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P-ADIC HEIGHTS AND p-ADIC HODGE THEORY

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Abstract. — Using the theory of (φ,Γ)-modules and the formalism of Selmer complexes we construct the p-adic height pairing for p-adic representations with coefficients in an affinoid algebra over $\mathbf{Q}_p$. For p-adic representations that are potentially semistable at p, we relate our contruction to universal norms and compare it to the p-adic height pairings of Nekovář and Perrin-Riou.

Résumé **(Hauteurs p-adiques et théorie de Hodge p-adique)**

En utilisant la théorie des (φ,Γ)-modules et le formalisme des complexes de Selmer nous construisons un accouplement de hauteur p-adique pour les représentations p-adiques à coefficients dans une algèbre affinoïde. Pour les représentations p-adiques potentiellement semistables en p nous ferons le lien de notre construction avec les normes universelles et les hauteurs p-adiques construites par Nekovář et Perrin-Riou.

CONTENTS

INTRODUCTION

0.1. Selmer complexes

0.1.1. — Let F be a number field. We denote by S_f and S_∞ the set of non-archimedean and archimedean places of F respectively. Fix a prime number p and denote by S_p the set of places $\mathfrak{q}$ above p.

Let S be a finite set of non-archimedean places of F containing S_p. To simplify notation, set $\Sigma_p = S \setminus S_p$. We denote by $G_{F,S}$ the Galois group of the maximal algebraic extension F_S of F unramified outside $S \cup S_\infty$. For each $\mathfrak{q} \in S$ we denote by $F_\mathfrak{q}$ the completion of F with respect to $\mathfrak{q}$ and by $G_{F_\mathfrak{q}}$ the absolute Galois group of $F_\mathfrak{q}$. We will write $I_\mathfrak{q}$ for the inertia subgroup of $G_{F_\mathfrak{q}}$ and $\mathrm{Fr}_\mathfrak{q}$ for the relative Frobenius over $F_\mathfrak{q}$. Fix an extension of $\mathfrak{q}$ to F_S and identify $G_{F_\mathfrak{q}}$ with the corresponding decomposition group at $\mathfrak{q}$.

We denote by $\chi : G_{F,S} \to \mathbf{Z}_p^*$ the p-adic cyclotomic character and, for each $\mathfrak{q} \in S_p$, write $\chi_\mathfrak{q}$ for the restriction of χ on $G_{F_\mathfrak{q}}$. If M is a topological $\mathbf{Z}_p$-module equipped with a continuous linear action of $G_{F,S}$ (resp. $G_{F_\mathfrak{q}}$) we denote by $M(\chi)$ (resp. $M(\chi_\mathfrak{q})$) or alternatively by $M(1)$ its Tate twist.

If G is a topological group and M is a topological G-module, we denote by $C^\bullet(G,M)$ the complex of continuous cochains of G with coefficients in M. If $X = M^\bullet$ is a complex of topological G-modules, we denote by $C^\bullet(G,X)$ the total complex associated to the double complex $C^n(G,M^m)$.

0.1.2. — Let A be a complete local noetherian ring with a finite residue field of characteristic p. An *admissible $A[G_{F,S}]$-module of finite type* is a $A[G_{F,S}]$-module T of finite type over A and such that the map $G_{F,S} \to \mathrm{Aut}_A(T)$ is continuous [(1)].

1. In other words, T is a "big" Galois representation with coefficients in A in the sense of [**56**].

Let $X = T^\bullet$ be a bounded complex of admissible $A[G_{F,S}]$-modules of finite type. A local condition at $\mathfrak{q} \in S$ is a morphism of complexes

$$g_\mathfrak{q} : U_\mathfrak{q}^\bullet(X) \to C^\bullet(G_{F_\mathfrak{q}}, X).$$

To each collection $U^\bullet(X) = (U_\mathfrak{q}^\bullet(X), g_\mathfrak{q})_{\mathfrak{q}\in S}$ of local conditions one can associate the following diagram

$$\begin{array}{ccc} C^\bullet(G_{F,S}, X) & \longrightarrow & \bigoplus_{\mathfrak{q}\in S} C^\bullet(G_{F_\mathfrak{q}}, X) \\ & & \uparrow{\scriptstyle (g_\mathfrak{q})} \\ & & \bigoplus_{\mathfrak{q}\in S} U_\mathfrak{q}^\bullet(X), \end{array} \tag{1}$$

where the upper row is the restriction map. The Selmer complex associated to the local conditions $U^\bullet(X)$ is defined as the mapping cone

$$S^\bullet(X, U^\bullet(X)) = \mathrm{cone}\left(C^\bullet(G_{F,S}, X) \oplus \left(\bigoplus_{\mathfrak{q}\in S} U_\mathfrak{q}^\bullet(X) \right) \to \bigoplus_{\mathfrak{q}\in S} C^\bullet(G_{F_\mathfrak{q}}, X) \right)[-1].$$

This notion was introduced by Nekovář in [**56**], where the machinery of Selmer complexes was developed in full generality.

0.1.3. — The most important example of local conditions is provided by Greenberg's local conditions [**56**, Section 7.8]. If $\mathfrak{q} \in S$, we will denote by $X_\mathfrak{q}$ the restriction of X on $G_{F_\mathfrak{q}}$. For each $\mathfrak{q} \in S_p$ we fix a complex $M_\mathfrak{q}$ of admissible $A[G_{F_\mathfrak{q}}]$-modules of finite type together with a morphism $M_\mathfrak{q} \to X_\mathfrak{q}$ and define

$$U_\mathfrak{q}^\bullet(X) = C^\bullet(G_{F_\mathfrak{q}}, M_\mathfrak{q}) \qquad \mathfrak{q} \in S_p.$$

For $\mathfrak{q} \in \Sigma_p$ we consider the unramified local conditions

$$U_\mathfrak{q}^\bullet(X) = C_{\mathrm{ur}}^\bullet(X_\mathfrak{q})$$

(see [**56**, Section 7.6] for the precise definition). In particular, if $X = T[0]$ is concentrated in degree 0, then

$$C_{\mathrm{ur}}^\bullet(X) = \left(T^{I_\mathfrak{q}} \xrightarrow{\mathrm{Fr}_\mathfrak{q} - 1} T^{I_\mathfrak{q}} \right),$$

where the terms are placed in degrees 0 and 1. To simplify notation, we will write $S^\bullet(X, M)$ for the Selmer complex associated to these conditions and $\mathbf{R}\Gamma(X, M)$ for the corresponding object of the derived category of A-modules of finite type.

0.1.4. — Let ω_A denote the dualizing complex for A. The Grothendieck dualization functor

$$X \to \mathfrak{D}(X) := \mathbf{R}\mathrm{Hom}_A(X, \omega_A)$$

is an anti-involution on the bounded derived category of admissible $A[G_{F,S}]$-modules of finite type [**56**, Section 4.3.2]. Consider the complex $\mathfrak{D}(X)(1)$ equipped with Greenberg local conditions $N = (N_\mathfrak{q})_{\mathfrak{q}\in S_p}$ such that M and N are orthogonal to each other under the canonical duality $X \times \mathfrak{D}(X)(1) \to \omega_A(1)$. In this case, the general construction of cup products for cones gives a pairing

$$\cup\, : \mathbf{R}\Gamma(X,M) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(\mathfrak{D}(X)(1),N) \to \omega_A[-3]$$

(see [**56**, Section 6.3]). Nekovář constructed the p-adic height pairing

$$h^{\mathrm{sel}}\, : \mathbf{R}\Gamma(X,M) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(\mathfrak{D}(X)(1),N) \to \omega_A[-2]$$

as the composition of $\cup$ with the Bockstein map [2] $\beta_{X,M} : \mathbf{R}\Gamma(X,M) \to \mathbf{R}\Gamma(X,M)[1]$:

$$h^{\mathrm{sel}}(x,y) = \beta_{X,M}(x) \cup y.$$

Passing to cohomology groups $H^i(X,M) := \mathbf{R}^i\Gamma(X,M)$, we obtain a pairing

$$h^{\mathrm{sel}}_1\, : H^1(X,M) \otimes_A H^1(\mathfrak{D}(X)(1),N) \to H^0(\omega_A). \qquad (2)$$

0.1.5. — The relationship of these constructions to traditional treatements is the following. Let $A = \mathscr{O}_E$ be the ring of integers of a local field $E/\mathbf{Q}_p$ and let T be a Galois stable $\mathscr{O}_E$-lattice of a p-adic Galois representation V with coefficients in E. We consider T as a complex concentrated in degree 0. Then $\omega_A = \mathscr{O}_E[0]$ and $\mathfrak{D}(T)$ coincides with the classical dual $T^* = \mathrm{Hom}_{\mathscr{O}_E}(T, \mathscr{O}_E)$. Each choice of orthogonal local conditions provides

$$h^{\mathrm{sel}}_1\, : H^1(T,M) \otimes_A H^1(T^*(1),N) \to \mathscr{O}_E.$$

Assume, in addition, that V is semistable in the sense of p-adic Hodge theory at all $\mathfrak{q} \in S_p$. We say that V satisfies the Panchishkin condition at p if, for each $\mathfrak{q} \in S_p$, there exists a subrepresentation $V^+_\mathfrak{q} \subset V_\mathfrak{q}$ such that all Hodge–Tate weights [3] of $V_\mathfrak{q}/V^+_\mathfrak{q}$ are $\geqslant 0$. Set $T^+_\mathfrak{q} = T \cap V^+_\mathfrak{q}$, $T^+ = (T^+_\mathfrak{q})_{\mathfrak{q}\in S_p}$. The cohomology group $H^1(T, T^+)$ is very close to the Selmer group defined by Greenberg [**33, 34**] and therefore to the Bloch–Kato Selmer group [**28**]. It can be shown [**56**, Theorem 11.3.9] that, under some mild conditions, the pairing h^{sel}_1 coincides with the p-adic height pairing constructed by Schneider [**66**], Perrin-Riou [**59**] and Nekovář [**54**] using universal norms.

2. See [**56**, Section 11.1] or Section 3.2 below for the definition of the Bockstein map.

3. We call Hodge–Tate weights the jumps of the Hodge–Tate filtration on the associated de Rham module. In particular, the Hodge–Tate weight of $\mathbf{Q}_p(1)$ is -1.

0.1.6. — More generally, assume that A is a Gorenstein ring and T is an admissible module of finite type which is projective over A. Then ω_A is quasi-isomorphic to A and again $\mathfrak{D}(T) = T^*$ where $T^* = \mathrm{Hom}_A(T,A)$. Then (2) takes the form

$$h_1^{\mathrm{sel}} : H^1(T,M) \otimes_A H^1(T^*(1),N) \to A. \tag{3}$$

Note that Nekovář's construction has many advantages over the classical definitions. In particular, it allows to study the variation of the p-adic heights in ordinary families of p-adic representations (see [**56**, Section 0.16 and Chapter 11], for further discussion).

0.2. Selmer complexes and (φ,Γ)-modules

0.2.1. — In this paper we study Selmer complexes associated to p-adic representations with coefficients in an affinoid algebra and local conditions coming from the theory of (φ,Γ)-modules. Namely, let A be a $\mathbf{Q}_p$-affinoid algebra. We will work in the category $\mathscr{K}_{\mathrm{ft}}^{[a,b]}(A)$ of complexes of A-modules whose cohomologies are finitely generated over A and concentrated in degrees $[a,b]$ and in the corresponding derived category $\mathscr{D}_{\mathrm{ft}}^{[a,b]}(A)$. Let $\mathscr{D}_{\mathrm{perf}}^{[a,b]}(A)$ denote the category of $[a,b]$-bounded perfect complexes over A, i.e. the full subcategory of $\mathscr{D}_{\mathrm{ft}}^{[a,b]}(A)$ consisting of objects quasi-isomorphic to complexes of finitely generated projective A-modules concentrated in degrees $[a,b]$.

A p-adic representation of $G_{F,S}$ with coefficients in A is a finitely generated projective A-module V equipped with a continuous A-linear action of $G_{F,S}$. In [**62**], Pottharst studied Selmer complexes associated to the diagrams of the form (1) in this context. We will consider a slightly more general situation because, for the local conditions $U_{\mathfrak{q}}^\bullet(V)$ that we have in mind, the maps $g_{\mathfrak{q}} : U_{\mathfrak{q}}^\bullet(V) \to C^\bullet(G_{F_{\mathfrak{q}}},V)$ are not defined on the level of complexes but only in the derived category $\mathscr{D}_{\mathrm{ft}}^{[0,2]}(A)$.

For each $\mathfrak{q} \in S_p$ we denote by $\Gamma_{\mathfrak{q}}$ the Galois group of the cyclotomic p-extension of $F_{\mathfrak{q}}$. As before, we denote by $V_{\mathfrak{q}}$ the restriction of V on the decomposition group at $\mathfrak{q}$. The theory of (φ,Γ)-modules associates to $V_{\mathfrak{q}}$ a finitely generated projective module $\mathbf{D}_{\mathrm{rig},A}^\dagger(V)$ over the Robba ring $\mathscr{R}_{F_{\mathfrak{q}},A}$ equipped with a semilinear Frobenius map φ and a continuous action of $\Gamma_{\mathfrak{q}}$ which commute to each other [**29, 18, 22, 45**]. In [**46**], Kedlaya, Pottharst and Xiao extended the results of Liu [**49**] about the cohomology of (φ,Γ)-modules to the relative case. Their results play a key role in this paper.

Namely, to each $(\varphi,\Gamma_{\mathfrak{q}})$-module $\mathbf{D}$ over $\mathscr{R}_{F_{\mathfrak{q}},A}$ one can associate the Fontaine–Herr complex $C_{\varphi,\gamma_{\mathfrak{q}}}^\bullet(\mathbf{D})$ of $\mathbf{D}$. The cohomology $H^*(\mathbf{D})$ of $\mathbf{D}$ is defined as the cohomology of $C_{\varphi,\gamma_{\mathfrak{q}}}^\bullet(\mathbf{D})$. If $\mathbf{D} = \mathbf{D}_{\mathrm{rig},A}^\dagger(V)$, there exist isomorphisms $H^*(\mathbf{D}_{\mathrm{rig},A}^\dagger(V)) \simeq H^*(F_{\mathfrak{q}},V)$, but the complexes $C^\bullet_{\varphi,\gamma_{\mathfrak q}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ and $C^\bullet(G_{F_{\mathfrak q}},V_{\mathfrak q})$ are not quasi-isomorphic. A simple argument allows us to construct a complex $K^\bullet(V_{\mathfrak q})$ together with quasi-isomorphisms $\xi_{\mathfrak q}\,:\,C^\bullet(G_{F_{\mathfrak q}},V)\to K^\bullet(V_{\mathfrak q})$ and $\alpha_{\mathfrak q}\,:\,C^\bullet_{\varphi,\gamma_{\mathfrak q}}(\mathbf{D}^\dagger_{\mathrm{rig},A}(V_{\mathfrak q}))\to K^\bullet(V_{\mathfrak q})$ (4). For each $\mathfrak q\in S_p$, we choose a $(\varphi,\Gamma_{\mathfrak q})$-submodule $\mathbf{D}_{\mathfrak q}$ of $\mathbf{D}^\dagger_{\mathrm{rig},A}(V_{\mathfrak q})$ that is a $\mathscr{R}_{F_{\mathfrak q},A}$-module direct summand of $\mathbf{D}^\dagger_{\mathrm{rig},A}(V_{\mathfrak q})$ and set $\mathbf{D}=(\mathbf{D}_{\mathfrak q})_{\mathfrak q\in S_p}$. Set

$$K^\bullet(V)=\left(\bigoplus_{\mathfrak q\in\Sigma_p}C^\bullet(G_{F_{\mathfrak q}},V)\right)\bigoplus\left(\bigoplus_{\mathfrak q\in S_p}K^\bullet(V_{\mathfrak q})\right)$$

and

$$U^\bullet_{\mathfrak q}(V,\mathbf{D})=\begin{cases}C^\bullet_{\varphi,\gamma_{\mathfrak q}}(\mathbf{D}_{\mathfrak q}), & \text{if } \mathfrak q\in S_p,\\ C^\bullet_{\mathrm{ur}}(V_{\mathfrak q}), & \text{if } \mathfrak q\in\Sigma_p.\end{cases}$$

For each $\mathfrak q\in S_p$, we have morphisms

$$\begin{aligned} f_{\mathfrak q}\,&:\,C^\bullet(G_{F,S},V)\xrightarrow{\mathrm{res}_{\mathfrak q}}C^\bullet(G_{F_{\mathfrak q}},V)\xrightarrow{\xi_{\mathfrak q}}K^\bullet(V_{\mathfrak q}),\\ g_{\mathfrak q}\,&:\,U^\bullet_{\mathfrak q}(V,\mathbf{D})\to C^\bullet_{\varphi,\gamma_{\mathfrak q}}(\mathbf{D}^\dagger_{\mathrm{rig},A}(V_{\mathfrak q}))\xrightarrow{\alpha_{\mathfrak q}}K^\bullet(V_{\mathfrak q}).\end{aligned}$$

If $\mathfrak q\in\Sigma_p$, we define the maps $f_{\mathfrak q}\,:\,C^\bullet(G_{F,S},V)\to C^\bullet(G_{F_{\mathfrak q}},V)$ and $g_{\mathfrak q}\,:\,C^\bullet_{\mathrm{ur}}(V_{\mathfrak q})\to C^\bullet(G_{F_{\mathfrak q}},V)$ exactly as in the case of Greenberg local conditions. Consider the diagram

$$\begin{array}{ccc} C^\bullet(G_{F,S},V) & \xrightarrow{(f_{\mathfrak q})_{\mathfrak q\in S}} & K^\bullet(V)\\ & & \Big\uparrow{\scriptstyle \bigoplus_{\mathfrak q\in S} g_{\mathfrak q}}\\ & & \bigoplus_{\mathfrak q\in S}U^\bullet_{\mathfrak q}(V,\mathbf{D}). \end{array}$$

We denote by $S^\bullet(V,\mathbf{D})$ the Selmer complex associated to this diagram and by $\mathbf{R}\Gamma(V,\mathbf{D})$ the corresponding object in the derived category of A-modules. Mimicking the arguments of [**62**, Section 1E] we see that $\mathbf{R}\Gamma(V,\mathbf{D})$ belongs to $\mathscr{D}^{[0,3]}_{\mathrm{ft}}(A)$. If, in addition, local conditions at all $\mathfrak q\in\Sigma_p$ can be represented by perfect complexes, then $\mathbf{R}\Gamma(V,\mathbf{D})$ belongs to $\mathscr{D}^{[0,3]}_{\mathrm{perf}}(A)$ (see Section 3.1 for detail).

The functor

$$X\to X^*:=\mathbf{R}\mathrm{Hom}_A(X,A)$$

is an anti-involution on the derived category $\mathscr{D}_{\mathrm{perf}}(A)$ of perfect complexes which can be viewed as a simple analog of the Grothendick duality $\mathfrak{D}$ in our context. For any

4. This complex was first introduced in [**9**].

p-adic representation V we have $V^* = \mathrm{Hom}_A(V,A)$. We equip $V^*(1)$ with orthogonal local conditions $\mathbf{D}^{\perp}$ setting

$$\mathbf{D}^{\perp}_{\mathfrak{q}} = \mathrm{Hom}_{\mathscr{R}_{F_{\mathfrak{q}},A}}\big(\mathbf{D}^{\dagger}_{\mathrm{rig},A}(V_{\mathfrak{q}})/\mathbf{D}_{\mathfrak{q}}, \mathscr{R}_{F_{\mathfrak{q}},A}(\chi_{\mathfrak{q}})\big), \qquad \mathfrak{q} \in S_p.$$

The general machinery gives us a cup product pairing

$$\cup_{V,\mathbf{D}} \,:\, \mathbf{R}\Gamma(V,\mathbf{D}) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) \to A[-3].$$

If local conditions at all $\mathfrak{q} \in \Sigma_p$ can be represented by perfect complexes, this pairing gives a duality in $\mathscr{D}^{[0,3]}_{\mathrm{perf}}(A)$:

$$\mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) \simeq \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(V,\mathbf{D}),A)[-3]$$

(see Theorem 3.1.5 and Section 3.1.6).

0.3. p-adic height pairings

0.3.1. — The previous theory allows us to construct the p-adic height pairing exactly in the same way as in the case of Greenberg local conditions. Let V be a p-adic representation with coefficients in A and $V^*(1)$ the Tate dual of V.

Definition. — *The p-adic height pairing associated to the data $(V,\mathbf{D})$ is defined as the morphism*

$$h^{\mathrm{sel}}_{V,\mathbf{D}} \,:\, \mathbf{R}\Gamma(V,\mathbf{D}) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) \xrightarrow{\delta_{V,\mathbf{D}}} \\ \to \mathbf{R}\Gamma(V,\mathbf{D})[1] \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) \xrightarrow{\cup_{V,\mathbf{D}}} A[-2],$$

where $\delta_{V,\mathbf{D}}$ denotes the Bockstein map.

The height pairing $h^{\mathrm{sel}}_{V,\mathbf{D},M}$ induces a pairing on cohomology groups

$$h^{\mathrm{sel}}_{V,\mathbf{D},1} \,:\, H^1(V,\mathbf{D}) \times H^1(V^*(1),\mathbf{D}^{\perp}) \to A.$$

Applying the machinery of Selmer complexes, we obtain the following result (see Theorem 3.2.4 below).

Theorem I. — *We have a commutative diagram*

$$\begin{array}{ccc}
\mathbf{R}\Gamma(V,\mathbf{D}) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) & \xrightarrow{h^{\mathrm{sel}}_{V,\mathbf{D}}} & A[-2] \\
\downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle =} \\
\mathbf{R}\Gamma(V^*(1),\mathbf{D}^{\perp}) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V,\mathbf{D}) & \xrightarrow{h^{\mathrm{sel}}_{V^*(1),\mathbf{D}^{\perp}}} & A[-2],
\end{array}$$

where $s_{12}(a\otimes b) = (-1)^{\deg(a)\deg(b)} b\otimes a$. In particular, the pairing $h^{\mathrm{sel}}_{V,\mathbf{D},1}$ is skew symmetric.

0.3.2. — Assume that $A = E$, where E is a finite extension of $\mathbf{Q}_p$. Fix a system $\mathbf{D} = (\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ of submodules $\mathbf{D}_{\mathfrak{q}} \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$ and consider tautological exact sequences

$$0 \to \mathbf{D}_{\mathfrak{q}} \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}}) \to \mathbf{D}'_{\mathfrak{q}} \to 0, \qquad \mathfrak{q} \in S_p,$$

where $\mathbf{D}'_{\mathfrak{q}} = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})/\mathbf{D}_{\mathfrak{q}}$. Passing to duals, we have exact sequences

$$0 \to (\mathbf{D}'_{\mathfrak{q}})^*(\chi_{\mathfrak{q}}) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{\mathfrak{q}}(1)) \to \mathbf{D}^*_{\mathfrak{q}}(\chi_{\mathfrak{q}}) \to 0,$$

where $(\mathbf{D}'_{\mathfrak{q}})^*(\chi_{\mathfrak{q}}) = \mathbf{D}^{\perp}_{\mathfrak{q}}$. Consider the following conditions on the data $(V,\mathbf{D})$ (see Section 5.1):

N1) $H^0(F_{\mathfrak{q}},V) = H^0(F_{\mathfrak{q}},V^*(1)) = 0$ for all $\mathfrak{q} \in S_p$;

N2) $H^0(\mathbf{D}'_{\mathfrak{q}}) = H^0(\mathbf{D}^*_{\mathfrak{q}}(\chi_{\mathfrak{q}})) = 0$ for all $\mathfrak{q} \in S_p$.

For each data $(V,\mathbf{D})$ satisfying these conditions we construct a pairing

$$h^{\mathrm{norm}}_{V,\mathbf{D}} \,:\, H^1(V,\mathbf{D}) \times H^1(V^*(1),\mathbf{D}^{\perp}) \to E,$$

which can be seen as a direct generalization of the p-adic height pairing, constructed for representations satisfying the Panchishkin condition using universal norms [**66, 54, 59**]. The following theorem generalizes [**56**, Theorem 11.3.9] (see Theorem 5.2.2 below).

Theorem II. — *Let V be a p-adic representation of $G_{F,S}$ with coefficients in a finite extension E of $\mathbf{Q}_p$. Assume that the family $\mathbf{D} = (\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ satisfies conditions* **N1-2)**. *Then*

$$h^{\mathrm{norm}}_{V,\mathbf{D}} = -h^{\mathrm{sel}}_{V,\mathbf{D},1}.$$

0.3.3. — We denote by $\mathbf{D}_{\mathrm{dR}}$, $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{D}_{\mathrm{st}}$ Fontaine's classical functors [**30, 31**]. Let V be a p-adic representation with coefficients in $E/\mathbf{Q}_p$. Assume that the restriction of V on $G_{F_{\mathfrak{q}}}$ is potentially semistable for all $\mathfrak{q} \in S_p$, and that V satisfies the following condition:

S) $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} = \mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} = 0, \qquad \forall \mathfrak{q} \in S_p.$

For each $\mathfrak{q} \in S_p$ we fix a splitting $w_{\mathfrak{q}} : \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ of the canonical projection $\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ and set $w = (w_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$. In this situation, Nekovář [**54**] constructed a p-adic height pairing

$$h^{\mathrm{Hodge}}_{V,w} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

on the Bloch–Kato Selmer groups [**16**] of V and $V^*(1)$, which is defined using the Bloch–Kato exponential map and depends on the choice of splittings w.

Let $\mathfrak{q} \in S_p$, and let L be a finite extension of $F_{\mathfrak{q}}$ such that $V_{\mathfrak{q}}$ is semistable over L. The semistable module $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ is a finite dimensional vector space over the maximal unramified subextension L_0 of L, equipped with a Frobenius φ, a monodromy N, and an action of $G_{L/F_{\mathfrak{q}}} = \mathrm{Gal}(L/F_{\mathfrak{q}})$.

Definition. — *Let $\mathfrak{q} \in S_p$. We say that a $(\varphi, N, G_{L/F_{\mathfrak{q}}})$-submodule $D_{\mathfrak{q}}$ of $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ is a splitting submodule if*

$$\mathbf{D}_{\mathrm{dR}/L}(V_{\mathfrak{q}}) = D_{\mathfrak{q},L} \oplus \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}/L}(V_{\mathfrak{q}}), \qquad D_{\mathfrak{q},L} = D_{\mathfrak{q}} \otimes_{L_0} L$$

as L-vector spaces.

It is easy to see, that each splitting submodule $D_{\mathfrak{q}}$ defines a splitting of the Hodge filtration of $\mathbf{D}_{\mathrm{dR}}(V)$, which we denote by $w_{D,\mathfrak{q}}$. For each family $D = (D_{\mathfrak{q}})_{\mathfrak{q} \in S_p}$ of splitting submodules we construct a pairing

$$h^{\mathrm{spl}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

using the theory of (φ, Γ)-modules and prove that

$$h^{\mathrm{spl}}_{V,D} = h^{\mathrm{Hodge}}_{V,w_D}$$

(see Proposition 6.2.3). Let $\mathbf{D}_{\mathfrak{q}}$ denote the $(\varphi, \Gamma_{\mathfrak{q}})$-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$ associated to $D_{\mathfrak{q}}$ by Berger [**14**] and let $\mathbf{D} = (\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q} \in S_p}$. In the following theorem we compare this pairing with previous constructions (see Theorem 6.3.3 and Corollary 6.3.4).

Theorem III. — *Assume that (V, D) satisfies conditions* **S)** *and* **N2)**. *Then*
i) $H^1(V, \mathbf{D}) = H^1_f(V)$ and $H^1(V^*(1), \mathbf{D}^{\perp}) = H^1_f(V^*(1))$;
ii) We have

$$h^{\mathrm{norm}}_{V,\mathbf{D}} = h^{\mathrm{spl}}_{V,D} = -h^{\mathrm{sel}}_{V,\mathbf{D},1}.$$

0.3.4. — If $F = \mathbf{Q}$, we can relax condition **N2)**. Namely, for each splitting submodule $D = D_p$ of $\mathbf{D}_{\mathrm{st}/L}(V_p)$, we construct a canonical filtration

$$\{0\} \subset F_{-1}\mathbf{D}_{\mathrm{st}/L}(V) \subset F_0\mathbf{D}_{\mathrm{st}/L}(V) \subset F_1\mathbf{D}_{\mathrm{st}/L}(V) \subset \mathbf{D}_{\mathrm{st}/L}(V) \tag{4}$$

which is a direct generalization of the filtration constructed in [**7**] in the semistable case. In particular, $F_0\mathbf{D}_{\mathrm{st}/L}(V) = D$, and the quotients $M_0 = \mathrm{gr}_0\mathbf{D}_{\mathrm{st}/L}(V)$ and $M_1 = \mathrm{gr}_1\mathbf{D}_{\mathrm{st}/L}(V)$ are filtered Dieudonné modules such that

$$M_0^{\varphi = p^{-1}} = M_0, \quad \mathrm{Fil}^0 M_0 = \{0\},$$
$$M_1^{\varphi = 1} = M_1, \qquad \mathrm{Fil}^0 M_1 = M_1.$$

Let $W = F_1\mathbf{D}_{\mathrm{st}/L}(V)/F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)$. We denote by $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{W}$ the $(\varphi,\Gamma_{\mathbf{Q}_p})$-modules associated to M_0, M_1 and W respectively. The tautological exact sequence

$$0 \to \mathbf{M}_0 \to \mathbf{W} \to \mathbf{M}_1 \to 0$$

induces the coboundary map

$$\delta_0 : H^0(\mathbf{M}_1) \to H^1(\mathbf{M}_0).$$

We introduce the following conditions **F1a-b)** and **F2a-b)** which reflect the conjectural behavior of V at p in the presence of trivial zeros [**7, 10, 35**]

F1a) $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\varphi=1} = \mathscr{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)))/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)))^{\varphi=1} = 0.$

F1b) $\mathscr{D}_{\mathrm{cris}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\varphi=1} = \mathscr{D}_{\mathrm{cris}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)))^{\varphi=1} = 0.$

F2a) The composed map

$$\delta_{0,c} : H^0(\mathbf{M}_1) \xrightarrow{\delta_0} H^1(\mathbf{M}_0) \xrightarrow{\mathrm{pr}_c} H^1_c(\mathbf{M}_0),$$

where the second arrow denotes the canonical projection on $H^1_c(\mathbf{M}_0)$, is an isomorphism.

F2b) The composed map

$$\delta_{0,f} : H^0(\mathbf{M}_1) \xrightarrow{\delta_0} H^1(\mathbf{M}_0) \xrightarrow{\mathrm{pr}_f} H^1_f(\mathbf{M}_0),$$

where the second arrows denotes the canonical projection $H^1_f(\mathbf{M}_0)$, are isomorphisms.

One expects that these conditions hold if V is the p-adic realization of a pure motive over $\mathbf{Q}$ of weight -1 (see Sections 0.4 and 4.3). Note that **F1a-b)** and **F2a)** imply **S)**.

We show that, under conditions **F1a)** and **F2a)**, there exists a canonically splitting exact sequence

$$(5) \qquad 0 \longrightarrow H^0(\mathbf{D}') \underset{\longrightarrow}{\overset{\mathrm{spl}^c_{V,\mathbf{D}}}{\longleftarrow}} H^1(V,\mathbf{D}) \underset{\longrightarrow}{\overset{\mathfrak{s}^c_{V,\mathbf{D}}}{\longleftarrow}} H^1_f(V) \longrightarrow 0,$$

where $\mathbf{D}' = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)/\mathbf{D}$. We call $H^1(V,\mathbf{D})$ *the extended Selmer group* of V associated to $\mathbf{D}$. Note that

$$\dim_E H^0(\mathbf{D}') = \dim_E M_0 = \dim_E M_1.$$

If, in addition, condition **F2b)** is satisfied, there exists another canonical splitting of this sequence

$$0 \longrightarrow H^0(\mathbf{D}') \underset{\mathrm{spl}^f_{V,\mathbf{D}}}{\overset{\longrightarrow}{\longleftarrow}} H^1(V,\mathbf{D}) \underset{\mathfrak{s}^f_{V,\mathbf{D}}}{\overset{\longrightarrow}{\longleftarrow}} H^1_f(V) \longrightarrow 0.$$

The following result is a simplified form of Theorem 7.2.4 below.

Theorem IV. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p and satisfies conditions* **F1a-b)** *and* **F2a-b)**. *Then for all $x \in H^1_f(V)$ and $y \in H^1_f(V^*(1))$ we have*

$$h^{\mathrm{spl}}_{V,D}(x,y) = -h^{\mathrm{sel}}_{V,\mathbf{D}}(\mathfrak{s}^f_{V,\mathbf{D}}(x), \mathfrak{s}^f_{V^*(1),\mathbf{D}^\perp}(y)).$$

Assume now that, instead of **F1a-b)**, the data $(V,\mathbf{D})$ satisfies the following stronger condition

F3) For all $i \in \mathbf{Z}$

$$\mathscr{D}_{\mathrm{pst}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=p^i} = \mathscr{D}_{\mathrm{pst}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=p^i} = 0.$$

By modifying the construction of Section 0.3.2, we define a pairing

$$h^{\mathrm{norm}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E.$$

The following result is proved in Theorem 7.3.2.

Theorem V. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p and satisfies conditions* **F2a-b)** *and* **F3)**. *Then*

$$h^{\mathrm{norm}}_{V,D} = h^{\mathrm{spl}}_{V,D}.$$

Theorems IV and V imply

CorollaryVI. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p and satisfies conditions* **F2a-b)** *and* **F3)**. *Then for all $x \in H^1_f(V)$ and $y \in H^1_f(V^*(1))$ we have*

$$h^{\mathrm{norm}}_{V,D}(x,y) = -h^{\mathrm{sel}}_{V,\mathbf{D}}(\mathfrak{s}^f_{V,\mathbf{D}}(x), \mathfrak{s}^f_{V^*(1),\mathbf{D}^\perp}(y)).$$

This generalizes [**56**, Theorem 11.4.6].

0.4. General remarks

0.4.1. — Assume that V is the p-adic realization of a pure motive M/F of weight $\mathrm{wt}(M)$. Beilinson's conjectures (in the formulation of Bloch and Kato) predict that

$$H^1_f(V) = 0, \quad \text{if} \quad \mathrm{wt}(M) \geqslant 0,$$

and therefore the pairings $h^{\mathrm{norm}}_{V,D}$ and $h^{\mathrm{spl}}_{V,D}$ are interesting only if $\mathrm{wt}(M) = \mathrm{wt}(M^*(1)) = -1$.

0.4.2. — Let $M = h^i(X)(m)$, where X is a smooth projective variety over F and $0 \leqslant i \leqslant 2\dim(X)$. The p-adic realization of M is $V = H^i_p(X)(m)$, where $H^i_p(X)$ denotes the p-adic étale cohomology of $X_{\overline{F}}$. The Poincaré duality and the hard Lefschetz theorem give a canonical isomorphism

$$H^i_p(X)^* \simeq H^i_p(X)(i). \tag{6}$$

Then $\mathrm{wt}(M) = -1$ if i is odd and $m = \dfrac{i+1}{2}$. In this case the representation V is self dual and we have a canonical isomorphism $V \simeq V^*(1)$ induced by (6). If, in addition, X has good reduction at $\mathfrak{q} \in S_p$, then $V_{\mathfrak{q}}$ is crystalline and $\mathbf{D}_{\mathrm{cris}}(V_{\mathfrak{q}})^{\varphi=1} = 0$ by a result of Katz–Messing [**43**]. Therefore, conditions **S)** and **N1-2)** hold if X has good reduction at all $\mathfrak{q} \in S_p$.

0.4.3. — We continue to assume that $V = H^i_p(m)$, where X is a smooth projective variety over F. For all $\mathfrak{q} \in S_p$ the representation $V_{\mathfrak{q}}$ is potentially semistable by the main result of Tsuji [**70**]. Let $L/F_{\mathfrak{q}}$ be a finite extension such that $V_{\mathfrak{q}}$ is semistable over L. The module $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ is equipped with a monodromy N and a Frobenius operator φ. The monodromy filtration $\mathfrak{M}_i\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ on $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ is an increasing filtration defined by

$$\mathfrak{M}_i\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}}) = \sum_{k-l=i} \ker(N^{k+1}) \cap \mathrm{Im}(N^l).$$

It is expected that φ acts semisimply on $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ and the p-adic analog of the monodromy-weight conjecture formulated by Jannsen [**40**] says that the absolute value of eigenvalues of φ acting on $\mathrm{gr}^{\mathfrak{M}}_i\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$ is $p^{(i+\mathrm{wt}(M))/2}$. Since

$$\mathbf{D}_{\mathrm{cris}}(V_{\mathfrak{q}})^{\varphi=1} \subset \mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})^{N=0} \subset \mathfrak{M}_0\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}}),$$

conditions **S)** and **N1)** conjecturally always hold if $\mathrm{wt}(M) = -1$.

On the other hand, condition **N2)** depends on the choice of $\mathbf{D}_{\mathfrak{q}}$ and does not hold in general in the bad reduction case. If it holds, then $h^{\mathrm{norm}}_{V,D} = h^{\mathrm{spl}}_{V,D} = -h^{\mathrm{sel}}_{V,\mathbf{D}}$, and composing this antisymmetric pairing with the isomorphism $H^1_f(V) \simeq H^1_f(V^*(1))$ we get a *symmetric* pairing

$$\mathfrak{h}_{V,D} : H^1_f(V) \times H^1_f(V) \to E. \tag{7}$$

0.4.4. — We maintain previous notation and assumptions. Let $\mathrm{wt}(M) = -1$. Assume, in addition, that $F = \mathbf{Q}$ and that V is *semistable* at p. Then conditions **F1a-b)** and **F2a)** follow from the p-adic analog of the monodromy-weight conjecture and therefore conjecturally always hold (see Proposition 4.3.7). The notion of splitting submodule coincides with the one of regular submodule from [**7, 60**] and condition **F2b)** is equivalent to the non-vanishing of the $\mathscr{L}$-invariant $\mathscr{L}(V,D)$ introduced in [**7**] (see Proposition 4.3.11). We also remark that condition **F3)** does not hold in general. A simple counter-example is given by the representation $V(E)^{\otimes 3}(-1)$, where $V(E)$ is the p-adic representation associated to an elliptic curve $E/\mathbf{Q}$ having split multiplicative reduction at p (see Remark 4.3.3 for more detail). We have two pairings

$$\mathfrak{h}_{V,D}^{\mathrm{spl}} : H^1_f(V) \times H^1_f(V) \to E,$$
$$\mathfrak{h}_{V,\mathbf{D}}^{\mathrm{sel}} : H^1(V,\mathbf{D}) \times H^1_f(V,\mathbf{D}^{\perp}) \to E,$$

provided by $h_{V,D}^{\mathrm{spl}}$ and $h_{V,\mathbf{D}}^{\mathrm{sel}}$ respectively and related by Theorem IV.

0.5. p-adic L-functions

0.5.1. — We keep the hypotheses and notation of Section 0.4.4. Let V be a semistable representation associated to a motive $M/\mathbf{Q}$ of weight -1. It is expected (see [**7, 41, 33, 34**] and especially Perrin-Riou's book [**60**]) that to each splitting submodule D of V_p one can associate a p-adic L-function $L_p(M,D,s)$ interpolating special values of the complex L-function $L(M,s)$. Namely, let r and r_p denote the orders of vanishing of $L(M,s)$ and $L_p(M,D,s)$ at $s=0$. Set $L^{(r)}(M,0)=\lim\limits_{s\to 0} s^{-r}L(M,s)$ and $L^{(r)}(M,D,0)=\lim\limits_{s\to 0} s^{-r}L(M,D,s)$. Beilinson's conjecture predicts that

$$r=\dim_{\mathbf{Q}_p} H^1_f(V)$$

and

$$\frac{L^{(r)}(M,0)}{R_\infty(M)\Omega_\infty(M)} \in \mathbf{Q}^*,$$

where $\Omega_\infty(M)$ is the Deligne period of M, and $R_\infty(M)$ is the determinant of the archimedean height on some fixed basis. The conjectural interpolation property of $L(M,D,s)$ at $s=0$ reads

$$L_p^{(r)}(M,D,0)=\mathscr{E}(M,D)R_p(M,D)\frac{L^{(r)}(M,0)}{R_\infty(M)\Omega_\infty(M)}, \tag{8}$$

where $R_p(V,D)$ is the determinant of the p-adic height $\mathfrak{h}_{V,D}^{\mathrm{spl}}$ taken on the same basis, and $\mathscr{E}(V,D)$ is some explicit Euler-like interpolation factor [**60**].

It is expected that if **N2)** holds (or equivalently $\mathbf{M}_0=\mathbf{M}_1=0$), then

$$r_p=r, \tag{9}$$

and (9) and (8) can be seen as a p-adic version of Beilinson's conjecture.

If condition **N2)** does not hold, we are in presence of extra-zeros. Generalizing the Mazur–Tate–Teitelbaum conjecture (for modular forms) and Greenberg's trivial zero conjecture [**35**] (in the general ordinary case), it is natural to expect that

$$r_p = r + e, \qquad e = \dim_{\mathbf{Q}_p} H^0(\mathbf{D}').$$

Taking into account (5) and (8), we can write this conjectural equality in the form

$$r_p = \dim_{\mathbf{Q}_p} H^1(V, \mathbf{D}). \tag{10}$$

The natural general conjecture for the special value of $L_p(V,D,s)$ at $s=0$ reads

$$L_p^{(r+e)}(V,D,0) = \mathscr{L}(V,D)\mathscr{E}^+(V,D)R_p(V,D)\frac{L^{(r)}(V,0)}{R_\infty(M)\Omega_\infty(M)}, \tag{11}$$

where $\mathscr{L}(V,D)$ is the $\mathscr{L}$-invariant constructed in [**7**] (see also Section 4.3.9) and $\mathscr{E}^+(V,D)$ is obtained from $\mathscr{E}(V,D)$ by removing linear zero factors (see [**7**] for further details). We remark that in (11), $R_p(V,D)$ is taken for the pairing $\mathfrak{h}^{\mathrm{spl}}_{V,D}$ and not for the extended height pairing $\mathfrak{h}^{\mathrm{sel}}_{V,\mathbf{D}}$. The comparision between these two pairings is given by Theorem 7.2.4, but does not make appear the $\mathscr{L}$-invariant. Formulas (10-11) can be seen as the p-adic version of Beilinson's conjecture in the presence of extra-zeros. We refer the reader to [**9**] for the formulation of the analog of this conjecture in the case $\mathrm{wt}(M) \neq -1$.

0.5.2. — We illustrate previous remarks with p-adic representations arising from modular forms. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k^{\mathrm{new}}(N)$ be a newform of even weight k for $\Gamma_0(N)$. Fix a prime p and denote by W_f the p-adic representation of $G_{\mathbf{Q}}$ associated to f by Deligne [**23**]. Its restriction on the decomposition group at p is potentially semistable with Hodge–Tate weights $(-k/2, k/2-1)$. It is crystalline if $(N,p)=1$ and semistable non-crystalline if $p \parallel N$. In the second case, $\mathbf{D}_{\mathrm{cris}}(W_f)$ is a one-dimensional subspace of $\mathbf{D}_{\mathrm{st}}(W_f)$. In the both cases

$$\det(1-\varphi X \mid \mathbf{D}_{\mathrm{cris}}(W_f)) = 1 - a_p X + \varepsilon_0(p)p^{k-1}X^2, \tag{12}$$

where ε_0 is the trivial Dirichlet character modulo N [**65, 67**].

Let M denote the motive associated to the central twist of f. Thus,

$$L(M,s) = L(f, s+k/2).$$

Its p-adic realization is the central twist $V_f = W_f(k/2)$ of W_f. The representation V_f is self dual. Fix an eigenvalue α of Frobenius acting on $\mathbf{D}_{\mathrm{cris}}(V_f)$. We will always assume [5] that $|\alpha|_p > (1/p)^{k/2-1}$. One expects that the corresponding eigenspace D_α

5. We exclude the critical case $|\alpha|_p = (1/p)^{k/2-1}$.

is one-dimensional [(6)]. It is easy to see that under this assumption D_α is a splitting submodule of $\mathbf{D}_{\mathrm{st}}(V_f)$, and we set

$$L(M,D_\alpha,s)=L_{p,\alpha}(f,s+k/2),$$

where $L_{p,\alpha}(f,s)$ is the classical p-adic L-function constructed in [**1, 50, 52, 73**]. As before, we write r and r_p for the orders of vanishing of $L(f,s)$ at $s=k/2$. Below we consider separately the following cases 0.5.2.1 and 0.5.2.2.

0.5.2.1. — $(p,N)=1$. The representation V_f is crystalline and from (12) it follows that $\mathbf{D}_{\mathrm{cris}}(V_f)^{\varphi=1}=0$. Therefore, V_f satisfies **S)** and **N2)**. The space $\mathbf{D}_{\mathrm{cris}}(V_f)$ is two-dimensional and we have two possible choices of α. The values of the complex and p-adic L-functions at $s=k/2$ are related by the formula

$$L_{p,\alpha}(f,k/2)=\left(1-\frac{1}{p\alpha}\right)^2\frac{L(f,k/2)}{\Omega_f}, \tag{13}$$

where Ω_f denotes Deligne's period of f. Since $|\alpha|=p^{(k-1)/2}$, the Euler-like interpolation factor does not vanish.

Assume first that $r=0$. Then $r_p=0$. By Kato [**42**], $H^1_f(V_f)=0$ and the p-adic height degenerates. Therefore, in this case, formula (8) reduces to (13).

If $r\geqslant 1$, the relation (13) says only that both $L(f,s)$ and $L_{p,\alpha}(f,s)$ vanish at $s=k/2$, but does not contain information about special values. In this case, (8) concides with the Mazur–Tate–Teitelbaum conjecture [**52**] in the nonexceptional case, namely

$$L_p^{(r)}(f,k/2)=\left(1-\frac{1}{p\alpha}\right)^2 R_p(f)\frac{L^{(r)}(f,k/2)}{R_\infty(f)\Omega_f},$$

where $R_\infty(f)$ and $R_p(f)$ are the determinants of the complex and the p-adic height pairings computed in the same basis. If $r=1$, this question is closely related to p-adic analogues of the Gross-Zagier formula [**55, 47, 58**]. Here one of the key points is the interpretation of the p-adic height pairing

$$\mathfrak{h}_{V_f,D_\alpha}\,:\,H^1_f(V_f)\times H^1_f(V_f)\to E$$

in terms of universal norms, and therefore the ordinarity condition appears naturally in [**55, 58**]. Kobayashi generalized Perrin–Riou's formula [**58**] to non-ordinary modular forms of higher weight [(7)].

Our theory provides a framework for working with universal norms in the completely general non-ordinary setting. In [**17**], combining the work of Kobayashi with

6. This obviously holds in the semistable case. In the cristalline case, this follows from the conjectural semisimplicity of φ acting on $\mathbf{D}_{\mathrm{cris}}(V_f)$.

7. Work in progress. See [**47**] for the elliptic curve case.

the methods of our paper, Büyükboduk, Pollack and Sasaki study the p-adic Gross–Zagier formula in families and deduce from it a p-adic Gross–Zagier formula for the critical slope stabilizations of modular forms.

0.5.2.2. — $p \parallel N$. The representation V_f is semistable non-crystalline. From (12) it follows that $\mathbf{D}_{\mathrm{cris}}(V_f)$ is one-dimensional and that φ acts on $\mathbf{D}_{\mathrm{cris}}(V_f)$ as multiplication by $\alpha = p^{-k/2}a_p$. By [**48**, Theorem 3], $a_p = \pm p^{k/2-1}$, and therefore $\alpha = \pm p^{-1}$. In both cases, condition **S)** holds. The only possible choice for splitting submodule is to take $D = \mathbf{D}_{\mathrm{cris}}(V_f)$. Denote by $\mathbf{D}$ the (φ,Γ)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f)$ associated to D. Set $\mathbf{D}' = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_f)/\mathbf{D}$. From the self-duality of V_f it follows that

$$H^0(\mathbf{D}') = H^0(\mathbf{D}^*(\chi_p)) = \begin{cases} D^* & \text{if } \alpha = p^{-1}, \\ 0 & \text{if } \alpha = -p^{-1}. \end{cases}$$

The values of the complex and p-adic L-functions at $s = k/2$ are related by the formula

$$L_p(f,k/2) = \left(1 - \frac{1}{p\alpha}\right)\frac{L(f,k/2)}{\Omega_f}. \tag{14}$$

If $\alpha = -p^{-1}$, condition **N2)** holds and Theorem III applies. The situation is quite similar to that we considered in Section 0.5.2.1 and we refer the reader to [**26, 25**] for the p-adic Gross–Zagier formula in this context and further references.

0.5.2.3. — We discuss in more detail the case $\alpha = p^{-1}$ which gives an archetypical example of the failure of condition **N2)**. In this case, conditions **F1a-b)**, **F2a)** and **F3)** hold. [8]. From [**7**, Formula (32), p. 1619] it follows that condition **F2b)** holds if and only if the Fontaine–Mazur $\mathscr{L}$-invariant $\mathscr{L}_{\mathrm{FM}}(f)$ [**51**] does not vanish. This is conjecturally always true, but is proved only for elliptic curves [**2**].

Set $\widetilde{H}^1_f(V_f) = H^1(V_f,\mathbf{D})$. Then the exact sequence (5) reads

$$0 \longrightarrow D^* \xrightarrow{\partial_0} \widetilde{H}^1_f(V_f) \longrightarrow H^1_f(V_f) \longrightarrow 0, \qquad \dim_E D^* = 1.$$

In this situation, we have the pairing $\mathfrak{h}_{V_f,D}$ on the Bloch–Kato Selmer group $H^1_f(V_f)$ induced by the pairing $h^{\mathrm{spl}}_{V_f,D}$ and the pairing

$$\widetilde{\mathfrak{h}}_{V_f,D} : \widetilde{H}^1_f(V_f) \times \widetilde{H}^1_f(V_f) \to E.$$

on the extended Selmer group provided by $h^{\mathrm{sel}}_{V_f,\mathbf{D}}$. If we assume, in addition, that $\mathscr{L}_{\mathrm{FM}}(f) \neq 0$, then we have the third pairing, induced by $h^{\mathrm{norm}}_{V_f,\mathbf{D}}$, which coincides with $\mathfrak{h}_{V_f,D}$ by Theorem V. Moreover, $\mathfrak{h}_{V_f,D}$ and $\widetilde{\mathfrak{h}}_{V_f,D}$ are related by Theorem IV.

