# The module action for isogeny based cryptography

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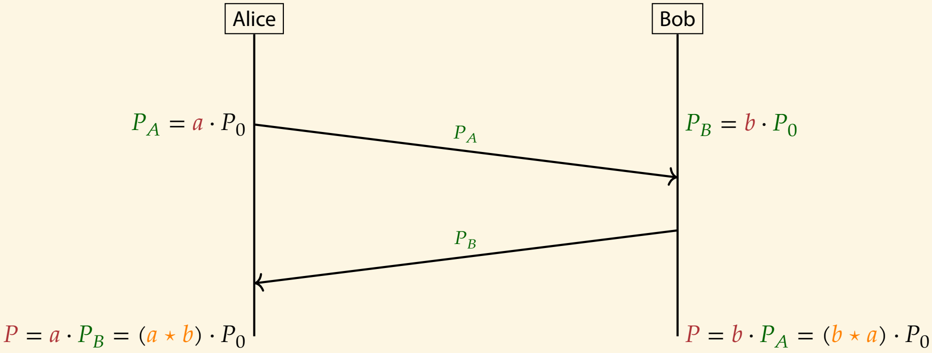
# NIKE: Non Interactive Key Exchange



# CRS Key Exchange ([Couveignes (1997)], [Rostovtsev–Stolbunov (2006)])

The ideal action on ordinary elliptic curves:

$$\begin{array}{ccc} E_0 & \longrightarrow & E_{[\mathfrak{a}]} = \mathfrak{a} \cdot E_0 \\ \downarrow & & \downarrow \\ E_{[\mathfrak{b}]} = \mathfrak{b} \cdot E_0 & \longrightarrow & E_{[\mathfrak{a}\mathfrak{b}]} \simeq \mathfrak{a}\mathfrak{b} \cdot E_0 \end{array}$$

☺ Commutative group action

- Restricted group action ⇝ Unrestricted group action:
  CSI-FiSh (2019), [Pearl-]Scallop[-HD] (2023–2024), [CKQ]lapoti[s]/Pegasis (2023–2025)

- Classical security $\approx \Delta^{1/4}$

☹ Susceptible to Kuperberg's subexponential quantum algorithm
$\Rightarrow$ need to work with $\Delta \gg 512$ bits

# The ordinary ideal action

- $E/\mathbb{F}_q$ ordinary elliptic curve
- $\mathfrak{a} \subset R := \mathrm{End}_{\mathbb{F}_q}(E)$ invertible ideal in a quadratic imaginary order

**Definition (The ideal action)**

$\mathfrak{a} \cdot E$ is the elliptic curve $E/E[\mathfrak{a}]$, where

$$E[\mathfrak{a}] := \{P \in E(\overline{\mathbb{F}}_q) \mid \alpha(P) = 0_E, \forall \alpha \in \mathfrak{a}\}$$

☹ This conflates the codomain $\mathfrak{a} \cdot E$ with the way we compute it as an isogeny $E \to E/E[\mathfrak{a}]$

- Not obvious that $\mathfrak{a} \cdot \mathfrak{b} \cdot E \simeq (\mathfrak{a}\mathfrak{b}) \cdot E$ (Can use that $\deg E[\mathfrak{a}] = N(\mathfrak{a})$)
  What happens at non invertible ideals?
- As in Deuring's correspondence, can kinda be reframed as an equivalence of category between (equivalence classes of) invertible ideals in $R$ and (isomorphism classes of) elliptic curves "horizontally" isogeneous to $E$
- An isogeny $\phi : \mathfrak{a} \cdot E \to \mathfrak{b} \cdot E$ corresponds to the invertible ideal $\mathfrak{b}\mathfrak{a}^{-1}$
- Not clear distinction of objects and morphisms
- **Question 1**: intrinsic characterisation of $\mathfrak{a} \cdot E$?

