\}} (x - x_P)$.

Theorem ([Vél71])

Given the equation h of the kernel $\operatorname{Ker} f$, Vélu's formula can compute the isogeny f in time linear in $\deg f$.

Modular polynomials

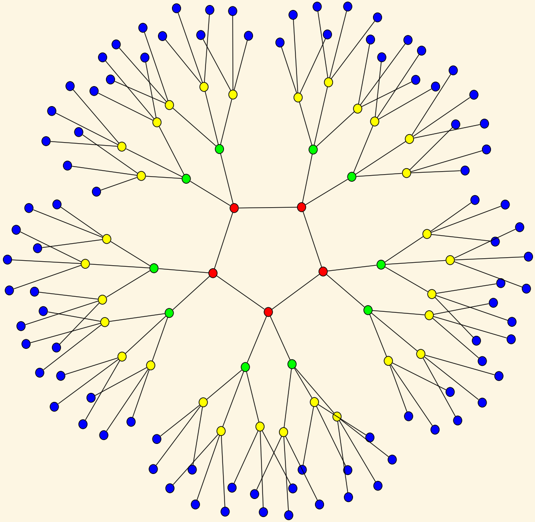
Here $k = \overline{k}$.

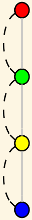
Definition (Modular polynomial)

The modular polynomial $\varphi_\ell(x,y) \in \mathbb{Z}[x,y]$ is a bivariate polynomial such that $\varphi_\ell(x,y) = 0 \Leftrightarrow x = j(E_1)$ and $y = j(E_2)$ with E_1 and E_2 ℓ-isogeneous.

- Roots of $\varphi_\ell(j(E_1),.) \Leftrightarrow$ elliptic curves ℓ-isogeneous to E_1. There are $\ell+1 = \#\mathbb{P}^1(\mathbb{F}_\ell)$ such roots if ℓ is prime.
- φ_ℓ is symmetric;
- The height of φ_ℓ grows as $\widetilde{O}(\ell)$;
- φ_ℓ has total size $\widetilde{O}(\ell^3)$.

A 3-isogeny graph in dimension 1 [Koh96; FM02]





Find elliptic curves with a prescribed number of points

- Let $E/\mathbb{F}_q$ be an ordinary elliptic curve, $\chi_\pi = X^2 - tX + q$ the characteristic polynomial of the Frobenius π;
- $\#E(\mathbb{F}_q) = 1 - t + q$.
- $\Delta_\pi = t^2 - 4q < 0$ (since $t \leqslant 2\sqrt{q}$ by Hasse) so $\mathrm{End}(E) \supset \mathbb{Z}[\pi]$ is an order in $K = \mathbb{Q}(\sqrt{\Delta_\pi})$ a quadratic imaginary field;
- Write $\Delta_\pi = \Delta_0 f^2$, where Δ is the discriminant of K, then f is the conductor of $\mathbb{Z}[\pi] \subset O_K$.

- Conversely fix N in the Hasse-Weil interval, and let $t = 1 + q - N$ and O_K be the maximal order in $\mathbb{Q}(\sqrt{\Delta_\pi})$;
- If $E/\mathbb{F}_q$ has endomorphism ring O_K (or an order in K containing $\mathbb{Z}[\pi]$), then $\#E(\mathbb{F}_q) = N$.

Complex Multiplication

Theorem (Fondamental theorem of Complex Multiplication)

Let K be a quadratic imaginary field, $E/\mathbb{C}$ an elliptic curve with $\mathrm{End}(E) = O_K$.

- *$j(E)$ is algebraic and $K(j(E))$ is the Hilbert class field $\mathfrak{H}_K$ of K (the maximal unramified abelian extension of K).*
- *The minimal polynomial of $j(E)$ is*

$$H_K(X) = \prod_{\sigma \in \mathrm{Gal}(\mathfrak{H}_K/K) \simeq \mathrm{Cl}(K)} (X - \sigma(j(E))) = \prod_{E_i/\mathbb{C} | \mathrm{End}(E_i) = O_K} (X - j(E_i)) \in \mathbb{Z}[X]$$

  where for $\sigma = [I] \in \mathrm{Gal}(\mathfrak{H}_K/K) \simeq \mathrm{Cl}(K)$, $\sigma(j(E)) = j(E/E[I])$;
- *If $p = \mathfrak{p}_1\mathfrak{p}_2$ splits in K, and $\mathfrak{P}$ is a prime above p in $\mathfrak{H}_K$ then E has good reduction at p and $E_{\mathbb{F}_{\mathfrak{P}}}$ is an ordinary elliptic curve over $\mathbb{F}_{\mathfrak{P}}$. The extension $\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_p$ has degree the order of $[\mathfrak{p}_i] \in \mathrm{Cl}(O_K)$ and $\mathrm{End}(E_{\mathbb{F}_{\mathfrak{P}}}) = O_K$*
- *In particular if p splits completely in $\mathfrak{H}_K$ (or equivalently if $\mathfrak{p}_i$ is principal), then H_K splits over $\mathbb{F}_p$:*