8. **F2a)** follows directly from the fact that V_f is not crystalline.

0.5.2.4. — The interpolation factor in (14) vanishes and $L_p(f,s)$ has an extra-zero at $s=k/2$. Conjectural formulas (10-11) reduce to the exceptional case of the Mazur–Tate–Teitelbaum conjecture

$$r_p = \dim_E H^1_f(V_f)+1, \tag{15}$$

$$L_p^{(r+1)}(f,k/2) = \mathscr{L}_{\mathrm{FM}}(f)R_p(f)\frac{L^{(r)}(f,k/2)}{R_\infty(f)\Omega_f}. \tag{16}$$

In the analytic rank zero case $r=0$, formula (16) takes the form

$$L'_p(f,k/2) = \mathscr{L}_{\mathrm{FM}}(f)\frac{L(f,k/2)}{\Omega_f}.$$

It was proved by different methods by Greenberg and Stevens [**36, 69**] and Kato, Kurihara and Tsuji (unpublished, but see [**8, 21**]). In particular, the validity of (15) in this case is equivalent to the non-vanishing of $\mathscr{L}_{\mathrm{FM}}(f)$.

0.5.2.5. — Assume that $r=1$. From [**42**] (see also [**20**] and [**10**]), it follows that in this case $\mathrm{ord}_{s=k/2}L_p(f,s)\geqslant 2$. For elliptic curves, a version of the Gross–Zagier formula involving the $\mathscr{L}$-invariant was proved by Venerucci [**71**]. Our theory of p-adic heights allows to generalize the method of Venerucci to modular forms of higher weights [9]. In [**11**], K. Büyükboduk and the author prove the following result. Let $z_f^{\mathrm{BK}}\in H^1_f(V_f)$ denote the first layer of the Beilinson–Kato Euler system constructed in [**42**]. Let

$$\mathfrak{z}_f^{\mathrm{BK}} = \mathfrak{s}^c_{V_f,\mathbf{D}}\left(z_f^{\mathrm{BK}}\right)\in \widetilde{H}^1_f(V_f)$$

be the canonical lift of z_f^{BK} under the splitting $\mathfrak{s}^c_{V_f,\mathbf{D}}$ defined in (5). Fix a basis b of the one-dimensional space $H^0(\mathbf{D}^*(\chi_p))$. Then

$$\Omega_p\cdot\frac{d^2}{ds^2}L_p(f,s)\big|_{s=k/2} = \det\begin{pmatrix} \widetilde{\mathfrak{h}}_{V_f,D}(\partial_0(b),\partial_0(b)) & \widetilde{\mathfrak{h}}_{V_f,D}\left(\partial_0(b),\mathfrak{z}_f^{\mathrm{BK}}\right) \\ \widetilde{\mathfrak{h}}_{V_f,D}\left(\mathfrak{z}_f^{\mathrm{BK}},\partial_0(b)\right) & \widetilde{\mathfrak{h}}_{V_f,D}\left(\mathfrak{z}_f^{\mathrm{BK}},\mathfrak{z}_f^{\mathrm{BK}}\right)\end{pmatrix}, \tag{17}$$

where Ω_p is some explicit "p-adic period" which depends on our choice of b (see [**11**, Section 7.2] for the precise definition). The key new ingredient of the proof of this formula is the interpretation of the height pairing in terms of universal norms which leads to non-ordinary versions of Rubin-style formulae.

9. Note that our results are weaker that the results of Venerucci, because the injectivity of the p-adic Abel–Jacobi map is an open question in the higher weight case.

If $\mathscr{L}_{\mathrm{FM}}(f) \neq 0$, formula (17) together with a standard argument (see, for example, the proof of [**54**, Theorem 7.13]) give an expression for $\frac{d^2}{ds^2}L_p(f,s)\big|_{s=k/2}$ in terms of $\mathscr{L}_{\mathrm{FM}}(f)$ and the height $\mathfrak{h}_{f,D}(z_f^{\mathrm{BK}}, z_f^{\mathrm{BK}})$ (see [**11**, Corollary B]).

0.5.2.6. — We maintain previous assumptions. Let $\mathbf{f}$ be the Coleman family of modular forms passing through f. Let $V_{\mathbf{f}}$ be the big Galois representation associated to this family which specializes to V_f at the weight k. A two-variable version of the Bockstein map which takes into account the deformation in the weight direction, gives a two-variable height pairing

$$\mathfrak{H}_{\mathbf{f}} : \widetilde{H}^1_f(V_f) \times \widetilde{H}^1_f(V_f) \to \mathfrak{J}/\mathfrak{J}^2,$$

where $\mathfrak{J} \subset E[[\kappa - k, s]]$ is the ideal of power series in $\kappa - k$ and s those vanish at $(k,0)$ [**11**, Section 4.3]. The specialization of $\mathfrak{H}_{\mathbf{f}}$ at $\kappa = k$ coincides with the height pairing $\mathfrak{h}_{V_f,D}$ and its restriction on the central critical line $s = (\kappa - k)/2$ coincides with the central critical height pairing constructed using the Cassels–Tate pairings [**11**, Section 3.3]. This pairing is closely related to the behavior of the two-variable p-adic L-function $L(\mathbf{f},s)$ at $(k,k/2)$ and we refer the reader to *op. cit.* for further detail and references.

0.6. The organization of this paper

This paper is very technical by the nature, and in Chapters 1-2 we assemble necessary preliminaries. In Chapter 1, we recall the formalism of cup products. In Section 1.1, to each complex $A^\bullet$ equipped with a morphism $\varphi : A^\bullet \to A^\bullet$ we associate the complex $T^\bullet(A^\bullet) = (A^\bullet \xrightarrow{\varphi-1} A^\bullet)$ and study cup products of these complexes. These results are used in Sections 2.5–2.7. In Section 1.2, we recall the the formalism of cup products for cones following [**56**] (see also [**57**]). These results play a key role in Chapter 3.

In Chapter 2, we consider local Galois representations with coefficients in an affinoid algebra. In Sections 2.1–2.2, we review the theory of (φ,Γ)-modules over affinoid algebras and its connection with p-adic representations and classical Fontaine's functors $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{D}_{\mathrm{st}}$ and $\mathbf{D}_{\mathrm{dR}}$. The reader familiar with (φ,Γ)-modules can skip them. In Section 2.3, we review local duality for Galois representations. In Section 2.4, we construct cup products for Fontaine–Herr complexes of (φ,Γ)-modules and review the computation of Galois cohomology in terms of these complexes. Sections 2.5–2.7 are the central parts of the chapter. They contain the most part of results we need to develop the theory of Selmer complexes with local conditions arising from (φ,Γ)-modules. In Sections 2.5–2.6, we introduce the complex $K^\bullet(V)$ which relates the Fontaine–Herr complex to the complex of continuous cochains with coefficients in V. Using results from Chapter 1, we prove some technical results about cup products of these complexes. These results are used to develop the duality theory for Selmer complexes in Section 3.1. In Section 2.7, we compute the Bockstein map for Fontaine–Herr complexes and for $K^{\bullet}(V)$. These results are used in Section 3.2 to generalize Nekovář's construction of the p-adic height pairing. In particular, Proposition 2.6.4 plays a key role in the proof of Theorem 3.2.4 (Theorem I of this Introduction) which asserts that the constructed p-adic height pairing is skew symmetric. In Section 2.8, we review Iwasawa cohomology of (φ,Γ)-modules and prove some auxiliary results. In Section 2.6, we review the definition and some properties of the Bloch–Kato group H^1_f of a (φ,Γ)-module. In particular, we review the canonical decomposition of H^1 of some "exceptional" isoclinic modules (φ,Γ)-modules into the direct sum of H^1_f and its canonical complement H^1_c. These results are used in Chapter 7 to study p-adic heights on extended Selmer groups.

Chapter 3 is the central part of the paper. It gathers the main constructions of our theory. Selmer complexes $\mathbf{R}\Gamma(V,\mathbf{D})$ are defined in Section 3.1. In Theorem 3.1.5, we construct the cup products. Theorem 3.1.7 gives a sufficient condition that the cup product be a duality. In Theorem 3.1.11 we prove that the cup product is skew symmetric following the method of Nekovář. The p-adic height pairing is defined is Section 3.2. In Theorem 3.2.4 (Theorem I of this Introduction), we deduce that it is skew symmetric from formal properties of cup products.

In the rest of the paper, we consider p-adic heights for p-adic representations with coefficients in a p-adic field. In Chapter 4, we study splitting submodules of potentially semistable representations. Sections 4.1-4.2 assembles technical results used to construct the pairing $h^{\mathrm{spl}}_{V,D}$. In Section 4.3, we assumme that the ground field is $\mathbf{Q}_p$. We construct the canonical filtration (4) and discuss in detail its properties. In particular, we show that conditions **F1a-b)** and **F2a)** follow from the semisimplicity of the Frobenius operator and the monodromy-weight conjecture.

In Chapters 5-6 we construct the pairings $h^{\mathrm{norm}}_{V,D}$ and $h^{\mathrm{spl}}_{V,D}$ and prove Theorems II and III.

In Chapter 7, we study extended Selmer groups and prove Theorems IV and V.

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CHAPTER 1

COMPLEXES AND PRODUCTS

1.1. The complex $T^\bullet(A^\bullet)$

1.1.1. — If R is a commutative ring, we write $\mathscr{K}(R)$ for the category of complexes of R-modules and $\mathscr{K}_{\mathrm{ft}}(R)$ for the subcategory of $\mathscr{K}(R)$ consisting of complexes $C^\bullet=(C^n,d^n_{C^\bullet})$ such that $H^n(C^\bullet)$ are finitely generated over R for all $n\in\mathbf{Z}$. We write $\mathscr{D}(R)$ and $\mathscr{D}_{\mathrm{ft}}(R)$ for the corresponding derived categories and denote by $[\cdot]\,:\,\mathscr{K}_*(R)\to\mathscr{D}_*(R)$, $(*\in\{\emptyset,\mathrm{ft}\})$ the obvious functors. We will also consider the subcategories $\mathscr{K}_{\mathrm{ft}}^{[a,b]}(R)$, $(a\leqslant b)$ consisting of objects of $\mathscr{K}_{\mathrm{ft}}(R)$ whose cohomologies are concentrated in degrees $[a,b]$. A perfect complex of R-modules is one of the form

$$0\to P_a\to P_{a+1}\to\ldots\to P_b\to 0,$$

where each P_i is a finitely generated projective R-module. If R is noetherian, we denote by $\mathscr{D}^{[a,b]}_{\mathrm{perf}}(R)$ the full subcategory of $\mathscr{D}_{\mathrm{ft}}(R)$ consisting of objects quasi-isomorphic to perfect complexes concentrated in degrees $[a,b]$.

If $C^\bullet=(C^n,d^n_{C^\bullet})_{n\in\mathbf{Z}}$ is a complex of R-modules and $m\in\mathbf{Z}$, we will denote by $C^\bullet[m]$ the complex defined by $C^\bullet[m]^n=C^{n+m}$ and $d^n_{C^\bullet[m]}(x)=(-1)^m d_{C^\bullet}(x)$. We will often write d^n or just simply d instead of $d^n_{C^\bullet}$. For each m, the truncation $\tau_{\geqslant m}C^\bullet$ of $C^\bullet$ is the complex

$$0\to\mathrm{coker}(d^{m-1})\to C^{m+1}\to C^{m+2}\to\cdots.$$

Therefore

$$H^i(\tau_{\geqslant m}C^\bullet)=\begin{cases}0, & \text{if } i<m,\\ H^i(C^\bullet), & \text{if } i\geqslant m\,.\end{cases}$$

The tensor product $A^\bullet \otimes B^\bullet$ of two complexes $A^\bullet$ and $B^\bullet$ is defined by

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{i\in\mathbf{Z}} \left(A^i \otimes B^{n-i}\right),$$
$$d(a_i \otimes b_{n-i}) = dx_i \otimes y_{n-i} + (-1)^i a_i \otimes b_{n-i}, \qquad a_i \in A^i, \quad b_{n-i} \in B^{n-i}.$$

We denote by $s_{12} : A^\bullet \otimes B^\bullet \to B^\bullet \otimes A^\bullet$ the transposition

$$s_{12}(a_n \otimes b_m) = (-1)^{nm} b_m \otimes a_n, \qquad a_n \in A^n, \quad b_m \in B^m.$$

It is easy to check that s_{12} is a morphism of complexes. We will also consider the map $s^*_{12} : A^\bullet \otimes B^\bullet \to B^\bullet \otimes A^\bullet$ given by

$$s^*_{12}(a_n \otimes b_m) = b_m \otimes a_n,$$

which is not a morphism of complexes in general.

Recall that a homotopy $h : f \rightsquigarrow g$ between two morphisms $f, g : A^\bullet \to B^\bullet$ is a family of maps $h = (h^n : A^{n+1} \to B^n)$ such that $dh + hd = g - f$. We will sometimes write h instead of h^n. A second order homotopy $H : h \rightsquigarrow k$ between homotopies $h, k : f \rightsquigarrow g$ is a collection of maps $H = (H^n : A^{n+2} \to B^n)$ such that $Hd - dH = k - h$.

If $f_i : A_1^\bullet \to B_1^\bullet$ $(i = 1, 2)$ and $g_i : A_2^\bullet \to B_2^\bullet$ $(i = 1, 2)$ are morphisms of complexes and $h : f_1 \rightsquigarrow f_2$ and $k : g_1 \rightsquigarrow g_2$ are homotopies between them, then the formula

$$(h \otimes k)_1(x_n \otimes y_m) = h(x_n) \otimes g_1(y_m) + (-1)^n f_2(x_n) \otimes k(y_m), \tag{18}$$

where $x_n \in A_1^n$, $y_m \in A_2^m$, defines a homotopy

$$(h \otimes k)_1 : f_1 \otimes g_1 \rightsquigarrow f_2 \otimes g_2.$$

1.1.2. — For the content of this subsection we refer the reader to [**72**, §3.1]. If $f : A^\bullet \to B^\bullet$ is a morphism of complexes, the cone of f is defined to be the complex

$$\mathrm{cone}(f) = A^\bullet[1] \oplus B^\bullet,$$

with differentials

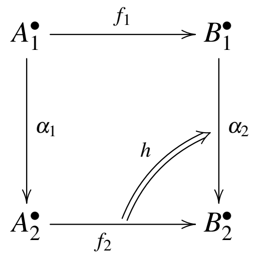
$$d^n(a_{n+1}, b_n) = (-d^{n+1}(a_{n+1}), f(a_{n+1}) + d^n(b_n)).$$

We have a canonical distinguished triangle

$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{cone}(f) \to A^\bullet[1].$$

We say that a diagram of complexes of the form

(19)



is commutative up to homotopy, if there exists a homotopy

$$h : f_2 \circ \alpha_1 \rightsquigarrow \alpha_2 \circ f_1.$$

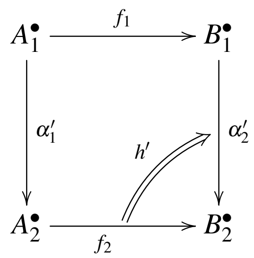
In this case, the formula

$$c(\alpha_1, \alpha_2, h)^n(a_{n+1}, b_n) = (\alpha_1(a_{n+1}), \alpha_2(b_n) + h^n(a_{n+1}))$$

defines a morphism of complexes

$$c(\alpha_1, \alpha_2, h) : \mathrm{cone}(f_1) \to \mathrm{cone}(f_2). \tag{20}$$

Assume that, in addition to (19), we have a diagram



together with homotopies

$$k_1 : \alpha_1 \rightsquigarrow \alpha_1'$$
$$k_2 : \alpha_2 \rightsquigarrow \alpha_2'$$

and a second order homotopy

$$H : f_2 \circ k_1 + h' \rightsquigarrow k_2 \circ f_1 + h.$$

Then the map

$$(a_{n+1}, b_n) \mapsto (-k_1(a_{n+1}), k_2(b_n) + H(a_{n+1})) \tag{21}$$

defines a homotopy $c(\alpha_1, \alpha_2, h) \rightsquigarrow c(\alpha_1', \alpha_2', h')$.

1.1.3. — Till the end of this section R is a commutative ring and all complexes are complexes of R-modules. Let $A^\bullet = (A^n, d^n)$ be a complex equipped with a morphism $\varphi : A^\bullet \to A^\bullet$. By definition, the total complex

$$T^\bullet(A^\bullet) = \mathrm{Tot}\,(A^\bullet \xrightarrow{\varphi-1} A^\bullet).$$

is given by $T^n(A^\bullet) = A^{n-1} \oplus A^n$ with differentials

$$d^n(a_{n-1}, a_n) = (d^{n-1}a_{n-1} + (-1)^n(\varphi - 1)a_n, d^n a_n), \quad (a_{n-1}, a_n) \in T^n(A^\bullet).$$

If $A^\bullet$ and $B^\bullet$ are two complexes equipped with morphisms $\varphi : A^\bullet \to A^\bullet$ and $\psi : B^\bullet \to B^\bullet$, and if $\alpha : A^\bullet \to B^\bullet$ is a morphism such that $\alpha \circ \varphi = \psi \circ \alpha$, then α induces a morphism $T(\alpha) : T^\bullet(A^\bullet) \to T^\bullet(B^\bullet)$. We will often write α instead of $T(\alpha)$ to simplify notation.

Lemma 1.1.4. — *Let $A^\bullet$ and $B^\bullet$ be two complexes equipped with morphisms $\varphi : A^\bullet \to A^\bullet$ and $\psi : B^\bullet \to B^\bullet$, and let $\alpha_i : A^\bullet \to B^\bullet$ ($i = 1,2$) be two morphisms such that*

$$\alpha_i \circ \varphi = \psi \circ \alpha_i \qquad i = 1,2.$$

If $h : \alpha_1 \rightsquigarrow \alpha_2$ is a homotopy between α_1 and α_2 such that $h \circ \varphi = \psi \circ h$, then the collection of maps $h_T = (h_T^n : T^{n+1}(A^\bullet) \to T^n(B^\bullet))$ defined by $h_T^n(a_n, a_{n+1}) = (h(a_n), h(a_{n+1}))$ is a homotopy between $T(\alpha_1)$ and $T(\alpha_2)$.

Proof. — The proof of this lemma is a direct computation and is omitted here. □

In the remainder of this subsection we will consider triples $(A_1^\bullet, A_2^\bullet, A_3^\bullet)$ of complexes of R-modules equipped with the following structures

A1) Morphisms $\varphi_i : A_i^\bullet \to A_i^\bullet$ ($i = 1,2,3$).

A2) A morphism $\cup_A : A_1^\bullet \otimes A_2^\bullet \to A_3^\bullet$ which satisfies

$$\cup_A \circ (\varphi_1 \otimes \varphi_2) = \varphi_3 \circ \cup_A.$$

Proposition 1.1.5. — *Assume that a triple $(A_i^\bullet, \varphi_i)$ ($1 \leqslant i \leqslant 3$) satisfies conditions* **A1-2)**. *Then the map*

$$\cup_A^T : T^\bullet(A_1^\bullet) \otimes T^\bullet(A_2^\bullet) \to T^\bullet(A_3^\bullet)$$

given by

$$(x_{n-1}, x_n) \cup_A^T (y_{m-1}, y_m) = (x_n \cup_A y_{m-1} + (-1)^m x_{n-1} \cup_A \varphi_2(y_m), x_n \cup_A y_m),$$

is a morphism of complexes.

Proof. — This proposition is well known to the experts (compare, for example, to [**57**, Proposition 3.1]). It follows from a direct computation which we recall for the convenience of the reader. Let $(x_{n-1},x_n)\in T^n(A_1^\bullet)$ and $(y_{m-1},y_m)\in T^m(A_2^\bullet)$. Then

$$\begin{aligned} &d((x_{n-1},x_n)\cup_A^T (y_{m-1},y_m) = \\ &\qquad = d(x_n\cup_A y_{m-1}+(-1)^m x_{n-1}\cup_A \varphi_2(y_m), x_n\cup_A y_m) = (z_{n+m}, z_{n+m+1}),\end{aligned}$$

where

$$\begin{aligned} z_{n+m} = {} & dx_n\cup_A y_{m-1}+(-1)^n x_n\cup_A dy_{m-1}+(-1)^m dx_{n-1}\cup_A \varphi_2(y_m)+ \\ & (-1)^{m+n-1}x_{n-1}\cup_A d(\varphi_2(y_m))+(-1)^{n+m}(\varphi_3-1)(x_n\cup_A y_m)\end{aligned}$$

and $z_{n+m+1}=d(x_n\cup_A y_m)$. On the other hand

$$\begin{aligned} &\cup_A^T\circ d((x_{n-1},x_n)\otimes(y_{m-1},y_m)) = \\ &= \cup_A^T\circ((dx_{n-1}+(-1)^n(\varphi_1-1)x_n, dx_n)\otimes(y_{m-1},y_m))+ \\ &+(-1)^n\cup_A^T\circ((x_{n-1},x_n)\otimes(dy_{m-1}+(-1)^m(\varphi_2-1)y_m, dy_m)) = \\ &= (u_{n+m}, u_{n+m+1}),\end{aligned}$$

where

$$\begin{aligned} u_{n+m} = {} & dx_n\cup_A y_{m-1}+(-1)^m(dx_{n-1}+(-1)^n(\varphi_1-1)x_n)\cup\varphi_2(y_m)+ \\ & (-1)^n x_n\cup(dy_{m-1}+(-1)^m(\varphi_2-1)y_m)+(-1)^{n+m-1}x_{n-1}\cup\varphi_2(dy_m),\end{aligned}$$

and $u_{n+m+1}=dx_n\cup_A y_m+(-1)^n x_n\cup_A dy_m$. Now the proposition follows from the formula

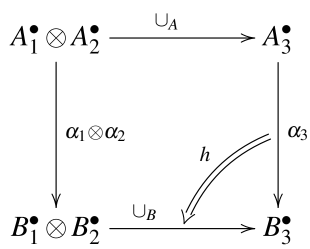
$$d(x_n\cup_A y_m)=dx_n\cup_A y_m+(-1)^n x_n\cup_A dy_m$$

and the assumption **A2)** that reads $\varphi_1(x_n)\cup_A\varphi_2(y_m)=\varphi_3(x_n\cup_A y_m)$. □

Proposition 1.1.6. — *Let $(A_i^\bullet,\varphi_i)$ and $(B_i^\bullet,\psi_i)$ $(1\leqslant i\leqslant 3)$ be two triples of complexes that satisfy conditions* **A1-2)**. *Assume that they are equipped with morphisms*

$$\alpha_i\,:\,A_i^\bullet\to B_i^\bullet,$$

such that $\alpha_i\circ\varphi_i=\psi_i\circ\alpha_i$ for all $1\leqslant i\leqslant 3$. Assume, in addition, that in the diagram



there exists a homotopy

$$h : \alpha_3 \circ \cup_A \rightsquigarrow \cup_B \circ (\alpha_1 \otimes \alpha_2).$$

such that $h \circ (\varphi_1 \otimes \varphi_2) = \psi_3 \circ h$. *Then the collection* h_T *of maps*

$$h_T^k : \bigoplus_{m+n=k+1} (T^n(A_1^\bullet) \otimes T^m(A_2^\bullet)) \to T^k(B_3^\bullet)$$

defined by

$$\begin{aligned} &h_T^k((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = \\ &\qquad = \big(h(x_n \otimes y_{m-1}) + (-1)^m h(x_{n-1} \otimes \varphi_2(y_m)), h(x_n \otimes y_m)\big), \end{aligned}$$

provides a homotopy $h_T : \alpha_3 \circ \cup_A^T \rightsquigarrow \cup_B^T \circ (\alpha_1 \otimes \alpha_2)$:

$$\begin{array}{ccc} T^\bullet(A_1^\bullet) \otimes T^\bullet(A_2^\bullet) & \xrightarrow{\cup_A^T} & T^\bullet(A_3^\bullet) \\ \downarrow{\scriptstyle \alpha_1 \otimes \alpha_2} & \swarrow\!\!\!\swarrow{\scriptstyle h_T} & \downarrow{\scriptstyle \alpha_3} \\ T^\bullet(B_1^\bullet) \otimes T^\bullet(B_2^\bullet) & \xrightarrow{\cup_B^T} & T^\bullet(B_3^\bullet). \end{array}$$

Proof. — Again, the proof is a routine computation. Let $(x_{n-1}, x_n) \in T^n(A_1^\bullet)$ and $(y_{m-1}, y_m) \in T^m(A_2^\bullet)$. We have

$$\begin{aligned} d((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) = (dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n, dx_n) \otimes (y_{m-1}, y_m) + \\ + (-1)^n (x_{n-1}, x_n) \otimes (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m, dy_m), \end{aligned}$$

and therefore

$$h_T \circ d((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) = (a, b),$$

where

$$\begin{aligned} a = {} & h(dx_n \otimes y_{m-1}) + (-1)^m h((dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n) \otimes \varphi_2(y_m)) + \\ & + (-1)^n (h(x_n \otimes (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m)) + \\ & + (-1)^{n+m-1} h(x_{n-1} \otimes \varphi_2(dy_m)) = \\ & = h \circ d(x_n \otimes y_{m-1}) + (-1)^m h \circ d(x_{n-1} \otimes \varphi_2(y_m)) + \\ & + (-1)^{n+m} (\psi_3 - 1) \circ h(x_n \otimes y_m) \end{aligned}$$

and

$$b = h(dx_n \otimes y_m) + (-1)^n h(x_n \otimes dy_m) = h \circ d(x_n \otimes y_m).$$

On the other hand

$$\begin{aligned}
&d\circ h_T((x_{n-1},x_n)\otimes(y_{m-1},y_m)) =\\
&= d(h(x_n\otimes y_{m-1})+(-1)^m h(x_{n-1}\otimes\varphi_2(y_m)), h(x_n\otimes y_m)) =\\
&= (d\circ h(x_n\otimes y_{m-1})+(-1)^m d\circ h(x_{n-1}\otimes\varphi_2(y_m))+\\
&+(-1)^{n+m-1}(\psi_3-1)h(x_n\otimes y_m), d\circ h(x_n\otimes y_m)).
\end{aligned}$$

Thus

$$\begin{aligned}
&(h_T d+dh_T)((x_{n-1},x_n)\otimes(y_{m-1},y_m)) =\\
&= ((hd+dh)(x_n\otimes y_{m-1})+(-1)^m(hd+dh)(x_{n-1}\otimes\varphi_2(y_m)),\\
&\qquad\qquad\qquad\qquad (hd+dh)(x_n\otimes y_m)) =\\
&= ((\alpha_1(x_n)\cup_B\alpha_2(y_{m-1})-\alpha_3(x_n\cup_A y_{m-1}))+\\
&\qquad (-1)^m(\alpha_1(x_{n-1})\cup_B\varphi_2(\alpha_2(y_m))-\alpha_3(x_{n-1}\cup_A\varphi_2(y_m)),\\
&\qquad\qquad\qquad \alpha_1(x_n)\cup_B\alpha_2(y_m)-\alpha_3(x_n\cup_A y_m)) =\\
&= (\cup_B^T\circ(\alpha_1\otimes\alpha_2)-\alpha_3\circ\cup_A^T)((x_{n-1},x_n)\otimes(y_{m-1},y_m)).
\end{aligned}$$

and the proposition is proved.

□

***Proposition 1.1.7.* —** *Let $A_i^\bullet$ ($1\leqslant i\leqslant 4$) be four complexes equipped with morphisms $\varphi_i : A_i^\bullet\to A_i^\bullet$ and such that*

a) The triples $(A_1^\bullet,A_2^\bullet,A_3^\bullet)$ and $(A_1^\bullet,A_2^\bullet,A_4^\bullet)$ satisfy **A1-2)**.

b) The complexes $A_i^\bullet$ ($i=1,2$) are equipped with morphisms $\mathscr{T}_i : A_i^\bullet\to A_i^\bullet$ which commute with morphisms φ_i

$$\mathscr{T}_i\circ\varphi_i=\varphi_i\circ\mathscr{T}_i,\qquad i=1,2.$$

c) There exists a morphism $\mathscr{T}_{34} : A_3^\bullet\to A_4^\bullet$ such that

$$\mathscr{T}_{34}\circ\varphi_3=\varphi_4\circ\mathscr{T}_{34}.$$

d) The diagram

$$\begin{array}{ccc}
A_1^\bullet\otimes A_2^\bullet & \xrightarrow{\ \cup_A\ } & A_3^\bullet\\
\Big\downarrow{\scriptstyle s_{12}\circ(\mathscr{T}_1\otimes\mathscr{T}_2)} & & \Big\downarrow{\scriptstyle \mathscr{T}_{34}}\\
A_2^\bullet\otimes A_1^\bullet & \xrightarrow{\ \cup_A\ } & A_4^\bullet.
\end{array}$$

commutes.

Let $\mathscr{T}_i : T^\bullet(A_i^\bullet) \to T^\bullet(A_i^\bullet)$ $(i = 1, 2)$ and $\mathscr{T}_{34} : T^\bullet(A_3^\bullet) \to T^\bullet(A_4^\bullet)$ be the morphisms (which we denote again by the same letter) defined by

$$\mathscr{T}_i(x_{n-1}, x_n) = (\mathscr{T}_i(x_{n-1}), \mathscr{T}_i(x_n)), \quad \mathscr{T}_{34}(x_{n-1}, x_n) = (\mathscr{T}_{34}(x_{n-1}), \mathscr{T}_{34}(x_n)).$$

Then in the diagram

$$\begin{array}{ccc} T^\bullet(A_1^\bullet) \otimes T^\bullet(A_2^\bullet) & \xrightarrow{\cup_A^T} & T^\bullet(A_3^\bullet) \\ \Big\downarrow{\scriptstyle s_{12}\circ(\mathscr{T}_1\otimes\mathscr{T}_2)} & \overset{h_{\mathscr{T}}}{\swarrow} & \Big\downarrow{\scriptstyle \mathscr{T}_{34}} \\ T^\bullet(A_2^\bullet) \otimes T^\bullet(A_1^\bullet) & \xrightarrow{\cup_A^T} & T^\bullet(A_4^\bullet) \end{array}$$

the maps $\mathscr{T}_{34} \circ \cup_A^T$ and $\cup_A^T \circ s_{12} \circ (\mathscr{T}_1 \otimes \mathscr{T}_2)$ are homotopic.

Proof. — Let $(x_{n-1}, x_n) \in T^n(A_1^\bullet)$ and $(y_{m-1}, y_m) \in T^m(A_2^\bullet)$. Then

$$\begin{aligned} (22) \quad & \mathscr{T}_{34}((x_{n-1}, x_n) \cup_A^T (y_{m-1}, y_m)) = \\ & = (\mathscr{T}_{34}(x_n \cup_A y_{m-1}) + (-1)^m \mathscr{T}_{34}(x_{n-1} \cup_A \varphi_2(y_m)), \mathscr{T}_{34}(x_n \cup_A y_m)) \end{aligned}$$

and

$$\begin{aligned} & \cup_A^T \circ s_{12} \circ (\mathscr{T}_1 \otimes \mathscr{T}_2)((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) = \\ & = (-1)^{mn} \mathscr{T}_2(y_{m-1}, y_m) \cup_A \mathscr{T}_1(x_{n-1}, x_n) = \\ (23) \quad & = (-1)^{mn} (\mathscr{T}_2(y_m) \cup_A \mathscr{T}_1(x_{n-1}) + (-1)^n \mathscr{T}_2(y_{m-1}) \cup_A \varphi_1(\mathscr{T}_1(x_n)), \\ & \qquad \mathscr{T}_2(y_m) \cup_A \mathscr{T}_1(x_n)) = \\ & = ((-1)^m \mathscr{T}_{34}(x_{n-1} \cup_A y_m) + \mathscr{T}_{34}(\varphi_1(x_n) \cup_A y_{m-1}), \mathscr{T}_{34}(x_n \cup_A y_m)). \end{aligned}$$

Define

$$h_{\mathscr{T}}^k : \bigoplus_{m+n=k+1} (T^n(A_1^\bullet) \otimes T^m(A_2^\bullet)) \to T^k(A_4^\bullet),$$

by

$$(24) \qquad h_{\mathscr{T}}^k((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = (-1)^{n-1} (\mathscr{T}_{34}(x_{n-1} \cup_A y_{m-1}), 0).$$

Then

$$\begin{aligned} (25) \quad & dh_{\mathscr{T}}((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = \\ & = (-1)^{n-1} d(\mathscr{T}_{34}(x_{n-1} \cup_A y_{m-1}), 0) = \\ & = (-1)^{n-1} (\mathscr{T}_{34}(dx_{n-1} \cup_A y_{m-1} + (-1)^{n-1} x_{n-1} \cup_A dy_{m-1}), 0) = \\ & = ((-1)^{n-1} \mathscr{T}_{34}(dx_{n-1} \cup_A y_{m-1}) + \mathscr{T}_{34}(x_{n-1} \cup_A dy_{m-1}), 0), \end{aligned}$$

and

(26)
$$\begin{aligned}
&h_{\mathscr{T}}d((x_{n-1},x_n)\otimes(y_{m-1}\otimes y_m)) =\\
&= h_{\mathscr{T}}((dx_{n-1}+(-1)^n(\varphi_1-1)x_n, dx_n)\otimes(y_{m-1},y_m)+\\
&+(-1)^n(x_{n-1},x_n)\otimes(dy_{m-1}+(-1)^m(\varphi_2-1)y_m, dy_m)) =\\
&= ((-1)^n\mathscr{T}_{34}(dx_{n-1}\cup_A y_{m-1})+\mathscr{T}_{34}(\varphi_1(x_n)\cup_A y_{m-1})-\\
&-\mathscr{T}_{34}(x_n\cup_A y_{m-1})-\mathscr{T}_{34}(x_{n-1}\cup_A dy_{m-1})-\\
&-(-1)^m\mathscr{T}_{34}(x_{n-1}\cup_A \varphi_2(y_m))+(-1)^m\mathscr{T}_{34}(x_{n-1}\cup_A y_m),0).
\end{aligned}$$

From (22-26) it follows that

$$\cup_A^T\circ s_{12}\circ(\mathscr{T}_1\otimes\mathscr{T}_2)-\mathscr{T}_{34}\circ\cup_A^T = dh_{\mathscr{T}}+h_{\mathscr{T}}d$$

and the proposition is proved. □

1.2. Products

1.2.1. — In this subsection we review the construction of products for cones following Nekovář [**56**] and Nizioł[**57**]. We will work with the following data:

P1) Diagrams

$$A_i^\bullet \xrightarrow{f_i} C_i^\bullet \xleftarrow{g_i} B_i^\bullet, \qquad i=1,2,3,$$

where $A_i^\bullet$, $B_i^\bullet$ and $C_i^\bullet$ are complexes of R-modules.

P2) Morphisms

$$\begin{aligned}
\cup_A &: A_1^\bullet\otimes A_2^\bullet \to A_3^\bullet,\\
\cup_B &: B_1^\bullet\otimes B_2^\bullet \to B_3^\bullet,\\
\cup_C &: C_1^\bullet\otimes C_2^\bullet \to C_3^\bullet.
\end{aligned}$$

P3) A pair of homotopies $h=(h_f,h_g)$

$$\begin{aligned}
h_f &: \cup_C\circ(f_1\otimes f_2)\rightsquigarrow f_3\circ\cup_A,\\
h_g &: \cup_C\circ(g_1\otimes g_2)\rightsquigarrow g_3\circ\cup_B.
\end{aligned}$$

Define

(27)
$$E_i^\bullet = \mathrm{cone}\left(A_i^\bullet\oplus B_i^\bullet \xrightarrow{f_i-g_i} C_i^\bullet\right)[-1].$$

Thus

$$E_i^n = A_i^n\oplus B_i^n\oplus C_i^{n-1}$$

with $d(a_n,b_n,c_{n-1}) = (da_n, db_n, -f_i(a_n)+g_i(b_n)-dc_{n-1})$.

Proposition 1.2.2. — *i) Given the data* **P1-3)***, for each $r \in R$ the formula*

$$(a_n, b_n, c_{n-1}) \cup_{r,h} (a'_m, b'_m, c'_{m-1}) =$$
$$(a_n \cup_A a'_m, b_n \cup_B b'_m, c_{n-1} \cup_C (rf_2(a'_m) + (1-r)g_2(b'_m)) +$$
$$(-1)^n((1-r)f_1(a_n) + rg_1(b_n)) \cup_C c'_{m-1} - (h_f(a_n \otimes a'_m) - h_g(b_n \otimes b'_m)))$$

defines a morphism in $\mathscr{K}(R)$

$$\cup_{r,h} : E_1^\bullet \otimes E_2^\bullet \to E_3^\bullet.$$

ii) If $r_1, r_2 \in R$, then the map

$$k : E_1^\bullet \otimes E_2^\bullet \to E_3^\bullet[-1],$$

given by

$$k((a_n, b_n, c_{n-1}) \otimes (a'_m, b'_m, c'_{m-1})) = (0, 0, (-1)^n (r_1 - r_2) c_{n-1} \cup_C c'_{m-1})$$

for all $(a_n, b_n, c_{n-1}) \in E_1^n$ and $(a'_m, b'_m, c'_{m-1}) \in E_2^m$, defines a homotopy $k : \cup_{r_1,h} \rightsquigarrow \cup_{r_2,h}$.

iii) If $h' = (h'_f, h'_g)$ is another pair of homotopies as in **P3)***, and if $\alpha : h_f \rightsquigarrow h'_f$ and $\beta : h_g \rightsquigarrow h'_g$ is a pair of second order homotopies, then the map*

$$s : E_1^\bullet \otimes E_2^\bullet \to E_3^\bullet[-1],$$
$$s((a_n, b_n, c_{n-1}) \otimes (a'_m, b'_m, c'_{m-1})) = (0, 0, \alpha(a_n \otimes a'_m) - \beta(b_n, b'_m))$$

defines a homotopy $s : \cup_{r,h} \rightsquigarrow \cup_{r,h'}$.

Proof. — See [**57**, Proposition 3.1]. □

1.2.3. — Assume that, in addition to **P1-3)**, we are given the following data:

T1) Morphisms of complexes

$$\mathscr{T}_A : A_i^\bullet \to A_i^\bullet,$$
$$\mathscr{T}_B : B_i^\bullet \to B_i^\bullet,$$
$$\mathscr{T}_C : C_i^\bullet \to C_i^\bullet,$$

for $i = 1, 2, 3$.

T2) Morphisms of complexes

$$\cup'_A : A_2^\bullet \otimes A_1^\bullet \to A_3^\bullet,$$
$$\cup'_B : B_2^\bullet \otimes B_1^\bullet \to B_3^\bullet,$$
$$\cup'_C : C_2^\bullet \otimes C_1^\bullet \to C_3^\bullet.$$

T3) A pair of homotopies $h' = (h'_f, h'_g)$

$$h'_f : \cup'_C \circ (f_2 \otimes f_1) \rightsquigarrow f_3 \circ \cup'_A,$$
$$h'_g : \cup'_C \circ (g_2 \otimes g_1) \rightsquigarrow g_3 \circ \cup'_B.$$

T4) Homotopies

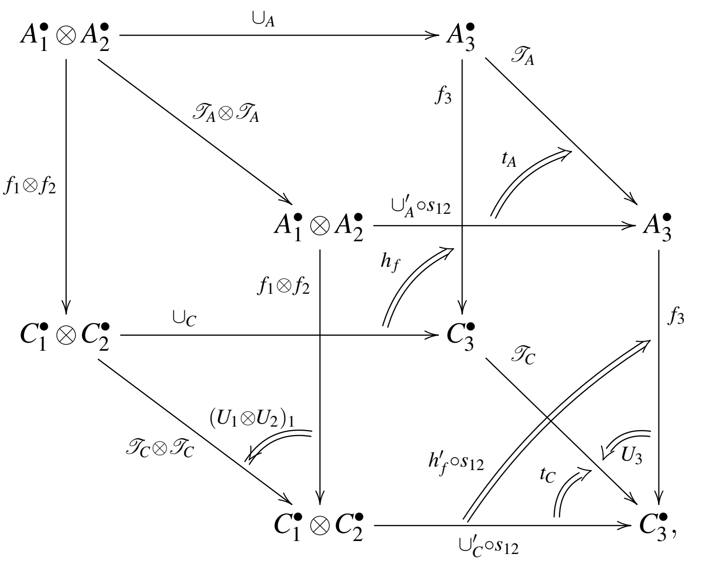
$$U_i : f_i \circ \mathscr{T}_A \rightsquigarrow \mathscr{T}_C \circ f_i,$$
$$V_i : g_i \circ \mathscr{T}_B \rightsquigarrow \mathscr{T}_C \circ g_i,$$

for $i = 1, 2, 3$.

T5) Homotopies

$$t_A : \cup'_A \circ s_{12} \circ (\mathscr{T}_A \otimes \mathscr{T}_A) \rightsquigarrow \mathscr{T}_A \circ \cup_A,$$
$$t_B : \cup'_B \circ s_{12} \circ (\mathscr{T}_B \otimes \mathscr{T}_B) \rightsquigarrow \mathscr{T}_B \circ \cup_B,$$
$$t_C : \cup'_C \circ s_{12} \circ (\mathscr{T}_C \otimes \mathscr{T}_C) \rightsquigarrow \mathscr{T}_C \circ \cup_C.$$

T6) A second order homotopy H_f trivializing the boundary of the cube

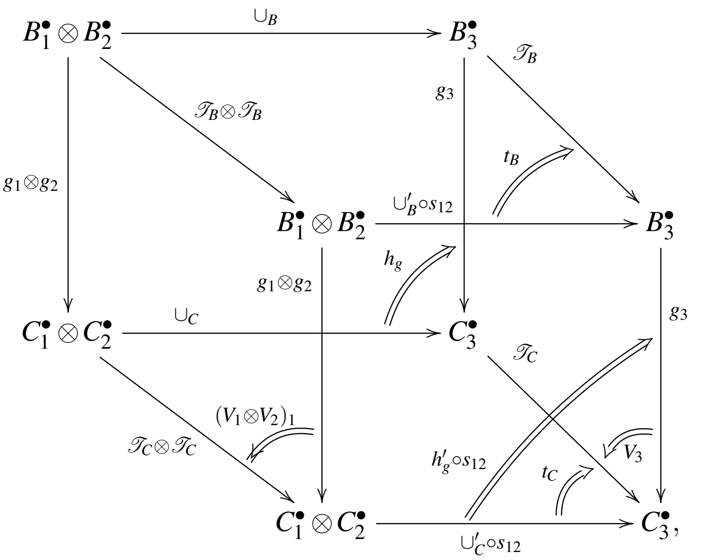


i.e. a system $H_f = (H_f^i)_{i \in \mathbf{Z}}$ of maps $H_f^i : (A_1 \otimes A_2)^i \to C_3^{i-2}$ such that

$$dH_f - H_f d = -t_C \circ (f_1 \otimes f_2) - \mathscr{T}_C \circ h_f + U_3 \circ \cup_A + \\ + f_3 \circ t_A + h'_f \circ (s_{12} \circ (\mathscr{T}_A \otimes \mathscr{T}_A)) - (\cup'_C \circ s_{12}) \circ (U_1 \otimes U_2)_1.$$

In this formula, $(U_1 \otimes U_2)_1$ denotes the homotopy defined by (18).

T7) A second order homotopy H_g trivializing the boundary of the cube



i.e. a system $H_g = (H_g^i)_{i \in \mathbf{Z}}$ of maps $H_g^i : (B_1 \otimes B_2)^i \to C_3^{i-2}$ such that

$$dH_g - H_g d = -t_C \circ (g_1 \otimes g_2) - \mathscr{T}_C \circ h_g + V_3 \circ \cup_B + \\ + g_3 \circ t_B + h'_g \circ (s_{12} \circ (\mathscr{T}_B \otimes \mathscr{T}_B)) - (\cup'_C \circ s_{12}) \circ (V_1 \otimes V_2)_1.$$

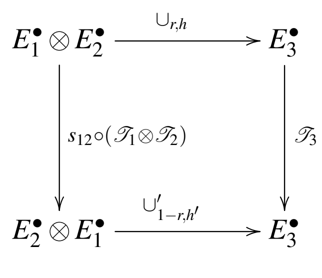
Proposition 1.2.4.* — *i) Given the data* P1-3) *and* T1-7)*, the formula

$$\mathscr{T}_i(a_n, b_n, c_{n-1}) = (\mathscr{T}_A(a_n), \mathscr{T}_B(b_n), \mathscr{T}_C(c_{n-1}) + U_i(a_n) - V_i(b_n))$$

defines morphisms of complexes

$$\mathscr{T}_i : E_i^\bullet \to E_i^\bullet, \qquad i = 1,2,3$$

such that, for any $r \in R$, the diagram



commutes up to homotopy.

Proof. — See [**56**, Proposition 1.3.6]. □

1.2.5. Bockstein maps. — Assume that, in addition to **P1-3)**, we are given the following data:

B1) Morphisms of complexes

$$\beta_{Z,i} : Z_i^\bullet \to Z_i^\bullet[1], \qquad Z_i^\bullet = A_i^\bullet, B_i^\bullet, C_i^\bullet, \quad i = 1,2.$$

B2) Homotopies

$$u_i : f_i[1] \circ \beta_{A,i} \rightsquigarrow \beta_{C,i} \circ f_i,$$
$$v_i : g_i[1] \circ \beta_{B,i} \rightsquigarrow \beta_{C,i} \circ g_i$$

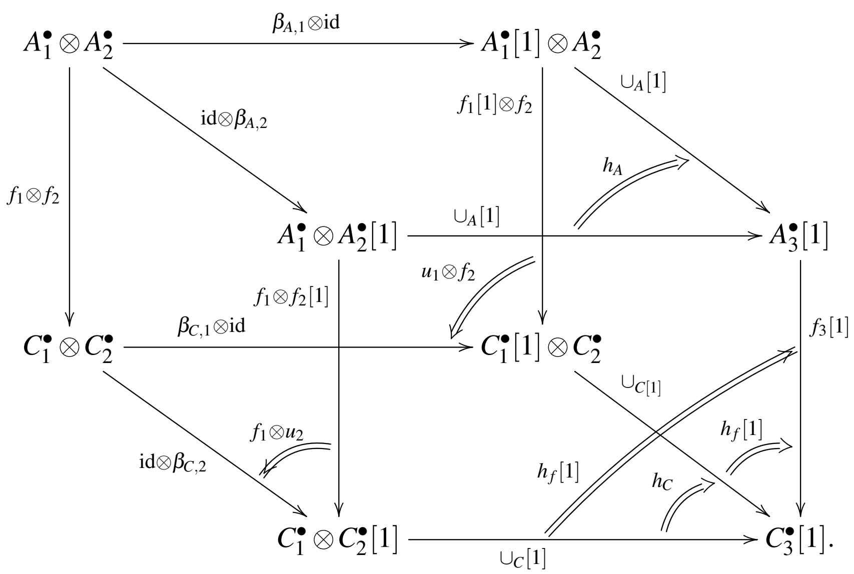
for $i = 1,2$.

B3) Homotopies

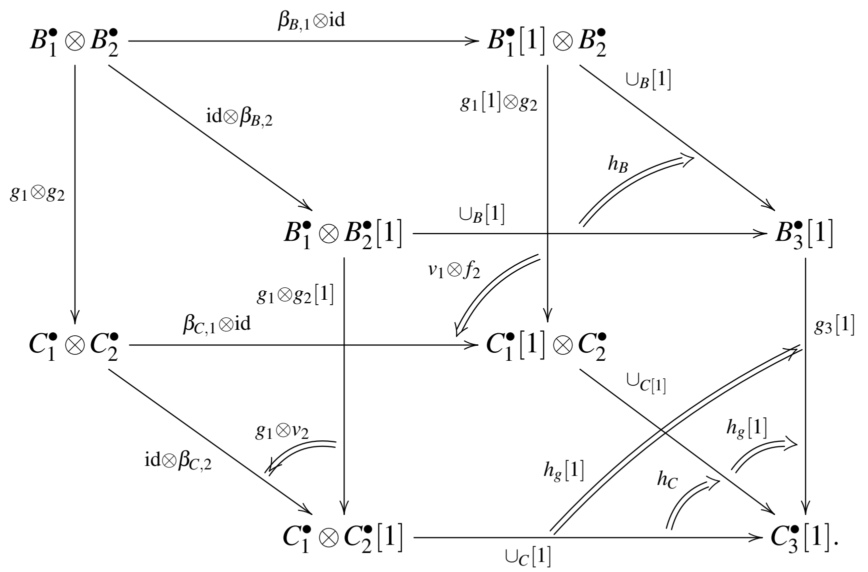
$$h_Z : \cup_Z[1] \circ (\mathrm{id} \otimes \beta_{Z,2}) \rightsquigarrow \cup_Z[1] \circ (\beta_{Z,1} \otimes \mathrm{id}),$$

for $Z^\bullet = A^\bullet, B^\bullet, C^\bullet$.