# SIDH/SIKE: supersingular isogeny key exchange ([De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)])

- **Idea**: Switch to maximal supersingular curves over $\mathbb{F}_{p^2}$
- No commutative group action ⇒ no Kuperberg attack

# SIDH/SIKE: supersingular isogeny key exchange ([De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)])

**Meme: Gru's plan**

- Isogeny based key exchange
- Use supersingular curves
- The graph is non commutative
- The graph is non commutative

## SIDH/SIKE: supersingular isogeny key exchange ([De Feo, Jao (2011)],[De Feo, Jao, Plût (2014)])

- **Observation:** The CRS diagram

$$\begin{array}{ccc} E_0 & \longrightarrow & E_{[\mathfrak{a}]} = \mathfrak{a} \cdot E_0 \\ \downarrow & & \downarrow \\ E_{[\mathfrak{b}]} = \mathfrak{b} \cdot E_0 & \longrightarrow & E_{[\mathfrak{a}\mathfrak{b}]} \simeq \mathfrak{a}\mathfrak{b} \cdot E_0 \end{array}$$

  is a pushforward if $N(\mathfrak{a})$ is coprime to $N(\mathfrak{b})$

- **SIDH**:

$$\begin{array}{ccc} E_0 & \longrightarrow & E_A = E_0/K_A \\ \downarrow & & \downarrow \\ E_B = E_0/K_B & \longrightarrow & E_{AB} \simeq E_0/(K_A + K_B) \end{array}$$

  where $K_A \subset E_0[2^a]$, $K_B \subset E_0[3^b]$ and $E_0/\mathbb{F}_{p^2}$ is a maximal supersingular curve

☹ To compute $E_{AB}$ from $E_A$ and $K_B$, Bob needs extra torsion information on $E_A$ from Alice

☹☹☹ SIDH attacks [Castryck-Decru; Maino-Martindale-Panny-Pope-Wesolowski; R. 2023]

# A commutative supersingular key exchange?

- There is also a supersingular ideal action [Deuring]
- $K_A = E_0[I_A], K_B = E_0[I_B], I_A, I_B \subset \mathfrak{O}_0 := \mathrm{End}_{\overline{\mathbb{F}}_p}(E_0)$

- **Problem**: the endomorphism ring $\mathfrak{O}_A$ of $E_A$ is distinct from $\mathfrak{O}_0$, so $I_B$ is not an ideal of it
- Instead, Bob needs to act by a different ideal $I'_B \subset \mathfrak{O}_A$ to get $E_{AB} = I'_B \cdot E_A$

- **Idea**: What if $I_A, I_B$ are generated by ideals $\mathfrak{a}, \mathfrak{b} \subset R$ of a commutative quadratic order $R \subset \mathfrak{O}$?
- Then $R \subset \mathfrak{O}_A$, and $I'_B$ is also generated by $\mathfrak{b}$ (Assume $R$ saturated in $\mathfrak{O}$ and the ideals $\mathfrak{a}, \mathfrak{b}$ invertible in $R$)
- And $E_A[I'_B] = E_A[\mathfrak{b}]$ can be computed as long as Bob knows how $R$ acts on $E_A$

- CSIDH [Castryck-Lange-Martindale-Panny-Renes 2018]: start with a supersingular $E_0/\mathbb{F}_p$ and $R = \mathbb{Z}[\sqrt{-p}] = \mathbb{Z}[\pi_p]$
- Oriented group actions [Colò-Kohel 2020], [Onuki 2020] on a (maximal) supersingular curve $E_0/\mathbb{F}_{p^2}$, with $R \subset \mathfrak{O}_0$ arbitrary

# Frobenius orientation (CSIDH) and arbitrary orientations (SCALLOP)

$$\begin{array}{ccc} E_0 & \longrightarrow & E_{[\mathfrak{a}]} = \mathfrak{a} \cdot E_0 \\ \downarrow & & \downarrow \\ E_{[\mathfrak{b}]} = \mathfrak{b} \cdot E_0 & \longrightarrow & E_{[\mathfrak{ab}]} \simeq \mathfrak{ab} \cdot E_0 \end{array}$$

- $E_0/\mathbb{F}_{p^2}$ supersingular curve
- $R \subset \mathfrak{O}_0$ orientation by a quadratic imaginary order; $\mathfrak{a}, \mathfrak{b} \subset R$ invertible ideals

CSIDH: $E_0/\mathbb{F}_p$ + natural Frobenius orientation $\pi_p \curvearrowright E_0$ (like in CRS)

- ☺ Great control on torsion (e.g. if $2^e \mid p+1$, the points in $E_0[2^e]$ are rational over $\mathbb{F}_{p^2}$)
- 😐 $\Delta_R = -4p$