$$H_K \equiv \prod_{E/\mathbb{F}_p | \mathrm{End}(E) = O_K} (X - j(E)) \mod p.$$

The CRT method to compute the class polynomial H_K

1. Find p completely split in $\mathfrak{H}_K$;
2. Find all $\#\mathrm{Cl}(K)$ elliptic curves E over $\mathbb{F}_p$ with $\mathrm{End}(E) = O_K$;
3. Recover $H_K \mod p = \prod_{E/\mathbb{F}_p \mid \mathrm{End}(E) = O_K} (X - j(E))$;
4. Iterate the process for several primes p_i and use the CRT to recover H_K from $H_K \mod p_i$.

Theorem ([Bel+08; Sut11])

Using isogenies in Step 3 to

- *Compute* $\mathrm{End}(E)$ *for a random* $E/\mathbb{F}_p$*;*
- *Go up in the volcano once a curve* E *in the right isogeny class is found;*
- *Once a curve* $E/\mathbb{F}_p$ *is found with* $\mathrm{End}(E) = O_K$ *compute all the others directly from the action of* $\mathrm{Cl}(K)$*;*

yields a quasi-linear algorithm.

Computing $\mathrm{End}(E)$ and going up in the volcano [Koh96; FM02]

- If $E/\mathbb{F}_q$ is ordinary, $\#E(\mathbb{F}_q)$ gives π and so $\mathbb{Z}[\pi] \subset \mathrm{End}(E) \subset O_K$;
- It remains to compute the conductor f of $\mathrm{End}(E)$;
- It suffices to compute $v_\ell(f)$ for ℓ dividing the conductor f_π of $\mathbb{Z}[\pi]$;
- In the ℓ-isogeny graph, following three paths allows to determine the height we are on, and from it the valuation $v_\ell(f)$.
- A similar method is used to go up in the volcano.

Polarised abelian varieties over $\mathbb{C}$

Definition

A complex abelian variety A of dimension g is isomorphic to a compact Lie group V/Λ with

- A complex vector space V of dimension g;
- A $\mathbb{Z}$-lattice Λ in V (of rank $2g$);
- An Hermitian form H on V with $E(\Lambda,\Lambda) \subset \mathbb{Z}$ where $E = \operatorname{Im} H$ is symplectic.

- Such an Hermitian form H is called a polarisation on A. Conversely, any symplectic form E on V such that $E(\Lambda,\Lambda) \subset \mathbb{Z}$ and $E(ix,iy) = E(x,y)$ for all $x,y \in V$ gives a polarisation H with $E = \operatorname{Im} H$.
- Over a symplectic basis of Λ, E is of the form.

$$\begin{pmatrix} 0 & D_\delta \\ -D_\delta & 0 \end{pmatrix}$$

  where D_δ is a diagonal positive integer matrix $\delta = (\delta_1, \delta_2, \ldots, \delta_g)$ and $\delta_1 \mid \delta_2 | \cdots \mid \delta_g$.
- $\deg H = \prod \delta_i$; H is a principal polarisation if $\deg H = 1$.

Principal polarisations

- If A is principally polarised, $A = \mathbb{C}^g/(\Omega\mathbb{Z}^g \oplus \mathbb{Z}^g)$ where the matrix Ω is in $\mathfrak{H}_g$, the Siegel space of symmetric matrices Ω with $\operatorname{Im}\Omega$ positive definite;
- The principal polarisation H is given by the matrix $(\operatorname{Im}\Omega)^{-1}$.
- The choice of a symplectic basis gives an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Omega = (a\Omega + b)(c\Omega + d)^{-1};$$

- The moduli space of principally polarised abelian varieties is isomorphic to $\mathfrak{H}_g/\mathrm{Sp}_{2g}(\mathbb{Z})$ and has dimension $g(g+1)/2$.

Examples

- In dimension 1 all abelian varieties are principally polarised and are exactly the elliptic curves;
- In dimension 2 the absolutely simple principally polarised abelian surfaces are a Jacobian of an hyperelliptic curve of genus 2;
- In dimension 3 the absolutely simple principally polarised abelian threefold are a Jacobian of a curve of genus 3.

Isogenies

Let $A = V/\Lambda$ and $B = V'/\Lambda'$.

Definition

An isogeny $f : A \to B$ is a bijective linear map $f : V \to V'$ such that $f(\Lambda) \subset \Lambda'$. The kernel of the isogeny is $f^{-1}(\Lambda')/\Lambda \subset A$ and its degree is the cardinal of the kernel.

- Two abelian varieties over a finite field are isogenous iff they have the same zeta function (Tate);
- A morphism of abelian varieties $f : A \to B$ (seen as varieties) is a group morphism iff $f(0_A) = 0_B$.