B4) A second order homotopy trivializing the boundary of the following diagram



B5) A second order homotopy trivializing the boundary of the cube



Proposition 1.2.6. — *i) Given the data* **P1-3)** *and* **B1-5)***, the formula*

$$\beta_{E,i}(a_n,b_n,c_{n-1}) = (\beta_{A,i}(a_n),\beta_{B,i}(b_n),-\beta_{C,i}(c_{n-1})-u_i(a_n)+v_i(b_n))$$

defines a morphism of complexes

$$\beta_{E,i}\,:\,E_i^\bullet \to E_i^\bullet[1]$$

such that for any $r\in R$ the diagram

$$\begin{array}{ccc} E_1^\bullet\otimes E_2^\bullet & \xrightarrow{\beta_{E,1}\otimes\mathrm{id}} & E_1^\bullet[1]\otimes E_2^\bullet \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\beta_{E,2}} & & \Big\downarrow{\scriptstyle \cup_{r,h}[1]} \\ E_1^\bullet\otimes E_2^\bullet[1] & \xrightarrow{\cup_{r,h}[1]} & E_3^\bullet[1] \end{array}$$

commutes up to homotopy.

ii) Given the data **P1-3)***,* **T1-7)** *and* **B1-5)***, for each $r\in R$ the diagram*

$$\begin{array}{ccccccc} E_1^\bullet\otimes E_2^\bullet & \xrightarrow{\beta_{E,1}\otimes\mathrm{id}} & E_1^\bullet[1]\otimes E_2^\bullet & \xrightarrow{\cup_{r,h}[1]} & E_3^\bullet[1] & \xrightarrow{\mathscr{T}_3[1]} & E_3^\bullet[1] \\ \Big\downarrow{\scriptstyle s_{12}} & & & & & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ E_2^\bullet\otimes E_1^\bullet & \xrightarrow{\beta_{E,2}\otimes\mathrm{id}} & E_2^\bullet[1]\otimes E_1^\bullet & \xrightarrow{\mathscr{T}_2[1]\otimes\mathscr{T}_1} & E_2^\bullet[1]\otimes E_1^\bullet & \xrightarrow{\cup'_{1-r,h'}[1]} & E_3^\bullet[1] \end{array}$$

is commutative up to a homotopy.

Proof. — See [**56**, Propositions 1.3.9 and 1.3.10]. □

CHAPTER 2

COHOMOLOGY OF (φ, Γ_K)-MODULES

2.1. (φ, Γ_K)-modules

2.1.1. — Throughout this section, K denotes a finite extension of $\mathbf{Q}_p$. Let k_K be the residue field of K, O_K its ring of integers and K_0 the maximal unramified subfield of K. We denote by K_0^{ur} the maximal unramified extension of K_0 and by σ the absolute Frobenius acting on K_0^{ur}. Fix an algebraic closure $\overline{K}$ of K and set $G_K = \mathrm{Gal}(\overline{K}/K)$. Let $\mathbf{C}_p$ be the p-adic completion of $\overline{K}$. We denote by $v_p : \mathbf{C}_p \to \mathbb{R} \cup \{\infty\}$ the p-adic valuation on $\mathbf{C}_p$ normalized so that $v_p(p) = 1$ and set $|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$. Write $A(r,1)$ for the p-adic annulus

$$A(r,1) = \{x \in \mathbf{C}_p \mid r \leqslant |x|_p < 1\}.$$

Fix a system of primitive p^n-th roots of unity $\varepsilon = (\zeta_{p^n})_{n \geqslant 0}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geqslant 0$. Let $K^{\mathrm{cyc}} = \bigcup_{n=0}^{\infty} K(\zeta_{p^n})$, $H_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{cyc}})$, $\Gamma_K = \mathrm{Gal}(K^{\mathrm{cyc}}/K)$ and let $\chi_K : \Gamma_K \to \mathbb{Z}_p^*$ denote the cyclotomic character.

Recall the constructions of some of Fontaine's rings of p-adic periods. Define

$$\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^p} O_{\mathbf{C}_p}/p\,O_{\mathbf{C}_p} = \{x = (x_0, x_1, \dots, x_n, \dots) \mid x_i^p = x_i, \quad \forall i \in \mathbf{N}\}.$$

Let $x = (x_0, x_1, \dots) \in \widetilde{\mathbf{E}}^+$. For each n, choose a lift $\hat{x}_n \in O_{\mathbf{C}_p}$ of x_n. Then, for all $m \geqslant 0$, the sequence $\hat{x}_{m+n}^{p^n}$ converges to $x^{(m)} = \lim_{n \to \infty} \hat{x}_{m+n}^{p^n} \in O_{\mathbf{C}_p}$, which does not depend on the choice of lifts. The ring $\widetilde{\mathbf{E}}^+$, equipped with the valuation $v_{\mathbf{E}}(x) = v_p(x^{(0)})$, is a complete local ring of characteristic p with residue field $\bar{k}_K$. Moreover, it is integrally closed in its field of fractions $\widetilde{\mathbf{E}} = \mathrm{Fr}(\widetilde{\mathbf{E}}^+)$.

Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ be the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}$. Denote by $[\cdot] : \widetilde{\mathbf{E}} \to W(\widetilde{\mathbf{E}})$ the Teichmüller lift. Each $u = (u_0, u_1, \ldots) \in \widetilde{\mathbf{A}}$ can be written in the form

$$u = \sum_{n=0}^{\infty} [u_n^{p^{-n}}] p^n.$$

Set $\pi = [\varepsilon] - 1$, $\mathbf{A}_{\mathbf{Q}_p}^{+} = \mathbf{Z}_p[[\pi]]$ and denote by $\mathbf{A}_{\mathbf{Q}_p}$ the p-adic completion of $\mathbf{A}_{\mathbf{Q}_p}^{+}[1/\pi]$ in $\widetilde{\mathbf{A}}$.

Let $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$, $\mathbf{B}_{\mathbf{Q}_p} = \mathbf{A}_{\mathbf{Q}_p}[1/p]$ and let $\mathbf{B}$ denote the completion of the maximal unramified extension of $\mathbf{B}_{\mathbf{Q}_p}$ in $\widetilde{\mathbf{B}}$. All these rings are endowed with natural actions of the Galois group G_K and the Frobenius operator φ, and we set $\mathbf{B}_K = \mathbf{B}^{H_K}$. Note that

$$\begin{aligned} \gamma(\pi) &= (1+\pi)^{\chi_K(\tau)} - 1, \qquad \gamma \in \Gamma_K, \\ \varphi(\pi) &= (1+\pi)^p - 1. \end{aligned}$$

For any $r > 0$ define

$$\widetilde{\mathbf{B}}^{\dagger,r} = \left\{ x \in \widetilde{\mathbf{B}} \mid \lim_{k \to +\infty} \left(v_{\mathbf{E}}(x_k) + \frac{pr}{p-1} k \right) = +\infty \right\}.$$

Set $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}_K^{\dagger,r} = \mathbf{B}_K \cap \mathbf{B}^{\dagger,r}$, $\mathbf{B}^{\dagger} = \bigcup_{r>0} \mathbf{B}^{\dagger,r}$ and $\mathbf{B}_K^{\dagger} = \bigcup_{r>0} \mathbf{B}_K^{\dagger,r}$.

Let L denote the maximal unramified subextension of $K^{\mathrm{cyc}}/\mathbf{Q}_p$ and let $e_K = [K^{\mathrm{cyc}} : L^{\mathrm{cyc}}]$. It can be shown (see [**18**]) that there exists $r_K \geqslant 0$ and $\pi_K \in \mathbf{B}_K^{\dagger,r_K}$ such that for all $r \geqslant r_K$ the ring $\mathbf{B}_K^{\dagger,r}$ has the following explicit description

$$\mathbf{B}_K^{\dagger,r} = \left\{ f(\pi_K) = \sum_{k \in \mathbb{Z}} a_k \pi_K^k \mid a_k \in L \text{ and } f \text{ is holomorphic} \right.$$
$$\left. \text{and bounded on } A(p^{-1/e_K r}, 1) \right\}.$$

Note that, if $K/\mathbf{Q}_p$ is unramified, $L = K_0$ and one can take $\pi_K = \pi$.

Define

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r} = \left\{ f(\pi_K) = \sum_{k \in \mathbb{Z}} a_k \pi_K^k \mid a_k \in L \text{ and } f \text{ is holomorphic} \right.$$
$$\left. \text{on } A(p^{-1/e_K r}, 1) \right\}.$$

The rings $\mathbf{B}_K^{\dagger,r}$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ are stable under Γ_K, and the Frobenius φ sends $\mathbf{B}_K^{\dagger,r}$ into $\mathbf{B}_K^{\dagger,pr}$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ into $\mathbf{B}_{\mathrm{rig},K}^{\dagger,pr}$. The ring

$$\mathscr{R}_K = \bigcup_{r\geqslant r_K} \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$$

is isomorphic to the Robba ring over L. Note that it is stable under Γ_K and φ. As usual, we set

$$t = \log(1+\pi) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\pi^n}{n} \in \mathscr{R}_{\mathbf{Q}_p}.$$

Note that $\varphi(t) = pt$ and $\gamma(t) = \chi_K(\gamma)t$, $\gamma \in \Gamma_K$.

To simplify notation, for each $r \geqslant r_K$ we set $\mathscr{R}_K^{(r)} = \mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$. The ring $\mathscr{R}_K^{(r)}$ is equipped with a canonical Fréchet topology (see [**12**]). Let A be an affinoid algebra over $\mathbf{Q}_p$. Define

$$\mathscr{R}_{K,A}^{(r)} = A\widehat{\otimes}_{\mathbf{Q}_p}\mathscr{R}_K^{(r)}, \qquad \mathscr{R}_{K,A} = \underset{r\geqslant r_K}{\cup} \mathscr{R}_{K,A}^{(r)}.$$

If the field K is clear from the context, we will often write $\mathscr{R}_A^{(r)}$ instead of $\mathscr{R}_{K,A}^{(r)}$ and $\mathscr{R}_A$ instead of $\mathscr{R}_{K,A}$.

***Definition.* —** *i) A (φ,Γ_K)-module over $\mathscr{R}_A^{(r)}$ is a finitely generated projective $\mathscr{R}_A^{(r)}$-module $\mathbf{D}^{(r)}$ equipped with the following structures:*

a) A φ-semilinear map

$$\mathbf{D}^{(r)} \to \mathbf{D}^{(r)} \otimes_{\mathscr{R}_A^{(r)}} \mathscr{R}_A^{(pr)}$$

such that the induced linear map

$$\varphi^* \ : \ \mathbf{D}^{(r)} \otimes_{\mathscr{R}_A^{(r)},\varphi} \mathscr{R}_A^{(pr)} \to \mathbf{D}^{(r)} \otimes_{\mathscr{R}_A^{(r)}} \mathscr{R}_A^{(pr)}$$

is an isomorphism of $\mathscr{R}_A^{(pr)}$-modules;

b) A semilinear continuous action of Γ_K on $\mathbf{D}^{(r)}$.

ii) $\mathbf{D}$ is a (φ,Γ_K)-module over $\mathscr{R}_A$ if $\mathbf{D} = \mathbf{D}^{(r)} \otimes_{\mathscr{R}_A^{(r)}} \mathscr{R}_A$ for some (φ,Γ_K)-module $\mathbf{D}^{(r)}$ over $\mathscr{R}_A^{(r)}$, with $r \geqslant r_K$.

If $\mathbf{D}$ is a (φ,Γ_K)-module over $\mathscr{R}_A$, we write $\mathbf{D}^* = \mathrm{Hom}_{\mathscr{R}_A}(\mathbf{D},A)$ for the dual (φ,Γ)-module. Let $\mathbf{M}_{\mathscr{R}_A}^{\varphi,\Gamma}$ denote the $\otimes$-category of (φ,Γ_K)-modules over $\mathscr{R}_A$.

2.1.2. — A p-adic representation of G_K with coefficients in an affinoid $\mathbf{Q}_p$-algebra A is a finitely generated projective A-module equipped with a continuous A-linear action of G_K. Note that, as A is a noetherian ring, a finitely generated A-module is projective if and only if it is flat. Let $\mathbf{Rep}_A(G_K)$ denote the $\otimes$-category of p-adic representations with coefficients in A. The relationship between p-adic representations and (φ,Γ_K)-modules first appeared in the pioneering paper of Fontaine [**29**]. The key result of this theory is the following theorem.

Theorem 2.1.3 **(Fontaine, Cherbonnier–Colmez, Kedlaya)**
Let A be an affinoid algebra over $\mathbf{Q}_p$.
i) There exists a fully faithul functor

$$\mathbf{D}^{\dagger}_{\mathrm{rig},A} \,:\, \mathbf{Rep}_A(G_K) \to \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_A},$$

which commutes with base change. More precisely, let $\mathscr{X} = \mathrm{Spm}(A)$. For each $x \in \mathscr{X}$, denote by $\mathfrak{m}_x$ the maximal ideal of A associated to x and set $E_x = A/\mathfrak{m}_x$. If V (resp. $\mathbf{D}$) is an object of $\mathbf{Rep}_A(G_{\mathbf{Q}_p})$ (resp. of $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_A}$), set $V_x = V \otimes_A E_x$ (resp. $\mathbf{D}_x = \mathbf{D} \otimes_A E_x$). Then the diagram

$$\begin{array}{ccc}
\mathbf{Rep}_A(G_{\mathbf{Q}_p}) & \xrightarrow{\mathbf{D}^{\dagger}_{\mathrm{rig},A}} & \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_A} \\
\downarrow{\scriptstyle \otimes E_x} & & \downarrow{\scriptstyle \otimes E_x} \\
\mathbf{Rep}_{E_x}(G_{\mathbf{Q}_p}) & \xrightarrow{\mathbf{D}^{\dagger}_{\mathrm{rig},E_x}} & \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{E_x}}
\end{array}$$

commutes, i.e. $\mathbf{D}^{\dagger}_{\mathrm{rig},A}(V)_x \simeq \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_x)$.
ii) If E is a finite extension of $\mathbf{Q}_p$, then the essential image of $\mathbf{D}^{\dagger}_{\mathrm{rig},E}$ is the subcategory of (φ,Γ_K)-modules of slope 0 in the sense of Kedlaya [**44**].

Proof. — This follows from the main results of [**29**], [**18**] and [**44**]. See also [**22**]. □

***Remark 2.1.4.* —** Note that in general the essential image of $\mathbf{D}^{\dagger}_{\mathrm{rig},A}$ does not coincide with the subcategory of étale modules. See [**15, 46, 37**] for further discussion.

2.2. Relation to p-adic Hodge theory

2.2.1. — In [**29**], Fontaine proposed to classify p-adic representations arising in p-adic Hodge theory in terms of (φ,Γ_K)-modules (Fontaine's program). More precisely, the problem is to recover classical Fontaine's functors $\mathbf{D}_{\mathrm{dR}}(V)$, $\mathbf{D}_{\mathrm{st}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ (see, for example, [**31**]) from $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. The complete solution was obtained by Berger in [**12, 14**]. His theory also allowed him to prove that each de Rham representation is potentially semistable. In this subsection, we review some of results of Berger. See also [**20**] for introduction and relation to the theory of p-adic differential equations. Let E be a fixed finite extension of $\mathbf{Q}_p$.

Definition. — *i) A filtered module over K with coefficients in E is a free $K\otimes_{\mathbf{Q}_p} E$-module M of finite rank equipped with a decreasing exhaustive filtration $(\mathrm{Fil}^i M)_{i\in\mathbf{Z}}$. We denote by $\mathbf{MF}_{K,E}$ the $\otimes$-category of such modules.*

ii) A filtered (φ,N)-module over K with coefficients in E is a free $K_0\otimes_{\mathbf{Q}_p} E$-module M of finite rank equipped with the following structures:

a) An exhaustive decreasing filtration $(\mathrm{Fil}^i M_K)_{i\in\mathbf{Z}}$ on $M_K = M\otimes_{K_0} K$;

b) A σ-semilinear bijective operator $\varphi : M\to M$;

c) A $K_0\otimes_{\mathbf{Q}_p} E$-linear operator N such that $N\varphi = p\,\varphi N$.

iii) A filtered φ-module over K with coefficients in E is a filtered (φ,N)-module such that $N=0$.

We denote by $\mathbf{MF}^{\varphi,N}_{K,E}$ the $\otimes$-category of filtered (φ,N)-module over K with coefficients in E and by $\mathbf{MF}^{\varphi}_{K,E}$ the category of filtered φ-modules.

iv) If L/K is a finite Galois extension and $G_{L/K}=\mathrm{Gal}(L/K)$, then a filtered $(\varphi,N,G_{L/K})$-module is a filtered (φ,N)-module M over L equipped with a semilinear action of $G_{L/K}$ which commutes with φ and N and such that the filtration $(\mathrm{Fil}^i M_L)_{i\in\mathbf{Z}}$ is stable under the action of $G_{L/K}$.

v) We say that M is a filtered (φ,N,G_K)-module if $M=K_0^{\mathrm{ur}}\otimes_{L_0} M'$, where M' is a filtered $(\varphi,N,G_{L/K})$-module for some L/K. We denote by $\mathbf{MF}^{\varphi,N,G_K}_{K,E}$ the $\otimes$-category of (φ,N,G_K)-modules.

Let $K^{\mathrm{cyc}}((t))$ denote the ring of formal Laurent power series with coefficients in K^{cyc} equipped with the filtration $\mathrm{Fil}^i K^{\mathrm{cyc}}((t)) = t^i K^{\mathrm{cyc}}[[t]]$ and the action of Γ_K given by

$$\gamma\left(\sum_{k\in\mathbf{Z}} a_k t^k\right) = \sum_{k\in\mathbf{Z}} \gamma(a_k)\chi_K(\gamma)^k t^k, \qquad \gamma\in\Gamma_K.$$

The ring $\mathscr{R}_{K,E}$ can not be naturally embedded in $E\otimes_{\mathbf{Q}_p} K^{\mathrm{cyc}}((t))$, but for any $r\geqslant r_K$ there exists a Γ_K-equivariant embedding $i_n : \mathscr{R}^{(r)}_{K,E}\to E\otimes_{\mathbf{Q}_p} K^{\mathrm{cyc}}((t))$ which sends π to $\zeta_{p^n}e^{t/p^n}-1$. Let $\mathbf{D}$ be a (φ,Γ_K)-module over $\mathscr{R}_{K,E}$ and let $\mathbf{D}=\mathbf{D}^{(r)}\otimes_{\mathscr{R}^{(r)}_{K,E}}\mathscr{R}_{K,E}$ for some $r\geqslant r_K$. Then

$$\mathscr{D}_{\mathrm{dR}/K}(\mathbf{D}) = \left(E\otimes_{\mathbf{Q}_p} K^{\mathrm{cyc}}((t))\otimes_{i_n}\mathbf{D}^{(r)}\right)^{\Gamma_K}$$

is a free $E\otimes_{\mathbf{Q}_p} K$-module of finite rank equipped with a decreasing filtration

$$\mathrm{Fil}^i \mathscr{D}_{\mathrm{dR}/K}(\mathbf{D}) = \left(E\otimes_{\mathbf{Q}_p} \mathrm{Fil}^i K^{\mathrm{cyc}}((t))\otimes_{i_n} \mathbf{D}^{(r)}\right)^{\Gamma_K},$$

which does not depend on the choice of r and n.

Let $\mathscr{R}_{K,E}[\log \pi]$ denote the ring of power series in variable $\log\pi$ with coefficients in $\mathscr{R}_{K,E}$. Extend the actions of φ and Γ_K to $\mathscr{R}_{K,E}[\log\pi]$ setting

$$\varphi(\log\pi) = p\log\pi + \log\left(\frac{\varphi(\pi)}{\pi^p}\right),$$
$$\gamma(\log\pi) = \log\pi + \log\left(\frac{\gamma(\pi)}{\pi}\right), \quad \gamma\in\Gamma_K.$$

(Note that $\log(\varphi(\pi)/\pi^p)$ and $\log(\tau(\pi)/\pi)$ converge in $\mathscr{R}_{K,E}$.) Define a monodromy operator $N : \mathscr{R}_{K,E}[\log\pi]\to\mathscr{R}_{K,E}[\log\pi]$ by

$$N = -\left(1-\frac{1}{p}\right)^{-1}\frac{d}{d\log\pi}.$$

For any (φ,Γ_K)-module $\mathbf{D}$ define

$$\mathscr{D}_{\mathrm{st}/K}(\mathbf{D}) = \left(\mathbf{D}\otimes_{\mathscr{R}_{K,E}}\mathscr{R}_{K,E}[\log\pi,1/t]\right)^{\Gamma_K}, \quad t=\log(1+\pi),$$
$$\mathscr{D}_{\mathrm{cris}/K}(\mathbf{D}) = \mathscr{D}_{\mathrm{st}}(\mathbf{D})^{N=0} = (\mathbf{D}[1/t])^{\Gamma_K}.$$

Then $\mathscr{D}_{\mathrm{st}}(\mathbf{D})$ is a free $E\otimes_{\mathbf{Q}_p} K_0$-module of finite rank equipped with natural actions of φ and N such that $N\varphi = p\,\varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration induced by the embeddings i_n. If L/K is a finite extension and $\mathbf{D}$ is a (φ,Γ_K)-module, the tensor product $\mathbf{D}_L = \mathscr{R}_{L,E}\otimes_{\mathscr{R}_{K,E}}\mathbf{D}$ has a natural structure of a (φ,Γ_L)-module, and we define

$$\mathscr{D}_{\mathrm{pst}/K}(\mathbf{D}) = \varinjlim_{L/K}\mathscr{D}_{\mathrm{st}/L}(\mathbf{D}_L).$$

Then $\mathscr{D}_{\mathrm{pst}/K}(\mathbf{D})$ is a free $E\otimes_{\mathbf{Q}_p} K_0^{\mathrm{ur}}$-module equipped with natural actions of φ and N and a discrete action of G_K. Therefore, we have four functors

$$\mathscr{D}_{\mathrm{dR}/K} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E}} \to \mathbf{MF}_{K,E},$$
$$\mathscr{D}_{\mathrm{st}/K} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E}} \to \mathbf{MF}^{\varphi,N}_{K,E},$$
$$\mathscr{D}_{\mathrm{pst}/K} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E}} \to \mathbf{MF}^{\varphi,N,G_K}_{K,E},$$
$$\mathscr{D}_{\mathrm{cris}/K} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E}} \to \mathbf{MF}^{\varphi}_{K,E}.$$

If the field K is fixed and understood from context, we will omit it and simply write $\mathscr{D}_{\mathrm{dR}}$, $\mathscr{D}_{\mathrm{st}}$, $\mathscr{D}_{\mathrm{pst}}$ and $\mathscr{D}_{\mathrm{cris}}$.

***Theorem 2.2.2* (Berger). —** *Let V be a p-adic representation of G_K. Then*

$$\mathbf{D}_{*/K}(V) \simeq \mathscr{D}_{*/K}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)), \qquad * \in \{\mathrm{dR}, \mathrm{st}, \mathrm{pst}, \mathrm{cris}\}.$$

Proof. — See [**12**]. □

For any (φ, Γ_K)-module over $\mathscr{R}_{K,E}$ one has

$$\mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{cris}/K}(\mathbf{D}) \leqslant \mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{st}/K}(\mathbf{D}) \leqslant \mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{dR}/K}(\mathbf{D}) \leqslant \mathrm{rk}_{\mathscr{R}_{K,E}}(\mathbf{D}).$$

***Definition*. —** *One says that $\mathbf{D}$ is de Rham (resp. semistable, resp. potentially semistable, resp. crystalline) if*

$$\mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{dR}/K}(\mathbf{D}) = \mathrm{rk}_{\mathscr{R}_{K,E}}(\mathbf{D})$$

(resp. $\mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{st}/K}(\mathbf{D}) = \mathrm{rk}_{\mathscr{R}_{K,E}}(\mathbf{D})$, resp. $\mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{pst}/K}(\mathbf{D}) = \mathrm{rk}_{\mathscr{R}_{K,E}}(\mathbf{D})$, resp. $\mathrm{rk}_{E\otimes K_0}\mathscr{D}_{\mathrm{cris}/K}(\mathbf{D}) = \mathrm{rk}_{\mathscr{R}_{K,E}}(\mathbf{D})$).

Let $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_E,\mathrm{st}}$, $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_E,\mathrm{pst}}$ and $\mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_E,\mathrm{cris}}$ denote the categories of semistable, potentially semistable and crystalline (φ, Γ)-modules respectively. If $\mathbf{D}$ is de Rham, the jumps of the filtration $\mathrm{Fil}^i\mathscr{D}_{\mathrm{dR}}(\mathbf{D})$ will be called the Hodge–Tate weights of $\mathbf{D}$.

***Theorem 2.2.3* (Berger). —** *i) The functors*

$$\mathscr{D}_{\mathrm{st}} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E},\mathrm{st}} \to \mathbf{MF}^{\varphi,N}_{K,E},$$
$$\mathscr{D}_{\mathrm{pst}} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E},\mathrm{pst}} \to \mathbf{MF}^{\varphi,N,G_K}_{K,E},$$
$$\mathscr{D}_{\mathrm{cris}} : \mathbf{M}^{\varphi,\Gamma}_{\mathscr{R}_{K,E},\mathrm{cris}} \to \mathbf{MF}^{\varphi}_{K,E}$$

are equivalences of $\otimes$-categories.

ii) Let $\mathbf{D}$ be a (φ, Γ_K)-module. Then $\mathbf{D}$ is potentially semistable if and only if $\mathbf{D}$ is de Rham.

Proof. — These results are proved in [**14**]. See Theorem A, Theorem III.2.4 and Theorem V.2.3 of *op. cit.*. □

2.3. Local Galois cohomology

2.3.1. — For the content of this section we refer the reader to [**62**]. Let V be a p-adic representation of G_K with coefficients in an affinoid algebra A. Consider the complex $C^{\bullet}(G_K, V)$ of continuous cochains of G_K with coefficients in A and the corresponding object $\mathbf{R}\Gamma(K, V)$ of $\mathscr{D}(A)$. For the Tate module $A(1)$, the base change (see [**62**, Proof of Theorem 1.14]) and the classical computation of $H^2(K, \mathbf{Z}_p(1))$ together give

$$\tau_{\geq 2}\mathbf{R}\Gamma(K, A(1)) \simeq A[-2].$$

In particular, we have a canonical isomorphism

$$\mathrm{inv}_K\,:\,H^2(K,\mathbf{Z}_p(1))\simeq A. \tag{28}$$

Recall (see Section 0.2) that on the category $\mathscr{D}_{\mathrm{perf}}(A)$ of perfect complexes we have the contravariant dualization functor

$$X\to X^*=\mathbf{R}\mathrm{Hom}_A(X,A). \tag{29}$$

The natural pairing $V^*(1)\otimes V\to A(1)$ induces a pairing

$$\mathbf{R}\Gamma(K,V^*(1))\otimes^{\mathbf{L}}_A\mathbf{R}\Gamma(K,V^*(1))\to\tau_{\geq 2}\mathbf{R}\Gamma(K,A(1))\simeq A[-2]. \tag{30}$$

The following theorem is a version of classical results on local Galois cohomology in our context.

Theorem 2.3.2 **(Pottharst). —** *Let V be a p-adic Galois representation with coefficients in an affinoid algebra A.*

i) Finiteness. We have $\mathbf{R}\Gamma(K,V)\in\mathscr{D}^{[0,2]}_{\mathrm{perf}}(A)$.

ii) Euler–Poincaré characteristic. We have

$$\sum_{i=0}^{2}(-1)^i\mathrm{rk}_AH^i(K,V)=-[K:\mathbf{Q}_p]\cdot\mathrm{rk}_A(V).$$

iii) Duality. The pairing (30) induces an isomorphism

$$\mathbf{R}\Gamma(K,V^*(1))\simeq\mathbf{R}\Gamma(K,V)^*[-2]:=\mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(K,V),A)[-2].$$

Proof. — See [**62**, Corollary 1.2 and Theorem 1.14]. □

Remark 2.3.3. **—** Theorem 2.3.2 is inspired by Nekovář's duality theory for big Galois representations [**56**, Chapters 2-5].

2.4. The complex $C_{\varphi,\gamma_K}(\mathbf{D})$

2.4.1. — In this section we review the generalization of local Galois cohomology to (φ,Γ_K)-modules over a Robba ring. We keep previous notation and conventions. Set $\Delta_K=\mathrm{Gal}(K(\zeta_p)/K)$. Then $\Gamma_K=\Delta_K\times\Gamma_K^0$, where Γ_K^0 is a pro-p-group isomorphic to $\mathbf{Z}_p$. Fix a topological generator γ_K of Γ_K^0. For each (φ,Γ_K)-module $\mathbf{D}$ over $\mathscr{R}_A=\mathscr{R}_{K,A}$ define

$$C^\bullet_{\gamma_K}(\mathbf{D})\;:\;\mathbf{D}^{\Delta_K}\xrightarrow{\gamma_K-1}\mathbf{D}^{\Delta_K},$$

where the first term is placed in degree 0. If $\mathbf{D}'$ and $\mathbf{D}''$ are two (φ,Γ_K)-modules, we will denote by

$$\cup_\gamma\;:\;C^\bullet_{\gamma_K}(\mathbf{D}')\otimes C^\bullet_{\gamma_K}(\mathbf{D}'')\to C^\bullet_{\gamma_K}(\mathbf{D}'\otimes\mathbf{D}'')$$

the bilinear map

$$\cup_\gamma(x_n\otimes y_m)=\begin{cases} x_n\otimes \gamma_K^n(y_m) & \text{if } x_n\in C^n_{\gamma_K}(\mathbf{D}'),\, y_m\in C^m_{\gamma_K}(\mathbf{D}''),\\ & \text{and } n+m=0 \text{ or } 1,\\ 0 & \text{if } n+m\geqslant 2.\end{cases}$$

Consider the total complex

$$C^\bullet_{\varphi,\gamma_K}(\mathbf{D})=\mathrm{Tot}\left(C^\bullet_{\gamma_K}(\mathbf{D})\xrightarrow{\varphi-1}C^\bullet_{\gamma_K}(\mathbf{D})\right).$$

More explicitly,

$$C^\bullet_{\varphi,\gamma_K}(\mathbf{D})\ :\ 0\to \mathbf{D}^{\Delta_K}\xrightarrow{d_0}\mathbf{D}^{\Delta_K}\oplus\mathbf{D}^{\Delta_K}\xrightarrow{d_1}\mathbf{D}^{\Delta_K}\to 0,$$

where $d_0(x)=((\varphi-1)x,(\gamma_K-1)x)$ and $d_1(x,y)=(\gamma_K-1)x-(\varphi-1)y$. Note that $C^\bullet_{\varphi,\gamma_K}(\mathbf{D})$ coincides with the complex of Fontaine–Herr [**38, 39, 49**]. We will write $H^*(\mathbf{D})$ for the cohomology of $C^\bullet_{\varphi,\gamma}(\mathbf{D})$. If $\mathbf{D}'$ and $\mathbf{D}''$ are two (φ,Γ_K)-modules, the cup product $\cup_\gamma$ induces, by Proposition 1.1.5, a bilinear map

$$\cup_{\varphi,\gamma}\,:\,C^\bullet_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^\bullet_{\varphi,\gamma_K}(\mathbf{D}'')\to C^\bullet_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}'').$$

Explicitly

$$\cup_{\varphi,\gamma}((x_{n-1},x_n)\otimes(y_{m-1},y_m))=(x_n\cup_\gamma y_{m-1}+(-1)^m x_{n-1}\cup_\gamma\varphi(y_m),x_n\cup_\gamma y_m),$$

if $(x_{n-1},x_n)\in C^n_{\varphi,\gamma_K}(\mathbf{D}')=C^{n-1}_{\gamma_K}(\mathbf{D}')\oplus C^n_{\gamma_K}(\mathbf{D}')$ and $(y_{m-1},y_m)\in C^m_{\varphi,\gamma}(\mathbf{D}'')=C^{m-1}_{\gamma}(\mathbf{D}'')\oplus C^m_{\gamma}(\mathbf{D}'')$. An easy computation gives the following formulas

$$\begin{cases}C^0_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^0_{\varphi,\gamma_K}(\mathbf{D}'')\to C^0_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ x_0\otimes y_0\mapsto x_0\otimes y_0,\end{cases}$$

$$\begin{cases}C^0_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^1_{\varphi,\gamma_K}(\mathbf{D}'')\to C^1_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ x_0\otimes(y_0,y_1)\mapsto(x_0\otimes y_0,x_0\otimes y_1),\end{cases}$$

$$\begin{cases}C^1_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^0_{\varphi,\gamma_K}(\mathbf{D}'')\to C^1_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ (x_0,x_1)\otimes y_0\mapsto(x_0\otimes\varphi(y_0),x_1\otimes\gamma_K(y_0)),\end{cases}$$

$$\begin{cases}C^1_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^1_{\varphi,\gamma_K}(\mathbf{D}'')\to C^2_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ (x_0,x_1)\otimes(y_0,y_1)\mapsto(x_1\otimes\gamma_K(y_0)-x_0\otimes\varphi(y_1)),\end{cases}$$

$$\begin{cases}C^0_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^2_{\varphi,\gamma_K}(\mathbf{D}'')\to C^2_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ x_0\otimes y_1\mapsto x_0\otimes y_1,\end{cases}$$

$$\begin{cases}C^2_{\varphi,\gamma_K}(\mathbf{D}')\otimes C^0_{\varphi,\gamma_K}(\mathbf{D}'')\to C^2_{\varphi,\gamma_K}(\mathbf{D}'\otimes\mathbf{D}''),\\ x_1\otimes y_0\mapsto x_1\otimes\gamma_K(\varphi(y_1)).\end{cases}$$

Here the zero components are omitted.

2.4.2. — For each (φ,Γ_K)-module $\mathbf{D}$ we denote by

$$\mathbf{R}\Gamma(K,\mathbf{D})=\left[C^\bullet_{\varphi,\gamma_K}(\mathbf{D})\right]$$

the corresponding object of the derived category $\mathscr{D}(A)$. The cohomology of $\mathbf{D}$ is defined by

$$H^i(\mathbf{D})=\mathbf{R}^i\Gamma(K,\mathbf{D})=H^i(C^\bullet_{\varphi,\gamma_K}(\mathbf{D})),\qquad i\geqslant 0.$$

There exists a canonical isomorphism in $\mathscr{D}(A)$

$$\mathrm{TR_K}\,:\,\tau_{\geqslant 2}\mathbf{R}\Gamma(K,\mathscr{R}_A(\chi_K))\simeq A[-2]$$

(see [**39**], [**49**], [**46**]). Therefore, for each $\mathbf{D}$ we have morphisms

$$\begin{aligned}(31)\quad \mathbf{R}\Gamma(K,\mathbf{D})\otimes^{\mathbf{L}}_A\mathbf{R}\Gamma(K,\mathbf{D}^*(\chi_K))&\xrightarrow{\cup_{\varphi,\gamma}}\mathbf{R}\Gamma(K,\mathbf{D}\otimes\mathbf{D}^*(\chi_K))\\ &\xrightarrow{\text{duality}}\mathbf{R}\Gamma(K,\mathscr{R}_A(\chi_K))\to\tau_{\geqslant 2}\mathbf{R}\Gamma(K,\mathscr{R}_A(\chi_K))\simeq A[-2].\end{aligned}$$

The following theorem generalizes main results on the local Galois cohomology to (φ,Γ)-modules.

Theorem 2.4.3 **(Kedlaya–Pottharst–Xiao). —** *Let* $\mathbf{D}$ *be a* (φ,Γ_K)*-module over* $\mathscr{R}_{K,A}$*, where* A *is an affinoid algebra.*

i) Finiteness. We have $\mathbf{R}\Gamma(K,\mathbf{D}) \in \mathscr{D}^{[0,2]}_{\mathrm{perf}}(A)$.

ii) Euler–Poincaré characteristic formula. We have

$$\sum_{i=0}^{2}(-1)^i \mathrm{rk}_A H^i(\mathbf{D}) = -[K:\mathbf{Q}_p]\,\mathrm{rk}_{\mathscr{R}_{K,A}}(\mathbf{D}).$$

iii) Duality. The morphism (31) induces an isomorphism

$$\mathbf{R}\Gamma(K,\mathbf{D}^*(\chi_K)) \simeq \mathbf{R}\Gamma(K,\mathbf{D})^*[-2] := \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(K,\mathbf{D}),A)[-2].$$

In particular, we have cohomological pairings

$$\cup\ :\ H^i(\mathbf{D}) \otimes H^{2-i}(\mathbf{D}^*(\chi_K)) \to H^2(\mathscr{R}_A(\chi_K)) \simeq A, \qquad i \in \{0,1,2\}.$$

iv) Comparision with Galois cohomology. Let V *is a* p*-adic representation of* G_K *with coefficients in* A*. There exist canonical (up to the choice of* γ_K*) and functorial isomorphisms*

$$H^i(K,V) \xrightarrow{\sim} H^i(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$$

which are compatible with cup-products. In particular, we have a commutative diagram

$$\begin{array}{ccc} H^2(\mathscr{R}_A(\chi_K)) & \xrightarrow{\mathrm{TR_K}} & A \\ \downarrow \simeq & & \downarrow = \\ H^2(K,A(\chi_K)) & \xrightarrow{\mathrm{inv}_K} & A, \end{array}$$

where inv_K *is the canonical isomorphism of the local class field theory (28).*

Proof. — of See [**46**, Theorem 4.4.5] and [**62**, Theorem 2.8]. □

***Remark 2.4.4.* —** The explicit construction of the isomorphism TR_K is given in [**39**] and [**6**, Theorem 2.2.6].

2.5. The complex $K^\bullet(V)$

2.5.1. — In this section, we give the derived version of isomorphisms

$$H^i(K,V) \xrightarrow{\sim} H^i(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$$

of Theorem 2.4.3 iv). We write $C^\bullet_{\varphi,\gamma_K}(V)$ instead of $C^\bullet_{\varphi,\gamma_K}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ to simplify notation. Let K be a finite extension of $\mathbf{Q}_p$. Let V be a p-adic representation of G_K with coefficients in an affinoid algebra A.

In [**12**], Berger constructed, for each $r \geqslant r_K$, a ring $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ which is the completion of $\mathbf{B}^{\dagger,r}$ with respect to Frechet topology. Set $\widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger,r} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}\widehat{\otimes}_{\mathbf{Q}_p}A$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger} = \underset{r\geqslant r_K}{\cup} \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger,r}$. For each $r \geqslant r_K$ we have an exact sequence

$$0 \to \mathbf{Q}_p \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,rp} \to 0$$

(see [**13**, Lemma I.7]). Since the completed tensor product by an orthonormalizable Banach space is exact in the category of Frechet spaces (see, for example, [**3**, proof of Lemma 3.9]), the sequence

$$0 \to A \to \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger,r} \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger,rp} \to 0.$$

is also exact. Passing to the direct limit we obtain an exact sequence

$$0 \to A \to \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger} \xrightarrow{\varphi-1} \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger} \to 0. \tag{32}$$

Set $V_{\mathrm{rig}}^{\dagger} = V \otimes_A \widetilde{\mathbf{B}}_{\mathrm{rig},A}^{\dagger}$ and consider the complex $C^\bullet(G_K, V_{\mathrm{rig}}^{\dagger})$. Then (32) induces an exact sequence

$$0 \to C^\bullet(G_K,V) \to C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger}) \xrightarrow{\varphi-1} C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger}) \to 0.$$

Define

$$K^\bullet(V) = T^\bullet(C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger})) = \mathrm{Tot}\left(C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger}) \xrightarrow{\varphi-1} C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger})\right).$$

Consider the map

$$\alpha_V \ : \ C_{\gamma_K}^\bullet(V) \to C^\bullet(G_K,V_{\mathrm{rig}}^{\dagger})$$

defined by

$$\begin{cases} \alpha_V(x_0) = x_0, & x_0 \in C_{\gamma_K}^0(V), \\ \alpha_V(x_1)(g) = \dfrac{\gamma_K^{\kappa(g)}-1}{\gamma_K-1}(x_1), & x_1 \in C_{\gamma_K}^1(V), \end{cases}$$

where $g \in G_K$ and $\gamma_K^{\kappa(g)} = g|_{\Gamma_K^0}$. It is easy to check that α_V is a morphism of complexes which commutes with φ. By fonctoriality, we obtain a morphism (which we denote again by α_V):

$$\alpha_V \ : C_{\varphi,\gamma_K}^\bullet(V) \to K^\bullet(V).$$

Proposition 2.5.2. — *The map $\alpha_V \ : C_{\varphi,\gamma_K}^\bullet(V) \to K^\bullet(V)$ and the map*

$$\begin{aligned} \xi_V : \qquad & C^\bullet(G_K,V) \to K^\bullet(V), \\ & x_n \mapsto (0,x_n), \qquad x_n \in C^n(G_K,V) \end{aligned}$$

are quasi-isomorphisms.

Proof. — This is [**10**, Proposition 9]. □

2.5.3. — If M and N are two Galois modules, the cup-product

$$\cup_c \,:\, C^{\bullet}(M)\otimes C^{\bullet}(M)\to C^{\bullet}(M\otimes N)$$

defined by

$$(x_n\cup_c y_m)(g_1,g_2,\dots,g_{n+m})=$$
$$=x_n(g_1,\dots,g_n)\otimes(g_1g_2\cdots g_n)y_m(g_{n+1},\dots,g_{n+m}),$$

where $x_n\in C^n(G_K,M)$ and $y_m\in C^m(G_K,N)$, is a morphism of complexes. Let V and U be two Galois representations of G_K. Applying Proposition 1.1.5 to the complexes $C^{\bullet}(G_K,V_{\mathrm{rig}}^{\dagger})$ and $C^{\bullet}(G_K,U_{\mathrm{rig}}^{\dagger})$ we obtain a morphism

$$\cup_K \,:\, K^{\bullet}(V)\otimes K^{\bullet}(U)\to K^{\bullet}(V\otimes U).$$

The following proposition will not be used in the remainder of this paper, but we state it here for completeness.

***Proposition 2.5.4.* —** *In the diagram*

$$\begin{array}{ccc}
C^{\bullet}_{\varphi,\gamma_K}(V)\otimes C^{\bullet}_{\varphi,\gamma_K}(U) & \xrightarrow{\ \cup_{\varphi,\gamma_K}\ } & C^{\bullet}_{\varphi,\gamma}(V\otimes U)\\
\Big\downarrow{\scriptstyle \alpha_V\otimes\alpha_U} & \overset{h_{\varphi,\gamma}}{\swarrow} & \Big\downarrow{\scriptstyle \alpha_{V\otimes U}}\\
K^{\bullet}(V)\otimes K^{\bullet}(U) & \xrightarrow{\ \cup_K\ } & K^{\bullet}(V\otimes U)
\end{array}$$

the maps $\alpha_{V\otimes U}\circ\cup_{\varphi,\gamma}$ and $\cup_K\circ(\alpha_V\otimes\alpha_U)$ are homotopic.

We need the following lemma.

***Lemma 2.5.5.* —** *For any $x\in C^1_{\gamma_K}(V)$, $y\in C^1_{\gamma_K}(U)$, let $c_{x,y}\in C^1(\Gamma_K^0,\mathbf{D}^{\dagger}_{\mathrm{rig}}(V\otimes U))$ denote the 1-cochain defined by*

$$c_{x,y}(\gamma_K^n)=\sum_{i=0}^{n-1}\gamma_K^i(x)\otimes\left(\frac{\gamma_K^n-\gamma_K^{i+1}}{\gamma_K-1}\right)(y),\qquad \text{if } n\neq 0,1, \tag{33}$$

and $c_{x,y}(1)=c_{x,y}(\gamma_K)=0$. Then

i) For each $x\in C^1_{\varphi,\gamma_K}(V)$ and $y\in C^0_{\varphi,\gamma_K}(U)$

$$c_{x,(\gamma_K-1)y}=\alpha_V(x)\cup_c\alpha_U(y)-\alpha_{V\otimes U}(x\cup_{\gamma}y).$$

ii) If $x\in C^0_{\varphi,\gamma_K}(V)$ and $y\in C^1_{\varphi,\gamma_K}(U)$ then

$$c_{(\gamma_K-1)x,y}=\alpha_{V\otimes U}(x\cup_{\gamma}y)-\alpha_V(x)\cup_c\alpha_U(y).$$

iii) One has

$$d^1(c_{x,y})=-\alpha_V(x)\cup_c\alpha_U(y).$$

Proof of the lemma. — i) Note that Γ_K^0 is the profinite completion of the cyclic group $\langle\gamma_K\rangle$, and an easy computation shows that the map $c_{x,y}$, defined on $\langle\gamma_K\rangle$ by (33), extends by continuity to a unique cochain on Γ_K^0 which we denote again by $c_{x,y}$.

For any natural $n\neq 0,1$ one has

$$\begin{aligned}
c_{x,(\gamma-1)y}(\gamma_K^n) \quad &= \sum_{i=0}^{n-1}\gamma_K^i(x)\otimes(\gamma_K^n-\gamma_K^{i+1})(y)=\\
&=\sum_{i=0}^{n-1}\gamma_K^i(x)\otimes\gamma_K^n(y)-\sum_{i=0}^{n-1}\gamma_K^i(x)\otimes\gamma_K^{i+1}(y)=\\
&=\frac{\gamma_K^n-1}{\gamma_K-1}(x)\otimes\gamma_K^n(y)-\frac{\gamma_K^n-1}{\gamma_K-1}(x\otimes\gamma_K(y))=\\
&=(g_V(x)\cup_c g_U(y))(\gamma_K^n)-(g_{V\otimes U}(x\cup_\gamma y))(\gamma_K^n).
\end{aligned}$$

By continuity, this implies that $c_{x,(\gamma_K-1)y}=\alpha_V(x)\cup_c\alpha_U(y)-\alpha_{V\otimes U}(x\cup_\gamma y)$, and i) is proved.

ii) An easy induction proves the formula

$$\begin{aligned}
(34)\quad \sum_{i=0}^{m}\gamma_K^i(\gamma_K-1)(x)\otimes\frac{\gamma_K^{i+1}-1}{\gamma_K-1}(y)&=\\
&=\gamma_K^{m+1}(x)\otimes\frac{\gamma_K^{m+1}-1}{\gamma_K-1}(y)-\frac{\gamma_K^{m+1}-1}{\gamma_K-1}(x\otimes y).
\end{aligned}$$

Therefore

$$\begin{aligned}
&c_{(\gamma_K-1)x,y}(\gamma_K^n)=\sum_{i=0}^{n-1}(\gamma_K^{i+1}-\gamma_K^i)(x)\otimes\frac{\gamma_K^n-\gamma_K^{i+1}}{\gamma_K-1}(y)=\\
&=\sum_{i=0}^{n-1}(\gamma_K^{i+1}-\gamma_K^i)(x)\otimes\frac{\gamma_K^n-1}{\gamma_K-1}(y)-\sum_{i=0}^{n-1}\gamma_K^i(\gamma_K-1)(x)\otimes\frac{\gamma_K^{i+1}-1}{\gamma_K-1}(y)=\\
&\overset{\text{by (34)}}{=}(\gamma_K^n-1)(x)\otimes\frac{\gamma_K^n-1}{\gamma_K-1}(y)+\frac{\gamma_K^n-1}{\gamma_K-1}(x\otimes y)-\gamma_K^n(x)\otimes\frac{\gamma_K^n-1}{\gamma_K-1}(y)=\\
&=\frac{\gamma_K^n-1}{\gamma_K-1}(x\otimes y)-x\otimes\frac{\gamma_K^n-1}{\gamma_K-1}(y)=\\
&=(\alpha_{V\otimes U}(x\cup_\gamma y))(\gamma_K^n)-(\alpha_V(x)\cup_c\alpha_U(y))(\gamma_K^n),
\end{aligned}$$

and ii) is proved.

iii) One has

$$\begin{aligned}
&d^1c_{x,y}(\gamma_K^n,\gamma_K^m)=\gamma_K^n c_{x,y}(\gamma_K^m)-c_{x,y}(\gamma_K^{n+m})+c_{x,y}(\gamma_K^n)=\\
&=\sum_{i=0}^{m-1}\gamma_K^{n+i}(x)\otimes\frac{\gamma_K^{n+m}-\gamma_K^{i+n+1}}{\gamma_K-1}(y)-\\
&-\sum_{i=0}^{n+m-1}\gamma_K^{i}(x)\otimes\frac{\gamma_K^{n+m}-\gamma_K^{i+1}}{\gamma_K-1}(y)+\sum_{i=0}^{n-1}\gamma_K^{i}(x)\otimes\frac{\gamma_K^{n}-\gamma_K^{i+1}}{\gamma_K-1}(y)=\\
&=-\sum_{i=0}^{n-1}\gamma_K^{i}(x)\otimes\frac{\gamma_K^{n+m}-\gamma_K^{i+1}}{\gamma_K-1}(y)+\sum_{i=0}^{n-1}\gamma_K^{i}(x)\otimes\frac{\gamma_K^{n}-\gamma_K^{i+1}}{\gamma_K-1}(y)=\\
&=\sum_{i=0}^{n-1}\gamma_K^{i}(x)\otimes\frac{\gamma_K^{n}-\gamma_K^{n+m}}{\gamma_K-1}(y)=-\frac{\gamma_K^{n}-1}{\gamma_K-1}(x)\otimes\gamma_K^n\frac{\gamma_K^{m}-1}{\gamma_K-1}(y)=\\
&=-(\alpha_V(x)\cup_c\alpha_U(y))(\gamma_K^n,\gamma_K^m).
\end{aligned}$$

By continuity, $d^1c_{x,y}=-\alpha_V(x)\cup_c\alpha_U(y)$, and the lemma is proved. □

Proof of Proposition 2.5.4. — Let

$$h_\gamma : C^\bullet_{\gamma_K}(V)\otimes C^\bullet_{\gamma_K}(U)\to C^\bullet(G_K,V^\dagger_{\mathrm{rig}}\otimes U^\dagger_{\mathrm{rig}})[-1]$$

be the map defined by

$$h_\gamma(x,y)=\begin{cases}-c_{x,y} & \text{if } x\in C^1_{\gamma_K}(V),\ y\in C^1_{\gamma_K}(U),\\ 0 & \text{elsewhere.}\end{cases}$$

From Lemma 2.5.5 it follows that h_γ defines a homotopy

$$h_\gamma : \alpha_{V\otimes U}\circ\cup_\gamma \rightsquigarrow \cup_c\circ(\alpha_V\otimes\alpha_U).$$

By Proposition 1.1.6, h_γ induces a homotopy

$$h_{\varphi,\gamma} : \alpha_{V\otimes U}\circ\cup_{\varphi,\gamma} \rightsquigarrow \cup_K\circ(\alpha_V\otimes\alpha_U).$$

The proposition is proved. □

2.6. Transpositions

2.6.1. — Let M be a continuous G_K-module. The complex $C^\bullet(G_K,M)$ is equipped with a transposition

$$\mathscr{T}_{V,c} : C^\bullet(G_K,M)\to C^\bullet(G_K,M)$$

which is defined by

$$\mathscr{T}_{V,c}(x_n)(g_1,g_2,\dots,g_n)=(-1)^{n(n+1)/2}g_1g_2\cdots g_n(x_n(g_1^{-1},\dots,g_n^{-1}))$$

(see [**56**, Section 3.4.5.1]). We will often write $\mathscr{T}_c$ instead of $\mathscr{T}_{V,c}$. The map $\mathscr{T}_c$ satisfies the following properties (see [**56**, Section 3.4.5.3]) :

a) $\mathscr{T}_c$ is an involution, i.e. $\mathscr{T}_c^2 = \mathrm{id}$.

b) $\mathscr{T}_c$ is functorially homotopic to the identity map.

c) Let $s_{12}^* : C^\bullet(G_K, M\otimes N) \to C^\bullet(C_K, N\otimes M)$ denote the map induced by the involution $M\otimes N \to N\otimes M$ given by $x\otimes y \mapsto y\otimes x$ (see Section 1.1.1). Set $\mathscr{T}_{12} = \mathscr{T}_c \circ s_{12}^*$. Then for all $x_n \in C^n(G_K,M)$ and $y_m \in C^m(G_K,N)$ one has

$$\mathscr{T}_{12}(x_n \cup y_m) = (-1)^{nm}(\mathscr{T}_c y_m) \cup (\mathscr{T}_c x_n),$$

i.e. the diagram

(35)
$$\begin{array}{ccc} C^\bullet(G_K,M)\otimes C^\bullet(G_K,N) & \xrightarrow{\cup_c} & C^\bullet(G_K,M\otimes N) \\ \big\downarrow{\scriptstyle s_{12}} & & \big\downarrow{\scriptstyle \mathscr{T}_{12}} \\ C^\bullet(G_K,N)\otimes C^\bullet(G_K,M) & \xrightarrow{\cup_c} & C^\bullet(G_K,N\otimes M) \end{array}$$

commutes.