SCALLOP: arbitrary orientation $R \subset \mathfrak{O}_0$

- ☺ Decouple the arithmetic ($\mathbb{F}_p$) with the discriminant $\Delta_R$ (For an ordinary curve, $\Delta(\pi_p) \approx p$)
- 😐 Needs a way to represent the orientation

☹ Both still susceptible to Kuperberg's subexponential quantum algorithm

# A commutative supersingular key exchange (round 2)?

$$\begin{array}{ccc} E_0 & \longrightarrow & E_{I_A} = I_A \cdot E_0 \\ \big\downarrow & & \\ E_{I_B} = I_B \cdot E_0 & & \end{array}$$

- **Goal**: complete the diagram for $I_A, I_B$ arbitrary ideals of $\mathfrak{O}_0$

- **Idea**: if $R \subset \mathfrak{O}_0$ is an orientation by a quadratic order, $I_A, I_B$ are rank $2$ $R$-modules
- $I_A I_B$ is not a well defined ideal, but $I_A \otimes_R I_B$ is a well defined rank $4$ $R$-module
- Commutativity: $I_A \otimes_R I_B \simeq I_B \otimes_R I_A$

- **Question 2**: Can we make sense of a module action?

## The module action

- If $A_1, A_2/k$ are two abelian varieties oriented by $R$, then $\mathrm{Hom}_R(A_1, A_2)$ is a $R$-module

**Definition (The power object)**

If $A$ is an abelian variety oriented by $R$ and $M$ a (finite type) $R$-module, $M \cdot A := \mathcal{H}om_R(M, A)$ is the (unique) $R$-oriented abelian variety, if it exists, such that

$$\mathrm{Hom}_{R-\mathrm{Ab}}(X, \mathcal{H}om_R(M, A)) = \mathrm{Hom}_R(M, \mathrm{Hom}_{R-\mathrm{Ab}}(X, A)) \quad \forall X \in R - \mathrm{Ab}$$

$R - \mathrm{Ab}$: category of $R$-oriented abelian varieties and $R$-oriented morphisms

[Giraud 1968] (credits Serre+Tate), [Serre 1985]

- Functoriality: an $R$-linear map $\psi : M_2 \to M_1$ induces an oriented morphism

$$\phi : \mathcal{H}om_R(M_1, A) \to \mathcal{H}om_R(M_2, A)$$

- Left exactness: $M_1 \twoheadrightarrow M_2 \to 0 \quad \rightsquigarrow \quad 0 \to \mathcal{H}om_R(M_2, A) \hookrightarrow \mathcal{H}om_R(M_1, A)$
  $0 \to A_1 \hookrightarrow A_2 \quad \rightsquigarrow \quad 0 \to \mathcal{H}om_R(M, A_1) \hookrightarrow \mathcal{H}om_R(M, A_2)$
- Commutativity: if $R$ is commutative, $M_2 \cdot M_1 \cdot A = \mathcal{H}om_R(M_2, \mathcal{H}om_R(M_1, A)) = \mathcal{H}om_R(M_1 \otimes_R M_2, A) = (M_1 \otimes_R M_2) \cdot A = M_1 \cdot M_2 \cdot A$

# Construction of the module action

- Embed both categories into $\underline{R}$-modules for the (big) fppf-topos (sheafs for the fppf site of $\operatorname{Spec} k$)
- $\mathcal{H}om_R(M, A) := \mathcal{H}om_{R-fppf}(\underline{M}, A)$ is the $\underline{R}$-Hom sheaf (internal $\underline{R}$-Hom in the fppf-topos)
  $\underline{M}$ is the fppf-sheafification of the constant sheaf $M$

- Functor of points: If $S/k$ is a f.t. $k$-algebra,

$$\mathcal{H}om_R(M, A)(S) = \operatorname{Hom}_R(M, A(S))$$

  [Waterhouse 1969, Appendix A] (cites [Serre 1965, 1967])