The dual abelian variety

Definition

If $A=V/\Lambda$ is an abelian variety, its dual is $\widehat{A}=\mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})/\Lambda^*$. Here $\mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$ is the space of anti-linear forms and $\Lambda^*=\{f \mid f(\Lambda)\subset\mathbb{Z}\}$ is the orthogonal of Λ.

- If H is a polarisation on A, its dual H^* is a polarisation on $\widehat{A}$. Moreover, there is an isogeny $\Phi_H : A\to\widehat{A}$:
$$x\mapsto H(x,\cdot)$$
of degree $\deg H$. We note $K(H)$ its kernel.
- If $f:A\to B$ is an isogeny, then its dual is an isogeny $\widehat{f}:\widehat{B}\to\widehat{A}$ of the same degree.

Remark

The canonical pairing $A\times\widehat{A}\to\mathbb{C}, (x,f)\mapsto f(x)$ induces a canonical principal polarisation on $A\times\widehat{A}$, the Poincaré bundle:
$$E_P((x_1,f_1),(x_2,f_2))=f_1(x_2)-f_2(x_1).$$
The pullback $(\mathrm{Id},\varphi_H)^*E_P=2E$.

Isogenies and polarisations

Definition

- An isogeny $f:(A,H_1)\to(B,H_2)$ between polarised abelian varieties is an isogeny such that
$$f^*H_2 := H_2(f(\cdot),f(\cdot)) = H_1.$$
- f is an ℓ-isogeny between principally polarised abelian varieties if H_1 and H_2 are principal and $f^*H_2=\ell H_1$.

An isogeny $f:(A,H_1)\to(B,H_2)$ respect the polarisations iff the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Phi_{H_1}"] & B \arrow[d, "\Phi_{H_2}"] \\ \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"] \end{tikzcd}$$

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$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[dl, "{[\ell]}"'] \arrow[d, "\Phi_{\ell H_1}"] & B \arrow[d, "\Phi_{H_2}"] \\
A \arrow[r, "\Phi_{H_1}"'] & \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"]
\end{tikzcd}$$

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Proposition

If $K\subset A(\overline{k})$, H_1 descends to a polarisation H_2 on A/K (ie $f^*H_2 = H_1$) if and only if $\operatorname{Im} H_1(K+\Lambda_1, K+\Lambda_1)\subset\mathbb{Z}$ *iff K is isotropic for the E_1-pairing. The degree of H_2 is then* $\deg H_1/\deg f^2$.

Example

Let $\Lambda_1 = \Omega_1\mathbb{Z}^g + \mathbb{Z}^g$, $H_1 = \ell(\operatorname{Im}\Omega_1)^{-1}$, then A/K is principally polarised ($A/K = \mathbb{C}^g/(\Omega_2\mathbb{Z}^g+\mathbb{Z}^g)$) if $K = \frac{1}{\ell}\mathbb{Z}^g$ or $K = \frac{1}{\ell}\Omega\mathbb{Z}^g$.

Theta functions

- Let (A, H_0) be a principally polarised abelian variety over $\mathbb{C}$;
- $A = \mathbb{C}^g / (\Omega\mathbb{Z}^g + \mathbb{Z}^g)$ with $\Omega \in \mathfrak{H}_g$ and $H_0 = (\Im\Omega)^{-1}$.
- All automorphic forms corresponding to a multiple of H_0 come from the theta functions with characteristics:

$$\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \,{}^t(n+a)\Omega(n+a) + 2\pi i \,{}^t(n+a)(z+b)} \qquad a, b \in \mathbb{Q}^g$$

- Automorphic property:

$$\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z + m_1\Omega + m_2, \Omega) = e^{2\pi i({}^t a \cdot m_2 - {}^t b \cdot m_1) - \pi i \,{}^t m_1 \Omega m_1 - 2\pi i \,{}^t m_1 \cdot z} \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \Omega).$$

- Define $\vartheta_i = \vartheta\left[\begin{smallmatrix} 0 \\ \frac{i}{n} \end{smallmatrix}\right](., \frac{\Omega}{n})$ for $i \in Z(\overline{n}) = \mathbb{Z}^g / n\mathbb{Z}^g$
- $(\vartheta_i)_{i \in Z(\overline{n})} = \begin{cases} \text{coordinates system} & n \geqslant 3 \\ \text{coordinates on the Kummer variety } A/\pm 1 & n = 2 \end{cases}$

