Applications of isogenies between abelian varieties to elliptic curves

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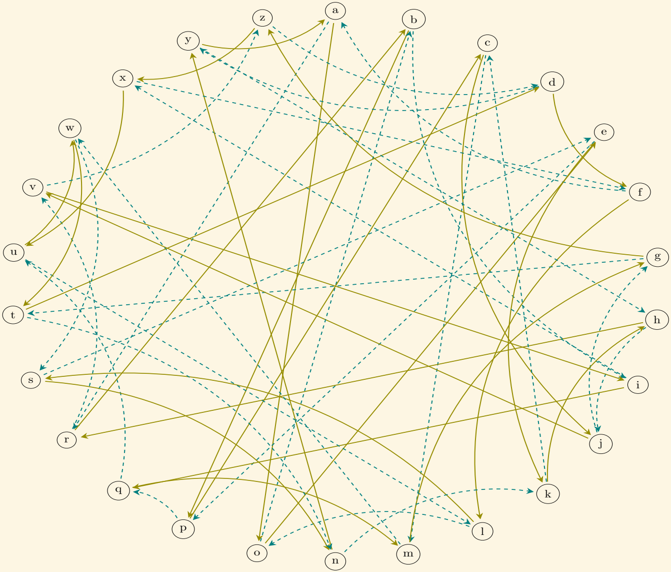
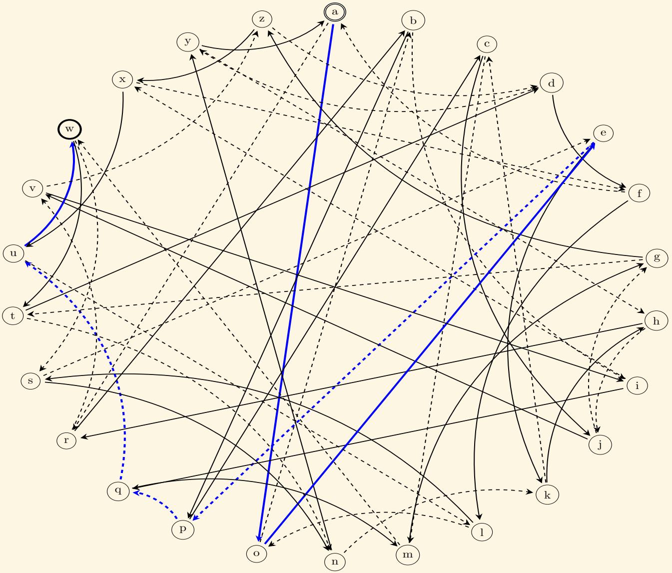
Key exchange on a (commutative) graph



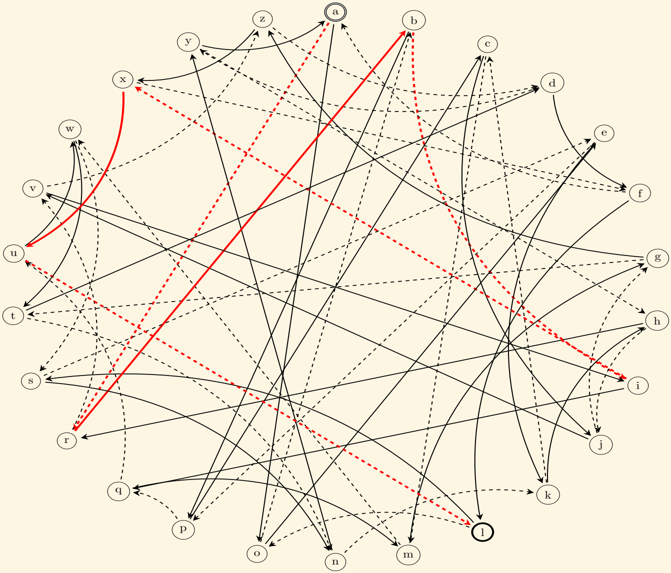
Key exchange on a (commutative) graph

Alice starts from 'a', follows the path 001110, and get 'w'.