- This is always the (sheaf associated to) a proper commutative group scheme, of dimension

$$\dim \mathcal{H}om_R(M, A) = \operatorname{rank} M \times \dim A$$

- $\mathcal{H}om_R(M, A)$ is an abelian variety if $M$ is projective [Serre]
- Exactness: if $0 \to M_2 \to M_1 \to M_1/M_2 \to 0$ is exact, and $\mathcal{H}om_R(M_2, A)$ is an abelian variety, then

$$0 \to \mathcal{H}om_R(M_1/M_2, A) \to \mathcal{H}om_R(M_1, A) \to \mathcal{H}om_R(M_2, A) \to 0$$

  is exact

# An equivalence of category

Oriented case: $E_0/k$ elliptic curve primitively oriented by $R$ quadratic imaginary

### Theorem (Module anti-equivalence of category)

*The action $M \mapsto M \cdot E_0 = \mathcal{H}om_R(M, E_0)$ gives an antiequivalence of category between the category of $R$-oriented abelian varieties [a] $A$ $k$-isogenous to $E_0^g$ and $R$-oriented $k$-morphisms; and the category of f.p. torsion free $R$-modules $M$ of rank $g$ and $R$-module morphisms.*

*Inverse map: $A \mapsto \mathrm{Hom}_R(A, E_0)$: module of (oriented) morphisms from $A$ to $E_0$*

[a] with the technical condition $\rho_R(A) \simeq \oplus_{i=1}^{g} \rho_R(E_0)$, where $\rho_R(A)$ is the representation of $R/pR$ on Lie $A$

[Kani 2011], [Jordan, Keeton, Poonen, Rains, Shepherd-Barron, Tate 2018], [Page-R. 2023]

### Example

- Frobenius orientation for $E_0/\mathbb{F}_p$: all $\mathbb{F}_p$-rational isogenies at level above $E_0^g$
- If $p$ is inert in $R$, the Frobenius isogeny $\pi_p : E_0 \to E_0^{(p)}$ cannot be represented by an $R$-module morphism $\Rightarrow$ Needs extra “Dieudonné” information to handle general inseparable isogenies, see [Centeleghe-Stix 2015, 2023; Bergström-Karemaker-Marseglia 2024]

- Symmetric monoidal structure: $(M_1 \cdot E_0) \otimes_{E_0} (M_2 \cdot E_0) := (M_1 \otimes_R M_2) \cdot E_0 = M_1 \cdot M_2 \cdot E_0$
This is an abelian variety if $M_1 \otimes_R M_2$ is torsion free.

# Computing the module action

- Needs to work with polarised abelian varieties. For simplicity: stick to ppavs.
- Since the Rosati involution on $E_0$ induces the complex conjugation on $R$, a principal polarisation on $M \cdot E_0$ corresponds to a unimodular $R$-Hermitian form on $M$ [Serre 1985, 2001], [Kirschmer, Narbonne, Ritzenthaler, R. 2021],
- If $(M_1, H_1), (M_2, H_2)$ are unimodular torsion free Hermitian $R$-modules of rank $g$ then $(A_i, \lambda_i) = (M_i, H_i) \cdot (E_0, \lambda_0)$ are principally polarised abelian varieties of dimension $g$
- We have a $M_1$-module orientation on $A_1$: if $m_1 \in M_1$, the map $R \to M_1, r \mapsto r m_1$ induces

$$m_1 : A_1 \to E_0.$$

**Proposition ([Kirschmer, Narbonne, Ritzenthaler, R. 2021])**

*If $\psi : (M_2, H_2) \hookrightarrow (M_1, H_1)$ is an $N$-similitude (i.e. $\psi^* H_1 = N H_2$), then $\phi : (A_1, \lambda_1) \to (A_2, \lambda_2)$ is an $N$-isogeny of ppavs, with kernel*

$$\operatorname{Ker} \phi = M_1/M_2 \cdot A = A_1[M_2] = \{P \in A_1(\overline{k}) \mid m(P) = 0_{E_0} \forall m \in M_2\}$$

**Corollary (Clapoti for the module action)**

*If we can find two $N_i$-similitudes $(M, H_M) \to (R^g, H_{R^g})$, with $N_1$ coprime to $N_2$, we can compute $(M, H_M) \cdot E_0$ in polynomial time.*