2.6.2. — There exists a homotopy

(36)
$$a = (a^n) : \mathrm{id} \rightsquigarrow \mathscr{T}_c$$

which is functorial in M ([**56**], Section 3.4.5.5). We remark, that from the discussion in *op. cit.* it follows, that one can take a such that $a^0 = a^1 = 0$.

2.6.3. — Let V be a p-adic representation of G_K. We denote by $\mathscr{T}_{K(V)}$, or simply by $\mathscr{T}_K$, the transposition induced on the complex $K^\bullet(V)$ by $\mathscr{T}_c$, thus

$$\mathscr{T}_{K(V)}(x_{n-1},x_n) = (\mathscr{T}_c(x_{n-1}), \mathscr{T}_c(x_n)).$$

From Proposition 1.1.7 it follows that in the diagram

(37)
$$\begin{array}{ccc} K^\bullet(V)\otimes K^\bullet(U) & \xrightarrow{\cup_K} & K^\bullet(V\otimes U) \\ \big\downarrow{\scriptstyle s_{12}\circ(\mathscr{T}_{K(V)}\otimes\mathscr{T}_{K(U)})} & {\scriptstyle h_{\mathscr{T}}} & \big\downarrow{\scriptstyle \mathscr{T}_{K(V\otimes U)}\circ s_{12}^*} \\ K^\bullet(U)\otimes K^\bullet(V) & \xrightarrow{\cup_K} & K^\bullet(U\otimes V) \end{array}$$

the morphisms $\mathscr{T}_{K(V\otimes U)} \circ s_{12}^* \circ \cup_K$ and $\cup_K \circ s_{12} \circ (\mathscr{T}_{K(V)} \otimes \mathscr{T}_{K(U)})$ are homotopic.

Proposition 2.6.4. — *i) The diagram*

$$\begin{array}{ccc} C^\bullet(G_K,V) & \xrightarrow{\xi_V} & K^\bullet(V) \\ \big\downarrow{\scriptstyle \mathscr{T}_c} & & \big\downarrow{\scriptstyle \mathscr{T}_{K(V)}} \\ C^\bullet(G_K,V) & \xrightarrow{\xi_V} & K^\bullet(V). \end{array}$$

is commutative. The map $a_{K(V)}=(a,a)$ *defines a homotopy* $a_{K(V)}\,:\,\mathrm{id}_{K(V)}\rightsquigarrow \mathscr{T}_{K(V)}$ *such that* $a_{K(V)}\circ\xi_V=\xi_V\circ a$.

ii) We have a commutative diagram

$$\begin{array}{ccc} C^\bullet_{\varphi,\gamma_K}(V) & \xrightarrow{\alpha_V} & K^\bullet(V) \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathscr{T}_K} \\ C^\bullet_{\varphi,\gamma_K}(V) & \xrightarrow{\alpha_V} & K^\bullet(V). \end{array}$$

If $a\,:\,\mathrm{id}\rightsquigarrow\mathscr{T}_c$ *is a homotopy such that* $a^0=a^1=0$, *then* $a_{K(V)}\circ\alpha_V=0$.

Proof. — i) The first assertion follows from Lemma 1.1.6.

ii) If $x_1\in C^1_{\gamma_K}(V)$ then $\alpha_V(x_1)\in C^\bullet(G_K,V^\dagger_{\mathrm{rig}})$ satisfies

$$\mathscr{T}_c(\alpha_V(x_1))(g)=-g(\alpha_V(x_1)(g^{-1}))=$$
$$=-\gamma_K^{\kappa(g)}\left(\frac{\gamma_K^{-\kappa(g)}-1}{\gamma_K-1}(x_1)\right)=\frac{\gamma_K^{\kappa(g)}-1}{\gamma_K-1}(x_1)=(\alpha_V(x_1))(g).$$

Thus $\mathscr{T}_c\circ\alpha_V=\alpha_V$. By functoriality, $\mathscr{T}_K\circ\alpha_V=\alpha_V$. Finally, the identity $a_{K(V)}\circ\alpha_V=0$ follows directly from the definition of ξ_V and the assumption that $a^0=a^1=0$. □

2.7. The Bockstein map

2.7.1. — Consider the completed group algebra $\Lambda_A=A[[\Gamma_K^0]]$ of Γ_K^0 over A. Note that $\Lambda_A=A\widehat{\otimes}_{\mathbf{Z}_p}\Lambda$, where $\Lambda=\mathbf{Z}_p[[\Gamma_K^0]]$ is the classical Iwasawa algebra. Let $\iota\,:\,\Lambda_A\to\Lambda_A$ denote the A-linear involution given by $\iota(\gamma)=\gamma^{-1}$, $\gamma\in\Gamma_K^0$. We equip Λ_A with the following structures:

a) The natural Galois action given by $g(x)=\bar g x$, where $g\in G_K$, $x\in\Lambda_A$ and $\bar g$ is the image of g under canonical projection of $G_K\to\Gamma_K^0$.

b) The Λ_A-module structure Λ_A^ι given by the involution ι, namely $\lambda\star x=\iota(\lambda)x$ for $\lambda\in\Lambda_A$, $x\in\Lambda_A^\iota$.

Let J_A denote the kernel of the augmentation map $\Lambda_A \to A$. Then the element

$$\widetilde{X} = \log^{-1}(\chi_K(\gamma))(\gamma-1) \pmod{J_A^2} \in J_A/J_A^2$$

does not depend on the choice of $\gamma \in \Gamma_K^0$ and we have an isomorphism of A-modules

$$\theta_A : A \to J_A/J_A^2,$$
$$\theta_A(a) = a\widetilde{X}.$$

The action of G_K on the quotient $\widetilde{A}_K^\iota = \Lambda_A^\iota/J_A^2$ is given by

$$g(1) = 1 + \log(\chi_K(g))\widetilde{X}, \qquad g \in G_K.$$

We have an exact sequence of G_K-modules

$$0 \to A \xrightarrow{\theta_K} \widetilde{A}_K^\iota \to A \to 0. \tag{38}$$

Let V be a p-adic representation of G_K with coefficients in A. Set $\widetilde{V}_K = V \otimes_A \widetilde{A}_K^\iota$. Then the sequence (38) induces an exact sequence of p-adic representations

$$0 \to V \to \widetilde{V}_K \to V \to 0.$$

Therefore, we have an exact sequence of complexes

$$0 \to C^\bullet(G_K,V) \to C^\bullet(G_K,\widetilde{V}_K) \to C^\bullet(G_K,V) \to 0$$

which gives a distinguished triangle

$$\mathbf{R}\Gamma(K,V) \to \mathbf{R}\Gamma(K,\widetilde{V}_K) \to \mathbf{R}\Gamma(K,V) \to \mathbf{R}\Gamma(K,V)[1]. \tag{39}$$

The map $s : A \to \widetilde{A}_K^\iota$ that sends a to $a \pmod{J_A^2}$ induces a canonical non-equivariant section $s_V : V \to \widetilde{V}_K$ of the projection $\widetilde{V}_K \to V$. Define a morphism $\beta_{V,c} : C^\bullet(G_K,V) \to C^\bullet(G_K,V)[1]$ by

$$\beta_{V,c}(x_n) = \frac{1}{\widetilde{X}}(d \circ s_V - s_V \circ d)(x_n), \qquad x_n \in C^\bullet(G_K,V).$$

We will write β_c instead of $\beta_{V,c}$ if the representation V is clear from the context.

Proposition 2.7.2. **—** *i) The distinguished triangle (39) can be represented by the following distinguished triangle of complexes*

$$C^\bullet(G_K,V) \to C^\bullet(G_K,\widetilde{V}_K) \to C^\bullet(G_K,V) \xrightarrow{\beta_{V,c}} C^\bullet(G_K,V)[1].$$

ii) For any $x_n \in C^n(G_K,V)$ one has

$$\beta_{V,c}(x_n) = -\log\chi_K \cup_c x_n.$$

Proof. — See [**56**, Lemma 11.2.3]. □

2.7.3. — We will prove analogs of this proposition for the complexes $C^{\bullet}_{\varphi,\gamma_K}(\mathbf{D})$ and $K^{\bullet}(V)$. Let $\mathbf{D}$ be a (φ,Γ_K)-module with coefficients in A. Set $\widetilde{\mathbf{D}}=\mathbf{D}\otimes_A \widetilde{A}^{\iota}_K$. The splitting s induces a splitting of the exact sequence

$$0 \longrightarrow \mathbf{D} \longrightarrow \widetilde{\mathbf{D}} \overset{s_{\mathbf{D}}}{\underset{}{\rightleftarrows}} \mathbf{D} \longrightarrow 0 \tag{40}$$

which we denote by $s_{\mathbf{D}}$. Define

$$\begin{aligned} &\beta_{\mathbf{D}}\,:\, C^{\bullet}_{\varphi,\gamma_K}(\mathbf{D}) \to C^{\bullet}_{\varphi,\gamma_K}(\mathbf{D})[1],\\ &\beta_{\mathbf{D}}(x)=\frac{1}{\widetilde{X}}(d\circ s_{\mathbf{D}}-s_{\mathbf{D}}\circ d)(x), \qquad x\in C^n_{\varphi,\gamma_K}(\mathbf{D}). \end{aligned} \tag{41}$$

Proposition 2.7.4. **—** *i) The map $\beta_{\mathbf{D}}$ induces the connecting maps $H^n(\mathbf{D}) \to H^{n+1}(\mathbf{D})$ of the long cohomology sequence associated to the short exact sequence (40).*

ii) For any $x\in C^n_{\varphi,\gamma_K}(\mathbf{D})$ one has

$$\beta_{\mathbf{D}}(x)=-(0,\log\chi_K(\gamma_K))\cup_{\varphi,\gamma}x,$$

where $(0,\log\chi_K(\gamma_K))\in C^1_{\varphi,\gamma_K}(\mathbf{Q}_p(0))$.

Proof. — The first assertion follows directly from the definition of the connecting map. Now, let $x=(x_{n-1},x_n)\in C^n_{\varphi,\gamma_K}(\mathbf{D})$. Then

$$\begin{aligned} &(ds_{\mathbf{D}}-s_{\mathbf{D}}d)(x)=\\ &=d(x_{n-1}\otimes 1, x_n\otimes 1)-s_{\mathbf{D}}((\gamma_K-1)x_{n-1}+(-1)^n(\varphi-1)x_n,(\gamma_K-1)x_n)=\\ &=(\gamma_K(x_{n-1})\otimes\gamma_K-x_{n-1}\otimes 1+(-1)^n(\varphi-1)x_n\otimes 1,\gamma_K(x_n)\otimes\gamma_K-x_n\otimes 1)-\\ &-((\gamma_K-1)(x_{n-1})\otimes 1+(-1)^n(\varphi-1)x_n\otimes 1,(\gamma_K-1)(x_n)\otimes 1)=\\ &=(\gamma_K(x_{n-1})\otimes(\gamma_K-1),\gamma_K(x_n)\otimes(\gamma_K-1)). \end{aligned}$$

From $\gamma_K=1+\widetilde{X}\log\chi_K(\gamma_K)$ it follows that $\gamma_K^{-1}-1\equiv -\widetilde{X}\log\chi_K(\gamma_K) \pmod{J_A^2}$ and we obtain

$$\begin{aligned} \beta_{\mathbf{D}}(x)=\frac{1}{\widetilde{X}}\left((\gamma_K(x_{n-1}),\gamma_K(x_n))\otimes(\gamma_K-1)\right)&=\\ =-\log\chi_K(\gamma_K)(\gamma_K(x_{n-1}),\gamma_K(x_n))&\in C^{n+1}_{\varphi,\gamma_K}(\mathbf{D}). \end{aligned}$$

On the other hand,

$$(0,\log\chi_K(\gamma_K))\cup_{\varphi,\gamma}(x_{n-1},x_n)=\log\chi_K(\gamma_K)(\gamma_K(x_{n-1}),\gamma_K(x_n))$$

and ii) is proved. □

The exact sequence

$$0\to C^\bullet(G_K,V^\dagger_{\mathrm{rig}})\to C^\bullet(G_K,(\widetilde{V}_K)^\dagger_{\mathrm{rig}})\to C^\bullet(G_K,V^\dagger_{\mathrm{rig}})\to 0,$$

induces an exact sequence

$$0\to K^\bullet(V)\to K^\bullet(\widetilde{V}_K)\to K^\bullet(V)\to 0. \qquad (42)$$

Again, the splitting $s_V\,:\,V\to \widetilde{V}_K$ induces a splitting $s_K\,:\,K^\bullet(V)\to K^\bullet(\widetilde{V}_K)$ of (42) and we have a distinguished triangle of complexes

$$K^\bullet(V)\to K^\bullet(\widetilde{V})\to K^\bullet(V)\xrightarrow{\beta_{K(V)}} K^\bullet(V)[1].$$

We will often write β_K instead of $\beta_{K(V)}$.

Proposition 2.7.5. — *i) One has*

$$\beta_K(x)=-(0,\log\chi_K)\cup_K x,\qquad x\in K^n(V).$$

ii) The following diagrams commute

$$\begin{array}{ccc} C^\bullet(G_K,V) & \xrightarrow{\beta_c} & C^\bullet(G_K,V)[1] \\ \downarrow{\scriptstyle \xi_V} & & \downarrow{\scriptstyle \xi_V[1]} \\ K^\bullet(V) & \xrightarrow{\beta_K} & K^\bullet(V)[1] \end{array}\;,\qquad \begin{array}{ccc} C^\bullet_{\varphi,\gamma_K}(V) & \xrightarrow{\beta_{\mathbf{D}^\dagger_{\mathrm{rig}}(V)}} & C^\bullet_{\varphi,\gamma_K}(V)[1] \\ \downarrow{\scriptstyle \alpha_V} & & \downarrow{\scriptstyle \alpha_V[1]} \\ K^\bullet(V) & \xrightarrow{\beta_K} & K^\bullet(V)[1]. \end{array}$$

Proof. — i) The proof is a routine computation. Let $x=(x_{n-1},x_n)\in K^n(V)$, where $x_{n-1}\in C^{n-1}(G_K,V^\dagger_{\mathrm{rig}})$, $x_n\in C^n(G_K,V^\dagger_{\mathrm{rig}})$. Since s_K commutes with φ one has

$$(ds_K-s_Kd)x=((ds_V-s_Vd)x_{n-1},(ds_V-s_Vd)x_n).$$

On the other hand,

$$((ds_V-s_Vd)x_{n-1})\,(g_1,g_2,\dots,g_n)=g_1x_{n-1}(g_2,\dots,g_n)\otimes(\bar g_1-1),$$

where $\bar g_1$ denote the image of $g_1\in G_K$ in Γ_K. As in the proof of Proposition 2.7.4, we can write $\bar g_1-1 \pmod{J_A^2}=\widetilde{X}\log\chi_K(g_1)$. Therefore

$$(d\circ s_V-s_V\circ d)x_{n-1}(g_1,g_2,\dots,g_n)=\log\chi_K(g_1)g_1x_{n-1}(g_2,\dots,g_n)\otimes\widetilde{X}.$$

and

$$(d\circ s_V-s_V\circ d)x_n(g_1,g_2,\dots,g_n,g_{n+1})=$$
$$=\log\chi_K(g_1)g_1x_{n-1}(g_2,\dots,g_n,g_{n+1})\otimes\widetilde{X}.$$

Since $\iota(g_1-1)=-\widetilde{X}\log\chi_K(g_1)$, we have

$$\beta_K(x)(g_1,\dots,g_n)=\frac{1}{\widetilde{X}}(d\circ s_K-s_K\circ d)x(g_1,g_2,\dots,g_n)=$$
$$=-\log\chi_K(g_1)(g_1x_{n-1}(g_2,\dots,g_n,g_n),g_1x_{n-1}(g_2,\dots,g_n,g_{n+1})).$$

On the other hand, $(0,\log\chi_K)\cup_K(x_{n-1},x_n)=(z_n,z_{n+1})$, where

$$z_i(g_1,g_2,\dots,g_i)=\log\chi_K(g_1)g_1x_i(g_2,\dots,g_i),\qquad i=n,n+1,$$

and i) is proved.

ii) The second statement follows from the compatibility of the Bockstein morphisms β_c, $\beta_{\mathbf{D}^\dagger_{\mathrm{rig}}(V)}$ and β_K with the maps α_V and β_V. This can be also proved using i) and Propositions 2.7.2 and 2.7.4. □

2.8. Iwasawa cohomology

2.8.1. — We keep previous notation and conventions. Set $K_\infty=(K^{\mathrm{cyc}})^{\Delta_K}$, where $\Delta_K=\mathrm{Gal}(K(\zeta_p)/K)$. Then $\mathrm{Gal}(K_\infty/K)\simeq\Gamma_K^0$ and we denote by K_n the unique subextension of K_∞ of degree $[K_n:K]=p^n$. Let E be a finite extension of $\mathbf{Q}_p$ and let $\mathscr{O}_E$ be its ring of integers. We denote by $\Lambda_{\mathscr{O}_E}=\mathscr{O}_E[[\Gamma_K^0]]$ the Iwasawa algebra of Γ_K^0 with coefficients in $\mathscr{O}_E$. The choice of a generator γ_K of Γ_K^0 fixes an isomorphism $\Lambda_{\mathscr{O}_E}\simeq\mathscr{O}_E[[X]]$ such that $\gamma_K\mapsto X+1$. Let $\mathscr{H}_E$ denote the algebra of formal power series $f(X)\in E[[X]]$ that converge on the open unit disk $A(0,1)=\{x\in\mathbf{C}_p\mid |x|_p<1\}$ and let

$$\mathscr{H}_E(\Gamma_K^0)=\{f(\gamma_K-1)\mid f(X)\in\mathscr{H}_E\}.$$

We consider $\Lambda_{\mathscr{O}_E}$ as a subring of $\mathscr{H}_E(\Gamma_K^0)$. The involution $\iota:\Lambda_{\mathscr{O}_E}\to\Lambda_{\mathscr{O}_E}$ extends to $\mathscr{H}_E(\Gamma_K^0)$. Let $\Lambda_{\mathscr{O}_E}^\iota$ (resp. $\mathscr{H}_E(\Gamma_K^0)^\iota$) denote $\Lambda_{\mathscr{O}_E}$ (resp. $\mathscr{H}_E(\Gamma_K^0)$) equipped with the $\Lambda_{\mathscr{O}_E}$-module (resp. $\mathscr{H}_E(\Gamma_K^0)$-module) structure given by $\alpha\star\lambda=\iota(\alpha)\lambda$.

Let V be a p-adic representation of G_K with coefficients in E. Fix a $\mathscr{O}_E$-lattice T of V stable under the action of G_K and set $\mathrm{Ind}_{K_\infty/K}(T)=T\otimes_{\mathscr{O}_E}\Lambda_{\mathscr{O}_E}^\iota$. We equip $\mathrm{Ind}_{K_\infty/K}(T)$ with the following structures:

a) The diagonal action of G_K, namely $g(x\otimes\lambda)=g(x)\otimes\bar{g}\lambda$, for all $g\in G_K$ and $x\otimes\lambda\in\mathrm{Ind}_{K_\infty/K}(T)$;

b) The structure of $\Lambda_{\mathscr{O}_E}$-module given by $\alpha(x\otimes\lambda)=x\otimes\lambda\iota(\alpha)$ for all $\alpha\in\Lambda_{\mathscr{O}_E}$ and $x\otimes\lambda\in\mathrm{Ind}_{K_\infty/K}(T)$.

Let $\mathbf{R}\Gamma_{\mathrm{Iw}}(K,T)$ denote the class of the complex $C^\bullet(G_K,\mathrm{Ind}_{K_\infty/K}(T))$ in the derived category $\mathscr{D}(\Lambda_{\mathscr{O}_E})$ of $\Lambda_{\mathscr{O}_E}$-modules. The augmentation map $\Lambda_{\mathscr{O}_E}\to\mathscr{O}_E$ induces an isomorphism

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(K,T)\otimes^{\mathbf{L}}_{\Lambda_{\mathscr{O}_E}}\mathscr{O}_E\simeq\mathbf{R}\Gamma(K,T).$$

We write $H^i_{\mathrm{Iw}}(K,T)=\mathbf{R}^i\Gamma_{\mathrm{Iw}}(K,T)$ for the cohomology of $\mathbf{R}\Gamma_{\mathrm{Iw}}(K,T)$. From Shapiro's lemma it follows that

$$H^i_{\mathrm{Iw}}(K,T)=\varprojlim_{\mathrm{cores}} H^i(K_n,T)$$

(see, for example, [**56**, Sections 8.1–8.3]).

We review the Iwasawa cohomology of (φ,Γ_K)-modules [**19, 46**]. The map $\varphi:\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}\to\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}$ equips $\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}$ with the structure of a free $\varphi:\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$-module of rank p. Define

$$\psi:\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}\to\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\qquad \psi(x)=\frac{1}{p}\varphi^{-1}\circ\mathrm{Tr}_{\mathbf{B}^{\dagger,pr}_{\mathrm{rig},K}/\varphi(\mathbf{B}^{\dagger,r}_{\mathrm{rig},K})}(x).$$

Since $\mathscr{R}_{K,\mathbf{Q}_p}=\underset{r\geqslant r_K}{\cup}\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$, the operator ψ extends by linearity to an operator $\psi:\mathscr{R}_{K,E}\to\mathscr{R}_{K,E}$ such that $\psi\circ\varphi=\mathrm{id}$.

Let $\mathbf{D}$ is a (φ,Γ_K)-module over $\mathscr{R}_{K,E}=\mathscr{R}_K\otimes_{\mathbf{Q}_p}E$. If $e_1,e_2,\dots,e_d$ is a base of $\mathbf{D}$ over $\mathscr{R}_{K,E}$, then $\varphi(e_1),\varphi(e_2),\dots,\varphi(e_d)$ is again a base of $\mathbf{D}$, and we define

$$\psi:\mathbf{D}\to\mathbf{D},$$
$$\psi\left(\sum_{i=1}^d a_i\varphi(e_i)\right)=\sum_{i=1}^d\psi(a_i)e_i.$$

The action of Γ^0_K on $\mathbf{D}^{\Delta_K}$ extends to a natural action of $\mathscr{H}_E(\Gamma^0_K)$ and we consider the complex of $\mathscr{H}_E(\Gamma^0_K)$-modules

$$C^\bullet_{\mathrm{Iw}}(\mathbf{D})\,:\,\mathbf{D}^{\Delta_K}\xrightarrow{\psi-1}\mathbf{D}^{\Delta_K},$$

where the terms are concentrated in degrees 1 and 2. Let $\mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{D})=[C^\bullet_{\mathrm{Iw}}(\mathbf{D})]$ denote the class of $C^\bullet_{\mathrm{Iw}}(\mathbf{D})$ in the derived category $\mathscr{D}(\mathscr{H}_E(\Gamma^0_K))$. We also consider the complex $C^\bullet_{\varphi,\gamma_K}(\mathrm{Ind}_{K_\infty/K}(\mathbf{D}))$, where $\mathrm{Ind}_{K_\infty/K}(\mathbf{D})=\mathbf{D}\otimes_E\mathscr{H}_E(\Gamma^0_K)^\iota$, and set $\mathbf{R}\Gamma(K,\mathrm{Ind}_{K_\infty/K}(\mathbf{D}))=\left[C^\bullet_{\varphi,\gamma_K}(\overline{\mathbf{D}})\right]$.

Theorem 2.8.2 **(Pottharst).** — *Let* $\mathbf{D}$ *be a* (φ,Γ_K)*-module over* $\mathscr{R}_{K,E}$*. Then*

i) The complexes $C^\bullet_{\mathrm{Iw}}(\mathbf{D})$ *and* $C_{\varphi,\gamma_K}(\overline{\mathbf{D}})$ *are quasi-isomorphic and therefore*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{D})\simeq\mathbf{R}\Gamma(K,\mathrm{Ind}_{K_\infty/K}(\mathbf{D})).$$

ii) The cohomology groups $H^i_{\mathrm{Iw}}(\mathbf{D})=\mathbf{R}^i\Gamma_{\mathrm{Iw}}(\mathbf{D})$ *are finitely-generated* $\mathscr{H}_E(\Gamma^0_K)$*-modules. Moreover,* $\mathrm{rk}_{\mathscr{H}_E(\Gamma^0_K)}H^1_{\mathrm{Iw}}(\mathbf{D})=[K:\mathbf{Q}_p]\,\mathrm{rk}_{\mathscr{R}_{K,E}}\mathbf{D}$ *and* $H^1_{\mathrm{Iw}}(\mathbf{D})_{\mathrm{tor}}$ *and* $H^2_{\mathrm{Iw}}(\mathbf{D})$ *are finite-dimensional* E*-vector spaces.*

iii) We have an isomorphism

$$C^\bullet_{\varphi,\gamma_K}(\mathrm{Ind}_{K_\infty/K}(\mathbf{D}))\otimes_{\mathscr{H}_E(\Gamma^0_K)}E\xrightarrow{\sim}C^\bullet_{\varphi,\gamma_K}(\mathbf{D})$$

which induces the Hochschild–Serre exact sequences

$$0 \to H^i_{\mathrm{Iw}}(\mathbf{D})_{\Gamma^0_K} \to H^i(\mathbf{D}) \to H^{i+1}_{\mathrm{Iw}}(\mathbf{D})^{\Gamma^0_K} \to 0.$$

iv) Let $\omega = \mathrm{cone}\left[\mathscr{K}_E(\Gamma^0_K) \to \mathscr{K}_E(\Gamma^0_K)/\mathscr{H}_E(\Gamma^0_K)\right][-1]$, *where* $\mathscr{K}_E(\Gamma^0_K)$ *is the field of fractions of* $\mathscr{H}_E(\Gamma^0_K)$. *Then the functor* $\mathscr{D} = \mathrm{Hom}_{\mathscr{H}_E(\Gamma^0_K)}(-,\omega)$ *furnishes a duality*

$$\mathscr{D}\mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{D}) \simeq \mathbf{R}\Gamma_{\mathrm{Iw}}(\mathbf{D}^*(\chi_K))^\iota[2].$$

v) If V *is a* p*-adic representation of* G_K, *then there are canonical and functorial isomorphisms*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(K,T) \otimes^{\mathbf{L}}_{\Lambda_{\mathscr{O}_E}} \mathscr{H}_E(\Gamma^0_K) \simeq \mathbf{R}\Gamma\left(K, T \otimes_{\mathscr{O}_E} \mathscr{H}_E(\Gamma^0_K)^\iota\right) \simeq$$
$$\simeq \mathbf{R}\Gamma(K, \mathrm{Ind}_{K_\infty/K}(\mathbf{D}^\dagger_{\mathrm{rig}}(V))).$$

Proof. — See [**61**, Theorem 2.6]. □

We will need the following lemma.

Lemma 2.8.3. — *Let* E *be a finite extension of* $\mathbf{Q}_p$ *and let* $\mathbf{D}$ *be a potentially semistable* (φ,Γ_K)*-module over* $\mathscr{R}_{K,E}$. *Then*

i) $H^1_{\mathrm{Iw}}(\mathbf{D})_{\mathrm{tor}} \simeq \left(\mathbf{D}^{\Delta_K}\right)^{\varphi=1}$.

ii) Assume that

$$\mathscr{D}_{\mathrm{pst}}(\mathbf{D}^*(\chi_K))^{\varphi=p^i} = 0, \qquad \forall i \in \mathbf{Z}.$$

Then $H^2_{\mathrm{Iw}}(\mathbf{D}) = 0$.

Proof. — i) Consider the exact sequence

$$0 \to \mathbf{D}^{\varphi=1} \to \mathbf{D}^{\psi=1} \xrightarrow{\varphi-1} \mathbf{D}^{\psi=0}.$$

Since $\left(\mathbf{D}^{\Delta_K}\right)^{\psi=1} \simeq H^1_{\mathrm{Iw}}(\mathbf{D})$ and, by [**46**, Theorem 3.1.1], $\mathbf{D}^{\psi=0}$ is $\mathscr{H}_E(\Gamma^0_K)$-torsion free, we have $H^1_{\mathrm{Iw}}(\mathbf{D})_{\mathrm{tor}} \subset \left(\mathbf{D}^{\Delta_K}\right)^{\varphi=1}$. On the other hand, $\mathbf{D}^{\varphi=1}$ is a finitely dimensional E-vector space (see, for example, [**46**, Lemma 4.3.5]) and therefore is $\mathscr{H}_E(\Gamma^0_K)$-torsion. This proves the first statement.

ii) By Theorem 2.8.2 iv), $H^2_{\mathrm{Iw}}(\mathbf{D})$ and $H^1_{\mathrm{Iw}}(\mathbf{D}^*(\chi_K))_{\mathrm{tor}}$ are dual to each other and it is enough to show that $\mathbf{D}^*(\chi_K)^{\varphi=1} = 0$. Since $\dim_E \mathbf{D}^*(\chi_K)^{\varphi=1} < \infty$, there exists r such that $\mathbf{D}^*(\chi_K)^{\varphi=1} \subset \mathbf{D}^*(\chi_K)^{(r)}$, and for $n \gg 0$ the map $i_n = \varphi^{-n} : \mathscr{R}^{(r)}_{K,E} \to E \otimes_{\mathbf{Q}_p} K^{\mathrm{cyc}}[[t]]$ gives an injection

$$\mathbf{D}^*(\chi_K)^{\varphi=1} \to \mathbf{D}^*(\chi_K)^{(r)} \otimes_{i_n} \left(E \otimes_{\mathbf{Q}_p} K^{\mathrm{cyc}}[[t]]\right) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \mathrm{Fil}^0\left(\mathscr{D}_{\mathrm{dR}}(\mathbf{D}^*(\chi_K)) \otimes_K \otimes K^{\mathrm{cyc}}((t))\right).$$

Looking at the action of Γ_K on $\mathrm{Fil}^0\left(\mathscr{D}_{\mathrm{dR}}(\mathbf{D}^*(\chi_K)) \otimes_K K^{\mathrm{cyc}}((t))\right)$ and using the fact that $\mathbf{D}^*(\chi_K)^{\varphi=1}$ is finite-dimensional over E, it is easy to prove, that there exists a finite extension L/K such that $\mathbf{D}^*(\chi_K)^{\varphi=1}$, viewed as G_L-module, is isomorphic to a finite direct sum of modules $\mathbf{Q}_p(i)$, $i\in\mathbf{Z}$. Therefore

$$\mathbf{D}^*(\chi_K)^{\varphi=1}\simeq\left(\mathbf{D}^*(\chi_K)^{\varphi=1}\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(-i)\right)^{\Gamma_L}\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(i)$$

as G_L-modules. Since

$$\left(\mathbf{D}^*(\chi_K)^{\varphi=1}\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(-i)\right)^{\Gamma_L}\subset\left(\mathbf{D}^*(\chi_K)\otimes_{\mathscr{R}_{K,E}}\mathscr{R}_{L,E}[1/t,\ell_\pi]\right)^{\varphi=p^{-i},\Gamma_L}=$$
$$=\mathscr{D}_{\mathrm{st}/L}(\mathbf{D}^*(\chi_K))^{\varphi=p^{-i}}=0,$$

we obtain that $\mathbf{D}^*(\chi_K)^{\varphi=1}=0$, and the lemma is proved. □

2.9. The group $H^1_f(\mathbf{D})$

2.9.1. — For the content of this section we refer the reader to [**7**, Sections 1.4-1.5]. Let $\mathbf{D}$ be a potentially semistable (φ,Γ_K)-module over $\mathscr{R}_{K,E}$, where E is a finite extension of $\mathbf{Q}_p$. As usual, we have the isomorphism

$$H^1(\mathbf{D})\simeq\mathrm{Ext}^1_{\mathscr{R}_{K,E}}(\mathscr{R}_{K,E},\mathbf{D})$$

which associates to each cocycle $x=(a,b)\in C^1_{\varphi,\gamma_K}(\mathbf{D})$ the extension

$$0\to\mathbf{D}\to\mathbf{D}_x\to\mathscr{R}_{K,E}\to 0$$

such that $\mathbf{D}_x=\mathbf{D}\oplus\mathscr{R}_{K,E}e$ with $\varphi(e)=e+a$ and $\gamma_K(e)=e+b$. We say that $[x]=\mathrm{class}(x)\in H^1(\mathbf{D})$ is crystalline if

$$\mathrm{rk}_{E\otimes K_0}(\mathscr{D}_{\mathrm{cris}}(\mathbf{D}_x))=\mathrm{rk}_{E\otimes K_0}(\mathscr{D}_{\mathrm{cris}}(\mathbf{D}))+1$$

and define

$$H^1_f(\mathbf{D})=\{[x]\in H^1(\mathbf{D})\mid \mathrm{cl}(x)\text{ is crystalline}\}.$$

This definition agrees with the definition of Bloch and Kato [**16**]. Namely, if V is a potentially semistable representation of G_K, then

$$H^1_f(\mathbf{D}^\dagger_{\mathrm{rig}}(V))\simeq H^1_f(K,V)$$

(see [**7**, Proposition 1.4.2]).

***Proposition 2.9.2.* —** *Let* $\mathbf{D}$ *be a potentially semistable* (φ,Γ_K)*-module over* $\mathscr{R}_{K,E}$*. Then*

i) $H^0(\mathbf{D})=\mathrm{Fil}^0(\mathscr{D}_{\mathrm{pst}}(\mathbf{D}))^{\varphi=1,N=0,G_K}$ *and* $H^1_f(\mathbf{D})$ *is a* E*-subspace of* $H^1(\mathbf{D})$ *of dimension*

$$\dim_E H^1_f(\mathbf{D})=\dim_E\mathscr{D}_{\mathrm{dR}}(\mathbf{D})-\dim_E\mathrm{Fil}^0\mathscr{D}_{\mathrm{dR}}(\mathbf{D})+\dim_E H^0(\mathbf{D}).$$

ii) There exists an exact sequence

$$0 \to H^0(\mathbf{D}) \to \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \xrightarrow{(\mathrm{pr},1-\varphi)} t_{\mathbf{D}}(K) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \to H^1_f(\mathbf{D}) \to 0,$$

where $t_{\mathbf{D}}(K) = \mathscr{D}_{\mathrm{dR}}(\mathbf{D})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{dR}}(\mathbf{D})$.

iii) $H^1_f(\mathbf{D}^*(\chi_K))$ *is the orthogonal complement to* $H^1_f(\mathbf{D})$ *under the duality* $H^1(\mathbf{D}) \times H^1(\mathbf{D}^*(\chi_K)) \to E$.

iv) Let

$$0 \to \mathbf{D}_1 \to \mathbf{D} \to \mathbf{D}_2 \to 0$$

be an exact sequence of potentially semistable (φ,Γ_K)*-modules. Assume that one of the following conditions holds*

a) $\mathbf{D}$ *is crystalline;*

b) $\mathrm{Im}((H^0(\mathbf{D}_2) \to H^1(\mathbf{D}_1)) \subset H^1_f(\mathbf{D}_1)$.

Then one has an exact sequence

$$0 \to H^0(\mathbf{D}_1) \to H^0(\mathbf{D}) \to H^0(\mathbf{D}_2) \to H^1_f(\mathbf{D}_1) \to H^1_f(\mathbf{D}) \to H^1_f(\mathbf{D}_2) \to 0.$$

Proof. — This proposition is proved in Proposition 1.4.4, and Corollaries 1.4.6 and 1.4.10 of [**7**]. For an another approach to $H^1_f(\mathbf{D})$ and an alternative proof see [**53**, Section 2]. □

2.9.3. — In this subsection we assume that $K = \mathbf{Q}_p$. We review the computation of the cohomology of some isoclinic $(\varphi,\Gamma_{\mathbf{Q}_p})$-modules given in [**7**]. To simplify notation, we write χ_p and Γ^0_p instead of $\chi_{\mathbf{Q}_p}$ and $\Gamma^0_{\mathbf{Q}_p}$ respectively.

***Proposition 2.9.4.* —** *Let* $\mathbf{D}$ *be a semistable* $(\varphi,\Gamma_{\mathbf{Q}_p})$*-module of rank d over* $\mathscr{R}_{\mathbf{Q}_p,E}$ *such that* $\mathscr{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1} = \mathscr{D}_{\mathrm{st}}(\mathbf{D})$ *and* $\mathrm{Fil}^0\mathscr{D}_{\mathrm{st}}(\mathbf{D}) = \mathscr{D}_{\mathrm{st}}(\mathbf{D})$. *Then*

i) $\mathbf{D}$ *is crystalline and* $H^0(\mathbf{D}) = \mathscr{D}_{\mathrm{cris}}(\mathbf{D})$.

ii) One has $\dim_E H^0(\mathbf{D}) = d$, $\dim_E H^1(\mathbf{D}) = 2d$ *and* $H^2(\mathbf{D}) = 0$.

iii) The map

$$i_{\mathbf{D}} : \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \oplus \mathscr{D}_{\mathrm{cris}}(\mathbf{D}) \to H^1(\mathbf{D}),$$
$$i_{\mathbf{D}} = \mathrm{cl}(-x, \log\chi_p(\gamma_{\mathbf{Q}_p})y)$$

is an isomorphism of E-vector spaces. Let $i_{\mathbf{D},f}$ *and* $i_{\mathbf{D},c}$ *denote the restrictions of* $i_{\mathbf{D}}$ *on the first and the second summand respectively. Then* $\mathrm{Im}(i_{\mathbf{D},f}) = H^1_f(\mathbf{D})$ *and we have a decomposition*

$$H^1(\mathbf{D}) = H^1_f(\mathbf{D}) \oplus H^1_c(\mathbf{D}),$$

where $H^1_c(\mathbf{D}) = \mathrm{Im}(i_{\mathbf{D},c})$.

iv) Let $\mathbf{D}^*(\chi_p)$ *be the Tate dual of* $\mathbf{D}$. *Then*

$$\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))^{\varphi=p^{-1}} = \mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))$$

and $\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))=0$. *In particular,* $H^0(\mathbf{D}^*(\chi_p))=0$. *Let*

$$[\ ,\ ]_{\mathbf{D}}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))\times\mathscr{D}_{\mathrm{cris}}(\mathbf{D})\to E$$

denote the canonical duality. Define a morphism

$$i_{\mathbf{D}^*(\chi_p)}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))\oplus\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi_p))\to H^1(\mathbf{D}^*(\chi_p))$$

by

$$i_{\mathbf{D}^*(\chi_p)}(\alpha,\beta)\cup i_{\mathbf{D}}(x,y)=[\beta,x]_{\mathbf{D}}-[\alpha,y]_{\mathbf{D}}$$

and denote by $\mathrm{Im}(i_{\mathbf{D}^*(\chi_p),f})$ *and* $\mathrm{Im}(i_{\mathbf{D}^*(\chi_p),c})$ *the restrictions of* $i_{\mathbf{D}^*(\chi_p)}$ *on the first and the second summand respectively. Then* $\mathrm{Im}(i_{\mathbf{D}^*(\chi_p),f})=H^1_f(\mathbf{D}^*(\chi_p))$ *and again we have*

$$H^1(\mathbf{D}^*(\chi_p))=H^1_f(\mathbf{D}^*(\chi_p))\oplus H^1_c(\mathbf{D}^*(\chi_p)),$$

where $H^1_c(\mathbf{D}^*(\chi_p))=\mathrm{Im}(i_{\mathbf{D}^*(\chi_p),c})$.

Proof. — See [**7**, Proposition 1.5.9 and Section 1.5.10]. □

Lemma 2.9.5. — *Let* $\mathbf{D}$ *be a semistable* $(\varphi,\Gamma_{\mathbf{Q}_p})$*-module of rank* d *over* $\mathscr{R}_{\mathbf{Q}_p,E}$ *such that* $\mathscr{D}_{\mathrm{st}}(\mathbf{D})^{\varphi=1}=\mathscr{D}_{\mathrm{st}}(\mathbf{D})$ *and* $\mathrm{Fil}^0\mathscr{D}_{\mathrm{st}}(\mathbf{D})=\mathscr{D}_{\mathrm{st}}(\mathbf{D})$. *Let* $w_p=(0,\log\chi_p(\gamma_{\mathbf{Q}_p}))\in C^1_{\varphi,\gamma_{\mathbf{Q}_p}}(E(0))$. *Then*

$$H^1_c(\mathbf{D})=\ker\left(\cup w_p\,:\,H^1(\mathbf{D})\to H^2(\mathbf{D})\right),$$
$$H^1_c(\mathbf{D}^*(\chi_p))=\ker\left(\cup w_p\,:\,H^1(\mathbf{D}^*(\chi_p))\to H^2(\mathbf{D}^*(\chi_p))\right).$$

Proof. — This follows directly from the definition of the cup product. □

We also need the following result.

Proposition 2.9.6. — *Let* $\mathbf{D}$ *be a crystalline* $(\varphi,\Gamma_{\mathbf{Q}_p})$*-module over* $\mathscr{R}_{\mathbf{Q}_p,E}$ *such that* $\mathscr{D}_{\mathrm{cris}}(\mathbf{D})^{\varphi=p^{-1}}=\mathscr{D}_{\mathrm{cris}}(\mathbf{D})$ *and* $\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{D})=0$. *Then*

$$H^1_{\mathrm{Iw}}(\mathbf{D})_{\Gamma_p^0}=H^1_c(\mathbf{D}).$$

Proof. — See [**10**, Proposition 4]. □

CHAPTER 3

p-ADIC HEIGHT PAIRINGS I: SELMER COMPLEXES

3.1. Selmer complexes

3.1.1. — In this section we construct p-adic height pairings using Nekovář's formalism of Selmer complexes. Let F be a number field. We denote by S_f (resp. S_∞) the set of all non-archimedean (resp. archimedean) absolute values on F. Fix a prime number p and a compatible system of p^n-th roots of unity $\varepsilon = (\zeta_{p^n})_{n\geqslant 1}$. Let $S \subset S_f$ be a finite subset containing the set S_p of all $\mathfrak{q} \in S_f$ such that $\mathfrak{q} \mid p$. We will write Σ_p for the complement of S_p in S. Let $G_{F,S}$ denote the Galois group of the maximal algebraic extension of F unramified outside $S \cup S_\infty$. For each $\mathfrak{q} \in S$, we fix a decomposition group at $\mathfrak{q}$ which we identify with $G_{F_\mathfrak{q}}$. If $\mathfrak{q} \in S_p$, we denote by $\Gamma_\mathfrak{q} = \Gamma_{F_\mathfrak{q}}$ the p-cyclotomic Galois group of $F_\mathfrak{q}$ and fix a generator $\gamma_\mathfrak{q} \in \Gamma_\mathfrak{q}^0$.

3.1.2. — Let V be a p-adic representation of $G_{F,S}$ with coefficients in a $\mathbf{Q}_p$-affinoid algebra A. We will write $V_\mathfrak{q}$ for the restriction of V on the decomposition group at $\mathfrak{q}$. For each $\mathfrak{q} \in S_p$, we fix a $(\varphi, \Gamma_\mathfrak{q})$-submodule $\mathbf{D}_\mathfrak{q}$ of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_\mathfrak{q})$ that is a $\mathscr{R}_{F_\mathfrak{q},A}$-module direct summand of $\mathbf{D}^\dagger_{\mathrm{rig},A}(V_\mathfrak{q})$. Set $\mathbf{D} = (\mathbf{D}_\mathfrak{q})_{\mathfrak{q}\in S_p}$ and define

$$U_\mathfrak{q}^\bullet(V,\mathbf{D}) = \begin{cases} C^\bullet_{\varphi,\gamma_\mathfrak{q}}(\mathbf{D}_\mathfrak{q}), & \text{if } \mathfrak{q} \in S_p, \\ C^\bullet_{\mathrm{ur}}(V_\mathfrak{q}), & \text{if } \mathfrak{q} \in \Sigma_p, \end{cases}$$

where

$$C^\bullet_{\mathrm{ur}}(V_\mathfrak{q}) \ : \ V_\mathfrak{q}^{I_\mathfrak{q}} \xrightarrow{\mathrm{Fr}_\mathfrak{q}-1} V_\mathfrak{q}^{I_\mathfrak{q}}, \qquad \mathfrak{q} \in \Sigma_p,$$

and the terms are concentrated in degrees 0 and 1. In this section we consider these complexes as objects in $\mathscr{K}^{[0,2]}_{\mathrm{ft}}(A)$. Note that, if $\mathfrak{q} \in S_p$, the objects $\mathbf{R}\Gamma(F_\mathfrak{q}, V) = [C^\bullet(G_{F_\mathfrak{q}}, V)]$ and $\mathbf{R}\Gamma(F_\mathfrak{q}, \mathbf{D}_\mathfrak{q}) = [U_\mathfrak{q}^\bullet(V,\mathbf{D})]$ belong to $\mathscr{D}^{[0,2]}_{\mathrm{perf}}(A)$ by Theorems 2.3.2 and 2.4.3. On the other hand, if $\mathfrak{q} \in \Sigma_p$, then, in general, the module $V^{I_\mathfrak{q}}$ and the complex $U_\mathfrak{q}^\bullet(V,\mathbf{D})$ are not quasi-isomorphic to a perfect complex of A-modules. We discuss this in more detail in Sections 3.1.6–3.1.9 in relation with the duality theory for Selmer complexes.

First assume that $\mathfrak{q} \in \Sigma_p$. Then we have a canonical morphism

$$g_{\mathfrak{q}} : U_{\mathfrak{q}}^{\bullet}(V, \mathbf{D}) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, V) \tag{43}$$

defined by

$$\begin{aligned} g_{\mathfrak{q}}(x_0) &= x_0, && \text{if} \quad x_0 \in U_{\mathfrak{q}}^0(V, \mathbf{D}), \\ g_{\mathfrak{q}}(x_1)(\mathrm{Fr}_{\mathfrak{q}}) &= x_1, && \text{if} \quad x_1 \in U_{\mathfrak{q}}^1(V, \mathbf{D}) \end{aligned}$$

and the restriction map

$$f_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}} : C^{\bullet}(G_{F,S}, V) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, V). \tag{44}$$

Now assume that $\mathfrak{q} \in S_p$. The inclusion $\mathbf{D}_{\mathfrak{q}} \subset \mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{\mathfrak{q}})$ induces a morphism $U_{\mathfrak{q}}^{\bullet}(V, \mathbf{D}) = C_{\varphi,\gamma}^{\bullet}(\mathbf{D}_{\mathfrak{q}}) \to C_{\varphi,\gamma}^{\bullet}(V_{\mathfrak{q}})$. We denote by

$$g_{\mathfrak{q}} : U_{\mathfrak{q}}^{\bullet}(V, \mathbf{D}) \to K^{\bullet}(V_{\mathfrak{q}}), \qquad \mathfrak{q} \mid p \tag{45}$$

the composition of this morphism with the quasi-isomorphism $\alpha_{V_{\mathfrak{q}}} : C_{\varphi,\gamma}^{\bullet}(V_{\mathfrak{q}}) \simeq K^{\bullet}(V_{\mathfrak{q}})$ constructed in Section 2.5 and by

$$f_{\mathfrak{q}} : C^{\bullet}(G_{F,S}, V) \to K^{\bullet}(V_{\mathfrak{q}}), \qquad \mathfrak{q} \mid p \tag{46}$$

the composition of the restriction map $\mathrm{res}_{\mathfrak{q}} : C^{\bullet}(G_{F,S}, V) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, V)$ with the quasi-isomorphism $\xi_{V_{\mathfrak{q}}} : C^{\bullet}(G_{F_{\mathfrak{q}}}, V) \to K^{\bullet}(V_{\mathfrak{q}})$ constructed in Proposition 2.5.2. Set

$$K_{\mathfrak{q}}^{\bullet}(V) = \begin{cases} K^{\bullet}(V_{\mathfrak{q}}) & \text{if } \mathfrak{q} \in S_p, \\ C^{\bullet}(G_{F_{\mathfrak{q}}}, V) & \text{if } \mathfrak{q} \in \Sigma_p, \end{cases}$$

and

$$\begin{aligned} K^{\bullet}(V) &= \underset{\mathfrak{q} \in S}{\oplus} K_{\mathfrak{q}}^{\bullet}(V), \\ U^{\bullet}(V, \mathbf{D}) &= \underset{\mathfrak{q} \in S}{\oplus} U_{\mathfrak{q}}^{\bullet}(V, \mathbf{D}). \end{aligned}$$

We turn now to global Galois cohomology of V. By [**62**, Section 1], one has

$$C^{\bullet}(G_{F,S}, V) \in \mathscr{K}_{\mathrm{ft}}^{[0,3]}(A)$$

and the associated object of the derived category

$$\mathbf{R}\Gamma_S(V) := [C^{\bullet}(G_{F,S}, V)] \in \mathscr{D}_{\mathrm{perf}}^{[0,3]}(A).$$

Therefore, we have a diagram in $\mathscr{K}_{\mathrm{ft}}^{[0,3]}(A)$

$$\begin{array}{ccc} C^\bullet(G_{F,S},V) & \xrightarrow{f} & K^\bullet(V) \\ & & \Big\uparrow{\scriptstyle g} \\ & & U^\bullet(V,\mathbf{D}), \end{array}$$

where $f=(f_{\mathfrak{q}})_{\mathfrak{q}\in S}$ and $g=(g_{\mathfrak{q}})_{\mathfrak{q}\in S}$, and the corresponding diagram in $\mathscr{D}_{\mathrm{ft}}^{[0,3]}(A)$

$$\begin{array}{ccc} \mathbf{R}\Gamma_S(V) & \longrightarrow & \bigoplus_{\mathfrak{q}\in S}\mathbf{R}\Gamma(F_{\mathfrak{q}},V) \\ & & \Big\uparrow \\ & & \bigoplus_{\mathfrak{q}\in S}\mathbf{R}\Gamma(F_{\mathfrak{q}},V,\mathbf{D}), \end{array}$$

where we set $\mathbf{R}\Gamma(F_{\mathfrak{q}},V,\mathbf{D})=\left[U_{\mathfrak{q}}^\bullet(V,\mathbf{D})\right]$ for all $\mathfrak{q}\in S$. The associated Selmer complex is defined as

$$S^\bullet(V,\mathbf{D})=\mathrm{cone}\left[C^\bullet(G_{F,S},V)\oplus U^\bullet(V,\mathbf{D})\xrightarrow{f-g}K^\bullet(V)\right][-1].$$

We set $\mathbf{R}\Gamma(V,\mathbf{D}):=[S^\bullet(V,\mathbf{D})]$ and write $H^\bullet(V,\mathbf{D})$ for the cohomology of $S^\bullet(V,\mathbf{D})$. Since all complexes involved in this definition belong to $\mathscr{K}_{\mathrm{ft}}(A)$, it is easy to check that $S^\bullet(V,\mathbf{D})\in\mathscr{K}_{\mathrm{ft}}^{[0,3]}(A)$. If, in addition, $[C_{\mathrm{ur}}^\bullet(V_{\mathfrak{q}})]\in\mathscr{D}_{\mathrm{perf}}^{[0,1]}(A)$ for all $\mathfrak{q}\in\Sigma_p$, then $\mathbf{R}\Gamma(V,\mathbf{D})\in\mathscr{D}_{\mathrm{perf}}^{[0,3]}(A)$.