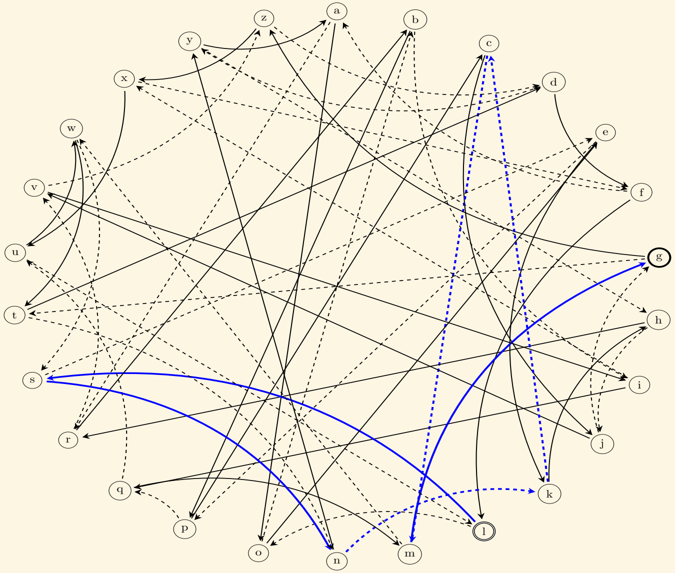
Key exchange on a (commutative) graph

Bob starts from 'a', follows the path 101101, and get 'l'.



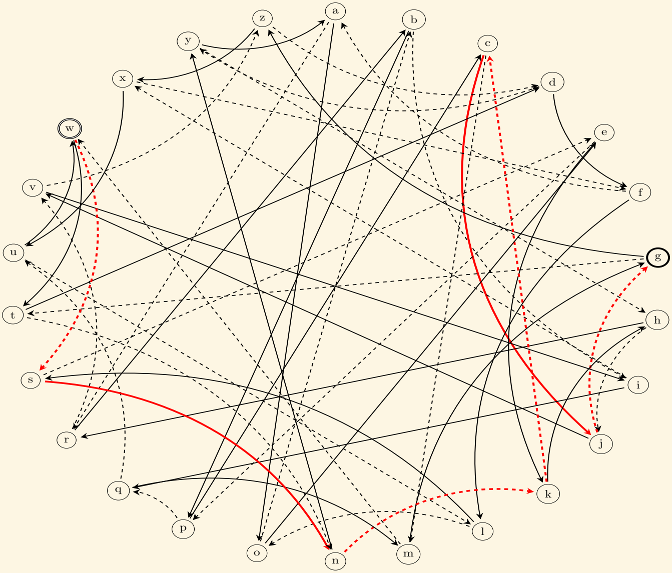
Key exchange on a (commutative) graph

Alice starts from 'l', follows the path 001110, and get 'g'.



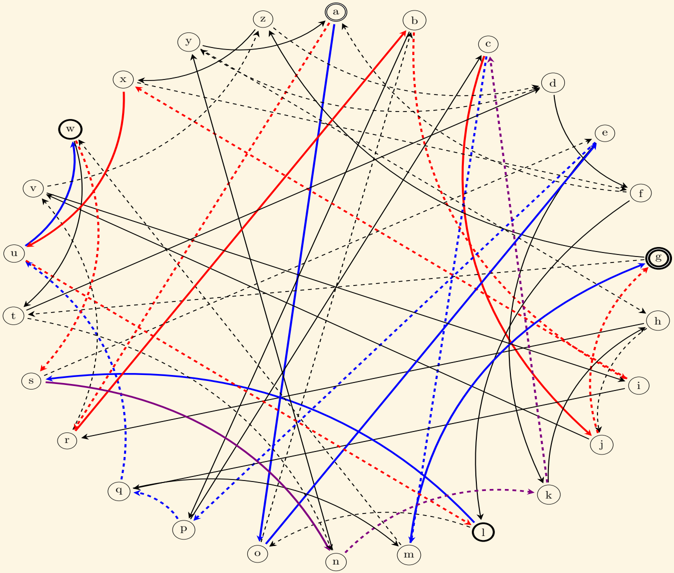
Key exchange on a (commutative) graph

Bob starts from 'w', follows the path 101101, and get 'g'.