# Computing the module action

**Proposition ([Kirschmer, Narbonne, Ritzenthaler, R. 2021])**

*If $\psi : (M_2, H_2) \hookrightarrow (M_1, H_1)$ is an $N$-similitude (i.e. $\psi^* H_1 = N H_2$), then $\phi : (A_1, \lambda_1) \to (A_2, \lambda_2)$ is an $N$-isogeny of ppavs, with kernel*

$$\operatorname{Ker} \phi = M_1/M_2 \cdot A = A_1[M_2] = \{P \in A_1(\overline{k}) \mid m(P) = 0_{E_0} \forall m \in M_2\}$$

**Example (The ideal action)**

If $\mathfrak{a} \subset R$, we have a canonical unimodular Hermitian form:

$$H_{\mathfrak{a}}(x, y) = \frac{x\overline{y}}{N(\mathfrak{a})}$$

The inclusion $(\mathfrak{a}, H_{\mathfrak{a}}) \subset (R, H_R)$ is a $N(\mathfrak{a})$-similitude, hence we obtain a $N(\mathfrak{a})$-isogeny

$$\phi_{\mathfrak{a}} : E = R \cdot E \to \mathfrak{a} \cdot E$$

with kernel $(R/\mathfrak{a}) \cdot E = E[\mathfrak{a}]$.

# Linking the supersingular ideal action with an oriented rank $2$ module action

$E_0/\mathbb{F}_p$ primitively oriented by $R = \mathbb{Z}[\pi_p]$.

## Proposition (Weil restriction)

*If $I \subset \mathfrak{O}_0$ and $E_I = I \cdot E_0$, then*

$$(M_I, H_I) \cdot (E_0, \lambda_0) = W_{\mathbb{F}_{p^2}/\mathbb{F}_p}(E_I, \lambda_I)$$

*where $W_{\mathbb{F}_{p^2}/\mathbb{F}_p}$ is the Weil restriction, $M_I$ is $I$ seen as an $R$-module, and $H_I$ is derived from the quaternionic Hermitian form*

$$H_{\mathfrak{O}_0, I} : x, y \in I \mapsto x\overline{y}/N(I).$$

## Corollary (Module inversion)

*The rank $2$ unimodular module supersingular action inversion problem over $\mathbb{F}_p$ is at least as hard as the supersingular isogeny path problem over $\mathbb{F}_{p^2}$.*

# ⊗-MIKE

$$\begin{array}{ccc} E_0' & \longrightarrow & E_{I_1} \\ \downarrow & & \rightsquigarrow \\ E_{I_2} & \rightsquigarrow & A_{12} = W'_{\mathbb{F}_{p^2}/\mathbb{F}_p} E_{I_1} \otimes_{E_0'} W'_{\mathbb{F}_{p^2}/\mathbb{F}_p} E_{I_2} \end{array}$$

- $E_0' : y^2 = x^3 - x / \mathbb{F}_p, \quad p = u 2^e - 1.$ (Ex: $p = 5 \cdot 2^{248} - 1$.)
- Alice and Bob each compute a $2^e$-isogeny from $E_0'$ over $\mathbb{F}_{p^2}$
- Then the common key $A_{12}$ requires computing a $2^e$-isogeny in dimension $4$ over $\mathbb{F}_p$
- No need for coprime degrees!
- Conjecture: 512 bits NIKE for 128 bits of quantum security
  This conjecture holds if:
  1. the module Diffie-Helmann problem is as hard as module action inversion;
  2. The difficulty of recovering the supersingular isogeny $E_0' \to E_{I_1}$ has $e/2$ bits of quantum security.

## Help needed!

Need good dimension $4$ modular invariants to represent $A_{12}$ (e.g. suitable symmetric polynomials in the theta constants?)

# Perspectives

- Implement this!
- Public Key Encryption via an ElGamal approach
- Signatures?
- Other protocols? (Problem: the dimension grows exponentially with the number of actions…)

- Can handle twists by looking at Galoisian $R[G]$-modules actions to encode descent data

**Example (Quadratic twists: $G = \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) = \langle \sigma \rangle$)**

- if $R' = R$ with $\sigma$ acting by $-1$, then $R' \cdot E_0 = E_0^t$ is the quadratic twist, and

$$R' \cdot I \cdot R' \cdot E_0 \simeq \bar{I} \cdot E_0$$

- $W_{\mathbb{F}_{p^2}/\mathbb{F}_p} E_0 = R[G] \cdot E_0$

- Extend the module equivalence of category to a ppav $(A_0, \lambda_0)$ primitively oriented by a CM order $\mathcal{O}$ with maximal real multiplication.