Each element $[x^{\mathrm{sel}}]\in H^i(V,\mathbf{D})$ can be represented by a triple

$$x^{\mathrm{sel}}=(x,(x_{\mathfrak{q}}^+),(\lambda_{\mathfrak{q}})), \tag{47}$$

where, for each $\mathfrak{q}\in S$,

$$\begin{array}{lll} x\in C^i(G_{F,S},V), & x_{\mathfrak{q}}^+\in U_{\mathfrak{q}}^i(V,\mathbf{D}), & \lambda_{\mathfrak{q}}\in K_{\mathfrak{q}}^{i-1}(V), \\ d(x)=0, & d(x_{\mathfrak{q}}^+)=0, & f_{\mathfrak{q}}(x)=g_{\mathfrak{q}}(x_{\mathfrak{q}}^+)-d(\lambda_{\mathfrak{q}}). \end{array}$$

3.1.3. — The previous construction can be slightly generalized. Fix a finite subset $\Sigma\subset\Sigma_p$ and, for each $\mathfrak{q}\in\Sigma$, a locally direct summand $M_{\mathfrak{q}}$ of the A-module $V_{\mathfrak{q}}$ stable under the action of $G_{F_{\mathfrak{q}}}$. Let $M=(M_{\mathfrak{q}})_{\mathfrak{q}\in\Sigma}$. Define

$$U_{\mathfrak{q}}^\bullet(V,\mathbf{D},M)=\begin{cases} C_{\varphi,\gamma_{\mathfrak{q}}}^\bullet(\mathbf{D}_{\mathfrak{q}}), & \text{if } \mathfrak{q}\in S_p, \\ C_{\mathrm{ur}}^\bullet(V_{\mathfrak{q}}), & \text{if } \mathfrak{q}\in\Sigma_p\setminus\Sigma, \\ C^\bullet(G_{F_{\mathfrak{q}}},M_{\mathfrak{q}}), & \text{if } \mathfrak{q}\in\Sigma. \end{cases}$$

In short, we replace unramified conditions at all $\mathfrak{q}\in\Sigma$ by Greenberg conditions defined by the family of subrepresentations $M=(M_{\mathfrak{q}})_{\mathfrak{q}\in\Sigma}$. We denote by $S_\bullet(V,\mathbf{D},M)$ the associated Selmer complex and set $\mathbf{R}\Gamma(V,\mathbf{D},M) := [S^\bullet(V,\mathbf{D},M)]$. This construction is a direct generalizaton of Selmer complexes considered in [**56**, Section 7.8] to the non-ordinary setting.

Consider two important particular cases. If $M_\mathfrak{q} = 0$ for all $\mathfrak{q} \in \Sigma$, we write $S^\bullet_\Sigma(V,\mathbf{D})$ and $\mathbf{R}\Gamma_\Sigma(V,\mathbf{D})$ for $S^\bullet(V,\mathbf{D},M)$ and $\mathbf{R}\Gamma(V,\mathbf{D},M)$ respectively. If $M_\mathfrak{q} = V_\mathfrak{q}$ for all $\mathfrak{q} \in \Sigma$, we write $S^{\Sigma,\bullet}(V,\mathbf{D})$ and $\mathbf{R}\Gamma^\Sigma(V,\mathbf{D})$ for $S^\bullet(V,\mathbf{D},M)$ and $\mathbf{R}\Gamma(V,\mathbf{D},M)$ respectively. These complexes are derived analogs of the *strict* and *relaxed* Selmer groups in the sense of [**63**, Section 1.5]. Note that $\mathbf{R}\Gamma_{\Sigma_p}(V,\mathbf{D})$ and $\mathbf{R}\Gamma^{\Sigma_p}(V,\mathbf{D})$ are objects of $\mathscr{D}^{[0,3]}_{\mathrm{perf}}(A)$. See Section 3.1.6 for further remarks concerning these complexes.

3.1.4. — We construct cup products for our Selmer complexes $\mathbf{R}\Gamma(V,\mathbf{D},M)$. Consider the dual representation $V^*(1)$ of V. We equip $V^*(1)$ with the dual local conditions setting

$$\mathbf{D}^\perp_\mathfrak{q} = \mathrm{Hom}_{\mathscr{R}_A}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/\mathbf{D}_\mathfrak{q}, \mathscr{R}_A(\chi_\mathfrak{q})), \qquad \forall \mathfrak{q} \in S_p,$$
$$M^\perp_\mathfrak{q} = \mathrm{Hom}_A(V_\mathfrak{q}/M_\mathfrak{q}, A(\chi_\mathfrak{q})), \qquad \forall \mathfrak{q} \in \Sigma,$$

and denote by $f^\perp_\mathfrak{q}$ and $g^\perp_\mathfrak{q}$ the morphisms (43-46) associated to $(V^*(1),\mathbf{D}^\perp,M^\perp)$. We also remark that the composition

$$C^\bullet_{\mathrm{ur}}(V_\mathfrak{q}) \otimes C^\bullet_{\mathrm{ur}}(V^*_\mathfrak{q}(1)) \xrightarrow{g_\mathfrak{q} \otimes g^\perp_\mathfrak{q}} C^\bullet(G_{F_\mathfrak{q}},V) \otimes C^\bullet(G_{F_\mathfrak{q}},V^*(1)) \xrightarrow{\cup_c} A[-2] \tag{48}$$

is the zero map [**56**, Lemma 7.5.2]. Consider the following data

1) The complexes $A^\bullet_1 = C^\bullet(G_{F,S},V)$, $B^\bullet_1 = U^\bullet(V,\mathbf{D},M)$, and $C^\bullet_1 = K^\bullet(V)$ equipped with the morphisms $f_1 = (f_\mathfrak{q})_{\mathfrak{q}\in S} : A^\bullet_1 \to C^\bullet_1$ and $g_1 = \underset{\mathfrak{q}\in S}{\oplus} g_\mathfrak{q} : B^\bullet_1 \to C^\bullet_1$;
2) The complexes $A^\bullet_2 = C^\bullet(G_{F,S},V^*(1))$, $B^\bullet_2 = U^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp)$, and $C^\bullet_2 = K^\bullet(V^*(1))$ equipped with the morphisms $f_2 = (f^\perp_\mathfrak{q})_{\mathfrak{q}\in S} : A^\bullet_2 \to C^\bullet_2$ and $g_2 = \underset{\mathfrak{q}\in S}{\oplus} g^\perp_\mathfrak{q} : B^\bullet_2 \to C^\bullet_2$;
3) The complexes $A^\bullet_3 = \tau_{\geqslant 2}C^\bullet(G_{F,S},A(1))$, $B^\bullet_3 = 0$ and $C^\bullet_3 = \tau_{\geqslant 2}K^\bullet(A(1))$ equipped with the map $f_3 : A^\bullet_3 \to C^\bullet_3$ given by
$$\tau_{\geqslant 2}C^\bullet(G_{F,S},A(1)) \xrightarrow{(\mathrm{res}_\mathfrak{q})_\mathfrak{q}} \bigoplus_\mathfrak{q} \tau_{\geqslant 2}C^\bullet(G_{F_\mathfrak{q}},A(1)) \to \tau_{\geqslant 2}K^\bullet(A(1))$$
and the zero map $g_3 : B^\bullet_3 \to C^\bullet_3$.
4) The cup product $\cup_A : A^\bullet_1 \otimes A^\bullet_2 \to A^\bullet_3$ defined as the composition

$$\cup_A : C^\bullet(G_{F,S},V) \otimes C^\bullet(G_{F,S},V^*(1)) \xrightarrow{\cup_c} C^\bullet(G_{F,S},V \otimes V^*(1)) \to$$
$$C^\bullet(G_{F,S},A^*(1)) \to \tau_{\geqslant 2}C^\bullet(G_{F,S},A^*(1)),$$

5) The zero cup product $\cup_B\,:\,B_1^\bullet \otimes B_2^\bullet \to B_3^\bullet$.

6) The cup product $\cup_C\,:\,C_1^\bullet \otimes C_2^\bullet \to C_3^\bullet$ defined as the composition

$$K^\bullet(V)\otimes K^\bullet(V^*(1)) \xrightarrow{\cup_K} K^\bullet(V\otimes V^*(1)) \to K^\bullet(A(1)) \to \tau_{\geqslant 2}K^\bullet(A(1)).$$

7) The zero maps $h_f\,:\,A_1^\bullet \otimes A_2^\bullet \to C_3^\bullet[-1]$ and $h_g\,:\,B_1^\bullet \otimes B_2^\bullet \to C_3^\bullet[-1]$.

Theorem 3.1.5. **—** *i) There exists a canonical, up to homotopy, quasi-isomorphism*

$$r_S\,:\,E_3^\bullet \to A[-2].$$

ii) The data 1-7) above satisfy conditions **P1-3)** *of Section 1.2 and therefore define, for each $a\in A$ and each quasi-isomorphism r_S, the cup product*

$$\cup_{a,r_S}\,:\,S^\bullet(V,\mathbf{D},M)\otimes_A S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp) \to A[-3].$$

iii) The homotopy class of $\cup_{a,r_S}$ does not depend on the choice of $r\in A$ and, therefore, defines a pairing

$$\cup_{V,\mathbf{D},M}\,:\,\mathbf{R}\Gamma(V,\mathbf{D},M)\otimes_A^{\mathbf{L}}\mathbf{R}\Gamma(V^*(1),\mathbf{D}^\perp,M^\perp) \to A[-3]. \tag{49}$$

Proof. — i) We repeat *verbatim* the argument of [**56**, Section 5.4.1]. For each $\mathfrak{q}\in S$, let $i_\mathfrak{q}$ denote the composition of the canonical isomorphism $A\simeq H^2(F_\mathfrak{q},A(1))$ of the local class field theory with the morphism $\tau_{\geqslant 2}C^\bullet(G_{F_\mathfrak{q}},A(1))\to K^\bullet(A_\mathfrak{q}(1))$. Then we have a commutative diagram

$$\begin{array}{ccccc}
\tau_{\geqslant 2}C^\bullet(G_{F,S},A(1)) & \xrightarrow{(\mathrm{res}_\mathfrak{q})_\mathfrak{q}} & \bigoplus\limits_{\mathfrak{q}\in S}\tau_{\geqslant 2}K^\bullet(A_\mathfrak{q}(1)) & \xrightarrow{\;j\;} & E_3^\bullet \\
 & & \Big\uparrow{\scriptstyle (i_\mathfrak{q})_\mathfrak{q}} & & \Big\uparrow{\scriptstyle i_S} \\
 & & \bigoplus\limits_{\mathfrak{q}\in S} A[-2] & \xrightarrow{\;\Sigma\;} & A[-2],
\end{array}$$

where $i_S=j\circ i_{\mathfrak{q}_0}$ for some fixed $\mathfrak{q}_0\in S$ and Σ denotes the summation over $\mathfrak{q}\in S$. By global class field theory, i_S is a quasi-isomorphism and, because $A[-2]$ is concentrated in degree 2, there exists a homotopy inverse r_S of i_S which is unique up to homotopy.

ii) We only need to show that condition **P3)** holds in our case. Note that $\cup_A=\cup_c$, $\cup_B=0$ and $\cup_C=\cup_K$. From the definition of $\cup_K$ it follows immediately that

$$\cup_K\circ(f_1\otimes f_2)=f_3\circ\cup_c. \tag{50}$$

If $\mathfrak{q}\in S_p$ (resp. if $\mathfrak{q}\in\Sigma$), from the orthogonality of $\mathbf{D}_\mathfrak{q}^\perp$ and $\mathbf{D}_\mathfrak{q}$ (resp. from the orthogonality of $M_\mathfrak{q}$ and $M_\mathfrak{q}^\perp$) it follows that $\cup_K\circ(g_\mathfrak{q}\otimes g_\mathfrak{q}^\perp)=0$. If $\mathfrak{q}\in\Sigma_p\setminus\Sigma$, we have $\cup_c\circ(g_\mathfrak{q}\otimes g_\mathfrak{q}^\perp)=0$ by (48). Since $g_3\circ\cup_B=0$, this gives

$$\cup_C\circ(g_1\otimes g_2)=g_3\circ\cup_B=0. \tag{51}$$

The equations (50) and (51) show that **P3)** holds with $h_f = h_g = 0$. We define $\cup_{a,r_S}$ as the composition of the cup product constructed in Proposition 1.2.2 with r_S. The rest of the theorem follows from Proposition 1.2.2. □

3.1.6. — In this subsection we discuss the duality theory for Selmer complexes. Recall that we have the anti-involution (29) on the category $\mathscr{D}_{\mathrm{perf}}(A)$ given by [1]

$$X \to X^* = \mathbf{R}\mathrm{Hom}_A(X, A).$$

The cup product $\cup_{V,\mathbf{D},M}$ induces a map in $\mathscr{D}^b_{\mathrm{ft}}(A)$:

$$\mathbf{R}\Gamma(V^*(1), \mathbf{D}^\perp, M^\perp) \to \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(V, \mathbf{D}, M), A)[-3]. \tag{52}$$

For each $\mathfrak{q} \in S$ define

$$\widetilde{U}^\bullet_\mathfrak{q}(V, \mathbf{D}, M) = \mathrm{cone}\left(U^\bullet_\mathfrak{q}(V, \mathbf{D}, M) \xrightarrow{g_\mathfrak{q}} K^\bullet(V_\mathfrak{q})\right)[-1]$$

and $\widetilde{\mathbf{R}\Gamma}(F_\mathfrak{q}, V, \mathbf{D}, M) = \left[\widetilde{U}^\bullet_\mathfrak{q}(V, \mathbf{D}, M)\right]$. From the orthogonality of $g_\mathfrak{q}$ and $g_\mathfrak{q}^\perp$ under the cup product $K^\bullet(V_\mathfrak{q}) \otimes K^\bullet(V^*_\mathfrak{q}(1)) \to A[-2]$ it follows that we have a pairing

$$\widetilde{U}^\bullet_\mathfrak{q}(V, \mathbf{D}, M) \otimes U^\bullet_\mathfrak{q}(V^*(1), \mathbf{D}^\perp, M^\perp) \to A[-2]$$

which gives rise to a morphism in $\mathscr{D}^b_{\mathrm{ft}}(A)$

$$\mathbf{R}\Gamma(F_\mathfrak{q}, V^*(1), \mathbf{D}^\perp, M^\perp) \to \mathbf{R}\mathrm{Hom}_A(\widetilde{\mathbf{R}\Gamma}(F_\mathfrak{q}, V, \mathbf{D}, M), A)[-2]. \tag{53}$$

Let $\mathfrak{q} \in \Sigma_p \setminus \Sigma$. Denote by $I^w_\mathfrak{q}$ the wild ramification subgroup of $I_\mathfrak{q}$. Fix a topological generator $t_\mathfrak{q}$ of $I_\mathfrak{q}/I^w_\mathfrak{q}$ such that for any uniformizer $\varpi_\mathfrak{q}$ of $F_\mathfrak{q}$

$$t_\mathfrak{q}(\varpi_\mathfrak{q}^{1/p^n}) = \zeta_{p^n} \varpi_\mathfrak{q}^{1/p^n}, \qquad n \geqslant 1,$$

where $\varepsilon = (\zeta_{p^n})_{n \geqslant 1}$ is our fixed system of p^n-th roots of unity. We also fix a lift $F_\mathfrak{q} \in G_\mathfrak{q}/I^w_\mathfrak{q}$ of the Frobenius $\mathrm{Fr}_\mathfrak{q}$. Define

$$C^\bullet_{\mathrm{tr}}(V_\mathfrak{q}) : V^{I^w_\mathfrak{q}} \xrightarrow{(F_\mathfrak{q}-1, t_\mathfrak{q}-1)} V^{I^w_\mathfrak{q}} \oplus V^{I^w_\mathfrak{q}} \xrightarrow{(1-t_\mathfrak{q}, \theta_\mathfrak{q}-1)} V^{I^w_\mathfrak{q}},$$

where $\theta_\mathfrak{q} = F_\mathfrak{q}(1 + t_\mathfrak{q} + \cdots + t_\mathfrak{q}^{q_\mathfrak{q}-1})$ and $q_\mathfrak{q}$ is the order of the residue field of F modulo $\mathfrak{q}$. We refer the reader to [**56**, Sections 7.1-7.6] for the proofs of the following results. The complex $C^\bullet_{\mathrm{tr}}(V_\mathfrak{q})$ is quasi-isomorphic to $C^\bullet(G_{F_\mathfrak{q}}, V)$. The natural inclusion $V^{I_\mathfrak{q}} \hookrightarrow V^{I^w_\mathfrak{q}}$ induces a monomorphism of complexes $C^\bullet_{\mathrm{ur}}(V_\mathfrak{q}) \to C^\bullet_{\mathrm{tr}}(V_\mathfrak{q})$. Let $\widetilde{C}^\bullet_{\mathrm{ur}}(V_\mathfrak{q}) = C^\bullet_{\mathrm{tr}}(V_\mathfrak{q})/C^\bullet_{\mathrm{ur}}(V_\mathfrak{q})$. Then the natural projections induce a quasi-isomorphism

$$\widetilde{C}^\bullet_{\mathrm{ur}}(V_\mathfrak{q}) \simeq \left(V^{I^w_\mathfrak{q}}/(t_\mathfrak{q}-1)V^{I^w_\mathfrak{q}} \xrightarrow{q_\mathfrak{q} F_\mathfrak{q} - 1} V^{I^w_\mathfrak{q}}/(t_\mathfrak{q}-1)V^{I^w_\mathfrak{q}}\right) \tag{54}$$

1. Note that the dualizaton functor is not defined on $\mathscr{D}^b_{\mathrm{ft}}(A)$.

where the terms are concentrated in degrees 1 and 2. We also remark that since $\mathfrak{q} \in \Sigma_p$, the group $I_{\mathfrak{q}}^w$ acts on V through a finite quotient H and we have a decomposition

$$V \simeq V^{I_{\mathfrak{q}}^w} \oplus I_H(V), \tag{55}$$

where $I_H = \ker(\mathbf{Z}[H] \to \mathbf{Z})$ is the augmentation ideal. In particular, the submodule $V^{I_{\mathfrak{q}}^w}$ is a direct factor of the projective A-module V and therefore is projective itself. From (55) we also get

$$V^*(1)^{I_{\mathfrak{q}}^w} = \mathrm{Hom}_A(V^{I_{\mathfrak{q}}^w}, A)(1). \tag{56}$$

For the representation $A(1)$ we have

$$C^\bullet_{\mathrm{tr}}(A(1))\,:\, A(1) \xrightarrow{(q_{\mathfrak{q}}^{-1}-1,0)} A(1) \oplus A(1) \xrightarrow{(0,0)} A(1).$$

The canonical isomorphism $\mathrm{inv}_{F_{\mathfrak{q}}}\,:\, H^2(F_{\mathfrak{q}}, A(1)) \to A$ has the following description in terms of this complex:

$$\begin{cases} H^2(C^\bullet_{\mathrm{tr}}(A(1))) \to A \\ x \otimes \varepsilon = x. \end{cases} \tag{57}$$

Now we can formulate the following result which is a more precise version of [**62**, Theorem 1.16] in our context.

Theorem 3.1.7. — *i) For all $\mathfrak{q} \in \Sigma \cup S_p$ the map (53) is an isomorphism in $\mathscr{D}^{[0,2]}_{\mathrm{perf}}(A)$.*

ii) Let $\mathfrak{q} \in \Sigma_p \setminus \Sigma$. If the A-module $V^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^w}$ is projective, then the A-modules $V^{I_{\mathfrak{q}}}$, $V^*(1)^{I_{\mathfrak{q}}}$ and $V^*(1)^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^*(1)^{I_{\mathfrak{q}}^w}$ are projectives and the map (53) is an isomorphism in $\mathscr{D}^{[0,2]}_{\mathrm{perf}}(A)$.

iii) If, for all $\mathfrak{q} \in \Sigma_p \setminus \Sigma$, the A-module $V^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^w}$ is projective, then the duality map (52) is an isomorphism in $\mathscr{D}^{[0,3]}_{\mathrm{perf}}(A)$:

$$\mathbf{R}\Gamma(V^*(1), \mathbf{D}^\perp, M^\perp) \simeq \mathbf{R}\Gamma(V, \mathbf{D}, M)^*[-3].$$

Proof. — i) For $\mathfrak{q} \in \Sigma$, the assertion i) is proved in [**56**, Section 6.7] in the context of admissible modules. Recall that it follows directly from the local duality for p-adic representations. Mimiking this proof and using Theorem 2.3.2 we obtain that (53) is an isomorphism for $\mathfrak{q} \in \Sigma$. The same proof applies to the case $\mathfrak{q} \in S_p$ if we use Theorem 2.4.3 instead Theorem 2.3.2. Namely, consider the tautological exact sequence

$$0 \to \mathbf{D}_{\mathfrak{q}} \to \mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{q}}) \to \widetilde{\mathbf{D}}_{\mathfrak{q}} \to 0,$$

where $\widetilde{\mathbf{D}}_{\mathfrak{q}} = \mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{q}})/\mathbf{D}_{\mathfrak{q}}$. Applying the functor $\mathbf{R}\Gamma(F_{\mathfrak{q}}, -)$ to this sequence, we obtain a distinguished triangle

$$\mathbf{R}\Gamma(F_{\mathfrak{q}}, \mathbf{D}_{\mathfrak{q}}) \to \mathbf{R}\Gamma(F_{\mathfrak{q}}, \mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathfrak{q}})) \to \mathbf{R}\Gamma(F_{\mathfrak{q}}, \widetilde{\mathbf{D}}_{\mathfrak{q}}) \to \mathbf{R}\Gamma(F_{\mathfrak{q}}, \mathbf{D}_{\mathfrak{q}})[1],$$

and therefore $\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V,\mathbf{D})\simeq \mathbf{R}\Gamma(F_{\mathfrak{q}},\widetilde{\mathbf{D}}_{\mathfrak{q}})$. From the definition of $\mathbf{D}_{\mathfrak{q}}^{\perp}$ we have $\mathbf{D}_{\mathfrak{q}}^{\perp}\simeq \widetilde{\mathbf{D}}_{\mathfrak{q}}^{*}(\chi)$. Using Theorem 2.4.3, we obtain that

$$\begin{aligned}\mathbf{R}\Gamma(F_{\mathfrak{q}},\mathbf{D}_{v}^{\perp})\simeq \mathbf{R}\Gamma(F_{\mathfrak{q}},\widetilde{\mathbf{D}}_{\mathfrak{q}}^{*}(\chi))\simeq \\ \simeq \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(F_{\mathfrak{q}},\widetilde{\mathbf{D}}_{\mathfrak{q}}),A)[-2]\simeq \mathbf{R}\mathrm{Hom}_A(\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V,\mathbf{D}),A)[-2],\end{aligned}$$

and therefore (53) holds for $\mathfrak{q}\in S_p$.

ii) Assume that $V^{I_{\mathfrak{q}}^{w}}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^{w}}$ is projective. Then the tautological exact sequence

$$0\to (t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^{w}}\to V^{I_{\mathfrak{q}}^{w}}\to V^{I_{\mathfrak{q}}^{w}}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^{w}}\to 0$$

splits and $(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^{w}}$ is projective as direct summand of the projective module $V^{I_{\mathfrak{q}}^{w}}$. The same argument applied to the exact sequence

$$0\to V^{I_{\mathfrak{q}}}\to V^{I_{\mathfrak{q}}^{w}}\xrightarrow{t_{\mathfrak{q}}-1}(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^{w}}\to 0 \tag{58}$$

shows that $V^{I_{\mathfrak{q}}}$ is projective. Dualizing the sequence (58) and taking into account (56) and the fact that $I_{\mathfrak{q}}$ acts trivially on $\mathbf{Q}_p(1)$ we get the sequence

$$0\to (t_{\mathfrak{q}}-1)V^*(1)^{I_{\mathfrak{q}}^{w}}\to V^*(1)^{I_{\mathfrak{q}}^{w}}\to \left(V^{I_{\mathfrak{q}}}\right)^*(1)\to 0.$$

This sequence is split exact because the sequence (58) splits. Therefore

$$V^*(1)^{I_{\mathfrak{q}}^{w}}/(t_{\mathfrak{q}}-1)V^*(1)^{I_{\mathfrak{q}}^{w}}\simeq \left(V^{I_{\mathfrak{q}}}\right)^*(1). \tag{59}$$

Since $V^{I_{\mathfrak{q}}}$ is projective, $V^*(1)^{I_{\mathfrak{q}}^{w}}/(t_{\mathfrak{q}}-1)V^*(1)^{I_{\mathfrak{q}}^{w}}$ is projective. This also implies the projectivity of $V^*(1)^{I_{\mathfrak{q}}^{w}}$.

Now we show that (53) is an isomorphism. Consider the following diagram in $\mathscr{D}_{\mathrm{perf}}^{[0,2]}(A)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left[C_{\mathrm{ur}}^{\bullet}(V_{\mathfrak{q}}^{*}(1))\right] & \longrightarrow & \mathbf{R}\Gamma(F_{\mathfrak{q}},V^*(1)) & \longrightarrow & \widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V^*(1)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & \widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V)^*[-2] & \longrightarrow & \mathbf{R}\Gamma(F_{\mathfrak{q}},V)^*[-2] & \longrightarrow & \left[C_{\mathrm{ur}}^{\bullet}(V_{\mathfrak{q}})\right]^*[-2] & \longrightarrow & 0 \end{array}$$

where we write $\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V):=\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V,\mathbf{D},M)$ to simplify notation. The upper row is exact by the definition of $\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V^*(1))$. The exactness of the bottom row follows from the definition of $\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V)$ and the exactness of the dualization functor. The middle vertical map μ is induced by the local duality and is an isomorphism by Theorem 2.3.2.

We show that ν is an isomorphism. This will imply that λ is an isomorphism. From (59) it follows that

$$\widetilde{\mathbf{R}\Gamma}(F_{\mathfrak{q}},V^*(1)) \simeq \left[(V^{I_{\mathfrak{q}}})^*(1) \xrightarrow{q_{\mathfrak{q}}\mathrm{Fr}_{\mathfrak{q}}-1} (V^{I_{\mathfrak{q}}})^*(1)\right]$$
$$\overset{\otimes\varepsilon^{-1}}{\simeq} \left[(V^{I_{\mathfrak{q}}})^* \xrightarrow{\mathrm{Fr}_{\mathfrak{q}}-1} (V^{I_{\mathfrak{q}}})^*\right] \simeq [C^\bullet_{\mathrm{ur}}(V_{\mathfrak{q}})]^*.$$

(Note that all involved modules are projective.) Using (57) it is easy to check that this isomorphism coincides with ν and ii) is proved.

iii) Repeating the arguments of [**56**] (see the proofs of Proposition 6.3.3 and Theorem 6.3.4 of *op. cit.*) it is easy to show that if $\mathbf{R}\Gamma(F_{\mathfrak{q}},V,\mathbf{D})$ and $\mathbf{R}\Gamma(F_{\mathfrak{q}},V^*(1),\mathbf{D}^{\perp})$ are perfect and (53) holds for all $\mathfrak{q}\in S$, then (52) is an isomorphism. Now the statement follows from i) and ii).

□

Corollary 3.1.8. — *Let* $\mathrm{WD}(V_{\mathfrak{q}})$ *denote the Weil-Deligne representation associated to* $V_{\mathfrak{q}}$ *equipped with the canonical monodromy* $N_{\mathfrak{q}}:\mathrm{WD}(V_{\mathfrak{q}})\to\mathrm{WD}(V_{\mathfrak{q}})$. *Assume that for all* $\mathfrak{q}\in\Sigma_p\setminus\Sigma$, *the A-module* $\mathrm{WD}(V_{\mathfrak{q}})/N_{\mathfrak{q}}\mathrm{WD}(V_{\mathfrak{q}})$ *is projective. Then the duality map (52) is an isomorphism.*

Proof. — We remark that Grothendieck's monodromy theorem holds for representations with coefficients in an affinoid algebra [**4**, Lemma 7.8.14]. Let $F'_{\mathfrak{q}}/F_{\mathfrak{q}}$ be a finite extension such that the action of the inertia subgroup $I'_{\mathfrak{q}}$ of $G_{F'_{\mathfrak{q}}}$ on $V_{\mathfrak{q}}$ factors through the p-part $T_K(p)$ of its tame quotient T_K. Recall that $\mathrm{WD}(V_{\mathfrak{q}})=V_{\mathfrak{q}}$ as A-module and that the monodromy $N_{\mathfrak{q}}$ is defined as the derivative of the action of $T_K(p)$ on $V_{\mathfrak{q}}$ at 1. The decomposition (55) is compatible with the action of $G_{F_{\mathfrak{q}}}$ and therefore with the monodromy $N_{\mathfrak{q}}$. Thus, $V^{I^w_{\mathfrak{q}}}/N_{\mathfrak{q}}\left(V^{I^w_{\mathfrak{q}}}\right)$ is a direct summand of $\mathrm{WD}(V_{\mathfrak{q}})/N_{\mathfrak{q}}\mathrm{WD}(V_{\mathfrak{q}})$.

From the definition of $N_{\mathfrak{q}}$ it follows that for $m\gg 0$

$$t_{\mathfrak{q}}^m\Big|_{V^{I^w_{\mathfrak{q}}}}=\exp(mN_{\mathfrak{q}}).$$

Since $\exp(mN_{\mathfrak{q}})-1=mN_{\mathfrak{q}}R_{\mathfrak{q}}$, where $R_{\mathfrak{q}}=1+mN_{\mathfrak{q}}/2!+(mN_{\mathfrak{q}})^2/3!+\cdots$ is invertible, we have

$$(t_{\mathfrak{q}}^m-1)V^{I^w_{\mathfrak{q}}}=N_{\mathfrak{q}}\left(V^{I^w_{\mathfrak{q}}}\right)$$

and

$$V^{I^w_{\mathfrak{q}}}/(t_{\mathfrak{q}}^m-1)V^{I^w_{\mathfrak{q}}}=V^{I^w_{\mathfrak{q}}}/N_{\mathfrak{q}}\left(V^{I^w_{\mathfrak{q}}}\right).$$

To simplify notation, set $W=V^{I^w_{\mathfrak{q}}}/(t_{\mathfrak{q}}^m-1)V^{I^w_{\mathfrak{q}}}$. Since $t_{\mathfrak{q}}^m$ acts trivially on W, we have

$$W=(t_{\mathfrak{q}}-1)W\oplus W',\qquad W'=(1+t_{\mathfrak{q}}+\cdots+t_{\mathfrak{q}}^{m-1})W.$$

Assume that $\mathrm{WD}(V_{\mathfrak{q}})/N_{\mathfrak{q}}\mathrm{WD}(V_{\mathfrak{q}})$ is projective. Then $W=V^{I_{\mathfrak{q}}^w}/N_{\mathfrak{q}}\left(V^{I_{\mathfrak{q}}^w}\right)$ is projective. Since

$$V^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^w}\simeq W/(t_{\mathfrak{q}}-1)W\simeq W'$$

and W' is a direct summand of W, the A-module $V^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^w}$ is projective. Now the corollary follows from Theorem 3.1.7 iii). □

Remarks 3.1.9. — 1) Let f be a primitive eigenform of level N and weight $k\geqslant 2$. Assume that $(p,N)=1$. Fix a p-stabilization f_α of f and denote by x_0 the corresponding point on the Coleman–Mazur eigencurve. Let $\mathbf{f}$ be the family of p-adic modular forms passing through f_α. Taking a sufficiently small affine neighborhood $U=\mathrm{Spm}(A)$ of x_0, we can associate to $\mathbf{f}$ a canonical p-adic Galois representation $W_{\mathbf{f}}$ over A. Let A_{x_0} and $W_{\mathbf{f},x_0}$ denote the localizations of A and $W_{\mathbf{f}}$ at x_0. Note that $W_f=W_{\mathbf{f},x_0}/\mathfrak{m}_{x_0}W_{\mathbf{f},x_0}$ is the p-adic representation associated to f by Deligne.

Consider the representation $V=W_{\mathbf{f}}(\psi)$, where ψ is a continuous Galois character unramified outside p with values in A^*. First assume that for all $\mathfrak{q}|N$ the following conditions hold:

a) If $\mathbf{f}$ is Steinberg at $\mathfrak{q}$, then $\psi_{x_0}(\mathrm{Fr}_{\mathfrak{q}})$ is *not* a Weil number of weight $-k$ or $2-k$;

b) If $\mathbf{f}$ is not Steinberg at $\mathfrak{q}$, then $\psi_{x_0}(\mathrm{Fr}_{\mathfrak{q}})$ is *not* a Weil number of weight $1-k$.

From the purity of p-adic representations associated to modular forms it follows that in this case, the complex $\mathbf{R}\Gamma(\mathbf{Q}_{\mathfrak{q}},V_{\mathfrak{q}})$ is locally acyclic at x_0 (see, for example, [**56**, Proposition 12.7.13.3]). Therefore, the duality map (52) is an isomorphism on a sufficiently small neighborhood of x_0.

In the general case, $\mathbf{R}\Gamma(\mathbf{Q}_{\mathfrak{q}},V_{\mathfrak{q}})$ is not locally acyclic and the argument is different. By [**27**, Proposition 2.2.4], for each $\mathfrak{q}|N$, the A_{x_0}-module $W_{\mathbf{f},x_0}^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)W_{\mathbf{f},x_0}^{I_{\mathfrak{q}}^w}$ is free (2). Replacing U by a smaller neighborhood if necessary, we obtain that $W_{\mathbf{f}}^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)W_{\mathbf{f}}^{I_{\mathfrak{q}}^w}$ is free. Since ψ is unramified outside p, the module $V^{I_{\mathfrak{q}}^w}/(t_{\mathfrak{q}}-1)V^{I_{\mathfrak{q}}^w}$ is free. Therefore Theorem 3.1.7 applies, and again the duality map is a local isomorphism at x_0.

2) In higher dimension, the situation is more complicated. See [**64**] for some related results.

2. In [**27**], the authors consider Hida families, but in the general case the proof is the same.

3.1.10. — Equip the complexes $A_i^\bullet$, $B_i^\bullet$ and $C_i^\bullet$ with the transpositions given by

$$
\begin{aligned}
&\mathscr{T}_{A_1} = \mathscr{T}_{V,c},\\
&\mathscr{T}_{B_1} = \Big(\bigoplus_{\mathfrak{q}\in S_p} \mathrm{id}_{C_{\varphi,\gamma}(\mathbf{D}_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p\setminus\Sigma} \mathrm{id}_{C_{\mathrm{ur}}(V_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in\Sigma} \mathscr{T}_{M_\mathfrak{q},c}\Big),\\
&\mathscr{T}_{C_1} = \Big(\bigoplus_{\mathfrak{q}\in S_p} \mathscr{T}_{K(V_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \mathscr{T}_{V_\mathfrak{q},c}\Big),\\
&\mathscr{T}_{A_2} = \mathscr{T}_{V^*(1),c},\\
&\mathscr{T}_{B_2} = \Big(\bigoplus_{\mathfrak{q}\in S_p} \mathrm{id}_{C_{\varphi,\gamma}(\mathbf{D}^\perp_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p\setminus\Sigma} \mathrm{id}_{C_{\mathrm{ur}}(V^*_\mathfrak{q}(1))}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in\Sigma} \mathscr{T}_{M^\perp_\mathfrak{q},c}\Big),\\
&\mathscr{T}_{C_2} = \Big(\bigoplus_{\mathfrak{q}\in S_p} \mathscr{T}_{K(V^*_\mathfrak{q}(1))}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \mathscr{T}_{V^*_\mathfrak{q}(1),c}\Big),\\
&\mathscr{T}_{A_3} = \mathscr{T}_{A(1),c},\\
&\mathscr{T}_{B_3} = \mathrm{id},\\
&\mathscr{T}_{C_3} = \Big(\bigoplus_{\mathfrak{q}\in S_p} \mathscr{T}_{K(A(1)_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \mathscr{T}_{A(1)_\mathfrak{q},c}\Big).
\end{aligned}
\tag{60}
$$

***Theorem 3.1.11.** — i) The data (60) satisfy conditions* **T1-7)** *of Section 1.2.*
ii) We have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{R}\Gamma(V,\mathbf{D},M) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1),\mathbf{D}^\perp,M^\perp) & \xrightarrow{\cup_{V,\mathbf{D}}} & A[-3]\\
\downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle =}\\
\mathbf{R}\Gamma(V^*(1),\mathbf{D}^\perp,M^\perp) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V,\mathbf{D},M) & \xrightarrow{\cup_{V^*(1),\mathbf{D}^\perp}} & A[-3].
\end{array}
$$

Proof. — i) We check conditions **T3-7)** taking $\cup'_A = \cup_c$, $\cup'_B = 0$ and $\cup'_C = \cup_K$. From (50) and (51) it follows that **T3)** holds if we take $h'_f = h'_g = 0$. To check condition **T4)** we remark that, by Proposition 2.6.4,i) we have $f_i \circ \mathscr{T}_A = \mathscr{T}_C \circ f_i$ and we can take $U_i = 0$. The existence of a homotopy V_i follows from Proposition 2.6.4 ii) and [**56**], Proposition 7.7.3. Note that again we can set $V_i = 0$.

We prove the existence of homotopies t_A, t_B and t_C satisfying **T5)**. From the commutativity of the diagram (35), it follows that $\cup_c \circ s_{12} \circ (\mathscr{T}_A \otimes \mathscr{T}_A) = \mathscr{T}_A \circ \cup_c$ and we can take $t_A = 0$. Since $\cup'_B = \cup_B = 0$, we can take $t_B = 0$. We construct t_C as a system of homotopies $(t_{C,\mathfrak{q}})_{\mathfrak{q}\in S}$ such that $t_{C,\mathfrak{q}} : \cup_c \circ s_{12} \circ (\mathscr{T}_{V_\mathfrak{q},c} \otimes \mathscr{T}_{V(1)_\mathfrak{q},c}) \rightsquigarrow \mathscr{T}_{A(1)_\mathfrak{q},c} \circ \cup_c$ for $\mathfrak{q} \in \Sigma_p$ and $t_{C,\mathfrak{q}} : \cup_K \circ s_{12} \circ (\mathscr{T}_{K(V_\mathfrak{q})} \otimes \mathscr{T}_{K(V(1)_\mathfrak{q})}) \rightsquigarrow \mathscr{T}_{K(A(1)_\mathfrak{q})} \circ \cup_K$ for $\mathfrak{q} \in S_p$. As before, from (35) it follows that for $\mathfrak{q} \in \Sigma_p$ one can take $t_{C,\mathfrak{q}} = 0$. If $\mathfrak{q} \in S_p$, by Proposition 1.1.7 we can set

$$t_{C,\mathfrak{q}}((x_{n-1},x_n) \otimes (y_{m-1} \otimes y_m)) = (-1)^n (\mathscr{T}_{A(1)_\mathfrak{q},c}(x_{n-1} \cup_c y_{m-1}), 0) \tag{61}$$

for $(x_{n-1},x_n) \in K^n(V_\mathfrak{q})$ and $(y_{m-1},y_m) \in K^m(V^*(1)_\mathfrak{q})$ (see (24)). This proves **T5)**. From (61) it follows that $t_C \circ (f_1 \otimes f_2) = 0$ and it is easy to see that **T6)** and **T7)** hold if we take $H_f = H_g = 0$.

ii) For each Galois module X, we denote by $a_X : \mathrm{id} \rightsquigarrow \mathscr{T}_{X,c}$ the homotopy (36). Recall that we can take a_X such that $a_X^0 = a_X^1 = 0$. Consider the following homotopies

$$\begin{aligned} & k_{A_1} = a_V : \mathrm{id} \rightsquigarrow \mathscr{T}_{A_1^\bullet}, && \text{on } A_1^\bullet, \\ & k_{B_1} = \left(\underset{\mathfrak{q} \in S_p \cup \Sigma_p \setminus \Sigma}{\oplus} 0_{U_\mathfrak{q}(V,\mathbf{D},M)} \right) \oplus \left(\underset{\mathfrak{q} \in \Sigma}{\oplus} a_{M_\mathfrak{q}} \right) : \mathrm{id} \rightsquigarrow \mathscr{F}_{B_1} && \text{on } B_1^\bullet, \\ & k_{C_1} = \left(\underset{\mathfrak{q} \in S_p}{\oplus} a_{K(V_\mathfrak{q})} \right) \oplus \left(\underset{\mathfrak{q} \in \Sigma_p}{\oplus} a_{V_\mathfrak{q}} \right) : \mathrm{id} \rightsquigarrow \mathscr{T}_{C_1^\bullet}, && \text{on } C_1^\bullet. \end{aligned} \tag{62}$$

We will denote by k_{A_2}, k_{B_2} k_{C_2} the homotopies on $A_2^\bullet$, $B_2^\bullet$ and $C_2^\bullet$ defined by the analogous formulas. From Proposition 2.6.4, ii) it follows that

$$f \circ k_{A_1} = k_{C_1} \circ f, \quad f^\perp \circ k_{A_2} = k_{C_2} \circ f^\perp,$$
$$g \circ k_{B_1} = k_{C_1} \circ g, \quad g^\perp \circ k_{B_2} = k_{C_2} \circ g^\perp.$$

By (20), these data induce transpositions $\mathscr{T}_V^{\mathrm{sel}}$ and $\mathscr{T}_{V^*(1)}^{\mathrm{sel}}$ on $S^\bullet(V,\mathbf{D},M)$ and $S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp)$, and the formula (21) of Subsection 1.1.2 defines homotopies $k_V^{\mathrm{sel}} : \mathrm{id} \rightsquigarrow \mathscr{T}_V^{\mathrm{sel}}$ and $k_{V^*(1)}^{\mathrm{sel}} : \mathrm{id} \rightsquigarrow \mathscr{T}_{V^*(1)}^{\mathrm{sel}}$. By Proposition 1.2.4, the following diagram commutes up to homotopy:

$$\begin{array}{ccc} S^\bullet(V,\mathbf{D},M) \otimes_A S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp) & \xrightarrow{\cup_{a,r_S}} & A[-3] \\ \Big\downarrow{\scriptstyle s_{12} \circ (\mathscr{T}_V^{\mathrm{sel}} \otimes \mathscr{T}_{V^*(1)}^{\mathrm{sel}})} & & \Big\downarrow{=} \\ S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp) \otimes_A S^\bullet(V,\mathbf{D},M) & \xrightarrow{\cup_{1-a,r_S}} & A[-3]. \end{array}$$

Now the theorem follows from the fact that the map $(k_V^{\mathrm{sel}} \otimes k_{V^*(1)}^{\mathrm{sel}})_1$, given by (18), furnishes a homotopy between id and $\mathscr{T}_V^{\mathrm{sel}} \otimes \mathscr{T}_{V^*(1)}^{\mathrm{sel}}$. □

3.2. p-adic height pairings

3.2.1. — We keep notation and conventions of the previous subsection. Let $F^{\mathrm{cyc}} = \overset{\infty}{\underset{n=1}{\cup}} F(\zeta_{p^n})$ denote the cyclotomic p-extension of F. The Galois group $\Gamma_F = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ decomposes into the direct sum $\Gamma_F = \Delta_F \times \Gamma_F^0$ of the group

$\Delta_F = \mathrm{Gal}(F(\zeta_p)/F)$ and a p-procyclic group Γ_F^0. We denote by $\chi : \Gamma_F \to \mathbf{Z}_p^*$ the cyclotomic character and by $\chi_{\mathfrak{q}}$ the restriction of χ on $\Gamma_{\mathfrak{q}}$, $\mathfrak{q} \in S$.

Consider the completed group algebra $\Lambda_A = A[[\Gamma_F^0]]$. As in Section 2.7, we equip Λ_A with the involution $\iota : \Lambda_A \to \Lambda_A$ such that $\iota(\gamma) = \gamma^{-1}$, $\gamma \in \Gamma_F^0$. Fix a generator γ_F of Γ_F^0. Set $\widetilde{A}_F^{\iota} = \Lambda_A^{\iota}/(J_A^2)$, where J_A is the augmentation ideal of $A[[\Gamma_F^0]]$. We have an exact sequence

$$0 \to A \xrightarrow{\theta_F} \widetilde{A}_F^{\iota} \to A \to 0, \tag{63}$$

where $\theta_F(a) = a\widetilde{X}$, and $\widetilde{X} = \log^{-1}(\chi(\gamma_F))(\gamma_F - 1)$ does not depend on the choice of $\gamma_F \in \Gamma_F^0$. For each p-adic representation V with coefficients in A, (63) induces an exact sequence

$$0 \to V \to \widetilde{V}_F \to V \to 0, \tag{64}$$

where $\widetilde{V}_F = \widetilde{A}_F^{\iota} \otimes_A V$. As in Section 2.7, passing to continuous Galois cohomology, we obtain a distinguished triangle

$$C^{\bullet}(G_{F,S}, V) \to C^{\bullet}(G_{F,S}, \widetilde{V}_F) \to C^{\bullet}(G_{F,S}, V) \xrightarrow{\beta_{V,c}} C^{\bullet}(G_{F,S}, V)[1].$$

For each $\mathfrak{q} \in S$, we have the local analog of the sequence (64) studied in Section 2.7

$$0 \to V \to \widetilde{V}_{F_{\mathfrak{q}}} \to V \to 0.$$

The inclusion $\Gamma_{\mathfrak{q}}^0 \hookrightarrow \Gamma_F^0$ induces a commutative diagram of $G_{F_{\mathfrak{q}}}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_{\mathfrak{q}} & \xrightarrow{\theta_{\mathfrak{q}}} & \widetilde{V}_{F_{\mathfrak{q}}} & \longrightarrow & V_{\mathfrak{q}} & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow{=} & & \\
0 & \longrightarrow & V_{\mathfrak{q}} & \xrightarrow{\theta_F} & (\widetilde{V}_F)_{\mathfrak{q}} & \longrightarrow & V_{\mathfrak{q}} & \longrightarrow & 0,
\end{array}$$

where the vertical middle arrow is an isomorphism by the five lemma. Taking into account Proposition 2.7.2, we see that the exact sequence (64) induces a distinguished triangle

$$C^{\bullet}(G_{F_{\mathfrak{q}}}, V) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, \widetilde{V}_F) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, V) \xrightarrow{\beta_{V_{\mathfrak{q}},c}} C^{\bullet}(G_{F_{\mathfrak{q}}}, V)[1].$$

where $\beta_{V_{\mathfrak{q}},c}(x) = -\log \chi_{\mathfrak{q}} \cup x$.