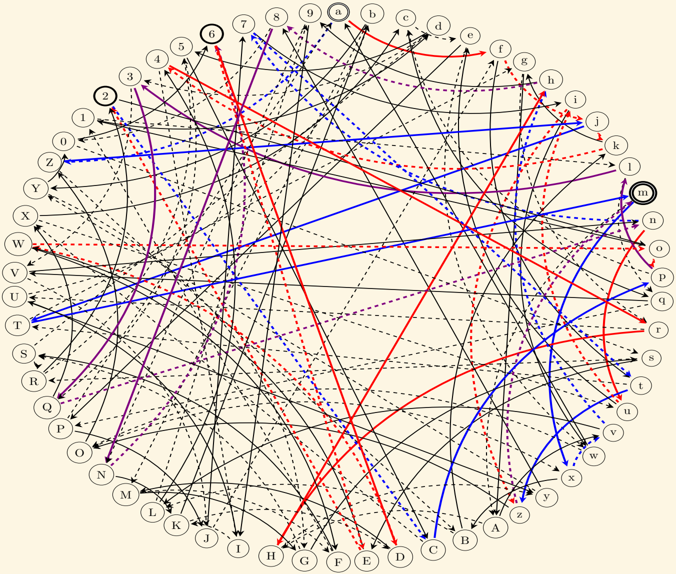
Key exchange on a (commutative) graph

The full exchange:



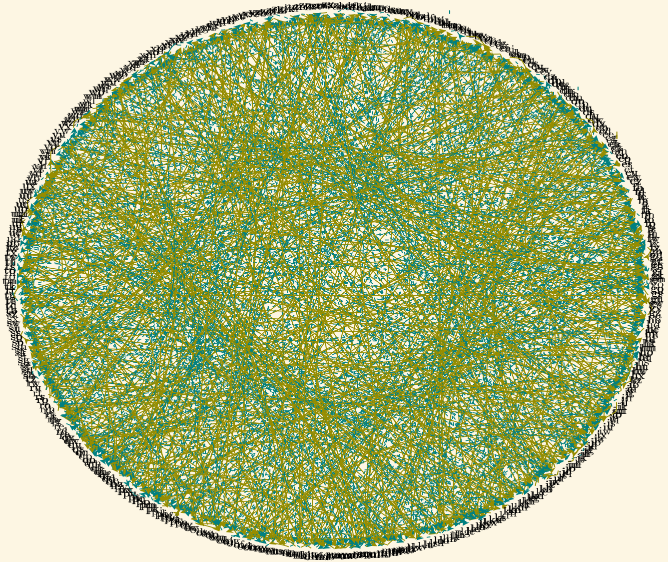
Key exchange on a (commutative) graph

Bigger graph (62 nodes)



Key exchange on a (commutative) graph

Even bigger graph (676 nodes)



Isogeny graphs for key exchange

- Needs a graph with good mixing properties:
  A path of length $O(\log N)$ gives a uniform node $\Rightarrow$ Ramanujan/expander graph.
- The graph does not fit in memory $(N = 2^{256})$.
- Needs an algorithm taking a node as input and giving the neighbour nodes as output.

- Isogeny graph of ordinary elliptic curves $E/\mathbb{F}_p$ [Couveignes (1997)], [Rostovtsev–Stolbunov (2006)]
- Graph of size $N \approx \sqrt{p}$.
- Torsor (principal homogeneous space) under the class group $\mathrm{Cl}(\mathrm{End}(E_0))$.

☺ Commutative graph!

☹ Hidden shift problem solvable in quantum subexponential $L(1/2)$ time for an abelian group action via Kuperberg's algorithm.

- SIDH: supersingular elliptic curve Diffie-Helmann [De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)]
- Use the isogeny graph of a supersingular elliptic curve E over $\mathbb{F}_{p^2}$ $(N \approx p)$.

Isogeny graphs for key exchange

Meme: Gru's plan

- Isogeny based key exchange
- Use supersingular curves
- The graph is non commutative
- The graph is non commutative

SIDH in practice

- $p = 2^a 3^b - 1$. $N_A = 2^a, N_B = 3^b$, N_A prime to N_B.
- $E_0 : y^2 = x^3 + x$ (supersingular when $a \geq 2$)
- $E_0[N_A] = \langle P_A, Q_A \rangle, E_0[N_B] = \langle P_B, Q_B \rangle$.
- Alice's secret isogeny: ϕ_A of kernel $\langle P_A + s_A Q_A \rangle$.
- Bob's secret isogeny: ϕ_B of kernel $\langle P_B + s_B Q_B \rangle$.
- Key exchange:

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi'_A} \\ E_A & \xrightarrow{\phi'_B} & E_{AB} \end{array}$$