Computing isogenies in dimension 2

- Richelot formuluae [Ric36; Ric37] allows to compute 2-isogenies between Jacobians of hyperelliptic curves of genus 2 (ie maximal isotropic kernels in $A[2]$);
- The duplication formulae for theta functions

$$\vartheta\left[\begin{smallmatrix}\chi\\0\end{smallmatrix}\right](0, 2\frac{\Omega}{n})^2 = \frac{1}{2^g} \sum_{t \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g} e^{-2i\pi 2\, {}^t\chi \cdot t} \vartheta\left[\begin{smallmatrix}0\\t\end{smallmatrix}\right](0, \frac{\Omega}{n})^2$$

$$\vartheta\left[\begin{smallmatrix}0\\i/2\end{smallmatrix}\right](0, 2\Omega)^2 = \frac{1}{2^g} \sum_{i_1+i_2=0 \pmod 2} \vartheta\left[\begin{smallmatrix}0\\i_1/2\end{smallmatrix}\right](0,\Omega)\vartheta\left[\begin{smallmatrix}0\\i_2/2\end{smallmatrix}\right](0,\Omega) \quad \text{(for all } \chi \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g\text{);}$$

  allows to generalize Richelot formulae to any dimension;
- Dupont compute modular polynomials of level 2 in [Dup06] and started the computation of modular polynomials of level 3.
- Low degree formulae [DL08] effective for $\ell = 3$ and made explicit in [Smi12];
- Via constructing functions on the Jacobian from functions on the curve [CE14].

The isogeny formula

$$\ell \wedge n = 1, \quad A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g), \quad B = \mathbb{C}^g/(\mathbb{Z}^g + \ell\Omega\mathbb{Z}^g)$$

$$\vartheta_b^A := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\Omega}{n}\right), \quad \vartheta_b^B := \vartheta\begin{bmatrix} 0 \\ \frac{b}{n} \end{bmatrix}\left(\cdot, \frac{\ell\Omega}{n}\right)$$

Theorem ([CR14; LR15])

Let F be a matrix of rank r such that ${}^tFF = \ell \,\mathrm{Id}_r$, $X = (\ell x, 0, \ldots, 0)$ in $(\mathbb{C}^g)^r$ and $Y = YF^{-1} = (x, 0, \ldots, 0)F^T \in (\mathbb{C}^g)^r$, $i \in (Z(\overline{n}))^r$ and $j = iF^{-1}$.

$$\vartheta_{i_1}^A(\ell z)\ldots\vartheta_{i_r}^A(0) = \sum_{\substack{t_1,\ldots,t_r \in \frac{1}{\ell}\mathbb{Z}^g/\mathbb{Z}^g \\ F(t_1,\ldots,t_r)=(0,\ldots,0)}} \vartheta_{j_1}^B(Y_1+t_1)\ldots\vartheta_{j_r}^B(Y_r+t_r),$$

This can be computed given only the equations (in a suitable form) of the kernel K. When K is rational, the complexity is $\widetilde{O}(\ell^g)$ or $\widetilde{O}(\ell^{2g})$ operations in $\mathbb{F}_q$ according to whether $\ell \equiv 1$ or 3 modulo 4.

- “Record” isogeny computation: $\ell = 1321$.

Birational invariants for $\mathfrak{H}_g/\mathrm{Sp}_4(\mathbb{Z})$

Definition

- The Igusa invariants are Siegel modular functions j_1, j_2, j_3 for $\Gamma = \mathrm{Sp}_4(\mathbb{Z})$ defined by

$$j_1 := \frac{h_{12}^5}{h_{10}^6}, \quad j_2 := \frac{h_4 h_{12}^3}{h_{10}^4}, \quad j_3 := \frac{h_{16} h_{12}^2}{h_{10}^4}$$

  where the h_i are modular forms of weight i given by explicit polynomials in terms of theta constants.

- Invariants derived by Streng are better suited for computations:

$$i_1 := \frac{h_4 h_6}{h_{10}}, \quad i_2 := \frac{h_4^2 h_{12}}{h_{10}^2}, \quad i_3 := \frac{h_4^5}{h_{10}^2}.$$

- The three invariants $j_{i,\ell}(\Omega) = j_i(\ell\Omega)$ encode a principally polarised abelian surface ℓ-isogeneous to $A = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$;
- All others ppav ℓ-isogenous to A comes from the action of $\Gamma/\Gamma_0(\ell)$ on Ω. The index is $\ell^3 + \ell^2 + \ell + 1$.

Modular polynomials in dimension 2

Definition (ℓ-modular polynomials)

$$\Phi_{1,\ell}(X, j_1, j_2, j_3) = \prod_{\gamma \in \Gamma/\Gamma_0(\ell)} (X - j_{1,\ell}^{\gamma})$$

$$\Psi_{i,\ell}(X, j_1, j_2, j_3) = \sum_{\gamma \in \Gamma/\Gamma_0(\ell)} j_{i,\ell}^{\gamma} \prod_{\gamma' \in \Gamma/\Gamma_0(\ell) \setminus \{\gamma\}} (X - j_{1,\ell}^{\gamma'}) \quad (i = 2, 3)$$

$$\Phi_{1,\ell}, \Psi_{2,\ell}, \Psi_{3,\ell} \in \mathbb{Q}(j_1, j_2, j_3)[X].$$

- Computed via an evaluation–interpolation approach;
- Evaluation requires evaluating the modular invariants on Ω at high precision;

⇒ Uses a generalized version of the AGM to compute theta functions in quasi-linear time in the precision [Dup06];

⇒ Need to interpolate rational functions;

- Denominator describes the Humbert surface of discriminant ℓ^2 [BL09; Gru10];
- Quasi-linear algorithm [Dup06; Mil14];
- Can be generalized to smaller modular invariants [Mil14].