Let $\mathbf{D}_{\mathfrak{q}}$ be a $(\varphi, \Gamma_{\mathfrak{q}})$-submodule of $\mathbf{D}_{\mathrm{rig}}^{\dagger}(V_{\mathfrak{q}})$ and let $\widetilde{\mathbf{D}}_{F,\mathfrak{q}} = \widetilde{A}_F^{\iota} \otimes_A \mathbf{D}_{\mathfrak{q}}$. As in Section 2.7, we have an exact sequence

$$0 \to \mathbf{D}_{\mathfrak{q}} \to \widetilde{\mathbf{D}}_{F,\mathfrak{q}} \to \mathbf{D}_{\mathfrak{q}} \to 0 \tag{65}$$

which sits in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D}_{\mathfrak{q}} & \xrightarrow{\theta_{\mathfrak{q}}} & \widetilde{\mathbf{D}}_{\mathfrak{q}} & \longrightarrow & \mathbf{D}_{\mathfrak{q}} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & \mathbf{D}_{\mathfrak{q}} & \xrightarrow{\theta_F} & \widetilde{\mathbf{D}}_{F,\mathfrak{q}} & \longrightarrow & \mathbf{D}_{\mathfrak{q}} & \longrightarrow & 0. \end{array}$$

Taking into account Proposition 2.7.4, we obtain that (65) induces the distingushed triangle

$$C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q}}) \to C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\widetilde{\mathbf{D}}_{F,\mathfrak{q}}) \to C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q}}) \xrightarrow{\beta_{\mathbf{D}_{\mathfrak{q}}}} C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q}})[1],$$

where $\beta_{\mathbf{D}_{\mathfrak{q}}}(x) = -(0, \log \chi_{\mathfrak{q}}(\gamma_{\mathfrak{q}})) \cup x$. Finally, replacing in the exact sequence (42) $\widetilde{V}$ by $(\widetilde{V}_F)_{\mathfrak{q}}$, and taking into account Proposition 2.7.5 we obtain the distinguished triangle

$$K^{\bullet}(V_{\mathfrak{q}}) \to K^{\bullet}((\widetilde{V}_F)_{\mathfrak{q}}) \to K^{\bullet}(V_{\mathfrak{q}}) \xrightarrow{\beta_{K(V_{\mathfrak{q}})}} K^{\bullet}(V_{\mathfrak{q}})[1],$$

where $\beta_{K(V_{\mathfrak{q}})}(x) = -(0, \log \chi_{\mathfrak{q}}) \cup x$.

If $\mathfrak{q} \in \Sigma_p$, we construct the Bockstein map for $U^{\bullet}_{\mathfrak{q}}(V, \mathbf{D}, M)$ following [**56**], Section 11.2.4. Namely, if $\mathfrak{q} \in \Sigma$, then $U^{\bullet}_{\mathfrak{q}}(V, \mathbf{D}, M) = C^{\bullet}(G_{F_{\mathfrak{q}}}, M_{\mathfrak{q}})$ and the exact sequence

$$0 \to M_{\mathfrak{q}} \to \widetilde{M}_{F,\mathfrak{q}} \to M_{\mathfrak{q}} \to 0 \qquad (66)$$

gives rise to a map $\beta_{M_{\mathfrak{q}},c} : C^{\bullet}(G_{F_{\mathfrak{q}}}, M_{\mathfrak{q}}) \to C^{\bullet}(G_{F_{\mathfrak{q}}}, M_{\mathfrak{q}})$. If $\mathfrak{q} \in \Sigma_p \setminus \Sigma$, then $(\widetilde{V}_F)^{I_{\mathfrak{q}}} = V^{I_{\mathfrak{q}}} \otimes \widetilde{A}^{\iota}_F$ and we denote by $s : V^{I_{\mathfrak{q}}} \to (\widetilde{V}_F)^{I_{\mathfrak{q}}}$ the section given by $s(x) = x \otimes 1$. There exists a distingushed triangle

$$C^{\bullet}_{\mathrm{ur}}(V_{\mathfrak{q}}) \to C^{\bullet}_{\mathrm{ur}}((\widetilde{V}_F)_{\mathfrak{q}}) \to C^{\bullet}_{\mathrm{ur}}(V_{\mathfrak{q}}) \xrightarrow{\beta_{V_{\mathfrak{q}},\mathrm{ur}}} C^{\bullet}_{\mathrm{ur}}(V_{\mathfrak{q}})[1],$$

where $\beta_{V_{\mathfrak{q}},\mathrm{ur}} : C^0_{\mathrm{ur}}(V_{\mathfrak{q}}) \to C^1_{\mathrm{ur}}(V_{\mathfrak{q}})$ is given by

$$\beta^F_{V_{\mathfrak{q}},\mathrm{ur}}(x) = \frac{1}{\widetilde{X}}(ds - sd)(x) = -\log \chi_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}})x.$$

***Proposition 3.2.2.* —** *In addition to (60), equip the complexes $A_i^\bullet$, $B_i^\bullet$ and $C_i^\bullet$ ($1 \leqslant i \leqslant 3$) with the Bockstein maps given by*

$$\begin{aligned}
\beta_{A_1} &= \beta_{V,c},\\
\beta_{B_1} &= \Big(\bigoplus_{\mathfrak{q}\in S_p} \beta_{\mathbf{D}_\mathfrak{q}}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \beta_{V_\mathfrak{q},\mathrm{ur}}\Big),\\
\beta_{C_1} &= \Big(\bigoplus_{\mathfrak{q}\in S_p} \beta_{K(V_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \beta_{V_\mathfrak{q},c}\Big),\\
\beta_{A_2} &= \beta_{V^*(1),c},\\
\beta_{B_2} &= \Big(\bigoplus_{\mathfrak{q}\in S_p} \beta_{\mathbf{D}_\mathfrak{q}^\perp}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \beta_{V_\mathfrak{q}^*(1),\mathrm{ur}}\Big),\\
\beta_{C_2} &= \Big(\bigoplus_{\mathfrak{q}\in S_p} \beta_{K(V_\mathfrak{q}^*(1))}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \beta_{V_\mathfrak{q}^*(1),c}\Big),\\
\beta_{A_3} &= \beta_{A(1),c},\\
\beta_{B_3} &= 0,\\
\beta_{C_3} &= \Big(\bigoplus_{\mathfrak{q}\in S_p} \beta_{K(A(1)_\mathfrak{q})}\Big) \oplus \Big(\bigoplus_{\mathfrak{q}\in \Sigma_p} \beta_{A(1)_\mathfrak{q},c}\Big).
\end{aligned}$$

Then these data satisfy conditions **B1-5)** *of Section 1.2.*

Proof. — We check **B2-5)** for our Bockstein maps. For each $\mathfrak{q}\in\Sigma_p$, Nekovář constructed homotopies

$$\begin{aligned}
v_{V,\mathfrak{q}} &: g_\mathfrak{q}\circ\beta_{V_\mathfrak{q},\mathrm{ur}} \rightsquigarrow \beta_{V_\mathfrak{q},c}\circ g_\mathfrak{q},\\
v_{V^*(1),\mathfrak{q}} &: g_\mathfrak{q}^\perp\circ\beta_{V_\mathfrak{q}^*(1),\mathrm{ur}} \rightsquigarrow \beta_{V_\mathfrak{q}^*(1),c}\circ g_\mathfrak{q}^\perp.
\end{aligned}$$

From Proposition 2.7.5, ii) it follows that for all $\mathfrak{q}\in S_p$

$$\begin{aligned}
g_\mathfrak{q}\circ\beta_{\mathbf{D}_\mathfrak{q}} &= \beta_{K(V_\mathfrak{q})}\circ g_\mathfrak{q},\\
g_\mathfrak{q}^\perp\circ\beta_{\mathbf{D}_\mathfrak{q}} &= \beta_{K(V_\mathfrak{q}^*(1))}\circ g_\mathfrak{q}^\perp.
\end{aligned}$$

Set $v_{V,\mathfrak{q}} = v_{V^*(1),\mathfrak{q}} = 0$ for all $\mathfrak{q}\in S_p$. Then condition **B2)** holds for $u_i = 0$ and $v_i = (v_{i,\mathfrak{q}})_{\mathfrak{q}\in S}$.

In **B3)**, we can set $h_B = 0$ because $\cup_B = 0$. The existence of a homotopy h_A between $\cup_A[1]\circ(\mathrm{id}\otimes\beta_{A,2})$ and $\cup_A[1]\circ(\beta_{A,1}\otimes\mathrm{id})$ is proved in [**56**], Section 11.2.6 and the same method allows to construct h_C. Namely, we construct a system $h_C = (h_{C,\mathfrak{q}})_{\mathfrak{q}\in S}$ of homotopies such that $h_{C,\mathfrak{q}} : \cup_c[1]\circ(\mathrm{id}\otimes\beta_{V_\mathfrak{q}^*(1),c}) \rightsquigarrow \cup_c[1]\circ(\beta_{V_\mathfrak{q},c}\otimes\mathrm{id})$ for $\mathfrak{q}\in\Sigma_p$ and $h_{C,\mathfrak{q}} : \cup_K[1]\circ(\mathrm{id}\otimes\beta_{K(V_\mathfrak{q}^*(1))}) \rightsquigarrow \cup_K[1]\circ(\beta_{K(V_\mathfrak{q})}\otimes\mathrm{id})$ for $\mathfrak{q}\in S_p$. For $\mathfrak{q}\in\Sigma_p$, the construction of $h_{C,\mathfrak{q}}$ is the same as those of h_A. Now, let $\mathfrak{q}\in S_p$. By Proposition 2.7.5, one has $\beta_{K(V_\mathfrak{q})}(x)=-(0,\log\chi_\mathfrak{q})\cup_K x$. Consider the following diagram, where $z_\mathfrak{q}=(0,\log\chi_\mathfrak{q})$

(67)
$$\begin{array}{ccc}
K^\bullet(V_\mathfrak{q})\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) & \xrightarrow{\mathrm{id}} & K^\bullet(V_\mathfrak{q})\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\
K^\bullet(V_\mathfrak{q})\otimes_A A\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) & \xrightarrow{\mathrm{id}} & A\otimes_A K^\bullet(V_\mathfrak{q})\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) \\
\big\downarrow{\scriptstyle \mathrm{id}\otimes(-z_\mathfrak{q})\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle (-z_\mathfrak{q})\otimes\mathrm{id}\otimes\mathrm{id}} \\
K^\bullet(V_\mathfrak{q})\otimes_A K^\bullet(A)[1]\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) & \xrightarrow{s_{12}\otimes\mathrm{id}} & K^\bullet(A)[1]\otimes_A K^\bullet(V_\mathfrak{q})\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) \\
\big\downarrow{\scriptstyle \cup_K\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle \cup_K\otimes\mathrm{id}} \\
K^\bullet(V_\mathfrak{q})[1]\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) & \xrightarrow{\mathrm{id}} & K^\bullet(V_\mathfrak{q})[1]\otimes_A K^\bullet(V_\mathfrak{q}^*(1)) \\
\big\downarrow{\scriptstyle \cup_K} & & \big\downarrow{\scriptstyle \cup_K} \\
K^\bullet(V_\mathfrak{q}\otimes V_\mathfrak{q}^*(1))[1] & \xrightarrow{\mathrm{id}} & K^\bullet(V_\mathfrak{q}\otimes V_\mathfrak{q}^*(1))[1].
\end{array}$$

The first, second and fourth squares of this diagram are commutative. From Proposition 1.1.7 (see also (37)) it follows that the diagram

$$\begin{array}{ccc}
K^\bullet(V_\mathfrak{q})\otimes K^\bullet(A)[1] & \xrightarrow{s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K)} & K^\bullet(A)[1]\otimes K^\bullet(V_\mathfrak{q}) \\
\big\downarrow{\scriptstyle \cup_K} & & \big\downarrow{\scriptstyle \cup_K} \\
K^\bullet(V_\mathfrak{q})[1] & \xrightarrow{\mathscr{T}_K} & K^\bullet(V_\mathfrak{q})[1]
\end{array}$$

is commutative up to some homotopy $k_1\,:\,\mathscr{T}_K\circ\cup_K\rightsquigarrow\cup_K\circ s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K)$. Since $\mathscr{T}_K^2=\mathrm{id}$, we have a homotopy

$$\mathscr{T}_K\circ k_1\,:\,\cup_K\rightsquigarrow\mathscr{T}_K\circ\cup_K\circ s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K).$$

By [**56**], Section 3.4.5.5 (see also Section 2.6.2), for any topological $G_{F_\mathfrak{q}}$-module M there exists a functorial homotopy $a\,:\,\mathrm{id}\rightsquigarrow\mathscr{T}_c$. By Proposition 2.6.4, a induces a homotopy between $\mathrm{id}\,:\,K^\bullet(V_\mathfrak{q})\to K^\bullet(V_\mathfrak{q})$ and $\mathscr{T}_K\,:\,K^\bullet(V_\mathfrak{q})\to K^\bullet(V_\mathfrak{q})$ which we denote by a_K. Let $(a_K\otimes a_K)_1\,:\,\mathrm{id}\rightsquigarrow\mathscr{T}_K\otimes\mathscr{T}_K$ denote the homotopy between the maps id and $\mathscr{T}_K\otimes\mathscr{T}_K\,:\,K^\bullet(V_\mathfrak{q})\otimes K^\bullet(\mathbf{Q}_p)[1]\to K^\bullet(V_\mathfrak{q})\otimes K^\bullet(\mathbf{Q}_p)[1]$ given by (18). Then

$$d(a_K\circ\mathscr{T}_K\circ\cup_K\circ s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K))+(a_K\circ\mathscr{T}_K\circ\cup_K\circ s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K))\,d=$$
$$=(\mathscr{T}_K-\mathrm{id})\circ\cup_K\circ s_{12}\circ(\mathscr{T}_K\otimes\mathscr{T}_K),$$

and

$$d(\cup_K \circ s_{12} \circ (a_K \otimes a_K)_1) + (\cup_K \circ s_{12} \circ (a_K \otimes a_K)_1) = \\ = \cup_K \circ s_{12} \circ (\mathscr{T}_K \otimes \mathscr{T}_K - \mathrm{id}).$$

Therefore the formula

$$k_2 = a_K \circ \mathscr{T}_K \circ \cup_K \circ s_{12} \circ (\mathscr{T}_K \otimes \mathscr{T}_K) + \cup_K \circ s_{12} \circ (a_K \otimes a_K)_1 \tag{68}$$

defines a homotopy

$$k_2 : \cup_K \circ s_{12} \rightsquigarrow \mathscr{T}_K \circ \cup_K \circ s_{12} \circ (\mathscr{T}_K \otimes \mathscr{T}_K).$$

Then $k_{C,\mathfrak{q}} = \mathscr{F}_K \circ k_1 - k_2$ defines a homotopy

$$k_{C,\mathfrak{q}} : \cup_K \rightsquigarrow \cup_K \circ s_{12}$$

and we proved that the third square of the diagram (67) commutes up to a homotopy. We define the homotopy

$$h_{C,\mathfrak{q}} : \cup_K[1] \circ (\mathrm{id} \otimes \beta_{K(V_\mathfrak{q}^*(1)),c}) \rightsquigarrow \cup_K[1] \circ (\beta_{K(V_\mathfrak{q})} \otimes \mathrm{id})$$

by

$$h_{C,\mathfrak{q}} = \cup_K \circ (k_{C,\mathfrak{q}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (-z_\mathfrak{q}) \otimes \mathrm{id}). \tag{69}$$

This proves **B3)**.

Since $u_1 = u_2 = h_f = 0$, condition **B4)** reads

$$dK_f - K_f d = -h_C \circ (f_1 \otimes f_2) + f_3[1] \circ h_A \tag{70}$$

for some second order homotopy K_f. It is proved in [**56**], Section 11.2.6, that if $\mathfrak{q} \in \Sigma_p$, then

$$h_{C,\mathfrak{q}} \circ (f_1 \otimes f_2) = \mathrm{res}_\mathfrak{q} \circ h_A. \tag{71}$$

Assume that $\mathfrak{q} \in S_p$. Recall (see [**56**], Section 11.2.6) that the homotopy h_A is given by

$$h_A = \cup_c \circ (k_A \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (-z) \otimes \mathrm{id}), \tag{72}$$

where $z = \log \chi$ and

$$k_A = -a \circ (\cup_c \circ s_{12} \circ (\mathscr{T}_c \otimes \mathscr{T}_c)) - (\mathscr{T}_c \circ \cup_c \circ s_{12}) \circ (a \otimes a)_1. \tag{73}$$

From (24), it follows that for all $x \in C^n(G_{F,S}, V)$ and $y \in C^m(G_{F,S}, V^*(1))$ we have

$$(k_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (-z_\mathfrak{q}) \otimes \mathrm{id}) \circ (f_1 \otimes f_2)(x \otimes y) = \\ = (k_1 \otimes \mathrm{id})((0, -\log \chi_\mathfrak{q}) \otimes (0, x_\mathfrak{q}) \otimes (0, y_\mathfrak{q})) = \\ = k_1((0, -\log \chi_\mathfrak{q}) \otimes (0, x_\mathfrak{q})) \otimes (0, y_\mathfrak{q}) = 0, \tag{74}$$

where $x_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(x)$, $y_{\mathfrak{q}} = \mathrm{res}_{\mathfrak{q}}(y)$. On the other hand, comparing (68) and (73) we see that

$$(75)\quad \begin{aligned}(k_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (-z_{\mathfrak{q}}) \otimes \mathrm{id}) \circ (f_1 \otimes f_2)(x \otimes y) = \\ = k_2((0, -\log \chi_{\mathfrak{q}}) \otimes (0, x_{\mathfrak{q}})) \otimes (0, y_{\mathfrak{q}}) = \\ = -(0, \mathrm{res}_{\mathfrak{q}}(k_A(-z \otimes x))) \otimes (0, y_{\mathfrak{q}}).\end{aligned}$$

From (74), (75), (69) and (73) we obtain that

$$(76)\quad \begin{aligned}h_{C,\mathfrak{q}} \circ (f_1 \otimes f_2)(x \otimes y) = \\ = (0, \mathrm{res}_{\mathfrak{q}}(k_A(-z \otimes x))) \cup_K (0, y_{\mathfrak{q}}) = \\ = (0, \mathrm{res}_{\mathfrak{q}}(k_A(-z \otimes x)) \cup_c y) = (0, \mathrm{res}_{\mathfrak{q}}(h_A(x \otimes y))).\end{aligned}$$

From (76) and (71) it follows that $h_C \circ (f_1 \otimes f_2) = f_3[1] \circ h_A$ and therefore we can set $K_f = 0$ in (70). Thus, **B4)** is proved.

It remains to check **B5)**. Since $v_1 = v_2 = h_g = 0$, this condition reads

$$(77)\qquad dK_g - K_g d = -h_C \circ (g_1 \otimes g_2) + \cup_{C[1]} \circ (v_1 \otimes g_2) - \cup_{C[1]} \circ (g_1 \otimes v_2)$$

for some second order homotopy K_g. Write $K_g = (K_{g,\mathfrak{q}})_{\mathfrak{q} \in S}$. For $\mathfrak{q} \in \Sigma_p$, Nekovář proved that the $\mathfrak{q}$-component of the right hand side of (77) is equal to zero. For $\mathfrak{q} \in S_p$, we have $v_{1,\mathfrak{q}} = v_{2,\mathfrak{q}} = 0$ and $h_{C,v} \circ (g_1 \otimes g_2) = 0$ because of orthogonality of $\mathbf{D}$ and $\mathbf{D}^{\perp}$, and again we can set $K_{g,\mathfrak{q}} = 0$. To sum up, condition (77) holds for $K_g = 0$. The proposition is proved. □

3.2.3. — The exact sequences (64), (65) and (66) give rise to a distinguished triangle

$$\mathbf{R}\Gamma(V, \mathbf{D}, M) \to \mathbf{R}\Gamma(\widetilde{V}_F, \widetilde{\mathbf{D}}_F, \widetilde{M}_F) \to \mathbf{R}\Gamma(V, \mathbf{D}, M) \xrightarrow{\delta_{V,\mathbf{D},M}} \mathbf{R}\Gamma(V, \mathbf{D}, M)[1]$$

***Definition*. —** *The p-adic height pairing associated to the data $(V, \mathbf{D}, M)$ is defined as the morphism*

$$\begin{aligned}h^{\mathrm{sel}}_{V,\mathbf{D},M} : \mathbf{R}\Gamma(V, \mathbf{D}, M) \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1), \mathbf{D}^{\perp}, M^{\perp}) \xrightarrow{\delta_{V,\mathbf{D},M}} \\ \to \mathbf{R}\Gamma(V, \mathbf{D}, M)[1] \otimes^{\mathbf{L}}_A \mathbf{R}\Gamma(V^*(1), \mathbf{D}^{\perp}, M^{\perp}) \xrightarrow{\cup_{V,\mathbf{D},M}} A[-2],\end{aligned}$$

where $\cup_{V,\mathbf{D},M}$ is the pairing (49).

The height pairing $h^{\mathrm{sel}}_{V,\mathbf{D},M}$ induces a pairing

$$(78)\qquad h^{\mathrm{sel}}_{V,\mathbf{D},M,1} : H^1(V, \mathbf{D}, M) \otimes_A H^1(V^*(1), \mathbf{D}^{\perp}, M^{\perp}) \to A.$$

Theorem 3.2.4. — *The diagram*

$$\begin{array}{ccc}
\mathbf{R}\Gamma(V,\mathbf{D},M)\otimes_A^{\mathbf{L}}\mathbf{R}\Gamma(V^*(1),\mathbf{D}^\perp,M^\perp) & \xrightarrow{h^{\mathrm{sel}}_{V,\mathbf{D},M}} & A[-2]\\
\downarrow{\scriptstyle s_{12}} & & \downarrow{\scriptstyle =}\\
\mathbf{R}\Gamma(V^*(1),\mathbf{D}^\perp,M^\perp)\otimes_A^{\mathbf{L}}\mathbf{R}\Gamma(V,\mathbf{D},M) & \xrightarrow{h^{\mathrm{sel}}_{V^*(1),\mathbf{D}^\perp,M^\perp}} & A[-2]
\end{array}$$

is commutative. In particular, the pairing $h^{\mathrm{sel}}_{V,\mathbf{D},1}$ is skew-symmetric.

Proof. — From Propositions 1.2.6 and 3.2.2 it follows, that the diagram

$$\begin{array}{ccc}
S^\bullet(V,\mathbf{D},M)\otimes_A S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp) & \xrightarrow{h^{\mathrm{sel}}_{V,\mathbf{D},M}} & E_3\\
\downarrow{\scriptstyle s_{12}\circ(\mathscr{T}^{\mathrm{sel}}_V\otimes\mathscr{T}^{\mathrm{sel}}_{V^*(1)})} & & \downarrow{\scriptstyle =}\\
S^\bullet(V^*(1),\mathbf{D}^\perp,M^\perp)\otimes_A S^\bullet(V,\mathbf{D},M) & \xrightarrow{h^{\mathrm{sel}}_{V^*(1),\mathbf{D}^\perp,M^\perp}} & E_3
\end{array}$$

is commutative up to homotopy. Now the theorem follows from the fact, that $(\mathscr{T}^{\mathrm{sel}}_V\otimes\mathscr{T}^{\mathrm{sel}}_{V^*(1)})$ is homotopic to the identity map (see the proof of Theorem 3.1.11). □

CHAPTER 4

SPLITTING SUBMODULES

4.1. Splitting submodules

4.1.1. — Let K be a finite extension of $\mathbf{Q}_p$, and let V be a potentially semistable representation of G_K with coefficients in a finite extension E of $\mathbf{Q}_p$. For each finite extension L/K we set $\mathbf{D}_{*/L}(V)=(\mathbf{B}_*\otimes V)^{G_L}$, where $*\in\{\mathrm{cris},\mathrm{st},\mathrm{dR}\}$ and write $\mathbf{D}_*(V)=\mathbf{D}_{*/K}(V)$ if $L=K$. We will use the same convention for the functors $\mathscr{D}_{*/L}$.

Fix a finite Galois extension L/K such that the restriction of V on G_L is semistable. Then $\mathbf{D}_{\mathrm{st}/L}(V)$ is a free filtered $(\varphi,N,G_{L/K})$-module over $E\otimes_{\mathbf{Q}_p}L_0$ and $\mathbf{D}_{\mathrm{dR}/L}(V)=\mathbf{D}_{\mathrm{st}/L}(V)\otimes_{L_0}L$. A $(\varphi,N,G_{L/K})$-submodule of $\mathbf{D}_{\mathrm{st}/L}(V)$ is a free $E\otimes_{\mathbf{Q}_p}L_0$-subspace D of $\mathbf{D}_{\mathrm{st}/L}(V)$ stable under the action of φ, N and $G_{L/K}$.

***Definition*. —** *We say that a $(\varphi,N,G_{L/K})$-submodule D of $\mathbf{D}_{\mathrm{st}/L}(V)$ is a splitting submodule if*

$$\mathbf{D}_{\mathrm{dR}/L}(V)=D_L\oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}/L}(V),\qquad D_L=D\otimes_{L_0}L$$

as $E\otimes_{\mathbf{Q}_p}L$-modules.

From this definition it follows that if D is a splitting submodule, then

$$D^{\perp}=\mathrm{Hom}_{E\otimes_{\mathbf{Q}_p}L_0}(\mathbf{D}_{\mathrm{st}/L}(V)/D,\mathbf{D}_{\mathrm{st}/L}(E(1))$$

is a splitting submodule of $\mathbf{D}_{\mathrm{st}/L}(V^*(1))$.

In Subsections 4.1–4.2 we will always assume that V satisfies the following condition:

S) $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}=0.$

One expects that this condition always holds for representations associated to pure motives of weight -1 (see Section 0.4). Namely, consider the Deligne–Jannsen monodromy filtration $\left(\mathfrak{M}_i\mathbf{D}_{\mathrm{st}/L}(V)\right)_{i\in\mathbf{Z}}$ on $\mathbf{D}_{\mathrm{st}/L}(V)$ given by

$$\mathfrak{M}_i\mathbf{D}_{\mathrm{st}/L}(V)=\sum_{k-l=i}\ker(N^{k+1})\cap\mathrm{Im}(N^l) \tag{79}$$

(see [**40**]). Denote by $\left(\mathrm{gr}_i^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V)\right)_{i\in\mathbf{Z}}$ its quotients. Assume for simplicity that $E=\mathbf{Q}_p$. Set $h=[L_0:\mathbf{Q}_p]$ and $q=p^h$. Then $\Phi=\varphi^h$ acts L_0-linearly on $\mathbf{D}_{\mathrm{st}/L}(V)$.

Lemma 4.1.2. — *Assume that Φ acts semisimply on $\mathbf{D}_{\mathrm{st}/L}(V)$ and that the absolute value of eigenvalues of Φ acting on $\mathrm{gr}_i^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V)$ is $q^{(i-1)/2}$. Then condition* **S)** *holds.*

Proof. — From our assumptions it follows that $\mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1}\cap\mathfrak{M}_0\mathbf{D}_{\mathrm{st}/L}(V)=0$. Since $\mathbf{D}_{\mathrm{cris}}(V)\subset\mathbf{D}_{\mathrm{st}/L}(V)^{N=0}\subset\mathfrak{M}_0\mathbf{D}_{\mathrm{st}/L}(V)$, this implies that

$$\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}\subset\mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1}\cap\mathbf{D}_{\mathrm{cris}}(V)=0.$$

Note that our assumptions are invariant under passing to the dual representation, and therefore we also get $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}=0$. □

4.1.3. — If D is a splitting submodule, we denote by $\mathbf{D}$ the (φ,Γ_K)-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ associated to D by Theorem 2.2.3. The natural embedding $\mathbf{D}\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ induces a map $H^1(\mathbf{D})\to H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))\overset{\sim}{\to}H^1(K,V)$. Passing to duals, we obtain a map $H^1(K,V^*(1))\to H^1(\mathbf{D}^*(\chi))$.

Proposition 4.1.4. — *Assume that V satisfies condition* **S)**. *Let D be a splitting submodule. Then*

i) $H^1_f(K,V^*(1))\to H^1_f(\mathbf{D}^*(\chi))$ is the zero map.

ii) $\mathrm{Im}(H^1(\mathbf{D})\to H^1(K,V))=H^1_f(K,V)$ and the map $H^1_f(\mathbf{D})\to H^1_f(K,V)$ is an isomorphism.

iii) If, in addition, $\mathrm{Fil}^0(\mathbf{D}_{\mathrm{st}/L}(V)/D)^{\varphi=1,N=0,G_{L/K}}=0$, then $H^1(\mathbf{D})=H^1_f(K,V)$.

Proof. — i) By Proposition 2.9.2 we have a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{cris}}(V^*(1)) & \longrightarrow & H^1_{\mathrm{cris}}(\mathbf{D}^*(\chi)) \\ \big\downarrow{=} & & \big\downarrow{=} \\ H^1_f(K,V^*(1)) & \longrightarrow & H^1_f(\mathbf{D}^*(\chi)), \end{array} \tag{80}$$

where we set

$$H^1_{\mathrm{cris}}(V^*(1))=\mathrm{coker}\left(\mathbf{D}_{\mathrm{cris}}(V)\xrightarrow{(1-\varphi,\mathrm{pr})}\mathbf{D}_{\mathrm{cris}}(V)\oplus t_V(K)\right)$$

and

$$H^1_{\mathrm{cris}}(\mathbf{D}^*(\chi)) = \mathrm{coker}\left(\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi)) \xrightarrow{(1-\varphi,\mathrm{pr})} \mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi))) \oplus t_{\mathbf{D}^*(\chi)}(K) \right)$$

to simplify notation.

Since $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} = 0$, the map $1-\varphi : \mathbf{D}_{\mathrm{cris}}(V^*(1)) \to \mathbf{D}_{\mathrm{cris}}(V^*(1))$ is an isomorphism and $H^1_{\mathrm{cris}}(V^*(1)) = t_{V^*(1)}(K)$. On the other hand, all Hodge–Tate weights of $\mathbf{D}^*(\chi)$ are $\geqslant 0$ and $t_{\mathbf{D}^*(\chi)}(K) = 0$. Hence

$$H^1_{\mathrm{cris}}(\mathbf{D}^*(\chi)) = \mathrm{coker}(1-\varphi \mid \mathscr{D}_{\mathrm{cris}}(\mathbf{D}^*(\chi)))$$

and the upper map in (80) is zero because it is induced by the canonical projection of $t_{V^*(1)}(K)$ on $t_{\mathbf{D}^*(\chi)}(K)$. This proves i).

Now we prove ii). Using i) together with the orthogonality property of H^1_f we obtain that the map

$$\mathrm{Hom}_E(H^1(K,V)/H^1_f(K,V),E) \to \mathrm{Hom}_E(H^1(\mathbf{D})/H^1_f(\mathbf{D}),E),$$

induced by $H^1(\mathbf{D}) \to H^1(K,V)$, is zero. This implies that the image of $H^1(\mathbf{D})$ is $H^1(K,V)$ is contained in $H^1_f(K,V)$. Finally one has a diagram

$$\begin{array}{ccc} H^1_{\mathrm{cris}}(\mathbf{D}) & \longrightarrow & H^1_{\mathrm{cris}}(V) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1_f(\mathbf{D}) & \longrightarrow & H^1_f(K,V). \end{array}$$

From **S)** it follows that the top arrow can be identified with the natural map $t_{\mathbf{D}}(K) \to t_V(K)$ which is an isomorphism by the definition of a splitting submodule.

iii) Taking into account ii), we only need to prove that the natural map $H^1(\mathbf{D}) \to H^1(K,V)$ is injective. This follows from the exact sequence

$$0 \to \mathbf{D} \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D}' \to 0, \qquad \mathbf{D}' = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/\mathbf{D}$$

and the fact that $H^0(\mathbf{D}') = \mathrm{Fil}^0(\mathbf{D}_{\mathrm{st}/L}(V)/D)^{\varphi=1,N=0,G_{L/K}} = 0$ (see Proposition 2.9.2, i)). □

4.2. The canonical splitting

4.2.1. — Let

$$y: \qquad 0 \to V^*(1) \to Y_y \to E \to 0$$

be an extension of E by $V^*(1)$.

Passing to (φ,Γ_K)-modules, we obtain an extension

$$0 \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y) \to \mathscr{R}_{K,E} \to 0.$$

By duality, we have exact sequences

$$0 \to E(1) \to Y_y^*(1) \to V \to 0$$

and

$$0 \to \mathscr{R}_{K,E}(\chi) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y^*(1)) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to 0.$$

We denote by $[y]$ the class of y in $\mathrm{Ext}^1_{E[G_K]}(E, V^*(1)) \xrightarrow{\sim} H^1(K, V^*(1))$. Assume that y is crystalline, i.e. that $[y] \in H^1_f(K, V^*(1))$. Let D be a splitting submodule of $\mathbf{D}_{\mathrm{st}/L}(V)$. Consider the commutative diagram

$$\begin{array}{rccccccccc}
y: & 0 & \longrightarrow & \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)) & \longrightarrow & \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y) & \longrightarrow & \mathscr{R}_{K,E} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & & \downarrow{\scriptstyle =} & & \\
\mathrm{pr}(y): & 0 & \longrightarrow & \mathbf{D}^*(\chi) & \longrightarrow & \mathbf{D}^*_y(\chi) & \longrightarrow & \mathscr{R}_{K,E} & \longrightarrow & 0
\end{array}$$

where $\mathbf{D}_y$ is the inverse image of $\mathbf{D}$ in $\mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y^*(1))$. The class of $\mathrm{pr}(y)$ in $H^1(\mathbf{D}^*(\chi))$ is the image of $[y]$ under the map

$$\mathrm{Ext}^1(\mathscr{R}_{K,E}, \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))) \to \mathrm{Ext}^1(\mathscr{R}_{K,E}, \mathbf{D}^*(\chi))$$

which coincides with the map

$$H^1(K, V^*(1)) = H^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))) \to H^1(\mathbf{D}^*(\chi))$$

after the identification of $\mathrm{Ext}^1(\mathscr{R}_{K,E}, -)$ with the cohomology group $H^1(-)$. Since we are assuming that $[y] \in H^1_f(K, V^*(1))$, by Proposition 4.1.4 i), we obtain that $[\mathrm{pr}(y)] = 0$. Thus the sequence $\mathrm{pr}(y)$ splits.

4.2.2. — We will construct a canonical splitting of $\mathrm{pr}(y)$ using the idea of Nekovář [**54**]. Since $\dim_E \mathbf{D}_{\mathrm{cris}}(Y_y) = \dim_E \mathbf{D}_{\mathrm{cris}}(V^*(1)) + 1$, the sequence

$$0 \to \mathbf{D}_{\mathrm{cris}}(V^*(1)) \to \mathbf{D}_{\mathrm{cris}}(Y_y) \to \mathbf{D}_{\mathrm{cris}}(E) \to 0$$

is exact by the dimension argument. From $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1} = 0$ and the snake lemma it follows that $\mathbf{D}_{\mathrm{cris}}(Y_y)^{\varphi=1} = \mathbf{D}_{\mathrm{cris}}(E)$ and we obtain a canonical φ-equivariant morphism of K_0-vector spaces $\mathbf{D}_{\mathrm{cris}}(E) \to \mathbf{D}_{\mathrm{cris}}(Y_y)$. By linearity, this map extends to a $(\varphi, N, G_{L/K})$-equivariant morphism of L_0-vector spaces $\mathbf{D}_{\mathrm{st}/L}(E) \to \mathbf{D}_{\mathrm{st}/L}(Y_y)$. Therefore we have a canonical splitting

$$\mathbf{D}_{\mathrm{st}/L}(Y_y) \xrightarrow{\sim} \mathbf{D}_{\mathrm{st}/L}(V^*(1)) \oplus \mathbf{D}_{\mathrm{st}/L}(E)$$

of the sequence

$$0 \to \mathbf{D}_{\mathrm{st}/L}(V^*(1)) \to \mathbf{D}_{\mathrm{st}/L}(Y_y) \to \mathbf{D}_{\mathrm{st}/L}(E) \to 0$$

in the category of $(\varphi,N,G_{L/K})$-modules. This splitting induces a $(\varphi,N,G_{L/K})$-equivariant isomorphism

$$\mathscr{D}_{\mathrm{st}/L}(\mathbf{D}^*_y(\chi)) \xrightarrow{\sim} \mathscr{D}_{\mathrm{st}/L}(\mathbf{D}^*(\chi)) \oplus \mathscr{D}_{\mathrm{st}/L}(E). \tag{81}$$

Moreover, since all Hodge–Tate weights of $\mathbf{D}^*(\chi)$ are positive, we have

$$\mathrm{Fil}^i \mathscr{D}_{\mathrm{dR}/L}(\mathbf{D}^*_y(\chi)) \xrightarrow{\sim} \mathrm{Fil}^i \mathscr{D}_{\mathrm{dR}/L}(\mathbf{D}^*(\chi)) \oplus \mathrm{Fil}^i \mathscr{D}_{\mathrm{dR}/L}(E)$$

and therefore the isomorphism

$$\mathscr{D}_{\mathrm{dR}/L}(\mathbf{D}^*_y(\chi)) \xrightarrow{\sim} \mathscr{D}_{\mathrm{dR}/L}(\mathbf{D}^*(\chi)) \oplus \mathscr{D}_{\mathrm{dR}/L}(E)$$

is compatible with filtrations. Thus, we obtain that (81) is an isomorphism in the category of filtered $(\varphi,N,G_{L/K})$-modules. This gives a canonical splitting

$$\mathrm{pr}(y): \qquad 0 \longrightarrow \mathbf{D}^*(\chi) \longrightarrow \mathbf{D}^*_y(\chi) \overset{\longleftarrow}{\longrightarrow} \mathscr{R}_{K,E} \longrightarrow 0$$

of the extension $\mathrm{pr}(y)$. Passing to duals, we obtain a splitting

$$0 \longrightarrow \mathscr{R}_{K,E}(\chi) \longrightarrow \mathbf{D}_y \overset{s_{\mathbf{D},y}}{\underset{\longrightarrow}{\longleftarrow}} \mathbf{D} \longrightarrow 0. \tag{82}$$

Taking cohomology, we get a splitting

$$0 \longrightarrow H^1_f(K,E(1)) \longrightarrow H^1_f(\mathbf{D}_y) \overset{s_y}{\underset{\longrightarrow}{\longleftarrow}} H^1_f(\mathbf{D}) \longrightarrow 0. \tag{83}$$

Our constructions can be summarized in the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1_f(K,E(1)) & \longrightarrow & H^1_f(\mathbf{D}_y) & \overset{s_y}{\underset{\longrightarrow}{\longleftarrow}} & H^1_f(\mathbf{D}) & \longrightarrow 0 \\
& \downarrow = & & \downarrow \simeq & & \downarrow \simeq & \\
0 \longrightarrow & H^1_f(K,E(1)) & \longrightarrow & H^1_f(K,Y^*_y(1)) & \longrightarrow & H^1_f(K,V) & \longrightarrow 0.
\end{array}$$

Here the vertical maps are isomorphisms by Proposition 4.1.4 and the five lemma.

Remark 4.2.3. — Assume that $H^0(\mathbf{D}^*(\chi)) = 0$. Then each crystalline extension of $\mathbf{D}$ by $\mathscr{R}_K(\chi)$ splits uniquely. This follows from Proposition 2.9.2 i) which implies that $H^1_f(\mathbf{D}^*(\chi)) = 0$ and from the fact that various splittings are parametrized by $H^0(\mathbf{D}^*(\chi))$.

4.3. Filtration associated to a splitting submodule

4.3.1. — In this subsection we assume that $K = \mathbf{Q}_p$. Let V be a potentially semistable representation of $G_{\mathbf{Q}_p}$ with coefficients in a finite extension E of $\mathbf{Q}_p$. As before, we fix a finite Galois extension $L/\mathbf{Q}_p$ such that V is semistable over L and denote by $\mathbf{D}_{\mathrm{st}/L}(V)$ the semistable module of the restriction of V on G_L. Let $G_{L/\mathbf{Q}_p} = \mathrm{Gal}(L/\mathbf{Q}_p)$. To each splitting submodule D of $\mathbf{D}_{\mathrm{st}/L}(V)$ we associate a canonical filtration on $\mathbf{D}_{\mathrm{st}/L}(V)$ which is a direct generalization of the filtration constructed by Greenberg [**35**] in the ordinary case and in [**7**] in the semistable case.

Let D be a splitting submodule of $\mathbf{D}_{\mathrm{st}/L}(V)$. Set $D' = \mathbf{D}_{\mathrm{st}/L}(V)/D$. Then $\mathrm{Fil}^0 D' = D'$ and we define

$$M_1 = (D')^{\varphi=1, N=0, G_{L/\mathbf{Q}_p}} \otimes_{\mathbf{Q}_p} L_0.$$

Recall that $D^\perp = \mathrm{Hom}_{E\otimes_{\mathbf{Q}_p} L_0}(\mathbf{D}_{\mathrm{st}/L}(V)/D, \mathbf{D}_{\mathrm{st}/L}(E(1))$ and that in the tautological exact sequence

$$0 \to D^\perp \to \mathbf{D}_{\mathrm{st}/L}(V^*(1)) \to (D^\perp)' \to 0$$

we have

$$(D^\perp)' \simeq D^* = \mathrm{Hom}_{E\otimes_{\mathbf{Q}_p} L_0}(D, \mathbf{D}_{\mathrm{st}/L}(E(1)).$$

For the filtered $(\varphi, N, G_{L/\mathbf{Q}_p})$-module D^* we have $\mathrm{Fil}^0 D^* = D^*$ and we define

$$M_0 = \left((D^*)^{\varphi=1, N=0, G_{L/\mathbf{Q}_p}} \otimes_{\mathbf{Q}_p} L_0 \right)^*.$$

From Lemma 4.4.2 ii) it follows that M_1 can be seen as a submodule of D' and that M_0^* can be seen as a submodule of $(D^\perp)'$. Clearly we have

$$\mathrm{rk}_{E\otimes_{\mathbf{Q}_p} L_0}(M_1) = \dim_E (D')^{\varphi=1, N=0, G_{L/\mathbf{Q}_p}}, \quad \mathrm{rk}_{E\otimes_{\mathbf{Q}_p} L_0}(M_0) = \dim_E (D^*)^{\varphi=1, N=0, G_{L/\mathbf{Q}_p}}.$$

We have canonical projections $\mathrm{pr}_{D'} : \mathbf{D}_{\mathrm{st}/L}(V) \to D'$ and $\mathrm{pr}_{M_0} : D \to M_0$. Define a five-step filtration

$$\{0\} = F_{-2}\mathbf{D}_{\mathrm{st}/L}(V) \subset F_{-1}\mathbf{D}_{\mathrm{st}/L}(V) \subset F_0\mathbf{D}_{\mathrm{st}/L}(V) \subset F_1\mathbf{D}_{\mathrm{st}/L}(V) \subset F_2\mathbf{D}_{\mathrm{st}/L}(V) = \mathbf{D}_{\mathrm{st}/L}(V)$$

by

$$F_i\mathbf{D}_{\mathrm{st}/L}(V) = \begin{cases} \ker(\mathrm{pr}_{M_0}) & \text{if } i = -1, \\ D & \text{if } i = 0, \\ \mathrm{pr}_{D'}^{-1}(M_1) & \text{if } i = 1. \end{cases}$$

Set $W=F_1\mathbf{D}_{\mathrm{st}/L}(V)/F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)$. These data can be represented by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \longrightarrow & \mathbf{D}_{\mathrm{st}/L}(V) & \xrightarrow{\mathrm{pr}_{D'}} & D' & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mathrm{pr}_{M_0}} & & & & \Big\uparrow & & \\
0 & \longrightarrow & M_0 & \longrightarrow & W & \longrightarrow & M_1 & \longrightarrow & 0.
\end{array}$$

We denote by $\left(\mathrm{gr}_i\mathbf{D}_{\mathrm{st}/L}(V)\right)_{i=-2}^{2}$ the quotients of the filtration $(F_i\mathbf{D}_{\mathrm{st}/L}(V))_{i=-2}^{2}$. Thus, $M_0=\mathrm{gr}_0\mathbf{D}_{\mathrm{st}/L}(V)$ and $M_1=\mathrm{gr}_1\mathbf{D}_{\mathrm{st}/L}(V)$. By Theorem 2.2.3, the filtration $\left(F_i\mathbf{D}_{\mathrm{st}/L}(V)\right)_{i=-2}^{2}$ induces a filtration $\left(F_i\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)\right)_{i=-2}^{2}$ on the $(\varphi,\Gamma_{\mathbf{Q}_p})$-module $\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ such that

$$\mathscr{D}_{\mathrm{st}/L}(F_i\mathbf{D}_{\mathrm{rig}}^{\dagger}(V))=F_i\mathbf{D}_{\mathrm{st}/L}(V),\qquad -2\leqslant i\leqslant 2.$$

Note that $\mathbf{D}=F_0\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$. We set $\mathbf{M}_0=\mathrm{gr}_0\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$, $\mathbf{M}_1=\mathrm{gr}_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ and $\mathbf{W}=F_1\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)/F_{-1}\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$. We have a tautological exact sequence

$$0\to\mathbf{M}_0\xrightarrow{\alpha}\mathbf{W}\xrightarrow{\beta}\mathbf{M}_1\to 0. \tag{84}$$

By construction, $\mathbf{M}_0$ and $\mathbf{M}_1$ are crystalline $(\varphi,\Gamma_{\mathbf{Q}_p})$-modules such that

$$\mathscr{D}_{\mathrm{cris}/\mathbf{Q}_p}(\mathbf{M}_0)=M_0,\qquad \mathscr{D}_{\mathrm{cris}/\mathbf{Q}_p}(\mathbf{M}_1)=M_1.$$

Since

$$\begin{aligned}
&M_0^{\varphi=p^{-1}}=M_0, &&\mathrm{Fil}^0M_0=0,\\
&M_1^{\varphi=1}=M_1, &&\mathrm{Fil}^0M_1=M_1,
\end{aligned}$$

the structure of modules $\mathbf{M}_0$ and $\mathbf{M}_1$ is given by Proposition 2.9.4. In particular, we have canonical decompositions

$$H^1(\mathbf{M}_0)\overset{(\mathrm{pr}_f,\mathrm{pr}_c)}{\simeq}H^1_f(\mathbf{M}_0)\oplus H^1_c(\mathbf{M}_0),\qquad H^1(\mathbf{M}_1)\overset{(\mathrm{pr}_f,\mathrm{pr}_c)}{\simeq}H^1_f(\mathbf{M}_1)\oplus H^1_c(\mathbf{M}_1).$$

The exact sequence (84) induces the cobondary map $\delta_0\,:\,H^0(\mathbf{M}_1)\to H^1(\mathbf{M}_0)$. Passing to cohomology in the dual exact sequence

$$0\to\mathbf{M}_1^*(\chi)\to\mathbf{W}^*(\chi)\to\mathbf{M}_0^*(\chi)\to 0, \tag{85}$$

we obtain the coboundary map $\delta_0^*\,:\,H^0(\mathbf{M}_0^*(\chi))\to H^1(\mathbf{M}_1^*(\chi))$.

4.3.2. — We keep previous notation and denote by $(F_i\mathbf{D}_{\mathrm{st}/L}(V^*(1)))_{-2\leqslant i\leqslant 2}$ the filtration on $\mathbf{D}_{\mathrm{st}/L}(V^*(1))$ associated to $D^\perp$. This filtration is dual to the filtration $F_i\mathbf{D}_{\mathrm{st}/L}(V)$. In particular,

$$F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))^*(\chi)\simeq \mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V), \tag{86}$$

$$\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))\simeq (F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^*(\chi), \tag{87}$$

and the sequence (84) for $(V^*(1),D^\perp)$ coincides with (85).

We consider the following conditions on (V,D) :

F1a) $\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=1}=\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))^{\varphi=1}=0.$

F1b) $\mathscr{D}_{\mathrm{cris}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=1}=\mathscr{D}_{\mathrm{cris}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))^{\varphi=1}=0.$

F2a) The composed map

$$\delta_{0,c}\,:\,H^0(\mathbf{M}_1)\xrightarrow{\delta_0}H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_c}H^1_c(\mathbf{M}_0),$$

where the second arrow denotes the canonical projection on $H^1_c(\mathbf{M}_0)$, is an isomorphism.

F2b) The composed map

$$\delta_{0,f}\,:\,H^0(\mathbf{M}_1)\xrightarrow{\delta_0}H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_f}H^1_f(\mathbf{M}_0),$$

where the second arrows denotes the canonical projection $H^1_f(\mathbf{M}_0)$, are isomorphisms.

F3) For all $i\in\mathbf{Z}$

$$\mathscr{D}_{\mathrm{pst}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=p^i}=\mathscr{D}_{\mathrm{pst}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=p^i}=0.$$

We expect that conditions **F1a-b)** and **F2a-b)** hold for p-adic representations arising from pure motives over $\mathbf{Q}$ of weight -1 (see Sections 4.3.4-4.3.11). On the other hand, it is easy to give an example of a motive for which condition **F3)** does not hold (see Remark 4.3.3.5) below.

***Remarks 4.3.3.* —** 1) Since for any potentially semistable $(\varphi,\Gamma_{\mathbf{Q}_p})$-module $\mathbf{X}$ one has $H^0(\mathbf{X})=\mathrm{Fil}^0\mathscr{D}_{\mathrm{cris}}(\mathbf{X})^{\varphi=1}$ and the Hodge–Tate weights of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ and $\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1))$ are $\geqslant 0$, condition **F1a)** is equivalent to

$$H^0(\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))=H^0(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1)))=0.$$

2) All conditions introduced above are preserved under duality.

3) From (86-87) it follows that **F3)** implies **F1a-b)**.

4) **F1a-b)** and **F2a)** imply condition **S)** introduced in Section 4.1 (see Proposition 4.3.13 iv) below).

5) We give a simple example of a p-adic representation arising from a motive of weight -1 which does not satisfy condition **F3)**. Let $V(E)$ be the p-adic representation associated to an elliptic curve $E/\mathbf{Q}$ having split multiplicative reduction at p. The restriction of $V(E)$ on the decomposition group at p sits in an exact sequence

$$0 \to \mathbf{Q}_p(1) \to V_p(E) \to \mathbf{Q}_p \to 0.$$

Then $\mathbf{D}_{\mathrm{st}}(V_p(E))$ is generated by two vectors e_α and e_β such that $N(e_\beta) = e_\alpha$, $\varphi(e_\alpha) = p^{-1}e_\alpha$, $\varphi(e_\beta) = e_\beta$ and $\mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(1)) = \mathbf{Q}_p e_\alpha$. Let $W = V(E)^{\otimes 3}(-1)$. Then $\mathbf{D}_{\mathrm{st}}(W_p) = \mathbf{D}_{\mathrm{st}}(V_p(E))^{\otimes 3}[1]$, where [1] denotes the $\otimes$-multiplication by the canonical generator of $\mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(-1))$.