- E_{AB} is the shared secret.
- $\phi'_A \circ \phi_B = \phi'_B \circ \phi_A : E_0 \to E_{AB}$ has kernel $\operatorname{Ker} \phi_A + \operatorname{Ker} \phi_B$.
- ϕ'_A has kernel $\langle \phi_B(P_A + s_A Q_A) \rangle$, ϕ'_B has kernel $\langle \phi_A(P_B + s_B Q_B) \rangle$.
- Alice publishes: $P'_B = \phi_A(P_B), Q'_B = \phi_A(Q_B)$.
  Bob publishes: $P'_A = \phi_B(P_A), Q'_A = \phi_B(Q_A)$. ("Torsion points".)
- $\operatorname{Ker} \phi'_A = \langle P'_A + s_A Q'_A \rangle$, $\operatorname{Ker} \phi'_B = \langle P'_B + s_B Q'_B \rangle$.
- Key exchange in $\widetilde{O}(\log N_A \ell_A^{1/2} + \log N_B \ell_B^{1/2})$
  (Via fast smooth isogeny computation [De Feo, Jao, Plût (2014)] and Velusqrt [Bernstein, De Feo, Leroux, Smith (2020)]).

Isogeny evaluation and interpolation

- Evaluation: given an N-isogeny f and a point $Q \in E(\mathbb{F}_q)$, evaluate $f(Q)$.
- N-evaluation problem: f is an N-isogeny = $\operatorname{Ker} f$ is of degree N.

- Interpolation: given a tuple $(P, f(P))$, recover f.
- (N, N')-interpolation problem: given f an N-isogeny and P a point of N'-torsion, from $(P, f(P))$ and $Q \in E(\mathbb{F}_q)$, evaluate $f(Q)$ ($N' \geq N$).
- Weak interpolation: we are given $(P_1, f(P_1))$, $(P_2, f(P_2))$ for (P_1, P_2) a basis of $E[N']$.

- SIDH: the key exchange uses the N_A and N_B evaluation problems
- If we can solve the weak interpolation problem when $N = N_A, N' = N_B$ are smooth in polylogarithmic time, we can break SIDH.

Isogeny evaluation and interpolation

Meme: Anakin

- I have a nice key exchange protocol
- You don't use torsion points, right?
- …
- Right?

Evaluation

- $f : E_1 \to E_2$ an N-isogeny
- $f(x,y) = \left(\frac{g(x)}{h(x)}, cy\left(\frac{g(x)}{h(x)}\right)'\right), \deg g, \deg h \leq N$
- [Vélu 1971]: given $h(x)$ representing the kernel $\operatorname{Ker} f : \{P \in E \mid h(x(P)) = 0\}$, evaluate $f(Q)$ in $O(N)$ operations in $\mathbb{F}_q$.
- Velusqrt: special case $\operatorname{Ker} f = \langle T \rangle, T \in \mathbb{F}_q$, evaluate $f(Q)$ in $\widetilde{O}(\sqrt{N})$ operations in $\mathbb{F}_q$.
- Linear time.

- If N is smooth, f can be decomposed into a product of small isogenies.
- Evaluation in $O(\log N \ell_N)$ or $\widetilde{O}(\log N \sqrt{\ell_N})$.
- Logarithmic time.

- The decomposition cost is quasi-logarithmic if $\operatorname{Ker} f = \langle T \rangle$ with $T \in \mathbb{F}_q$; polylogarithmic if N is powersmooth; but linear if T lives in a large extension.

Interpolation

- Given $(P, f(P))$, P a point of order $N' \geq 2N$, recover the rational function $\frac{g(x)}{h(x)}$ in $\widetilde{O}(N)$ by interpolating the points $(x(mP), x(mf(P)))$, $m = 1, \ldots, N'-1$.
- Can evaluate on Q directly.
- Special case : $P \in T_{0_E}(E)$ a "fat point" of order $p \Rightarrow$ solve a differential equation [Elkies 1992] ($P \neq 0, p > 2N$).
- Quasi-linear time.

- Faster algorithm when N' is smooth?
- Yes if $f(P) = 0$. Then $N = N'$ and $\operatorname{Ker} f = \langle P \rangle$.
- If $N = N'$, the weak interpolation problem reduces via the DLP to the N'-evaluation problem.
- This is why the SIDH key exchange is fast: Bob uses the torsion point information published by Alice to find the kernel of his pushforward isogeny.
- No reason to expect a fast algorithm when N' is prime to N.

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Revisiting isogeny evaluation

- Can an N-isogeny be evaluated faster than linear time when N has a large prime factor?
- If $f = [\ell]$ (so $N = \ell^2$): double and add in $O(\log \ell)$ to evaluate ℓQ.

- $F : E^2 \to E^2, (P_1, P_2) \mapsto (P_1 + P_2, P_1 - P_2)$ is a 2-isogeny in dimension 2.
- Double: $F(T, T) = (2T, 0)$.
- Add: $F(T, Q) = (T + Q, T - Q)$.