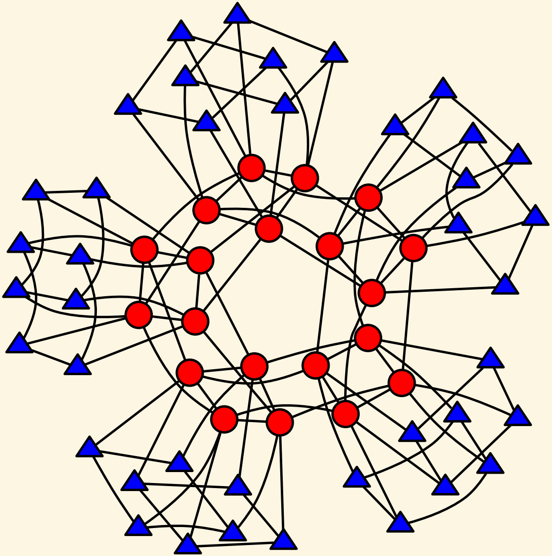
Example of modular polynomials in dimension 2 [Mil14]

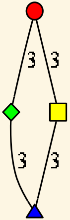
Invariant	ℓ	Size
Igusa	2	57 MB
Streng	2	2.1 MB
Streng	3	890 MB
Theta	3	175 KB
Theta	5	200 MB
Theta	7	29 GB

Example

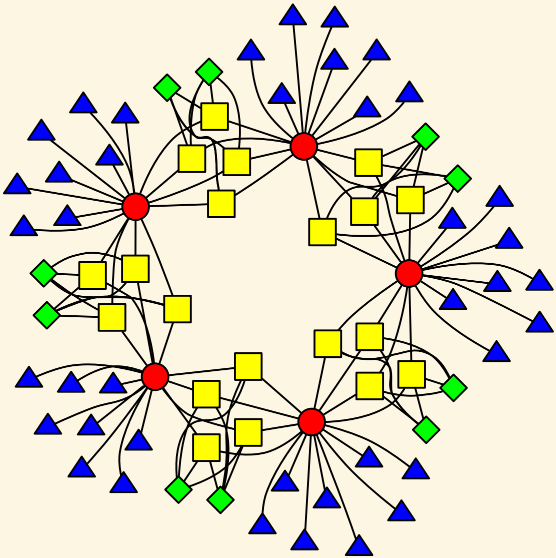
The denominator of $\Phi_{1,3}$ for modular functions b_1, b_2, b_3 derived from theta constant of level 2 is:
$1024b_3^6b_2^6b_1^{10} - ((768b_3^8 + 1536b_3^4 - 256)b_3^8 + 1536b_3^8b_3^4 - 256b_3^8)b_1^8 + (1024b_3^6b_2^{10} + (1024b_3^{10} + 2560b_3^6 - 512b_3^2)b_2^6 - (512b_3^6 - 64b_3^2)b_2^2)b_1^6 - (1536b_3^8b_2^8 + (-416b_3^4 + 32)b_2^4 + 32b_3^4)b_1^4 - ((512b_3^6 - 64b_3^2)b_2^6 - 64b_3^6b_2^2)b_1^2 + 256b_3^8b_2^8 - 32b_3^4b_2^4 + 1.$

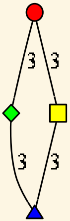
Isogeny graphs in dimension 2 ($\ell = q_1 q_2 = Q_1 \overline{Q_1} Q_2 \overline{Q_2}$)



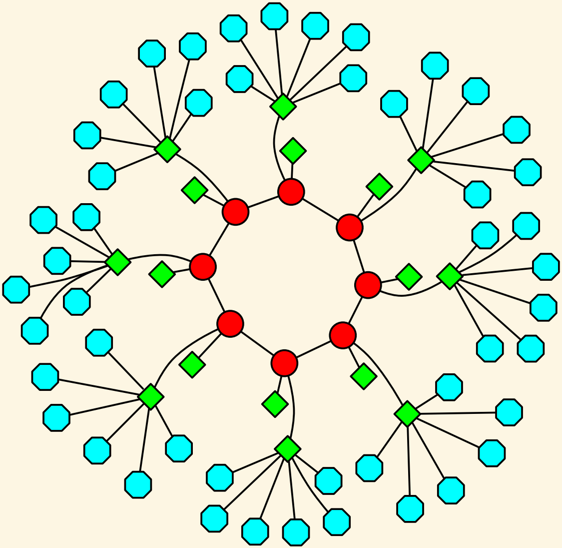


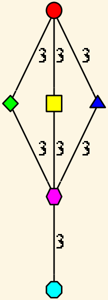
Isogeny graphs in dimension 2 ($\ell = q = Q\overline{Q}$)