It is easy to see that the $\mathbf{Q}_p$-vector space generated by the vectors

$$d_0 = (e_\alpha \otimes e_\alpha \otimes e_\alpha)[1], \quad d_1 = (e_\beta \otimes e_\alpha \otimes e_\alpha)[1],$$
$$d_2 = (e_\alpha \otimes e_\beta \otimes e_\alpha)[1], \quad d_0 = (e_\alpha \otimes e_\alpha \otimes e_\beta)[1]$$

is a splitting submodule of $\mathbf{D}_{\mathrm{st}}(W_p)$. Since $\varphi(d_0) = p^{-2}d_0$ and $\varphi(d_i) = p^{-1}d_i$ for $1 \leqslant i \leqslant 3$, we have $F_{-1}\mathbf{D}_{\mathrm{st}}(W_p) = \mathbf{Q}_p d_0$. An easy computation shows that $F_1\mathbf{D}_{\mathrm{st}}(W_p)$ is the six-dimensional subspace generated by $(d_i)_{1\leqslant i\leqslant 3}$ and $(d_i^+)_{1\leqslant i\leqslant 3}$, where

$$d_1^+ = (e_\alpha \otimes e_\beta \otimes e_\beta)[1], \quad d_2^+ = (e_\beta \otimes e_\alpha \otimes e_\beta)[1], \quad d_3^+ = (e_\beta \otimes e_\beta \otimes e_\alpha)[1].$$

Thus, $\mathbf{D}_{\mathrm{st}}(W_p)/F_1\mathbf{D}_{\mathrm{st}}(W_p) \simeq \mathbf{D}_{\mathrm{st}}(\mathbf{Q}_p)$ and $F_{-1}\mathbf{D}_{\mathrm{st}}(W_p) \simeq \mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(2))$ and condition **F3)** fails.

4) If V is semistable over $\mathbf{Q}_p$, and the linear map $\varphi : \mathbf{D}_{\mathrm{st}}(V) \to \mathbf{D}_{\mathrm{st}}(V)$ is semisimple at 1 and p^{-1}, the filtration $F_i\mathbf{D}_{\mathrm{st}}(V)$ coincides with the filtration defined in [**7**, Section 2.1.4] (see Proposition 4.3.5 below).

4.3.4. — In the next two sections we show that conditions **F1a-b)** and **F2a)** hold if the Frobenius operator acts semisimply on $\mathbf{D}_{\mathrm{st}/L}(V)$ and V satisfies the p-adic monodromy-weight conjecture. To simplify the exposition, we assume that the coefficient field $E = \mathbf{Q}_p$.

Let W be a finite-dimensional vector space over a vector space K. If f is a linear operator on W, then for each field extension K'/K we denote by the same letter f the linear extension of f to $W_{K'} = W \otimes_K K'$. If $\alpha \in K'$, we say that f is semisimple at α if

$$W_{K'} = (f - \alpha)W_{K'} \oplus W_{K'}^{f=\alpha}.$$

Note that f is semisimple if and only if it is semisimple at all its eigenvalues. Let

$$0 \to W_1 \to W \to W_2 \to 0$$

be an exact seguence of K-vector spaces equipped with compatible linear actions of f. If the action of f on W is semisimple at $\alpha \in K$, then the actions of f on W_1 and W_2

are semisimple at α and the sequence

$$0 \to W_1^{f=\alpha} \to W^{f=\alpha} \to W_2^{f=\alpha} \to 0 \tag{88}$$

is exact.

Let G be a finite group acting on W. Then W decomposes canonically into the direct sum $W = W^G \oplus W^0$, where $W^0 = \{w \in W | \mathrm{Tr}_G(w) = 0\}$. If

$$0 \to W_1 \to W \to W_2 \to 0$$

is an exact sequence of $K[G]$-modules, then the induced sequence of G-invariants

$$0 \to W_1^G \to W^G \to W_2^G \to 0 \tag{89}$$

is exact. In particular, the inertia subgroup $I_{L/\mathbf{Q}_p}$ acts on the splitting submodule D and we have

$$D = D^{I_{L/\mathbf{Q}_p}} \oplus D^0.$$

***Proposition 4.3.5.* —** *Let V be a potentially semistable representation of $G_{\mathbf{Q}_p}$ and let $L/\mathbf{Q}_p$ be a finite Galois extension such that V becomes semistable over L. Assume that $\varphi : \mathbf{D}_{\mathrm{st}/L}(V) \to \mathbf{D}_{\mathrm{st}/L}(V)$ is semisimple at 1 and p^{-1}. Then*

i) The filtration $(F_i\mathbf{D}_{\mathrm{st}/L}(V))_{i=-2}^{2}$ is explicitly given by

$$F_i\mathbf{D}_{\mathrm{st}/L}(V) = \begin{cases} D^0 + \left((1-p^{-1}\varphi^{-1})D^{G_{L/\mathbf{Q}_p}} + N(D^{G_{L/\mathbf{Q}_p}})\right) \otimes_{\mathbf{Q}_p} L_0 & \text{if } i=-1, \\ D & \text{if } i=0, \\ D + \left(\mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1, G_{L/\mathbf{Q}_p}} \cap N^{-1}(D^{\varphi=p^{-1}})\right) \otimes_{\mathbf{Q}_p} L_0 & \text{if } i=1. \end{cases}$$

ii) We have

$$M_0 = \left(\frac{D^{G_{L/\mathbf{Q}_p}, \varphi=p^{-1}}}{N(D^{G_{L/\mathbf{Q}_p}, \varphi=1})}\right) \otimes_{\mathbf{Q}_p} L_0, \quad M_1 = \left(\frac{\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p}, \varphi=1} \cap N^{-1}(D)}{D^{G_{L/\mathbf{Q}_p}, \varphi=1}}\right) \otimes_{\mathbf{Q}_p} L_0.$$

iii) Condition **F1a)** *holds.*

Proof. — i) Since φ is semisimple at 1, from the definition of M_1 and properties (88 - 89) it follows that

$$\begin{aligned} F_1\mathbf{D}_{\mathrm{st}/L}(V) &= D + \left(\mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1, G_{L/\mathbf{Q}_p}} \cap N^{-1}(D)\right) \otimes_{\mathbf{Q}_p} L_0 \\ &= D + \left(\mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1, G_{L/\mathbf{Q}_p}} \cap N^{-1}(D^{\varphi=p^{-1}})\right) \otimes_{\mathbf{Q}_p} L_0 \end{aligned}$$

Let $D^\perp$ be the orthogonal complement of D under the canonical pairing

$$[\,,\,] \ : \ \mathbf{D}_{\mathrm{st}/L}(V) \times \mathbf{D}_{\mathrm{st}/L}(V^*(1)) \to \mathbf{D}_{\mathrm{st}/L}(\mathbf{Q}_p(1))$$

and let $\left(F_i\mathbf{D}_{\mathrm{st}/L}(V^*(1))\right)_{i=-2}^2$ denote the associated filtration. Then $F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)$ is the orthogonal complement of $F_1\mathbf{D}_{\mathrm{st}/L}(V^*(1))$ under $[\ ,\ ]$ and we have

$$\begin{aligned} F_{-1}\mathbf{D}_{\mathrm{st}/L}(V) &= \left(D^{\perp}+\left(\mathbf{D}_{\mathrm{st}/L}(V^*(1))^{\varphi=1,G_{L/\mathbf{Q}_p}}\cap N^{-1}(D^{\perp})\right)\otimes_{\mathbf{Q}_p}L_0\right)^{\perp}= \\ &= D\cap\left(N^{-1}(D^{\perp})^{\perp}+\left(\mathbf{D}_{\mathrm{st}/L}(V^*(1))^{\varphi=1,G_{L/\mathbf{Q}_p}}\otimes_{\mathbf{Q}_p}L_0\right)^{\perp}\right).\end{aligned}$$

If $f\in N^{-1}(D^{\perp})$ and $x\in\mathbf{D}_{\mathrm{st}/L}(V)$, then $f(Nx)=(Nf)(x)$, where $Nf\in D^{\perp}$. This implies that $N^{-1}(D^{\perp})^{\perp}=N(D)$. Since $N(D)\subset D$, we get

$$F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)=N(D)+D\cap\left(\mathbf{D}_{\mathrm{st}/L}(V^*(1))^{\varphi=1,G_{L/\mathbf{Q}_p}}\otimes_{\mathbf{Q}_p}L_0\right)^{\perp}. \tag{90}$$

From Lemma 4.4.7 we have that

$$\begin{aligned}\left(\mathbf{D}_{\mathrm{st}/L}(V^*(1))^{\varphi=1,G_{L/\mathbf{Q}_p}}\otimes_{\mathbf{Q}_p}L_0\right)^{\perp}&= \\ &=\left((1-p^{-1}\varphi^{-1})\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p}}\right)\otimes_{\mathbf{Q}_p}L_0+\mathbf{D}_{\mathrm{st}/L}(V)^0.\end{aligned} \tag{91}$$

Set $X=D\cap\left((1-p^{-1}\varphi^{-1})\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p}}\right)\otimes_{\mathbf{Q}_p}L_0$. Since X is an L_0-vector space equipped with a semilinear action of $\mathrm{Gal}(L_0/\mathbf{Q}_p)$, by Hilbert's Theorem 90

$$X=X^{G_{L/\mathbf{Q}_p}}\otimes_{\mathbf{Q}_p}L_0=\left(D^{G_{L/\mathbf{Q}_p}}\cap\left((1-p^{-1}\varphi^{-1})\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p}}\right)\right)\otimes_{\mathbf{Q}_p}L_0.$$

Since φ is semisimple at p^{-1}, we have

$$D^{G_{L/\mathbf{Q}_p}}\cap\left((1-p^{-1}\varphi^{-1})\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p}}\right)=(1-p^{-1}\varphi^{-1})D^{G_{L/\mathbf{Q}_p}}.$$

Together with (90) and (91) this gives

$$F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)=\left((1-p^{-1}\varphi^{-1})D^{G_{L/\mathbf{Q}_p}}\right)\otimes_{\mathbf{Q}_p}L_0+N(D)+D^0.$$

Write

$$D=\left(D^{G_{L/\mathbf{Q}_p}}\otimes_{\mathbf{Q}_p}L_0\right)\oplus D^0 \tag{92}$$

and

$$D^{G_{L/\mathbf{Q}_p}}=D^{G_{L/\mathbf{Q}_p},\varphi=1}\oplus\left((1-\varphi)D^{G_{L/\mathbf{Q}_p}}\right).$$

Then $N(D^0)\subset D^0$ and

$$N\left(D^{G_{L/\mathbf{Q}_p}}\right)=N\left(D^{G_{L/\mathbf{Q}_p},\varphi=1}\right)\oplus\left((1-p^{-1}\varphi^{-1})N\left(D^{G_{L/\mathbf{Q}_p},\varphi=1}\right)\right).$$

Therefore

$$F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)=\left((1-p^{-1}\varphi^{-1})D^{G_{L/\mathbf{Q}_p}}\right)\otimes_{\mathbf{Q}_p}L_0+N\left(D^{G_{L/\mathbf{Q}_p},\varphi=1}\right)\otimes_{\mathbf{Q}_p}L_0+D^0 \tag{93}$$

and i) is proved.

ii) From the definition of M_1 and the semisimplicity of φ at 1 if follows immediately that

$$M_1 = \left(\frac{\mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p},\varphi=1} \cap N^{-1}(D)}{D^{G_{L/\mathbf{Q}_p},\varphi=1}} \right) \otimes_{\mathbf{Q}_p} L_0.$$

Using (93), the decomposition (92) and the semisimplicity of φ at p^{-1} we have

$$\begin{aligned} M_0 &= \frac{D^{G_{L/\mathbf{Q}_p}}}{\left((1-p^{-1}\varphi^{-1})D^{G_{L/\mathbf{Q}_p}}\right) + N\left(D^{G_{L/\mathbf{Q}_p},\varphi=1}\right)} \otimes_{\mathbf{Q}_p} L_0 = \\ &= \left(\frac{D^{G_{L/\mathbf{Q}_p},\varphi=p^{-1}}}{N\left(D^{G_{L/\mathbf{Q}_p},\varphi=1}\right)} \right) \otimes_{\mathbf{Q}_p} L_0 \end{aligned}$$

and ii) is proved.

iii) The statement iii) follows from ii). The proof repeats *verbatim* the proof of the property **D2)** from [**7**, Lemma 2.1.5]. □

4.3.6. — Set $h = [L_0 : \mathbf{Q}_p]$, $q = p^h$ and $\Phi = \varphi^h$. Then Φ is an L_0-linear operator on $\mathbf{D}_{\mathrm{st}/L}(V)$. Let $\mathfrak{M}_i\mathbf{D}_{\mathrm{st}/L}(V)$ denote the Deligne –Jannsen monodromy filtration (79). By [**24**, Section 1.6], the monodromy N induces an isomorphism

$$\overline{N}^{\mathfrak{M}} : \mathrm{gr}_1^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V) \to \mathrm{gr}_{-1}^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V). \tag{94}$$

***Proposition 4.3.7.* —** *i) Assume that $\Phi : \mathbf{D}_{\mathrm{st}/L}(V) \to \mathbf{D}_{\mathrm{st}/L}(V)$ is semisimple at 1 and q^{-1}. Then φ is semisimple at 1 and p^{-1}.*

ii) If, in addition, for all $i \in \mathbf{Z}$ the absolute value of eigenvalues of Φ acting on $\mathrm{gr}_i^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V)$ is $q^{(i-1)/2}$, then conditions **F1a-b)**, **F2a)** *and* **S)** *hold.*

Proof. — i) This is a particular case of Proposition 4.4.5.

ii) The proof will be divided into several steps.

a) From the semisimplicity of φ and Proposition 4.3.5 iii) it follows that **F1a)** holds. Next, **S)** holds by Lemma 4.1.2. Since **S)** implies **F1b)**, we only need to show that **F2a)** holds.

b) From the semisimplicity of Φ and our assumption about the action of Φ on $\mathrm{gr}_1^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}}(V)$, it follows that the canonical inclusions induce isomorphisms

$$\mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1} \simeq \mathrm{gr}_1^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V), \quad \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=q^{-1}} \simeq \mathrm{gr}_{-1}^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V).$$

Using (94), we see that the operator N induces an isomorphism

$$N : \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1} \to \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=q^{-1}}.$$

Since $\mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1} \subset \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1}$ and $N(\mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1}) \subset \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=p^{-1}}$, the map $N : \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1} \to \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=p^{-1}}$ is injective. Let $y \in \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=p^{-1}}$. Then there exists $x \in \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=1}$ such that $N(x) = y$. Set $z = \varphi(x) - x$. Then

$$N(z) = N(\varphi(x)) - N(x) = p\varphi N(x) - N(x) = 0$$

and therefore $z \in \mathbf{D}_{\mathrm{st}/L}(V)^{\Phi=0,N=0} = \{0\}$. This implies that $x \in \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1}$ and we proved that the map

$$N : \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=1} \to \mathbf{D}_{\mathrm{st}/L}(V)^{\varphi=p^{-1}}.$$

is an isomorphism of $\mathbf{Q}_p$-vector spaces. Taking $G_{L/\mathbf{Q}_p}$-invariants, we also get an isomorphism (which we denote by the same letter N)

$$N : \mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p},\varphi=1} \to \mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p},\varphi=p^{-1}}. \tag{95}$$

c) From Proposition 4.3.13, we have

$$M_1 = \left(N^{-1}(D) \cap \mathbf{D}_{\mathrm{st}/L}(V)^{G_{L/\mathbf{Q}_p},\varphi=1} / D^{G_{L/\mathbf{Q}_p},\varphi=1}\right) \otimes_{\mathbf{Q}_p} L_0,$$
$$M_0 = \left(D^{G_{L/\mathbf{Q}_p},\varphi=p^{-1}} / N(D^{G_{L/\mathbf{Q}_p},\varphi=1})\right) \otimes_{\mathbf{Q}_p} L_0.$$

The isomorphism (95) shows that the monodromy map N induces an isomorphism

$$\overline{N} : M_1 \to M_0.$$

d) Recall (see Section 4.3.1) that we set $W = F_1\mathbf{D}_{\mathrm{st}/L}(V)/F_{-1}\mathbf{D}_{\mathrm{st}/L}(V)$ and denote by $\mathbf{M}_0$, $\mathbf{M}_1$ and $\mathbf{W}$ the $(\varphi,\Gamma_{\mathbf{Q}_p})$-modules associated to M_0, M_1 and W respectively. Set $e = \dim_{L_0} M_0 = \dim_{L_0} M_1$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & W & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & & & & \searrow^{N} & \downarrow{\scriptstyle \overline{N}}\,\simeq & & \\ & & & & & & M_0. & & \end{array}$$

Then

$$H^0(\mathbf{W}) = W^{N=0,\varphi=1} = M_0^{\varphi=1} = 0.$$

and the coboundary map $\delta_0 : H^0(\mathbf{M}_1) \to H^1(\mathbf{M}_0)$ is injective. Since $\dim_E H^0(\mathbf{M}_1) = \dim_E H^1_c(\mathbf{M}_0) = e$, we only need to show that $\mathrm{Im}(\delta_0) \cap H^1_f(\mathbf{M}_0) = 0$. For each semistable (φ,Γ)-module $\mathbf{A}$ we denote by $C_{\mathrm{st}}(\mathbf{A})$ the complex

$$0 \to \mathscr{D}_{\mathrm{st}}(\mathbf{A}) \xrightarrow{g} \left(\mathscr{D}_{\mathrm{st}}(\mathbf{A})/\mathrm{Fil}^0\mathscr{D}_{\mathrm{st}}(\mathbf{A})\right) \oplus \mathscr{D}_{\mathrm{st}}(\mathbf{A}) \oplus \mathscr{D}_{\mathrm{st}}(\mathbf{A}) \xrightarrow{h} \mathscr{D}_{\mathrm{st}}(\mathbf{A}),$$

where

$$g(x) = (x \pmod{\mathrm{Fil}^0\mathscr{D}_{\mathrm{st}}(\mathbf{A})}), (\varphi-1)(x), N(x)), \quad h(x,y,z) = N(y) - (p\varphi-1)(z).$$

We refer to [**7**, Sections 1.4-1.5] for the proofs of the following facts. The cohomology group $H^0(C_{\mathrm{st}}(\mathbf{A}))$ is canonically isomorphic to $H^0(\mathbf{A})$. The group $H^1(C_{\mathrm{st}}(\mathbf{A}))$ is canonically isomorphic to the subgroup $H^1_{\mathrm{st}}(\mathbf{A})$ of $H^1(\mathbf{A})$ classifying semistable extensions. One has $H^1_{\mathrm{st}}(\mathbf{M}_0)=H^1(\mathbf{M}_0)$ and the subgroups $H^1_f(\mathbf{M}_0)$ and $H^1_c(\mathbf{M}_0)$ have the following description in terms of $C_{\mathrm{st}}(\mathbf{A})$

$$H^1_f(\mathbf{M}_0)=\{\mathrm{cl}(x,0,0)\mid x\in M_0\},$$
$$H^1_c(\mathbf{M}_0)=\{\mathrm{cl}(0,0,z)\mid x\in M_0\}.$$

We have a commutative diagram

$$\begin{array}{ccc}
H^0(\mathbf{M}_1) & \xrightarrow{\delta_0} & H^1(\mathbf{M}_0)\\
\simeq\uparrow & & \simeq\uparrow\\
H^0(C_{\mathrm{st}}(\mathbf{M}_1)) & \xrightarrow{\Delta_0} & H^1(C_{\mathrm{st}}(\mathbf{M}_0)),
\end{array}$$

where Δ_0 is induced by the exact sequence

$$0\to C_{\mathrm{st}}(\mathbf{M}_0)\to C_{\mathrm{st}}(\mathbf{W})\to C_{\mathrm{st}}(\mathbf{M}_1)\to 0.$$

Let $x\in H^0(\mathbf{M}_1)=M_1^{\varphi=1}$. By the snake lemma, $W^{\varphi=1}\simeq M_1^{\varphi=1}$ and we denote by $y\in W^{\varphi=1}$ the lift of x under this isomorphism. It is easy to check that $\Delta_0(x)=\mathrm{cl}(y,0,\overline{N}(x))$. This implies that if $\Delta_0(x)\in H^1_f(\mathbf{M}_0)$ then $\overline{N}(x)=0$. Since $\overline{N}$ is an isomorphism, this implies that $x=0$. The proposition is proved. □

Remark 4.3.8. — Assume that V is the p-adic realization of a pure motive M over $\mathbf{Q}$. The p-adic version of the Grothendieck semisimplicity conjecture says that Φ acts semisimply on $\mathbf{D}_{\mathrm{st}/L}(V)$. If, in addition, M is of weight -1, the p-adic monodromy conjecture of Deligne–Jannsen [**40**] asserts that the absolute value of eigenvalues of Φ acting on $\mathrm{gr}_i^{\mathfrak{M}}\mathbf{D}_{\mathrm{st}/L}(V)$ is $q^{\frac{i-1}{2}}$. Therefore conjecturally conditions **F1a-b)** and **F2a)** always hold in this case.

4.3.9. — We continue to assume that V is potentially semistable at p. If, in addition, condition **F2a)** holds, we have a diagram

$$\begin{array}{ccc}
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0) & \xrightarrow{i_{\mathbf{M}_0,f}} & H^1_f(\mathbf{M}_0)\\
\kappa_f\uparrow & \nearrow^{\delta_{0,f}} & \uparrow\mathrm{pr}_f\\
H^0(\mathbf{M}_1) & \xrightarrow{\delta_0} & H^1(\mathbf{M}_0)\\
\kappa_c\downarrow & \searrow^{\delta_{0,c}} & \downarrow\mathrm{pr}_c\\
\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0) & \xrightarrow{i_{\mathbf{M}_0,c}} & H^1_c(\mathbf{M}_0),
\end{array}$$

where $i_{\mathbf{M}_{0,c}}$ and $i_{\mathbf{M}_{0,f}}$ are the canonical isomorphisms defined in Proposition 2.9.4 and κ_c and κ_f are the unique maps making the resulting diagram commute.

Definition. — *The determinant*

$$\mathscr{L}(V,D)=\det_E\left(\kappa_f\circ\kappa_c^{-1}\mid \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1)\right) \tag{96}$$

is called the $\mathscr{L}$-invariant associated to V and D.

Remark 4.3.10. — This is a generalization of the $\mathscr{L}$-invariant defined in [**7**] in the semistable case. Note that in *op. cit.* we assume that V is the restriction on $G_{\mathbf{Q}_p}$ of a global Galois representation satisfying the additional condition $H^1_f(V)=H^1_f(V^*(1))=0$, but the definition of $\mathscr{L}(V,D)$ in the semistable case is purely local and does not use this assumption. We expect that $\mathscr{L}(V,D)\neq 0$ if V is associated to a pure motive of weight -1 (see Section 0.4).

The next proposition follows immediately from definitions.

Proposition 4.3.11. — *Assume that condition* **F2a)** *holds. Then* **F2b)** *holds if and only if $\mathscr{L}(V,D)\neq 0$.*

4.3.12. — Now we come back to the general setting described in Section 4.3.1 and summarize below some properties of the filtration $F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$.

Proposition 4.3.13. — *Let D be a regular submodule of $\mathbf{D}_{\mathrm{st}/L}(V)$. Then*

i) If (V,D) satisfies **F2a)***, then $\mathrm{rk}(\mathbf{M}_0)=\mathrm{rk}(\mathbf{M}_1)$ and $H^0(\mathbf{W})=0$*

ii) If (V,D) satisties **F1a)***, then*

$$H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))=H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)),$$
$$H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))=H^1_f(\mathbf{Q}_p,V).$$

iii) If (V,D) satisfies **F1a)** *and* **F2a)***, then we have exact sequences*

$$0\to H^0(\mathbf{M}_1)\to H^1(\mathbf{M}_0)\to H^1_f(\mathbf{W})\to 0 \tag{97}$$

and

$$0\to H^0(\mathbf{M}_1)\to H^1(\mathbf{D})\to H^1_f(\mathbf{Q}_p,V)\to 0. \tag{98}$$

iv) If (V,D) satisfies **F1a-b)** *and* **F2a)***, then the representation V satisfies* **S)***, namely*

$$\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}=\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}=0.$$

Proof. — i) From **F2a)** and the fact that $\dim_E H^0(\mathbf{M}_1)=\mathrm{rk}(\mathbf{M}_1)$ and $\dim_E H^1_c(\mathbf{M}_0)=\mathrm{rk}(\mathbf{M}_0)$ (see Proposition 2.9.4) we obtain that $\mathrm{rk}(\mathbf{M}_0)=\mathrm{rk}(\mathbf{M}_1)$.

By Proposition 2.9.4, iv), $H^0(\mathbf{M}_0)=0$, and we have an exact sequence

$$0\to H^0(\mathbf{W})\to H^0(\mathbf{M}_1)\xrightarrow{\delta_0} H^1(\mathbf{M}_0).$$

By **F2a)**, the map δ_0 is injective and therefore $H^0(\mathbf{W})=0$.

ii) By **F1a)** together with Proposition 2.9.2 and the Euler–Poincaré characteristic formula, we have

$$\dim_E H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) - \dim_E H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) =$$
$$= \dim_E H^0((F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^*(\chi)) = \dim_E H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1))/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*(1)))) = 0,$$

and therefore $H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. Since $H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = 0$, the exact sequence

$$0 \to F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \to 0$$

induces, by Proposition 2.9.2 iv), an exact sequence

$$0 \to H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)). \to H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to 0.$$

On the other hand, since

$$\mathscr{D}_{\mathrm{dR}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = \mathrm{Fil}^0\mathscr{D}_{\mathrm{dR}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)),$$

by Proposition 2.9.2, i) we have

$$\dim_E H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = \dim_E H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = 0,$$

and therefore $H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1_f(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1_f(\mathbf{Q}_p, V)$.

iii) To prove the exacteness of (97), we only need to show that the image of the map $\alpha : H^1(\mathbf{M}_0) \to H^1(\mathbf{W})$, induced by the exact sequence (84), coincides with $H^1_f(\mathbf{W})$. By **F2a)**, $\mathrm{Im}(\delta_0) \cap H^1_f(\mathbf{M}_0) = \{0\}$, and therefore the map $H^1_f(\mathbf{M}_0) \to H^1_f(\mathbf{W})$ is injective. Set $e = \mathrm{rk}(\mathbf{M}_0) = \mathrm{rk}(\mathbf{M}_1)$. Since

$$\dim_E H^1_f(\mathbf{W}) = \dim_E t_{\mathbf{W}}(\mathbf{Q}_p) - H^0(\mathbf{W}) = e = \dim_E H^1_f(\mathbf{M}_0),$$

we obtain that $H^1_f(\mathbf{M}_0) = H^1_f(\mathbf{W})$. On the other hand, the exact sequence

$$0 \to H^0(\mathbf{M}_1) \xrightarrow{\delta_0} H^1(\mathbf{M}_0) \xrightarrow{\alpha} H^1(\mathbf{W})$$

shows that $\dim_E \mathrm{Im}(\alpha) = \dim_E H^1(\mathbf{M}_0) - \dim_E H^0(\mathbf{M}_1) = e = \dim_E H^1_f(\mathbf{M}_0)$. Therefore $\mathrm{Im}(\alpha) = H^1_f(\mathbf{M}_0) = H^1_f(\mathbf{W})$, and the exacteness of (97) is proved.

Since $H^0(\mathbf{W}) = 0$ and $H^1_f(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$, by Proposition 2.9.2 iv) we have an exact sequence

$$0 \to H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1_f(\mathbf{W}) \to 0,$$

which shows that $H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ is the inverse image of $H^1_f(\mathbf{W})$ under the map $H^1(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to H^1(\mathbf{W})$. Therefore we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^0(\mathbf{M}_1) & \xrightarrow{=} & H^0(\mathbf{M}_1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) & \longrightarrow & H^1(\mathbf{D}) & \longrightarrow & H^1(\mathbf{M}_0) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) & \longrightarrow & H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) & \longrightarrow & H^1_f(\mathbf{W}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since the right column of this diagram is exact, the five lemma gives the exacteness of the middle column. Now the exacteness of (98) follows from the fact that $H^1_f(F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) = H^1_f(\mathbf{Q}_p, V)$ by ii).

iv) First prove that $\mathscr{D}_{\mathrm{cris}}(\mathbf{W}) = \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0)$. The exact sequence (84) gives an exact sequence

$$0 \to \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0) \xrightarrow{\alpha} \mathscr{D}_{\mathrm{cris}}(\mathbf{W}) \xrightarrow{\beta} \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1)$$

and we have immediately the inclusion $\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0) \subset \mathscr{D}_{\mathrm{cris}}(\mathbf{W})$. Thus, it is enough to check that $\dim_E \mathscr{D}_{\mathrm{cris}}(\mathbf{W}) = \dim_E \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0)$. Assume that $\dim_E \mathscr{D}_{\mathrm{cris}}(\mathbf{W}) > \dim_E \mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0)$. Then there exists $x \in \mathscr{D}_{\mathrm{cris}}(\mathbf{W})$ such that $m = \beta(x) \neq 0$. Since φ acts trivially on $\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1) = \mathbf{M}_1^{\Gamma_{\mathbf{Q}_p}}$, $\mathscr{R}_{\mathbf{Q}_p,E}m$ is a $(\varphi, \Gamma_{\mathbf{Q}_p})$-submodule of $\mathbf{M}_1$, and there exists a submodule $\mathbf{X} \subset \mathbf{W}$ which sits in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{M}_0 & \longrightarrow & \mathbf{X} & \longrightarrow & \mathscr{R}_{\mathbf{Q}_p,E}m & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{M}_0 & \longrightarrow & \mathbf{W} & \longrightarrow & \mathbf{M}_1 & \longrightarrow & 0.
\end{array}$$

Since $\mathscr{D}_{\mathrm{cris}}(\mathbf{W}) = (\mathbf{W}[1/t])^{\Gamma_{\mathbf{Q}_p}}$, there exists $n \geqslant 0$ such that $t^n x \in \mathbf{X}$, and therefore $x \in \mathscr{D}_{\mathrm{cris}}(\mathbf{X})$. This implies that $\mathbf{X}$ is crystalline, and by Proposition 2.9.2 iv) we have a commutative diagram

$$\begin{array}{ccc} Em & \longrightarrow & H^1_f(\mathbf{M}_0) \\ \downarrow & & \downarrow \\ H^0(\mathbf{M}_1) & \xrightarrow{\delta_0} & H^1(\mathbf{M}_0). \end{array}$$

Thus, $\mathrm{Im}(\delta_0)\cap H^1_f(\mathbf{M}_0)\neq\{0\}$ and condition **F2a)** is violated. This proves that $\mathscr{D}_{\mathrm{cris}}(\mathbf{W})=\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0)$.

Now we can finish the proof. Taking invariants, we have $\mathscr{D}_{\mathrm{cris}}(\mathbf{W})^{\varphi=1}=\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_0)^{\varphi=1}=0$. By **F1b)**,

$$\mathscr{D}_{\mathrm{cris}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\varphi=1}=\mathscr{D}_{\mathrm{cris}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\varphi=1}=0,$$

and, applying the functor $\mathscr{D}_{\mathrm{cris}}(-)^{\varphi=1}$ to the exact sequences

$$0\to F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to 0,$$
$$0\to F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{W}\to 0,$$

we obtain that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1}\subset\mathscr{D}_{\mathrm{cris}}(\mathbf{W})^{\varphi=1}=0$. The same argument shows that $\mathbf{D}_{\mathrm{cris}}(V^*(1))^{\varphi=1}=0$.

□

4.3.14. — Assume that (V,D) satisfies conditions **F1a-b)**. The tautological exact sequence

$$0\to\mathbf{D}\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{D}'\to 0.$$

induces the coboundary map

$$\partial_0\,:\,H^0(\mathbf{D}')\to H^1(\mathbf{D}),$$

Since $H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))=0$, we have that $H^0(\mathbf{D}')=H^0(\mathbf{M}_1)$, and the exact sequence (98) shows that the sequence

$$0\to H^0(\mathbf{D}')\xrightarrow{\partial_0}H^1(\mathbf{D})\to H^1_f(\mathbf{Q}_p,V)\to 0 \qquad (99)$$

is also exact.

Proposition 4.3.15.* —** *Let V be a p-adic representation of $G_{\mathbf{Q}_p}$ which satisfies conditions* **F2b)** *and* **F3). Then*

$$H^1(\mathbf{D})=H^1_{\mathrm{Iw}}(\mathbf{D})_{\Gamma^0_{\mathbf{Q}_p}}\oplus\partial_0\left(H^0(\mathbf{D}')\right).$$

Proof. — Since $\mathscr{D}_{\mathrm{pst}}\left(\left(\left(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)^*(\chi)\right)^{\varphi=p^i}\right)=0$ for all $i\in\mathbf{Z}$, by Lemma 2.8.3 we have $H^2_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))=0$. Then the tautological exact sequence

$$0\to F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to\mathbf{D}\to\mathbf{M}_0\to 0$$

induces an exact sequence

$$0\to H^1_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))\to H^1_{\mathrm{Iw}}(\mathbf{D})\to H^1_{\mathrm{Iw}}(\mathbf{M}_0)\to 0.$$

Since $H^1_{\mathrm{Iw}}(\mathbf{M}_0)^{\Gamma^0_{\mathbf{Q}_p}}=H^0(\mathbf{M}_0)=0$ by Proposition 4.3.13, the snake lemma gives an exact sequence

$$0\to H^1_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))_{\Gamma^0_{\mathbf{Q}_p}}\to H^1_{\mathrm{Iw}}(\mathbf{D})_{\Gamma^0_{\mathbf{Q}_p}}\to H^1_{\mathrm{Iw}}(\mathbf{M}_0)_{\Gamma^0_{\mathbf{Q}_p}}\to 0. \tag{100}$$

The Hochschild–Serre exact sequence

$$0\to H^1_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))_{\Gamma^0_{\mathbf{Q}_p}}\to H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))\to H^2_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\Gamma^0_{\mathbf{Q}_p}}\to 0$$

together with the fact that

$$\begin{aligned}\dim_E H^2_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\Gamma^0_{\mathbf{Q}_p}}&=\dim_E H^2_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))_{\Gamma^0_{\mathbf{Q}_p}}=\\&=\dim_E H^0\left((F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^*(\chi)\right)=0\end{aligned}$$

implies that $H^1_{\mathrm{Iw}}(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))_{\Gamma^0_{\mathbf{Q}_p}}=H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. On the other hand,

$$H^1_{\mathrm{Iw}}(\mathbf{M}_0)_{\Gamma^0_{\mathbf{Q}_p}}=H^1_c(\mathbf{M}_0)$$

by Proposition 2.9.6. Therefore, the sequence (100) reads

$$0\to H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))\to H^1_{\mathrm{Iw}}(\mathbf{D})_{\Gamma^0_{\mathbf{Q}_p}}\to H^1_c(\mathbf{M}_0)\to 0$$

and we have a commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))&\longrightarrow&H^1_{\mathrm{Iw}}(\mathbf{D})_{\Gamma^0_{\mathbf{Q}_p}}&\longrightarrow&H^1_c(\mathbf{M}_0)&\longrightarrow&0\\&&\downarrow{=}&&\downarrow&&\downarrow&&\\0&\longrightarrow&H^1(F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))&\longrightarrow&H^1(\mathbf{D})&\longrightarrow&H^1(\mathbf{M}_0)&\longrightarrow&0.\end{array} \tag{101}$$

Since $H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))=0$, the exact sequence

$$0\to\mathbf{M}_1\to\mathbf{D}'\to\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\to 0$$

gives $H^0(\mathbf{M}_1) = H^0(\mathbf{D}')$ and we have a commutative diagram

(102)
$$\begin{array}{ccc} H^0(\mathbf{D}') & \xrightarrow{\partial_0} & H^1(\mathbf{D}) \\ {\scriptstyle =}\uparrow & & \downarrow \\ H^0(\mathbf{M}_1) & \overset{\delta_0}{\hookrightarrow} & H^1(\mathbf{M}_0). \end{array}$$

Finally, from **F2b)** it follows that $H^1_c(\mathbf{M}_0) \cap \delta_0\left(H^0(\mathbf{M}_1)\right) = \{0\}$, and the dimension argument shows that

(103)
$$H^1(\mathbf{M}_0) = H^1_c(\mathbf{M}_0) \oplus \delta_0\left(H^0(\mathbf{M}_1)\right).$$

Now, the proposition follows from (103) and the diagrams (101) and (102). □

4.4. Appendix. Some semilinear algebra

4.4.1. — In this section we assemble auxiliary results used in Section 4.3. They are certainly known to experts, but we give detailed proofs for completeness.

Let L_0 be a finite unramified extension of $\mathbf{Q}_p$. We denote by σ the absolute Frobenius automorphism on L_0. Let W be a finite dimensional L_0-vector space equipped with a σ-semilinear bijective operator $\varphi : W \to W$. For each extension $E/\mathbf{Q}_p$, denote by the same letter φ the operator on $E \otimes_{\mathbf{Q}_p} W$ induced by φ by extension of scalars. Note that W is a free $E \otimes_{\mathbf{Q}_p} L_0$-module and that φ acts on $E \otimes_{\mathbf{Q}_p} L_0$ by $\varphi(a \otimes_{\mathbf{Q}_p} b) = a \otimes_{\mathbf{Q}_p} \sigma(b)$.

Lemma 4.4.2. — *Let L'_0/L_0 be a field extension and let $\varphi : L'_0 \otimes_{\mathbf{Q}_p} W \to L'_0 \otimes_{\mathbf{Q}_p} W$ be the L'_0-linear map induced by φ by extension of scalars. Then*

i) For each $\alpha \in E$, the natural map

$$\iota : L'_0 \otimes_{\mathbf{Q}_p} W \to L'_0 \otimes_{L_0} W, \qquad \iota(a \otimes_{\mathbf{Q}_p} x) = a \otimes_{L_0} x$$

induces an injection

$$\left(L'_0 \otimes_{\mathbf{Q}_p} W\right)^{\varphi=\alpha} \to L'_0 \otimes_{L_0} W.$$

ii) For any $\alpha \in \mathbf{Q}_p$, the natural map

$$L_0 \otimes_{\mathbf{Q}_p} W^{\varphi=\alpha} \to W$$

is injective.

Proof. — Set $d = \dim_{L_0} W$. Let $\{v_j\}_{1\leqslant j\leqslant d}$ be a basis of W over L_0 and $\{\theta_i\}_{1\leqslant i\leqslant h}$ be a basis of L_0 over $\mathbf{Q}_p$. Then $\{\theta_i v_j\}_{1\leqslant i\leqslant h, 1\leqslant j\leqslant d}$ is a basis of $L_0 \otimes_{\mathbf{Q}_p} W$ over L'_0. Let

$C^{(s)}=(c_{jk}^{(s)})_{1\leqslant j,k\leqslant d}$ be the matrix of φ^s in the basis $\{v_j\}_{1\leqslant j\leqslant d}$, i.e.

$$\varphi^s(v_j)=\sum_{k=1}^{d}c_{jk}^{(s)}v_k,\qquad 1\leqslant j\leqslant h.$$

Assume that

$$x=\sum_{i=1}^{h}\sum_{j=1}^{d}a_{ij}\otimes_{\mathbf{Q}_p}(\theta_i v_j)\in\ker(\iota),\qquad a_{ij}\in L_0'.$$

If, in addition, $\varphi(x)=\alpha x$, then

$$\varphi^s(x)=\sum_{i=1}^{h}\sum_{j=1}^{d}a_{ij}\otimes_{\mathbf{Q}_p}\varphi^s(\theta_i)\varphi^s(v_j)\in\ker(\iota)\quad\text{for all } 0\leqslant s\leqslant h-1.$$

Set

$$x_j^{(s)}=\sum_{i=1}^{h}a_{ij}\varphi^s(\theta_i),\qquad 1\leqslant j\leqslant d.$$

Then

$$\sum_{j=1}^{d}x_j^{(s)}c_{jk}^{(s)}=0,\qquad 1\leqslant j\leqslant d.$$

Since $\det(C^{(s)})\neq 0$, this implies that $x_j^{(s)}=0$ for all $1\leqslant j\leqslant d$ and $0\leqslant s\leqslant h-1$. Therefore for each $1\leqslant j\leqslant d$ we have

$$\sum_{i=1}^{h}a_{ij}\varphi^s(\theta_i)=0,\qquad 0\leqslant s\leqslant h-1.$$

Since $\det(\varphi^s(\theta_i)_{1\leqslant s,i\leqslant h})\neq 0$ by the linear independence of automorphisms, we get $a_{ij}=0$ for all $1\leqslant j\leqslant d$ and $1\leqslant i\leqslant h$. Thus $x=0$ and i) is proved.

ii) Take $L_0'=L_0$ (with the trivial action of φ). Since $\alpha\in\mathbf{Q}_p$, we have $(L_0\otimes_{\mathbf{Q}_p}W)^{\varphi=\alpha}=L_0\otimes_{\mathbf{Q}_p}W^{\varphi=\alpha}$ and by i) the map $L_0\otimes_{\mathbf{Q}_p}W^{\varphi=\alpha}\to W$ is injective. This proves ii). Note that the usual proof of this statement uses Artin's trick (see Lemma 4.4.3 below). □

Lemma 4.4.3. — *Let U be an L_0-subspace of W stable under the action of φ and let $\alpha\in\mathbf{Q}_p^*$. Then*

$$\left(L_0\otimes_{\mathbf{Q}_p}W^{\varphi=\alpha}\right)\cap U=L_0\otimes_{\mathbf{Q}_p}U^{\varphi=\alpha}.$$

In particular,

$$\left(L_0\otimes_{\mathbf{Q}_p}W^{\varphi=\alpha}\right)\cap U\neq\{0\}\implies W^{\varphi=\alpha}\cap U\neq\{0\}.$$

Proof. — First note that $L_0 \otimes_{\mathbf{Q}_p} W^{\varphi=\alpha} \subset W$ and $L_0 \otimes_{\mathbf{Q}_p} U^{\varphi=\alpha} \subset W$ by Lemma 4.4.2. Fix a $\mathbf{Q}_p$-basis $\{w_i\}_{i=1}^k$ of $U^{\varphi=\alpha}$ and complete it to a basis $\{w_i\}_{i=1}^n$ of $W^{\varphi=\alpha}$. We prove the lemma by contradiction. Assume that there exist a nonzero element

$$x = \sum_{i=1}^m a_i \otimes w_i \in \left(L_0 \otimes_{\mathbf{Q}_p} W^{\varphi=\alpha}\right) \cap U,$$

such that $x \notin L_0 \otimes_{\mathbf{Q}_p} U^{\varphi=\alpha}$. In the set of elements with this property we choose a "shortest" element which we denote again by x. Note that $m > k$ and that we can assume that $a_m = 1$. Then

$$\varphi(x) = \alpha \sum_{i=1}^m \sigma(a_i) \otimes w_i \in \left(L_0 \otimes_{\mathbf{Q}_p} W^{\varphi=\alpha}\right) \cap U,$$

and therefore

$$\alpha^{-1}\varphi(x) - x = \sum_{i=1}^{m-1} (\sigma(a_i) - a_i) \otimes w_i \in \left(L_0 \otimes_{\mathbf{Q}_p} W^{\varphi=\alpha}\right) \cap U.$$

By the choice of x, we have $\alpha^{-1}\varphi(x) - x \in L_0 \otimes_{\mathbf{Q}_p} U^{\varphi=\alpha}$. This implies that $\sigma(a_i) = a_i$ for all $k+1 \leqslant i \leqslant m$. Thus $a_i \in \mathbf{Q}_p$ for all $k+1 \leqslant i \leqslant m$. Therefore

$$x = x_0 + x_1, \qquad x_0 = \sum_{i=1}^k a_i \otimes w_i \in L_0 \otimes_{\mathbf{Q}_p} U^{\varphi=\alpha}, \quad x_1 = \sum_{i=k+1}^m a_i \otimes w_i \in W^{\varphi=\alpha}.$$

Thus $x_1 = x - x_0 \in U \cap W^{\varphi=\alpha} = U^{\varphi=\alpha}$ and by the construction of the basis $\{w_i\}_{i=1}^n$ we get that $x_1 = 0$. The lemma is proved. □

4.4.4. — Let $h = [L_0 : \mathbf{Q}_p]$ and $\Phi = \varphi^h$. We consider φ as a linear map on the $\mathbf{Q}_p$-vector space W and Φ as a L_0-linear map on the L_0-vector space W.

Proposition 4.4.5. — *i) Let L_0' be a finite extension of L_0 and $\alpha \in L_0'$. Assume that Φ is semisimple at α^h. Then φ is semisimple at α.*

ii) Φ is semisimple if and only if φ is semisimple.

Proof. — i) We prove i) by contradiction. Assume that φ is not semisimple at α. Then there exists a nonzero vector $y = (\varphi - \alpha)x$ such that $\varphi(y) = \alpha y$. Set

$$z = \sum_{i=0}^{h-1} \alpha^i \varphi^{h-i-1} y = (\Phi - \alpha^h)(x).$$

Then $z = h\alpha^{h-1}x \neq 0$ and $\Phi(z) = \alpha^h z$. The map

$$\iota : W \otimes_{\mathbf{Q}_p} L_0' \to W \otimes_{L_0} L_0', \qquad \iota(x \otimes_{\mathbf{Q}_p} a) = x \otimes_{L_0} a \tag{104}$$

is compatible with the action of Φ. Since ι is injective by Lemma 4.4.2, $\iota(z) \neq 0$ and

$$\iota(z) \in (\Phi - \alpha^h)W \cap W^{\Phi=\alpha^h}.$$

This proves i).

ii) From i) it follows that φ is semisimple if Φ is. Now we show that the converse holds. If φ is semisimple, there exist an extension L_0'/L_0 and a basis $\{w_i\}_{1\leqslant i\leqslant dh}$ of $W\otimes_{\mathbf{Q}_p} L_0'$ over L_0' such that $\varphi(w_i)=\lambda_i w_i$, $\lambda_i\in L_0'$ for all i. Since the map (104) is surjective, one can find a subsystem $\{v_i\}_{1\leqslant i\leqslant d}$ of $\{w_i\}_{1\leqslant i\leqslant dh}$ such that $\{\iota(v_i)\}_{1\leqslant i\leqslant d}$ is a basis of $W\otimes_{L_0} L_0'$. Since the map ι is compatible with Φ, this proves that the matrix of Φ in this basis is diagonal. □

4.4.6. — Let G be a finite group sitting in an exact sequence of the form

$$0\to I\to G\xrightarrow{\pi}\mathrm{Gal}(L_0/\mathbf{Q}_p)\to 0.$$

We write Tr_I for the trace operator $\mathrm{Tr}_I=\sum\limits_{g\in I} g$. Assume that W is equipped with a semilinear action of G *via* the projection π which commutes with the operator φ. Then I acts L_0-linearly on W and we have

$$W=W^I\oplus W^0,\quad W^0=\{x\in W\,|\,\mathrm{Tr}_I(x)=0\}.$$

Moreover, from Hilbert's Theorem 90 for GL_n we have

$$W^I=L_0\otimes_{\mathbf{Q}_p} W^G. \tag{105}$$

We denote by W^* the dual space $W^*=\mathrm{Hom}_{L_0}(W,L_0)$ equipped with the semilinear action of φ given by

$$(\varphi f)(w)=\sigma f(\varphi^{-1}(w)),\qquad f\in W^*, w\in W.$$

For any W we denote by $W[1]$ the space W equipped with the operator $\varphi_{W[1]}=p^{-1}\varphi$. The canonical duality gives a pairing of L_0-vector spaces

$$[\ ,\ ]: W\times W^*[1]\to L_0[1],\quad [x,f]=f(x).$$

We equip $W^*[1]$ with the natural action of G given by

$$(gf)(x)=gf(g^{-1}x),\qquad g\in G,\quad x\in W,\quad f\in W^*[1].$$

If Y is a L_0-subspace of $W^*[1]$, we denote by $Y^\perp$ the orthogonal complement of Y in W with respect to the pairing $[\ ,\ ]$.

Lemma 4.4.7. **—** *For any* $\alpha\in\mathbf{Q}_p^*$ *we have*

$$\left(L_0\otimes_{\mathbf{Q}_p} W^*[1]^{\varphi=\alpha,G}\right)^\perp=\left((\alpha-p^{-1}\varphi^{-1})W^G\right)\otimes_{\mathbf{Q}_p} L_0+W^0.$$

Proof. — The pairing $[\ ,\ ]$ induces non-degenerate pairings

$$[\ ,\ ]_I: W^I\times W^*[1]^I\to L_0[1],$$
$$[\ ,\ ]_G: W^G\times W^*[1]^G\to \mathbf{Q}_p[1].$$

From (105) it follows that $[\ ,\ ]_I$ is induced from $[\ ,\ ]_G$ by extension of scalars. Since $(\alpha - p^{-1}\varphi^{-1})W^G$ is the orthogonal complement of $W^*[1]^{\varphi=\alpha,G}$ under $[\ ,\ ]_G$, this implies the lemma.