- We can evaluate ℓQ as a composition of $O(\log \ell)$ evaluations of F, projections $E^2 \to E$ and embeddings $E \to E^2$.
- Double and add on E = 2-isogenies in dimension 2

Polarisations and isogenies on an abelian variety

- Polarisation on A = a (symmetric) isogeny $\lambda_A : A \to \widehat{A}$
- Principal polarisation: λ_A is an isomorphism.
- Warning: A may have several non equivalent principal polarisations if $g > 1$.

- $f : (A, \lambda_A) \to (B, \lambda_B)$ N-isogeny between ppav: $f^* \lambda_B = N \lambda_A$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \lambda_A^{-1}} \uparrow & & \downarrow {\scriptstyle \lambda_B} \\ \widehat{A} & \xleftarrow[\hat{f}]{} & \hat{B} \end{array}$$

- Dual isogeny: $\hat{f} : \hat{B} \to \widehat{A}$
- Contragredient isogeny: $\tilde{f} = \lambda_A^{-1} \hat{f} \lambda_B : B \to A$
- f N-isogeny $\Leftrightarrow \tilde{f} f = N \Leftrightarrow f \tilde{f} = N$.

- $\operatorname{Ker} f = \operatorname{Im} \left(\tilde{f} \mid B[N] \right)$.

Algorithms for N-isogenies in higher dimension

- [Cosset-R. (2014), Lubicz-R. (2012–2022)]: An N-isogeny in dimension g can be evaluated in linear time $O(N^g)$ arithmetic operations in the theta model given generators of its kernel.
- Warning: exponential dependency 2^g or 4^g in the dimension g.

- [Couveignes-Ezome (2015)]: Algorithm in $O(N^g)$ in the Jacobian model.
- Not hard to extend to product of Jacobians.
- Restricted to $g \leq 3$.

Kani's lemma [Kani 1997] ($g = 1$), [R. 2022] ($g > 1$)

- $\alpha : A \to B$ a a-isogeny, $\beta : A \to C$ a b-isogeny.
- $\alpha' : C \to D$ a a-isogeny, $\beta' : C \to D$ a b-isogeny with $\beta'\alpha = \alpha'\beta$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\ C & \xrightarrow{\alpha'} & D \end{array}$$

- If a prime to b, the pushforward α', β' of α by β satisfy these conditions.

- $F = \begin{pmatrix} \alpha & \widetilde{\beta'} \\ -\beta & \widetilde{\alpha'} \end{pmatrix} : A \times D \to B \times C.$
- $\tilde{F} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \beta' & \alpha' \end{pmatrix} : B \times C \to A \times D, \quad \tilde{F}F = a + b.$
- F is an $a + b$-isogeny with respect to the product polarisations.

- $\operatorname{Ker} F = \{\tilde{\alpha}(P), \beta'(P) \mid P \in B[a+b]\}$ (if a is prime to b)

Revisiting the interpolation

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ E_1' & \xrightarrow{f'} & E_2' \end{array}$$

- $f : E_1 \to E_2$ an N-isogeny.
- Goal: replace f by F an N'-isogeny.
- Find $\alpha : E_1 \to E_1'$ an m-isogeny, with $N' = N + m$.
- Kani's lemma: $F : E_1 \times E_2' \to E_1' \times E_2$ is an N'-isogeny.
- Since we know $f(E[N'])$, and we can evaluate α on $E[N']$, we recover Ker F (or Ker $\tilde{F}$)
- Evaluate F, hence f at any point: $F(P, 0) = (\alpha(P), -f(P))$.
- This evaluation is fast if N' is (power) smooth.

Examples:

- m smooth [Castryck–Decru 2022; Maino–Martindale 2022]
- $m = \ell^2$: take $\alpha = [\ell]$
- $\mathrm{End}(E)$ has an efficient endomorphism α of norm m [Castryck–Decru].

The general case

- $\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2$ on E^2 (Gaussian integers $\mathbb{Z}[i]$)

- $\alpha = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix}$ is always an endomorphism of norm $m = a_1^2 + a_2^2 + a_3^2 + a_4^2$ on E^4 (Hamilton's quaternion algebra)
- Evaluating α costs $O(\log m)$ arithmetic operations
- Every integer is a sum of four squares [Διόφαντος ὁ Ἀλεξανδρεύς, Lagrange].

$$\begin{array}{ccc} E_1^4 & \xrightarrow{f} & E_2^4 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ E_1^4 & \xrightarrow{f} & E_2^4 \end{array}$$

- $F : E_1^4 \times E_2^4 \to E_1^4 \times E_2^4$ is an N'-isogeny.