  (And such that the Rosati involution restricts to the complex conjugation on $\mathcal{O}$. Maximal real multiplication ensures that $\mathcal{O}$ is a Bass order)

# Constructing the power object

- Embed $R-\mathrm{Ab}$ into $R$-oriented proper commutative group schemes to get an abelian category
- Embed both categories ($R$-modules and $R$-oriented proper commutative group schemes) inside the (big) fppf-topos (sheafs for the fppf site of $\operatorname{Spec} k$)
- We obtain abelian subcategories of fppf $\underline{R}$-modules.
  More precisely we have exact fully faithful morphisms:
  - to an $R$-oriented proper commutative group scheme $G$ we associate its functor of points $S \mapsto G(S)$, which is an fppf sheaf
  - to an $R$-module $M$ we associate $\underline{M}$ is the fppf-sheafification of the constant (pre)sheaf $M$

- $\mathcal{H}om_R(M, A) := \mathcal{H}om_{R-fppf}(\underline{M}, A)$ is the $\underline{R}$-Hom sheaf (internal $\underline{R}$-Hom in the fppf-topos)

- This is only the power object in the larger category of $\underline{R}$-modules. Still, if this is (the sheaf associated to) an abelian variety, then it has to be the power object for ($R$-oriented) abelian varieties.
- If $M$ is f.p., this is always (the sheaf associated to) a proper commutative group scheme.

## Exactness properties

- Recall: if $0 \to M_2 \to M_1 \to M_1/M_2 \to 0$ is exact, and $\mathcal{H}om_R(M_2, A)$ is an abelian variety, then
$$0 \to \mathcal{H}om_R(M_1/M_2, A) \to \mathcal{H}om_R(M_1, A) \to \mathcal{H}om_R(M_2, A) \to 0$$
is exact

- In general, we have a long exact sequence
$$0 \to \mathcal{H}om_R(M_1/M_2, A) \to \mathcal{H}om_R(M_1, A) \to \mathcal{H}om_R(M_2, A) \to \mathcal{E}xt^1_R(M_1/M_2, A) \to \mathcal{E}xt^1_R(M_1, A) \to \mathcal{E}xt^1_R(M_1, A) \to \dots$$

There are different variants of $\mathcal{E}xt^1_R$ we can take here:

- $\mathcal{E}xt^1_R(M, A) := \mathcal{E}xt^1_{R-fppf}(\underline{M}, A) = H^1(\mathcal{R}\mathcal{H}om_{R-fppf}(\underline{M}, A))$
- $\mathcal{E}xt^1_R(M, A) := i^*_{fppf}\mathcal{E}xt^1_{R-PSh}(M, A)$ where $i^*_{fppf}$ is the fppf sheafification of presheaves

# Scholten's construction

- To have lots of $2^e$-torsion, we work with $p \equiv 7 \pmod 8$, so we have a non trivial 2-volcano
- For technical reasons, we will start with a curve $E'_0$ on the crater of the 2-volcano rather than on the floor
- $\mathrm{End}_{\mathbb{F}_p}(E'_0)$ is the maximal order $O_R$ of $R = \mathbb{Z}[\pi_p]$, and the conductor $\mathfrak{f} \subset \mathbb{Z}[\pi_p]$ is of index 2
- We use a slight variant of the Weil restriction: $W'_{\mathbb{F}_{p^2}/\mathbb{F}_p} = \mathfrak{f} \cdot_R W_{\mathbb{F}_{p^2}/\mathbb{F}_p}$
  (we can prove that $W'_{\mathbb{F}_{p^2}/\mathbb{F}_p}$ gives Scholten's construction)
- If $E_{I'} = I' \cdot E'_0$ for $I' \subset \mathfrak{O}'_0$, we still have $(M_{I'}, H_{I'}) \cdot_{O_R} (E'_0, \lambda'_0) = W'_{\mathbb{F}_{p^2}/\mathbb{F}_p}(E_{I'}, \lambda_{I'})$
- In practice: take $E'_0 : y^2 = x^3 - x / \mathbb{F}_p$, so that $W'_{\mathbb{F}_{p^2}/\mathbb{F}_p} E'_0 \simeq {E'_0}^2$