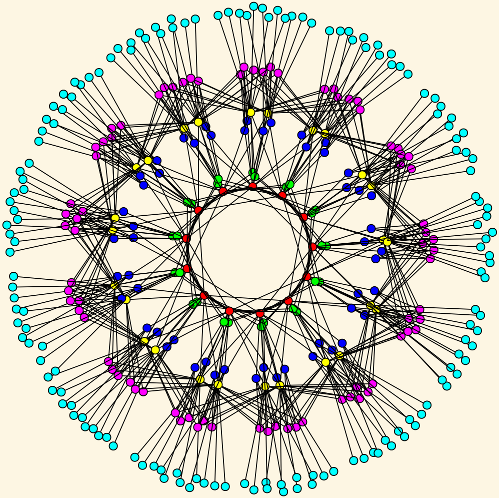


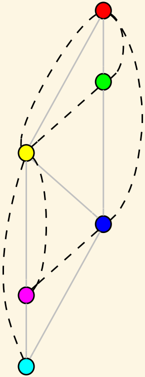
Isogeny graphs in dimension 2 ($\ell = q = Q\overline{Q}$)





Isogeny graphs and lattice of orders [Bisson, Cosset, R.]





Non principal polarisations

- Let $f : (A, H_1) \rightarrow (B, H_2)$ be an isogeny between principally polarised abelian varieties;
- When $\operatorname{Ker} f$ is not maximal isotropic in $A[\ell]$ then $f^* H_2$ is not of the form ℓH_1;
- How can we go from the principal polarisation H_1 to $f^* H_1$?

Non principal polarisations

Theorem (Birkenhake-Lange, Th. 5.2.4)

Let A be an abelian variety with a principal polarisation H_1;

- *Let $O_0 = \mathrm{End}(A)^s$ be the real algebra of endomorphisms symmetric under the Rosati involution;*
- *Let $\mathrm{NS}(A)$ be the Néron-Severi group of line bundles modulo algebraic equivalence.*

Then

- *$\mathrm{NS}(A)$ is isomorphic to O_0 via*

$$\beta \in O_0 \mapsto H_\beta = \beta H_1 = H_1(\beta\cdot,\cdot);$$

- *This induces a bijection between polarisations of degree d in $\mathrm{NS}(A)$ and totally positive symmetric endomorphisms of norm d in O_0^{++};*
- *The isomorphic class of a polarisation $H_\beta \in \mathrm{NS}(A)$ for $f \in O_0^{++}$ correspond to the action $\varphi \mapsto \varphi^* \beta \varphi$ of the automorphisms of A.*

Cyclic isogeny

- Let $f:(A,H_1)\to(B,H_2)$ be an isogeny between principally polarised abelian varieties with cyclic kernel of degree ℓ;
- There exists β such that the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[dl, "\beta"'] \arrow[d, "\Phi_{f^*H_2}"] & B \arrow[d, "\Phi_{H_2}"] \\
A \arrow[r, "\Phi_{H_1}"'] & \widehat{A} & \widehat{B} \arrow[l, "\widehat{f}"]
\end{tikzcd}$$

- β is an $(\ell,0,\dots,\ell,0,\dots)$-isogeny whose kernel is not isotropic for the H_1-Weil pairing on $A[\ell]$!
- β commutes with the Rosatti involution so is a real endomorphism (β is H_1-symmetric). Since H_1 is Hermitian, β is totally positive.
- $\operatorname{Ker} f$ is maximal isotropic for βH_1; conversely if K is a maximal isotropic kernel in $A[\beta]$ then $f: A\to A/K$ fits in the diagram above.

β-isogenies

Theorem ([Dudeanu, Jetchev, R.])

- *Let $(A, \mathscr{L})$ be a ppav and $\beta \in \mathrm{End}(A)^{++}$ be a totally positive real element of degree ℓ. Let $K \subset \mathrm{Ker}\,\beta$ be cyclic of degree ℓ (note that it is automatically isotropic). Then A/K is principally polarised.*
- *Conversely if there is a cyclic isogeny $f : A \to B$ of degree ℓ between ppav then there exists $\beta \in \mathrm{End}(A)^{++}$ such that $\mathrm{Ker}\, f \subset \mathrm{Ker}\,\beta$.*
- *Given the kernel $\ker f$ we have a polynomial time algorithm in $\deg f$ for computing the isogeny f.*

Corollary

- *If $NS(A) = \mathbb{Z}$ there are no cyclic isogenies to a ppav;*
- *For an ordinary abelian surface, if there is a cyclic isogeny of degree ℓ then ℓ splits into totally positive principal ideals in the real quadratic order which is locally maximal at ℓ. A cyclic isogeny does not change the real multiplication.*

Cyclic modular polynomials in dimension 2 [Milio-R.]