The embedding lemma [R. 2022]

- A N-isogeny $f : A \to B$ in dimension g can always be efficiently embedded into a N' isogeny $F : A' \to B'$ in dimension $8g$ (and sometimes $4g, 2g$) for any $N' \geq N$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \uparrow \\ A' & \xrightarrow{F} & B' \end{array}$$

- Considerable flexibility (at the cost of going up in dimension).

- Breaks SIDH ([Castryck–Decru], [Maino–Martindale] in dimension 2, [R.] in dimension 4 or 8)
- Reduces the (N, N')-weak interpolation problem to the N'-evaluation problem in higher dimension
- Only needs $N'^2 \geq N$ (uses the dual isogeny)

⇒ Solves the weak interpolation problem when N' is (power) smooth

- Amazing fact: does not requires $\operatorname{Ker} f$, works even if N is prime

- Open question: case N' prime? Need a fast N'-evaluation algorithm!

Efficient representation of isogenies [R. 2022]

- For the N-evaluation problem, once we have evaluated f on a basis of the N'-torsion this reduces to the N'-weak interpolation problem which reduces to the N'-evaluation problem (in higher dimension).

$\Rightarrow$ Can always embed an N-isogeny f into a N'-isogeny F with N' powersmooth

- Then decompose F as a product of small isogenies.

- Polylogarithmic space $O(\log^3 N)$
- Evaluation in polylogarithmic time $O(\log^7 N)$ arithmetic operations.
- Previously: no representation giving better than linear time for a generic isogeny.

- Representation: $(P_i, Q_i, f(P_i), f(Q_i))$ for (P_i, Q_i) basis of $E[\ell_i]$, small torsion points $(\ell_i \mid N')$
- We need to evaluate f on the N'-torsion: given the kernel, the decomposition step is quasi-linear.

Examples

f an N-isogeny with an efficient representation.

- Efficient division: evaluate f/D on any point.
- Contragredient isogeny: evaluate $\tilde{f}$ on any point.

$\Rightarrow$ Efficient evaluation of the Verschiebung $\hat{\pi}_p$.

- Efficient lifting of isogenies: embed f into F at precision $m = 1$, then lift F to precision $m > 1$.

Applications [R. 2022]

- $E/\mathbb{F}_q$ ordinary elliptic curve, $K = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Given the factorisation of $[O_K : \mathbb{Z}[\pi]]$, compute $\mathrm{End}(E)$ in polynomial time.
  Factorisation: quantum polynomial time, classical subexponential time
- Previously: no quantum polynomial time algorithm known.
  Classical algorithm in $L(1/2)$ under GRH [Bisson–Sutherland 2009].

- Compute the canonical lift $\hat{E}/\mathbb{Z}_q$ in polynomial time.
- Previously: $L(1/2)$ under GRH [Couveignes–Henocq 2002]

- Compute the modular polynomial Φ_ℓ in quasi-linear time $O(\ell^3 \log^3 \ell \log\log \ell)$ (no heuristics!).
- Compute $\Phi_\ell \mod p$ in quasi-linear time $\widetilde{O}(\ell^2 \log p)$.
- If E/K elliptic curve of height H over a number field, compute $\Phi_\ell(j(E), Y)$ in quasi-linear time $\widetilde{O}(H\ell^2)$.

- Generalisations to abelian varieties.
- Previously: no algorithm known to compute Φ_ℓ in quasi-linear time when $g > 2$.

Point counting and canonical lifts

$E/\mathbb{F}_q, q = p^n$.

- [Schoof 1985]: $\widetilde{O}(n^5 \log^5 p)$ (Étale cohomology)
- [SEA 1992]: $\widetilde{O}(n^4 \log^4 p)$ (Heuristic)

- [Kedlaya 2001]: $\widetilde{O}(n^3 p)$ (Rigid cohomology)
- [Harvey 2007]: $\widetilde{O}(n^{3.5} p^{1/2} + n^5 \log p)$

- [Satoh 2000] (canonical lifts of ordinary curves): $\widetilde{O}(n^2 p^2)$ (Crystalline cohomology)
- [Maiga – R. 2021]: $\widetilde{O}(n^2 p)$
- [R. 2022]: $\widetilde{O}(n^2 \log^8 p + n \log^{11} p)$