- Given $\beta \in O_{K_0}$ one can define the β-modular polynomial in terms of symmetric invariants of the Hilbert space $\mathfrak{H}_1^g/(\mathrm{Sl}_2(O_{K_0}) \oplus \mathrm{Sl}_2(O_{K_0})^\sigma)$;
- If $D=2$ or $D=5$ the symmetric Hilbert moduli space is rational and parametrized by two invariants: the Gundlach invariants;
- Use an evaluation–interpolation approach via the action of $\mathrm{Sl}_2(O_{K_0})/\Gamma_0(\beta_i)$ which give all the $\ell+1$ β_i-isogenies;

- For general D the Hilbert space is not unirational $\Rightarrow$ we need to interpolate three invariants (the pull back of three Siegel invariants);
- There is an algebraic relation between the invariants we interpolate $\Rightarrow$ need to normalise the modular polynomials by fixing a Gröbner basis.

Example of cyclic modular polynomials in dimension 2 [Milio-R.]

ℓ $(D=2)$	Size (Gundlach)	Theta	ℓ $(D=5)$	Size (Gundlach)	Theta
2	8.5KB		5	22KB	45KB
7	172KB		11	3.5MB	308KB
17	5.8MB	221KB	19	33MB	3.6MB
23	21 MB		29	188MB	
31	70 MB		31	248 MB	
41	225 MB	7.2MB			

Example

For $D=2$, $\beta = 5+2\sqrt{2} \,|\, 17$, using b_1, b_2, b_3 pullback of level 2 theta functions on the Hilbert space, the denominator of $\Phi_{1,\beta}$ is $b_3^6 b_2^{18} + (6b_3^8 6b_3^4 + 1)b_2^{16} + (15b_3^{10} 24b_3^6 + 7b_3^2)b_2^{14} + (20b_3^{12} 42b_3^8 + 9b_3^4 + 2)b_2^{12} + (15b_3^{14} 48b_3^{10} + 37b_3^6 + 4b_3^2)b_2^{10} + (6b_3^{16} 42b_3^{12} + 68b_3^8 26b_3^4 + 3)b_2^8 + (b_3^{18} 24b_3^{14} + 37b_3^{10} + 8b_3^6 b_3^2)b_2^6 + (6b_3^{16} + 9b_3^{12} 26b_3^8 24b_3^4 + 2)b_2^4 + (7b_3^{14} + 4b_3^{10} b_3^6)b_2^2 + (b_3^{16} + 2b_3^{12} + 3b_3^8 + 2b_3^4 + 1)$.

Abelian varieties with real and complex multiplication

- Let K be a CM field (a totally imaginary quadratic extension of a totally real field K_0 of dimension g);
- An abelian variety with RM by K_0 is of the form $\mathbb{C}^g/(\Lambda_1 \oplus \Lambda_2 \tau)$ where Λ_i is a lattice in K_0, K_0 is embedded into $\mathbb{C}^g$ via $K_0 \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^g \subset \mathbb{C}^g$, and $\tau \in \mathfrak{H}_1^g$;
- The polarisations are of the form

$$H(z_1, z_2) = \sum_{\varphi_i : K \to \mathbb{C}} \varphi_i(\lambda z_1 \overline{z_2}) / \Im \tau_i$$

  for a totally positive element $\lambda \in K_0^{++}$. In other words if $x_i, y_i \in K_0$, then $E(x_1 + y_1 \tau, x_2 + y_2 \tau) = \mathrm{Tr}_{K_0/\mathbb{Q}}(\lambda(x_2 y_1 - x_1 y_2))$.
- An abelian variety with CM by K is of the form $\mathbb{C}^g/\Phi(\Lambda)$ where Λ is a lattice in K and Φ is a CM-type.
- The polarisations are of the form

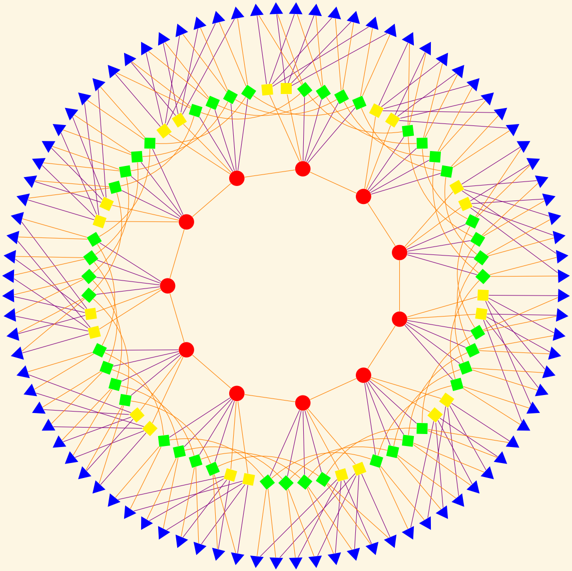
$$E(z_1, z_2) = \mathrm{Tr}_{K/Q}(\xi z_1 \overline{z_2})$$

  for a totally imaginary element $\xi \in K$. The polarisation is principal iff $\xi \overline{\Lambda} = \Lambda^\star$ where $\Lambda^\star$ is the dual of Λ for the trace.

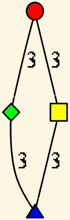
Cyclic isogeny graph in dimension 2 [IT14]

- Let A be a principally polarised abelian surface over $\mathbb{F}_q$ with CM by $O \subset O_K$ and RM by $O_0 \subset O_{K_0}$;
- If O_0 is maximal (locally at ℓ) and that we are in the split case: $(\ell) = (\beta_1)(\beta_2)$ in O_0, then $A[\ell] = A[\beta_1] \oplus A[\beta_2]$. Assume that β_i is totally positive.
- There are two kind of cyclic isogenies: β_1-isogenies ($K \subset A[\beta_1]$) and β_2-isogenies.
- Looking at β_1 isogenies, we recover the volcano structure: $O = O_0 + \mathfrak{f} O_K$ for a certain O_0-ideal $\mathfrak{f}$ such that the conductor of O is $\mathfrak{f} O_K$.
  - If $\mathfrak{f}$ is prime to β_1, there are 2, 1, or 0 horizontal isogenies according to whether β_1 splits, is ramified or is inert in O. The others are descending to $O_0 + \mathfrak{f}\beta_1 O_K$;
  - If $\mathfrak{f}$ is not prime to β_1 there is one ascending isogeny (to $O_0 + \mathfrak{f}/\beta_1 O_K$) and ℓ descending ones;
  - We are at the bottom when the β_1-valuation of $\mathfrak{f}$ is equal to the valuation of the conductor of $\mathbb{Z}[\pi, \overline{\pi}]$.
- ℓ-isogenies preserving O_0 are a composition of a β_1-isogeny with a β_2-isogeny.
- When ℓ is inert, ℓ-isogenies preserving the RM O_0 form a volcano.

Cyclic isogeny graph in dimension 2 [IT14]



β_1 is inert and β_2 is split in K.



Changing the real multiplication in dimension 2: moving between pancakes

Cyclic isogenies (that preserve principal polarisations) conserve real multiplication; so we need to look at ℓ-isogenies.

Proposition

- *Let O_ℓ be the order of conductor ℓ inside O_{K_0}. ℓ-isogenies going from O_ℓ to O_{K_0} are of the form*

$$\mathbb{C}^g/(O_\ell \oplus O_\ell^\vee \tau) \to \mathbb{C}^g/(O_{K_0} \oplus O_{K_0}^\vee \tau).$$

- $\mathrm{Sl}_2(O_{K_0} \oplus O_{K_0}^\vee)/\mathrm{Sl}_2(O_\ell \oplus O_\ell^\vee)$ *acts on such isogenies;*
- *When ℓ splits in O_{K_0}, $\mathrm{Sl}_2(O_{K_0} \oplus O_{K_0}^\vee)/\mathrm{Sl}_2(O_\ell \oplus O_\ell^\vee) \simeq \mathrm{Sl}_2(O_{K_0}/\ell O_{K_0})/\mathrm{Sl}_2(O_\ell/\ell O_\ell) \simeq \mathrm{SL}_2(\mathbb{F}_l^2)/\mathrm{Sl}_2(\mathbb{F}_l) \simeq \mathrm{Sl}_2(\mathbb{F}_l)$, so we find $\ell^3 - \ell$ ℓ-isogenies changing the real multiplication.*
- *On the other hand there is $(\ell+1)^2$ ℓ-isogenies preserving the real multiplication*
- *In total we find all $\ell^3 + \ell^2 + \ell + 1$ ℓ-isogenies.*

Changing the real multiplication in dimension 2: moving between pancakes

Corollary ([Ionica, Martindale, R., Streng])

If O is maximal at ℓ,

- *If ℓ is split there are $\ell^2+2\ell+1$ RM-horizontal ℓ-isogenies and $\ell^3-\ell$ RM-descending ℓ-isogenies;*
- *If ℓ is inert there are ℓ^2+1 RM-horizontal ℓ-isogenies and $\ell^3+\ell$ RM-descending ℓ-isogenies;*
- *If ℓ is ramified there are $\ell^2+\ell+1$ RM-horizontal ℓ-isogenies and ℓ^3 RM-descending ℓ-isogenies;*

If O is not maximal at ℓ, there are 1 RM-ascending ℓ-isogeny, $\ell^2+\ell$ RM-horizontal ℓ-isogenies and ℓ^3 RM-descending ℓ-isogenies.

AVIsogenies [Bisson, Cosset, R.]

- AVIsogenies: Magma code written by Bisson, Cosset and R. `http://avisogenies.gforge.inria.fr`
- Released under LGPL 2+.
- Implement isogeny computation (and applications thereof) for abelian varieties using theta functions.

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