□

CHAPTER 5

p-ADIC HEIGHT PAIRINGS II: UNIVERSAL NORMS

5.1. The pairing $h^{\mathrm{norm}}_{V,D}$

5.1.1. — In this section, we construct the pairing $h^{\mathrm{norm}}_{V,D}$, which is a direct generalization of the pairing constructed in [**66**] [**59**] and [**54**, Section 6]. Let V is a p-adic representation of $G_{F,S}$ with coefficients in a finite extension E of $\mathbf{Q}_p$. We fix a system $\mathbf{D}=(\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ of submodules $\mathbf{D}_{\mathfrak{q}}\subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$ and denote by $\mathbf{D}^{\perp}=(\mathbf{D}^{\perp}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ the orthogonal complement of $\mathbf{D}$. We have tautological exact sequences

$$0\rightarrow \mathbf{D}_{\mathfrak{q}}\rightarrow \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})\rightarrow \mathbf{D}'_{\mathfrak{q}}\rightarrow 0,\qquad \mathfrak{q}\in S_p,$$

where $\mathbf{D}'_{\mathfrak{q}}=\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})/\mathbf{D}_{\mathfrak{q}}$. Passing to duals, we have exact sequences

$$0\rightarrow (\mathbf{D}'_{\mathfrak{q}})^*(\chi_{\mathfrak{q}})\rightarrow \mathbf{D}^{\dagger}_{\mathrm{rig}}(V^*_{\mathfrak{q}}(1))\rightarrow \mathbf{D}^*_{\mathfrak{q}}(\chi_{\mathfrak{q}})\rightarrow 0,$$

where $(\mathbf{D}'_{\mathfrak{q}})^*(\chi_{\mathfrak{q}})=\mathbf{D}^{\perp}_{\mathfrak{q}}$. If the contrary is not explicitly stated, in this section we will assume that the following conditions hold

N1) $H^0(F_{\mathfrak{q}},V)=H^0(F_{\mathfrak{q}},V^*(1))=0$ for all $\mathfrak{q}\in S_p$;

N2) $H^0(\mathbf{D}'_{\mathfrak{q}})=H^0(\mathbf{D}^*_{\mathfrak{q}}(\chi_{\mathfrak{q}}))=0$ for all $\mathfrak{q}\in S_p$.

As we noticed in Section 0.4, if V is the p-adic realization of a pure motive of weight -1 condition **N1)** conjecturally always holds. Condition **N2)** means that the p-adic L-function $L(V,D,s)$ conjecturally associated to $\mathbf{D}$ has no extra-zeros at $s=0$. From **N2)**, it follows immediately that $H^1(\mathbf{D}_{\mathfrak{q}})$ injects into $H^1(F_{\mathfrak{q}},V)$. By our definition of Selmer complexes we have

$$H^1(V,\mathbf{D})\simeq \ker\left(H^1_S(V)\rightarrow \bigoplus_{\mathfrak{q}\in\Sigma_p}\frac{H^1(F_{\mathfrak{q}},V)}{H^1_f(F_{\mathfrak{q}},V)}\right)\bigoplus\left(\bigoplus_{v\in S_p}\frac{H^1(F_{\mathfrak{q}},V)}{H^1(\mathbf{D}_{\mathfrak{q}})}\right),\qquad (106)$$

and the same formula holds for $V^*(1)$ if we replace $\mathbf{D}_{\mathfrak{q}}$ by $\mathbf{D}^{\perp}_{\mathfrak{q}}$. Recall that each element of $H^1(V,\mathbf{D})$ can be written as the class $[x^{\mathrm{sel}}]$ of a triple $x^{\mathrm{sel}}=(x,(x^+_{\mathfrak{q}}),(\lambda_{\mathfrak{q}}))$

(see (47)). The isomorphism (106) identifies $[x^{\mathrm{sel}}]$ with the corresponding global cohomology class $[x]\in H^1_S(V)$.

5.1.2. — Let $[y^{\mathrm{sel}}]=[(y,(y_{\mathfrak{q}}^+),(\mu_{\mathfrak{q}}))]\in H^1(V^*(1),\mathbf{D}^{\perp})$ and let Y_y be the associated extention

$$0\to V^*(1)\to Y_y\to E\to 0.$$

Passing to duals, we have an exact sequence

$$0\to E(1)\to Y_y^*(1)\to V\to 0.$$

For each $\mathfrak{q}\in S_p$, this sequence induces an exact sequence of $(\varphi,\Gamma_{\mathfrak{q}})$-modules

$$0\to \mathscr{R}_{F_{\mathfrak{q}},E}(\chi_{\mathfrak{q}})\to \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y^*(1)_{\mathfrak{q}})\to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})\to 0.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1(\mathbf{D}_{\mathfrak{q},y}) & \xrightarrow{\pi_{\mathbf{D},\mathfrak{q}}} & H^1(\mathbf{D}_{\mathfrak{q}}) & \xrightarrow{\delta^1_{\mathbf{D},\mathfrak{q}}} & H^2(F_{\mathfrak{q}},E(1)) \\
 & & \downarrow{=} & & g_{\mathfrak{q},y}\downarrow & & g_{\mathfrak{q}}\downarrow & & \downarrow{=} \\
0 & \longrightarrow & H^1(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1(F_{\mathfrak{q}},Y_y^*(1)) & \xrightarrow{\pi_{\mathfrak{q}}} & H^1(F_{\mathfrak{q}},V) & \xrightarrow{\delta^1_{V,\mathfrak{q}}} & H^2(F_{\mathfrak{q}},E(1)) \\
 & & \uparrow & & \mathrm{res}_{\mathfrak{q}}\uparrow & & \mathrm{res}_{\mathfrak{q}}\uparrow & & \mathrm{res}_{\mathfrak{q}}\uparrow \\
0 & \longrightarrow & H^1_S(E(1)) & \longrightarrow & H^1_S(Y_y^*(1)) & \xrightarrow{\pi} & H^1_S(V) & \xrightarrow{\delta^1_V} & H^2_S(E(1)),
\end{array}$$

where $\mathbf{D}_{\mathfrak{q},y}$ denotes the inverse image of $\mathbf{D}_{\mathfrak{q}}$ in $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$.

In the following lemma we do not assume that condition **N2)** holds.

***Lemma 5.1.3.* —** *Assume that V is a p-adic representation satisfying condition* **N1)**. *Let $[x]=[(x,(x_{\mathfrak{q}}^+),(\lambda_{\mathfrak{q}}))]\in H^1(V,\mathbf{D})$ and let $x_{\mathfrak{q}}=\mathrm{res}_{\mathfrak{q}}(x)$. Then*

i) If $\mathfrak{q}\nmid p$, then $H^1_f(F_{\mathfrak{q}},E(1))=0$ and

$$H^1_f(F_{\mathfrak{q}},Y_y^*(1))\simeq H^1_f(F_{\mathfrak{q}},V).$$

ii) For each $\mathfrak{q}\in S_p$ one has $\delta^1_{V,\mathfrak{q}}([x_{\mathfrak{q}}])=\delta^1_{\mathbf{D},\mathfrak{q}}([x_{\mathfrak{q}}^+])=0.$

iii) $\delta^1_V([x])=0.$

iv) The sequence

$$0\to H^1(E(1),\mathscr{R}(\chi))\to H^1(Y_y^*(1),\mathbf{D}_y)\to H^1(V,\mathbf{D})\to 0,$$

where $\mathscr{R}(\chi)=(\mathscr{R}_{F_{\mathfrak{q}},E}(\chi_{\mathfrak{q}}))_{\mathfrak{q}\in S_p}$, is exact.

Proof. — i) If $\mathfrak{q}\nmid p$, then $E(1)$ is unramified at $\mathfrak{q}$, $H^0(F_{\mathfrak{q}}^{\mathrm{ur}}/F_{\mathfrak{q}},E(1))=0$ and

$$H^1_f(F_{\mathfrak{q}},E(1))=H^1(F_{\mathfrak{q}}^{\mathrm{ur}}/F_{\mathfrak{q}},E(1))=E(1)/(\mathrm{Fr}_{\mathfrak{q}}-1)E(1)=0.$$

Since $[y]$ is unramified at $\mathfrak{q}$, the sequence

$$0\to E(1)\to Y_y^*(1)^{I_\mathfrak{q}}\to V^{I_\mathfrak{q}}\to 0$$

is exact. Passing to the associated long exact cohomology sequence of $\mathrm{Gal}(F_\mathfrak{q}^{\rm ur}/F_\mathfrak{q})$ and taking into account that

$$H^1(F_\mathfrak{q}^{\rm ur}/F_\mathfrak{q},E(1))=H^2(F_\mathfrak{q}^{\rm ur}/F_\mathfrak{q},E(1))=0$$

we obtain that $H^1(F_\mathfrak{q}^{\rm ur}/F_\mathfrak{q},Y_y^*(1)^{I_\mathfrak{q}})\xrightarrow{\sim} H^1(F_\mathfrak{q}^{\rm ur}/F_\mathfrak{q},V^{I_\mathfrak{q}})$. This proves i).

ii) For each $\mathfrak{q}\in S_p$ we have $g_\mathfrak{q}([x_\mathfrak{q}^+])=[x_\mathfrak{q}]$. From the orthogonality of $\mathbf{D}_\mathfrak{q}$ and $\mathbf{D}_\mathfrak{q}^\perp$ it follows that

$$\delta^1_\mathbf{D}(x_\mathfrak{q}^+)=-x_\mathfrak{q}^+\cup y_\mathfrak{q}^+=0.$$

Therefore, $\delta^1_{V,\mathfrak{q}}([x_\mathfrak{q}])=\delta^1_{\mathbf{D},\mathfrak{q}}([x_\mathfrak{q}^+])=0$ for each $\mathfrak{q}\in S_p$.

iii) Let $\mathfrak{q}\in\Sigma_p$. Since $[x_\mathfrak{q}]\in H^1_f(F_\mathfrak{q},V)$, from i) it follows that again $\delta_{V,\mathfrak{q}}([x_\mathfrak{q}])=0$. As the localization map

$$H^2_S(E(1))\to\bigoplus_{v\in S}H^2(F_\mathfrak{q},E(1))$$

is injective, we obtain that $\delta^1_V(x)=0$.

iv) First prove the surjectivity of $\pi:H^1(Y_y^*(1),\mathbf{D}_y)\to H^1(V,\mathbf{D})$. We remark that $H^1(Y_y^*(1),\mathbf{D}_y)\subset H^1_S(Y_y^*(1))$ and therefore each element of $H^1(Y_y^*(1),\mathbf{D}_y)$ is completely defined by its global cohomology component. For each $\mathfrak{q}\in\Sigma_p$ we denote by

$$s_{y,\mathfrak{q}}:H^1_f(F_\mathfrak{q},V)\simeq H^1_f(F_\mathfrak{q},Y_y^*(1))$$

the inverse of the isomorphism i). Let $[x^{\rm sel}]=[(x,(x_\mathfrak{q}^+),(\lambda_\mathfrak{q}))]\in H^1(V,\mathbf{D})$. By ii), $\delta^1_V([x])=0$, and there exists $[a]\in H^1_S(Y_y^*(1))$ such that $\pi([a])=[x]$. For each $\mathfrak{q}\in\Sigma_p$ set $[a_\mathfrak{q}]=\mathrm{res}_\mathfrak{q}([a])$. Since $[x_\mathfrak{q}^+]\in H^1_f(F_\mathfrak{q},V)$, there exists $[b_\mathfrak{q}^+]\in H^1(F_\mathfrak{q},E(1))$ such that

$$[a_\mathfrak{q}]=s_{y,\mathfrak{q}}([x_\mathfrak{q}^+])+[b_\mathfrak{q}^+].$$

The localization map $H^1_S(E(1))\to\bigoplus_{\mathfrak{q}\in\Sigma_p}H^1(F_\mathfrak{q},E(1))$ is surjective, and there exists $[b]\in H^1_S(E(1))$ such that $\mathrm{res}_\mathfrak{q}([b])=[b_\mathfrak{q}^+]$ for each $\mathfrak{q}\in\Sigma_p$. Then $[a]-[b]\in H^1_S(Y_y^*(1))$ defines a class $[\widehat{x}^{\rm sel}][x]\in H^1(Y_y^*(1),\mathbf{D}_y)$ such that $\pi([\widehat{x}^{\rm sel}])=[x]$. Thus, the map π is surjective.

Finally, from i) we have

$$H^1(E(1),\mathscr{R}(\chi))=\ker\left(H^1_S(E(1))\to\bigoplus_{\mathfrak{q}\in\Sigma_p}H^1(F_\mathfrak{q},E(1))\right),$$

and it is easy to see that $H^1(E(1),\mathscr{R}(\chi))$ coincides with the kernel of π. The lemma is proved.

□

5.1.4. — Let $\log_p : \mathbf{Q}_p^* \to \mathbf{Q}_p$ denote the p-adic logarithm normalized by $\log_p(p) = 0$. For each finite place $\mathfrak{q}$ we define an homomorphism $\ell_\mathfrak{q} : F_\mathfrak{q}^* \to \mathbf{Q}_p$ by

$$\ell_\mathfrak{q}(x) = \begin{cases} \log_p(N_{F_\mathfrak{q}/\mathbf{Q}_p}(x)), & \text{if } \mathfrak{q} \mid p, \\ \log_p |x|_\mathfrak{q}, & \text{if } \mathfrak{q} \nmid p, \end{cases}$$

where $N_{F_\mathfrak{q}/\mathbf{Q}_p}$ denotes the norm map. By linearity, $\ell_\mathfrak{q}$ can be extended to a map $\ell_\mathfrak{q} : F_\mathfrak{q}^* \widehat{\otimes}_{\mathbf{Z}_p} E \to E$, and the isomorphism $F_\mathfrak{q}^* \widehat{\otimes}_{\mathbf{Z}_p} E \xrightarrow{\sim} H^1(F_\mathfrak{q}, E(1))$ allows to consider $\ell_\mathfrak{q}$ as a map $H^1(F_\mathfrak{q}, E(1)) \to E$ which we denote again by $\ell_\mathfrak{q}$.

From the product formula

$$|N_{F/\mathbf{Q}}(x)|_\infty \prod_{\mathfrak{q} \in S_f} |x|_\mathfrak{q} = 1$$

and the fact that $N_{F/\mathbf{Q}}(x) = \prod_{\mathfrak{q} \mid p} N_{F_\mathfrak{q}/\mathbf{Q}_p}(x)$ it follows that

$$\sum_{\mathfrak{q} \in S_f} \ell_\mathfrak{q}(x) = 0, \qquad \forall x \in F^*. \tag{107}$$

We set $\Lambda_{\mathscr{O}_E,\mathfrak{q}} = \mathscr{O}_E[[\Gamma_\mathfrak{q}^0]]$ and $\Lambda_{E,\mathfrak{q}} = \Lambda_{\mathscr{O}_E,\mathfrak{q}}[1/p]$.

***Lemma 5.1.5.** — Let V be a p-adic representation of $G_{F,S}$ that satisfies* **N1-2)** *and let $[y^{\mathrm{sel}}] \in H^1(V^*(1), \mathbf{D}^\perp)$. For each $\mathfrak{q} \in S_p$, the following diagram is commutative with exact rows and columns*

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\mathscr{H}(\Gamma_\mathfrak{q}^0) \otimes_{\Lambda_{E,\mathfrak{q}}} H^1_{\mathrm{Iw}}(F_\mathfrak{q}, E(1)) & \longrightarrow & H^1(F_\mathfrak{q}, E(1)) & \xrightarrow{\ell_\mathfrak{q}} & E \\
\downarrow & & \downarrow & & \\
H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q},y}) & \xrightarrow{\mathrm{pr}_{\mathfrak{q},y}} & H^1(\mathbf{D}_{\mathfrak{q},y}) & & \\
\downarrow{\scriptstyle \pi^{\mathrm{Iw}}_{\mathbf{D},\mathfrak{q}}} & & \downarrow{\scriptstyle \pi_{\mathbf{D},\mathfrak{q}}} & & \\
H^1_{\mathrm{Iw}}(\mathbf{D}_\mathfrak{q}) & \xrightarrow{\mathrm{pr}_\mathfrak{q}} & H^1(\mathbf{D}_\mathfrak{q}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^2(F_\mathfrak{q}, E(1)) & \xrightarrow{=} & H^2(F_\mathfrak{q}, E(1)). & &
\end{array} \tag{108}$$

Proof. — The exacteness of the left column is clear. The exactness of the right column follows from the fact that the diagram

$$\begin{array}{ccc} H^2(F_{\mathfrak{q},n},E(1)) & \xrightarrow{\mathrm{inv}_{F_{\mathfrak{q},n}}} & E \\ \Big\downarrow{\scriptstyle \mathrm{cores}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ H^2(F_{\mathfrak{q},n-1},E(1)) & \xrightarrow{\mathrm{inv}_{F_{\mathfrak{q},n-1}}} & E \end{array}$$

is commutative, and therefore

$$H^2_{\mathrm{Iw}}(F_{\mathfrak{q}},E(1)) \simeq H^2(F_{\mathfrak{q}},E(1)) \simeq E.$$

The diagram (108) is clearly commutative. Now, we prove that the projection map $H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}}) \to H^1(\mathbf{D}_{\mathfrak{q}})$ is surjective. We have an exact sequence

$$0 \to H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})_{\Gamma^0_{\mathfrak{q}}} \to H^1(\mathbf{D}_{\mathfrak{q}}) \to H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})^{\Gamma^0_{\mathfrak{q}}} \to 0,$$

and therefore it is enough to show that $H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})^{\Gamma^0_{\mathfrak{q}}} = 0$. Consider the exact sequence

$$0 \to H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})^{\Gamma^0_{\mathfrak{q}}} \to H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}}) \xrightarrow{\gamma_{\mathfrak{q}}-1} H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}}) \to H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})_{\Gamma^0_{\mathfrak{q}}} \to 0.$$

Since $H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})$ is a finite-dimensional E-vector space, we have

$$\dim_E H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})^{\Gamma^0_{\mathfrak{q}}} = \dim_E H^2_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}})_{\Gamma^0_{\mathfrak{q}}} = \dim_E H^2(\mathbf{D}_{\mathfrak{q}}) = \dim_E H^0(\mathbf{D}^*_{\mathfrak{q}}(\chi)) = 0.$$

Thus, the map $H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}}) \to H^1(\mathbf{D}_{\mathfrak{q}})$ is surjective. To prove the exactness of the first row, we remark that the sequence

$$H^1_{\mathrm{Iw}}(F_{\mathfrak{q}},E(1)) \to H^1(F_{\mathfrak{q}},E(1)) \xrightarrow{\ell_{\mathfrak{q}}} E$$

is known to be exact (see, for example, [**56**, Section 11.3.5]), and that the image of the projection $\mathscr{H}(\Gamma^0_{\mathfrak{q}}) \otimes_{\Lambda_{E,\mathfrak{q}}} H^1_{\mathrm{Iw}}(F_{\mathfrak{q}},E(1)) \to H^1(F_{\mathfrak{q}},E(1))$ coincides with the image of the projection $H^1_{\mathrm{Iw}}(F_{\mathfrak{q}},E(1)) \to H^1(F_{\mathfrak{q}},E(1))$. □

5.1.6. — By Lemma 5.1.5, for each $\mathfrak{q} \in S_p$ we have the following commutative diagram with exact rows, where the map $\mathrm{pr}_{\mathfrak{q}}$ is surjective

(109)

$$\begin{array}{ccccccccc} & & & & H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q},y}) & \xrightarrow{\pi^{\mathrm{Iw}}_{\mathbf{D},\mathfrak{q}}} & H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q}}) & \longrightarrow & H^2(F_{\mathfrak{q}},E(1)) \\ & & & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{q},y}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathfrak{q}}} & & \Big\downarrow{\scriptstyle =} \\ 0 & \longrightarrow & H^1(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1(\mathbf{D}_{\mathfrak{q},y}) & \xrightarrow{\pi_{\mathbf{D},\mathfrak{q}}} & H^1(\mathbf{D}_{\mathfrak{q}}) & \xrightarrow{\delta^1_{\mathbf{D},v}} & H^2(F_{\mathfrak{q}},E(1)) \\ & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle g_{\mathfrak{q},y}} & & \Big\downarrow{\scriptstyle g_{\mathfrak{q}}} & & \\ 0 & \longrightarrow & H^1(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1(F_{\mathfrak{q}},Y^*_y(1)) & \xrightarrow{\pi_{\mathfrak{q}}} & H^1(F_{\mathfrak{q}},V). & & \end{array}$$

Let $[x^{\mathrm{sel}}]=[(x,(x_{\mathfrak{q}}^{+}),(\lambda_{\mathfrak{q}}))]\in H^1(V,\mathbf{D})$. By Lemma 5.1.3 ii), for each $\mathfrak{q}\in S_p$ we have $\delta_{\mathbf{D},\mathfrak{q}}([x_{\mathfrak{q}}^{+}])=0$, and therefore there exists $[x_{\mathfrak{q},y}^{\mathrm{Iw}}]\in H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q},y})$ such that $\mathrm{pr}_{\mathfrak{q}}\circ \pi_{\mathbf{D},\mathfrak{q}}^{\mathrm{Iw}}\left([x_{\mathfrak{q},y}^{\mathrm{Iw}}]\right)=[x_{\mathfrak{q}}^{+}]$. By Lemma 5.1.3 iv), there exists a lift $[\widehat{x}^{\mathrm{sel}}]=[(\widehat{x},(\widehat{x}_{\mathfrak{q}}^{+}),(\widehat{\lambda}_{\mathfrak{q}}))]\in H^1(Y_y^*(1),\mathbf{D}_y)$ of $[x^{\mathrm{sel}}]$. Note that $\mathrm{res}_{\mathfrak{q}}([\widehat{x}])=g_{\mathfrak{q},y}([\widehat{x}_{\mathfrak{q}}^{+}])$ in $H^1(F_{\mathfrak{q}},Y_y^*(1))$. For each $\mathfrak{q}\in S_p$ we set

$$[u_{\mathfrak{q}}]=[\widehat{x}_{\mathfrak{q}}^{+}]-\mathrm{pr}_{\mathfrak{q},y}([x_{\mathfrak{q},y}^{\mathrm{Iw}}])=\mathrm{res}_{\mathfrak{q}}([\widehat{x}]-g_{\mathfrak{q},y}\circ \mathrm{pr}_{\mathfrak{q},y}([x_{\mathfrak{q},y}^{\mathrm{Iw}}])). \tag{110}$$

Then $\pi_{\mathfrak{q}}([u_{\mathfrak{q}}])=0$, and therefore $[u_{\mathfrak{q}}]\in H^1(F_{\mathfrak{q}},E(1))$.

Definition. — *Let V be a p-adic representation of $G_{F,S}$ equipped with a family $\mathbf{D}=(\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ of $(\varphi,\Gamma_{\mathfrak{q}})$-modules satisfying conditions* **N1-2)**. *The p-adic height pairing $h_{V,\mathbf{D}}^{\mathrm{norm}}$ associated to these data is defined to be the map*

$$h_{V,\mathbf{D}}^{\mathrm{norm}}\,:\,H^1(V,\mathbf{D})\times H^1(V^*(1),\mathbf{D}^{\perp})\to E,$$
$$h_{V,\mathbf{D}}^{\mathrm{norm}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])=\sum_{\mathfrak{q}\in S_p}\ell_{\mathfrak{q}}\left([u_{\mathfrak{q}}]\right).$$

Remarks 5.1.7. — 1) If $[\widetilde{x}^{\mathrm{sel}}]\in H^1(Y_y^*(1),\mathbf{D}_y)$ is another lift of $[x^{\mathrm{sel}}]$, then from (107) and the fact that $[\widehat{x}_{\mathfrak{q}}^{+}]=[\widetilde{x}_{\mathfrak{q}}^{+}]=s_{y,\mathfrak{q}}([x_{\mathfrak{q}}^{+}])$ for all $\mathfrak{q}\in\Sigma_p$, it follows that the definition of $h_{V,D}^{\mathrm{norm}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])$ does not depend on the choice of the lift $[\widehat{x}_{\mathfrak{q}}^{+}]$.

2) It is not indispensable to take $[\widehat{x}^{\mathrm{sel}}]$ in $H^1(Y_y^*(1),\mathbf{D}_y)$. If $[\widehat{x}]\in H^1_S(Y_y^*(1))$ is such that $\pi([\widehat{x}])=[x]$, we can again define $[u_{\mathfrak{q}}]$ by (110). For $\mathfrak{q}\in\Sigma_p$ we set

$$[u_{\mathfrak{q}}]=\mathrm{res}_{\mathfrak{q}}([\widehat{x}]-g_{\mathfrak{q},y}\circ s_{y,\mathfrak{q}}([x_{\mathfrak{q}}^{+}])),$$

where $s_{y,\mathfrak{q}}\,:\,H^1_f(F_{\mathfrak{q}},V)\xrightarrow{\sim}H^1_f(F_{\mathfrak{q}},Y_y^*(1))$ denotes the isomorphism from Lemma 5.1.3 i). Note that again $[u_{\mathfrak{q}}]\in H^1(F_{\mathfrak{q}},E(1))$. Then

$$h_{V,\mathbf{D}}^{\mathrm{norm}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])=\sum_{\mathfrak{q}\in S}\ell_{\mathfrak{q}}\left([u_{\mathfrak{q}}]\right).$$

3) The map $h_{V,\mathbf{D}}^{\mathrm{norm}}$ is bilinear. This can be shown directly, but follows from Theorem 5.2.2 below.

5.2. Comparision with $h_{V,\mathbf{D}}^{\mathrm{sel}}$

5.2.1. — In this subsection we compare $h_{V,\mathbf{D}}^{\mathrm{norm}}$ with the p-adic height pairing constructed in Subsection 3.2. We take $\Sigma=\emptyset$ and denote by

$$h_{V,\mathbf{D},1}^{\mathrm{sel}}\,:\,H^1(V,\mathbf{D})\times H^1(V^*(1),\mathbf{D}^{\perp})\to E$$

the associated height pairing (78).

Theorem 5.2.2. — *Let V be a p-adic representation of $G_{F,S}$ with coefficients in a finite extension E of $\mathbf{Q}_p$. Assume that the family $\mathbf{D}=(\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ satisfies conditions* **N1-2)**. *Then $h^{\mathrm{norm}}_{V,\mathbf{D}}$ is a bilinear map and*

$$h^{\mathrm{norm}}_{V,\mathbf{D}}=-h^{\mathrm{sel}}_{V,\mathbf{D},1}.$$

Proof. — The proof repeats the arguments of [**56**, Sections 11.3.9-11.3.12], where this statement is proved in the case of p-adic height pairings arising from Greenberg's local conditions. We remark that in this case our definition of $h^{\mathrm{norm}}_{V,\mathbf{D}}$ differs from Nekovář's h^{norm}_{π} by a sign.

Let $[x^{\mathrm{sel}}]\in H^1(V,\mathbf{D})$ and $[y^{\mathrm{sel}}]\in H^1(V^*(1),\mathbf{D}^{\perp})$. We use the notation of Section 3.1 and denote by $f_{\mathfrak{q}}$ and $g_{\mathfrak{q}}$ the morphisms defined by (43–46). As before, to simplify notation we set $x_{\mathfrak{q}}=f_{\mathfrak{q}}(x)$ and $y_{\mathfrak{q}}=f^{\perp}_{\mathfrak{q}}(y)$. We represent $[x^{\mathrm{sel}}]$ and $[y^{\mathrm{sel}}]$ by cocycles $x^{\mathrm{sel}}=(x,(x^+_{\mathfrak{q}}),(\lambda_{\mathfrak{q}}))\in S^1(V,\mathbf{D})$ and $y^{\mathrm{sel}}=(y,(y^+_{\mathfrak{q}}),(\mu_{\mathfrak{q}}))\in S^1(V^*(1),\mathbf{D}^{\perp})$, where

$$\begin{aligned} &x\in C^1(G_{F,S},V), && x^+_{\mathfrak{q}}\in U^1_{\mathfrak{q}}(V,\mathbf{D}), && \lambda_{\mathfrak{q}}\in K^0_{\mathfrak{q}}(V),\\ &y\in C^1(G_{F,S},V^*(1)), && y^+_{\mathfrak{q}}\in U^1_{\mathfrak{q}}(V^*(1),\mathbf{D}^{\perp}), && \mu_{\mathfrak{q}}\in K^0_{\mathfrak{q}}(V^*(1)) \end{aligned}$$

and for all $\mathfrak{q}\in S$

$$\begin{aligned} &dx=0, && dy=0,\\ &dx^+_{\mathfrak{q}}=0, && dy^+_{\mathfrak{q}}=0,\\ &g_{\mathfrak{q}}(x^+_{\mathfrak{q}})=f_{\mathfrak{q}}(x)+d\lambda_{\mathfrak{q}}, && g^{\perp}_{\mathfrak{q}}(y^+_{\mathfrak{q}})=f^{\perp}_{\mathfrak{q}}(x)+d\mu_{\mathfrak{q}}. \end{aligned}$$

For simplicity, we will use the same notation for the resulting maps on cohomologies, namely

$$f_{\mathfrak{q}}=\mathrm{res}_{\mathfrak{q}}\,:\,H^i_S(V)\to H^i(F_{\mathfrak{q}},V),\quad g_{\mathfrak{q}}\,:\,H^i(\mathbf{D}_{\mathfrak{q}})\to H^i(F_{\mathfrak{q}},V),\quad \mathfrak{q}\in S_p.$$

This agrees with the notation used in Section 5.1. Also, we will write $f_{\mathfrak{q},y}$ and $g_{\mathfrak{q},y}$ for the maps $f_{\mathfrak{q}}$ and $g_{\mathfrak{q},y}$ associated to the data $(Y^*_y(1),\mathbf{D}_y)$.

By Propositions 2.7.2, 2.7.4 and 2.7.5 we have

$$\beta_{V,\mathbf{D}}(x^{\mathrm{sel}})=(-z\cup x,(-w_{\mathfrak{q}}\cup x^+_{\mathfrak{q}}),(z_{\mathfrak{q}}\cup\lambda_{\mathfrak{q}}))\in S^2(V,\mathbf{D}), \tag{111}$$

where

$$\begin{aligned} &z=\log\chi\in C^1(G_{F,S},E(0)),\\ &w_{\mathfrak{q}}=\begin{cases}0, & \text{if } \mathfrak{q}\in\Sigma_p,\\ (0,\log\chi_{\mathfrak{q}}(\gamma_{\mathfrak{q}}))\in C^1_{\varphi,\gamma_{\mathfrak{q}}}(E(0)), & \text{if } \mathfrak{q}\in S_p,\end{cases}\\ &z_{\mathfrak{q}}=\begin{cases}\log\chi_{\mathfrak{q}}\in C^1(G_{F_{\mathfrak{q}}},E(0)), & \text{if } \mathfrak{q}\in\Sigma_p,\\ (0,\log\chi_{\mathfrak{q}})\in K^1(E(0)_{\mathfrak{q}}), & \text{if } \mathfrak{q}\in S_p.\end{cases} \end{aligned}\tag{112}$$

Let $[\widehat{x}] \in H^1_S(Y_y^*(1))$ be a lift of $[x] \in H^1_S(V)$. The diagram (109) shows, that there exist unique cohomology classes

$$[\widehat{x}_{\mathfrak{q}}^+] \in H^1(\mathbf{D}_{\mathfrak{q},y}), \qquad \mathfrak{q} \in S_p,$$
$$[\widehat{x}_{\mathfrak{q}}^+] \in H^1_f(F_{\mathfrak{q}}, Y_y^*(1)), \quad \mathfrak{q} \in \Sigma_p$$

represented by cocycles $\widehat{x} \in C^1(G_{F,S}, Y_y^*(1))$, $\widehat{x}_{\mathfrak{q}}^+ \in C^1_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y})$ (if $\mathfrak{q} \in S_p$), and $\widehat{x}_{\mathfrak{q}}^+ \in C^1_{\mathrm{ur}}(Y_y^*(1)_{\mathfrak{q}})$ (if $\mathfrak{q} \in \Sigma_p$) such that

$$g_{\mathfrak{q},y}\left([\widehat{x}_{\mathfrak{q}}^+]\right) = f_{\mathfrak{q},y}\left([\widehat{x}]\right), \qquad \mathfrak{q} \in S_p \cup \Sigma_p.$$

Since $g_{\mathfrak{q},y}(\widehat{x}_{\mathfrak{q}}^+) = f_{\mathfrak{q},y}(\widehat{x}) + d\widehat{\lambda}_{\mathfrak{q}}$ for some $\widehat{\lambda}_{\mathfrak{q}} \in K^0_{\mathfrak{q}}(Y_y^*(1))$, we obtain a cocycle $\widehat{x}^{\mathrm{sel}} = (\widehat{x}, (\widehat{x}_{\mathfrak{q}}^+), (\widehat{\lambda}_{\mathfrak{q}})) \in S^1(Y_y^*(1), \mathbf{D}_y)$.

Lemma 5.2.3. — *Suppose that for each $\mathfrak{q} \in S_p$ we are given a 1-cocycle $\xi_{\mathfrak{q}} \in C^1_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y})$ such that $\beta_{\mathbf{D}_{\mathfrak{q},y}}([\xi_{\mathfrak{q}}]) = 0$. Then $\beta_{Y_y^*(1),\mathbf{D}_y}(\widehat{x}^{\mathrm{sel}})$ is homologous to a cocycle of the form*

$$(\widehat{a}, (\widehat{b}_{\mathfrak{q}}), (\widehat{c}_{\mathfrak{q}})) \in S^2(Y_y^*(1), \mathbf{D}_y),$$

where

$$\widehat{b}_{\mathfrak{q}} = \begin{cases} 0, & \text{if } \mathfrak{q} \in \Sigma_p, \\ w_{\mathfrak{q}} \cup (\xi_{\mathfrak{q}} - \widehat{x}_{\mathfrak{q}}^+) \in C^2_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y}), & \text{if } \mathfrak{q} \in S_p. \end{cases}$$

Proof. — By (111), we have

$$\beta_{Y_y^*(1),\mathbf{D}_y}(\widehat{x}^{\mathrm{sel}}) = (-z \cup \widehat{x}, (-w_{\mathfrak{q}} \cup \widehat{x}_{\mathfrak{q}}^+), (z_{\mathfrak{q}} \cup \widehat{\lambda}_{\mathfrak{q}})).$$

If $\mathfrak{q} \in \Sigma_p$, we have $w_{\mathfrak{q}} = 0$ and $w_{\mathfrak{q}} \cup \widehat{x}_{\mathfrak{q}}^+ = 0$, If $\mathfrak{q} \in S_p$, we have

$$\widehat{b}_{\mathfrak{q}} = w_{\mathfrak{q}} \cup (\xi_{\mathfrak{q}} - \widehat{x}_{\mathfrak{q}}^+) = -w_{\mathfrak{q}} \cup \widehat{x}_{\mathfrak{q}}^+ + w_{\mathfrak{q}} \cup \xi_{\mathfrak{q}}.$$

Since $\beta_{\mathbf{D}_{\mathfrak{q},y}}([\xi_{\mathfrak{q}}]) = 0$, there exists $\nu_{\mathfrak{q}} \in C^1_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y})$ such that $w_{\mathfrak{q}} \cup \xi_{\mathfrak{q}} = d\nu_{\mathfrak{q}}$. Therefore,

$$\beta_{Y_y^*(1),\mathbf{D}_y}(\widehat{x}^{\mathrm{sel}}) = (-z \cup \widehat{x}, (\widehat{b}_{\mathfrak{q}}), (z_{\mathfrak{q}} \cup \widehat{\lambda}_{\mathfrak{q}} + g_{\mathfrak{q}}(\nu_{\mathfrak{q}}))) - d(0, (\nu_{\mathfrak{q}}), 0)$$

and we can set $\widehat{a} = -z \cup \widehat{x}$ and $\widehat{c}_{\mathfrak{q}} = z_{\mathfrak{q}} \cup \widehat{\lambda}_{\mathfrak{q}} + g_{\mathfrak{q}}(\nu_{\mathfrak{q}})$ for all $\mathfrak{q} \in S_p$. The lemma is proved. □

For each $\mathfrak{q} \in S_p$, we have the canonical isomorphism of local class field theory

$$\mathrm{inv}_{F_{\mathfrak{q}}} : H^2(F_{\mathfrak{q}}, E(1)) \xrightarrow{\sim} E.$$

Let $\kappa_{\mathfrak{q}} : F_{\mathfrak{q}}^* \widehat{\otimes} E \to H^1(F_{\mathfrak{q}}, E(1))$ denote the Kummer map. Then

$$\mathrm{inv}_{F_{\mathfrak{q}}}(\log \chi_{\mathfrak{q}} \cup \kappa_{\mathfrak{q}}(x)) = \log_p(N_{F_{\mathfrak{q}}/\mathbf{Q}_p}(x)) = \ell_{\mathfrak{q}}(\kappa_{\mathfrak{q}}(x))$$

([**68**, Chapitre 14], see also [**5**, Corollaire 1.1.3]) and therefore

$$(113) \qquad \mathrm{inv}_{F_{\mathfrak{q}}}(\log \chi_{\mathfrak{q}} \cup [b]) = \ell_{\mathfrak{q}}([b]), \qquad \text{for all } [b] \in H^1(F_{\mathfrak{q}}, E(1)).$$

Lemma 5.2.4. — *Assume that* $\beta_{V,\mathbf{D}}([x^{\mathrm{sel}}]) \in H^2(V,\mathbf{D})$ *is represented by a* 2*-cocycle* $e=(a,(b_{\mathfrak{q}}),(c_{\mathfrak{q}}))$ *of the form* $e=\pi(\widehat{e})$, *where*

$$\widehat{e}=(\widehat{a},(\widehat{b}_{\mathfrak{q}}),(\widehat{c}_{\mathfrak{q}}))\in S^2(Y_y^*(1),\mathbf{D}_y)$$

is also a 2*-cocycle and* $\pi\,:\,S^2(Y_y^*(1),\mathbf{D}_y)\to S^2(V,\mathbf{D})$ *denotes the canonical projection. Then*

$$[\beta_{V,\mathbf{D}}(x^{\mathrm{sel}})]\cup[y^{\mathrm{sel}}]=\sum_{\mathfrak{q}\in S_p}\mathrm{inv}_{F_{\mathfrak{q}}}([g_{\mathfrak{q},y}(\widehat{b}_{\mathfrak{q}})\cup f_{\mathfrak{q}}^{\perp}(\alpha_y)+g_{\mathfrak{q}}(b_{\mathfrak{q}})\cup\mu_{\mathfrak{q}}]),$$

where $\alpha_y\in C^0(G_{F,S},Y_y)$ *is an element that maps to* $1\in C^0(G_{F,S},E)=E$ *and satisfies* $d\alpha_y=y$. *If, in addition,*

$$\widehat{b}_{\mathfrak{q}}\in C^2_{\varphi,\gamma_{\mathfrak{q}}}(E(1)_{\mathfrak{q}}),\qquad \forall\mathfrak{q}\in S_p,$$

then

$$[\beta_{V,\mathbf{D}}(x^{\mathrm{sel}})]\cup[y^{\mathrm{sel}}]=\sum_{\mathfrak{q}\in S_p}\mathrm{inv}_{F_{\mathfrak{q}}}([\widehat{b}_{\mathfrak{q}}]),$$

where we identify $[\widehat{b}_{\mathfrak{q}}]\in H^2(\mathscr{R}_{F_{\mathfrak{q}},E}(\chi_{\mathfrak{q}}))$ *with an element of* $H^2(F_{\mathfrak{q}},E(1))$ *using Theorem 2.4.3.*

Proof. — The proof of this lemma is purely formal and follows *verbatim* the proof of [**56**, Lemma 11.3.11]. □

Now we can proof Theorem 5.2.2. Take $\xi_{\mathfrak{q}}=\mathrm{pr}_{\mathfrak{q},y}(x^{\mathrm{Iw}}_{\mathfrak{q},y})$. Then $[u_{\mathfrak{q}}]=[\widehat{x}^{+}_{\mathfrak{q}}]-[\xi_{\mathfrak{q}}]$ coincides with the cohomology class (110) used in the definition of $h^{\mathrm{norm}}_{V,\mathbf{D}}$. Since the map

$$C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\mathrm{Ind}_{F_{\mathfrak{q},\infty}/F_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y}))\to C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\mathbf{D}_{\mathfrak{q},y})$$

factors through $C^{\bullet}_{\varphi,\gamma_{\mathfrak{q}}}(\widetilde{\mathbf{D}}_{\mathfrak{q},y})$, where $\widetilde{\mathbf{D}}_{\mathfrak{q},y}=\mathbf{D}_{\mathfrak{q},y}\otimes\widetilde{A}^{\iota}_{F_{\mathfrak{q}}}$, from the distinguished triangle

$$\mathbf{R}\Gamma(F_{\mathfrak{q}},\mathbf{D}_{\mathfrak{q},y})\to\mathbf{R}\Gamma(F_{\mathfrak{q}},\widetilde{\mathbf{D}}_{\mathfrak{q},y})\to\mathbf{R}\Gamma(F_{\mathfrak{q}},\mathbf{D}_{\mathfrak{q},y})\xrightarrow{\beta_{\mathbf{D}_{\mathfrak{q},y}}}\mathbf{R}\Gamma(F_{\mathfrak{q}},\mathbf{D}_{\mathfrak{q},y})[1]$$

it follows that $\beta_{\mathbf{D}_{\mathfrak{q},y}}([\xi_{\mathfrak{q}}])=0$. In addition, $[u_{\mathfrak{q}}]\in H^1(F_{\mathfrak{q}},\mathscr{R}_{F_{\mathfrak{q}},E}(\chi_{\mathfrak{q}}))$ and adding a coboundary to $u_{\mathfrak{q}}$ we can assume that $u_{\mathfrak{q}}\in C^1_{\varphi,\gamma_{\mathfrak{q}}}(E(1))$. Combining Lemma 5.2.3 and Lemma 5.2.4 we have

$$\begin{aligned}h^{\mathrm{sel}}_{V,\mathbf{D},1}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])&=[\beta_{V,\mathbf{D}}(x^{\mathrm{sel}})]\cup[y^{\mathrm{sel}}]=\sum_{\mathfrak{q}\in S_p}\mathrm{inv}_{F_{\mathfrak{q}}}([\widehat{b}_{\mathfrak{q}}])=\\&=-\sum_{\mathfrak{q}\in S_p}\mathrm{inv}_{F_{\mathfrak{q}}}([w_{\mathfrak{q}}\cup u_{\mathfrak{q}}])=-\sum_{\mathfrak{q}\in S_p}\mathrm{inv}_{F_{\mathfrak{q}}}(\log\chi_{\mathfrak{q}}\cup[u_{\mathfrak{q}}])=\\&=-\sum_{\mathfrak{q}\in S_p}\ell_{\mathfrak{q}}([u_{\mathfrak{q}}])=-h^{\mathrm{norm}}_{V,\mathbf{D}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}]).\end{aligned}$$

□

CHAPTER 6

p-ADIC HEIGHT PAIRINGS III: SPLITTING OF LOCAL EXTENSIONS

6.1. The pairing $h^{\mathrm{spl}}_{V,D}$

6.1.1. — Let F be a finite extension of $\mathbf{Q}$. We keep notation of Chapters 3-5. In particular, we fix a finite set S of places of F such that $S_p \subset S$ and denote by $G_{F,S}$ the Galois group of the maximal algebraic extension of F which is unramified outside $S \cup S_\infty$. For each topological $G_{F,S}$-module M, we write $H^*_S(M)$ for the continuous cohomology of $G_{F,S}$ with coefficients in M.

Let V be a p-adic representation of $G_{F,S}$ with coefficients in a finite extension $E/\mathbf{Q}_p$ which is potentially semistable at all $\mathfrak{q} \mid p$. Following Bloch and Kato, for each $\mathfrak{q} \in S$ we define the subgroup $H^1_f(F_\mathfrak{q}, V)$ of $H^1(F_\mathfrak{q}, V)$ by

$$H^1_f(F_\mathfrak{q},V) = \begin{cases} \ker(H^1(F_\mathfrak{q},V) \to H^1(F_\mathfrak{q}, V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}})) & \text{if } \mathfrak{q} \mid p, \\ \ker(H^1(F_\mathfrak{q},V) \to H^1(F_\mathfrak{q}^{\mathrm{ur}},V)) & \text{if } \mathfrak{q} \nmid p. \end{cases}$$

The Bloch–Kato Selmer group [**16**] of V is defined as

$$H^1_f(V) = \ker \left(H^1_S(V) \to \bigoplus_{\mathfrak{q} \in S} \frac{H^1(F_\mathfrak{q},V)}{H^1_f(F_\mathfrak{q},V)} \right).$$

In this section, we assume that, for all $\mathfrak{q} \in S_p$, the representation $V_\mathfrak{q}$ satisfies condition **S)** of Section 4.1, namely that

S) $\mathbf{D}_{\mathrm{cris}}(V_\mathfrak{q})^{\varphi=1} = \mathbf{D}_{\mathrm{cris}}(V^*_\mathfrak{q}(1))^{\varphi=1} = 0$ for all $\mathfrak{q} \in S_p$.

As we noticed in Section 0.4, this condition conjecturally always holds if V is the p-adic realization of a pure motive of weight -1. For each $\mathfrak{q} \mid p$, we fix a splitting $(\varphi, N, G_{F_\mathfrak{q}})$-submodule $D_\mathfrak{q}$ of $\mathbf{D}_{\mathrm{pst}}(V_\mathfrak{q})$ (see Section 4.1). We will associate to these data a pairing

$$h^{\mathrm{spl}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

and compare it with the height pairing constructed in [**54**, Section 4] using the exponential map and splitting of the Hodge filtration.

Let $[y]\in H^1_f(V^*(1))$. Fix a representative $y\in C^1(G_{F,S},V^*(1))$ of y and consider the corresponding extension of Galois representations

$$0\to V^*(1)\to Y_y\to E\to 0. \tag{114}$$

Passing to duals, we obtain an extension

$$0\to E(1)\to Y_y^*(1)\to V\to 0.$$

From **S)**, it follows that $H^0_S(V)=0$, and the associated long exact sequence of global Galois cohomology reads

$$0\to H^1_S(E(1))\to H^1_S(Y_y^*(1))\to H^1_S(V)\xrightarrow{\delta^1_V} H^2_S(E(1))\to\dots.$$

Also, for each place $\mathfrak{q}\in S$ we have the long exact sequence of local Galois cohomology

$$H^0(F_\mathfrak{q},V)\to H^1(F_\mathfrak{q},E(1))\to H^1(F_\mathfrak{q},Y_y^*(1))\to$$
$$\to H^1(F_\mathfrak{q},V)\xrightarrow{\delta^1_{V,\mathfrak{q}}} H^2(F_\mathfrak{q},E(1))\to\dots.$$

The following results, which can be seen as an analog of Lemma 5.1.3, are well known but we recall them for the reader's convenience.

Lemma 6.1.2. — *Let V be a p-adic representation of $G_{F,S}$ that is potentially semistable at all $\mathfrak{q}\in S_p$ and satisfies condition* **S)**. *Assume that $[y]\in H^1_f(V^*(1))$. Then*

i) $\delta^1_V([x])=0$ for all $x\in H^1_f(V)$;

ii) There exists an exact sequence

$$0\to H^1_f(E(1))\to H^1_f(Y_y^*(1))\to H^1_f(V)\to 0.$$

Proof. — i) For any $x\in C^1(G_{F,S},V)$, let $x_\mathfrak{q}=\mathrm{res}_\mathfrak{q}(x)\in C^1(G_{F_\mathfrak{q}},V)$ denote the localization of x at $\mathfrak{q}$. If $[x]\in H^1_f(V)$, then for each $\mathfrak{q}$ one has $\delta^1_{V,\mathfrak{q}}([x_\mathfrak{q}])=-[x_\mathfrak{q}]\cup[y_\mathfrak{q}]=0$ because $H^1_f(F_\mathfrak{q},V)$ and $H^1_f(F_\mathfrak{q},V^*(1))$ are orthogonal to each other under the cup product. Since the map $H^2_S(E(1))\to\bigoplus_{\mathfrak{q}\in S}H^2(F_\mathfrak{q},E(1))$ is injective and the localization commutes with cup products, this shows that $\delta^1_V([x])=0$.

ii) This is a particular case of [**32**, Proposition II, 2.2.3]. □

6.1.3. — Let $[x]\in H^1_f(V)$ and $[y]\in H^1_f(V^*(1))$. In Section 4.2, for each $\mathfrak{q}\in S_p$ we constructed the canonical splitting (83) which sits in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1_f(\mathbf{D}_{\mathfrak{q},y}) & \overset{s_{y,\mathfrak{q}}}{\rightleftarrows} & H^1_f(\mathbf{D}_{\mathfrak{q}}) & \longrightarrow & 0\\
 & & \Big\downarrow = & & g_{\mathfrak{q},y}\Big\downarrow\simeq & & g_{\mathfrak{q}}\Big\downarrow\simeq & & \\
0 & \longrightarrow & H^1_f(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1_f(F_{\mathfrak{q}},Y_y^*(1)) & \longrightarrow & H^1_f(F_{\mathfrak{q}},V) & \longrightarrow & 0.
\end{array}$$

By Lemma 6.1.2 ii), we can lift $[x]\in H^1_f(V)$ to an element $[\widehat{x}]\in H^1_f(Y_y^*(1))$. Let $[\widehat{x}_{\mathfrak{q}}]=\mathrm{res}_{\mathfrak{q}}\left([\widehat{x}]\right)\in H^1_f(F_{\mathfrak{q}},Y_y^*(1))$. If $\mathfrak{q}\in S_p$, we denote by $[\widetilde{x}^+_{\mathfrak{q}}]$ the unique element of $H^1_f(\mathbf{D}_{\mathfrak{q}})$ such that $g_{\mathfrak{q}}([\widetilde{x}^+_{\mathfrak{q}}])=[x_{\mathfrak{q}}]$.

***Definition*. —** *The p-adic height pairing associated to splitting submodules $D=(D_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ is defined to be the map*

$$h^{\mathrm{spl}}_{V,D}\,:\,H^1_f(V)\times H^1_f(V^*(1))\to E$$

given by

$$h^{\mathrm{spl}}_{V,D}([x],[y])=\sum_{\mathfrak{q}\in S_p}\ell_{\mathfrak{q}}\left([\widehat{x}_{\mathfrak{q}}]-g_{\mathfrak{q},y}\circ s_{y,\mathfrak{q}}([\widetilde{x}^+_{\mathfrak{q}}])\right).$$

***Remarks 6.1.4*. —** 1) For each $\mathfrak{q}\in\Sigma_p$, denote by $s_{y,\mathfrak{q}}\,:\,H^1_f(F_{\mathfrak{q}},V)\xrightarrow{\sim}H^1_f(F_{\mathfrak{q}},Y_y^*(1))$ the isomorphism constructed in Lemma 5.1.3, i) and by $g_{\mathfrak{q}}\,:\,H^1_f(F_{\mathfrak{q}},V)\hookrightarrow H^1(F_{\mathfrak{q}},V)$ and $g_{\mathfrak{q},y}\,:\,H^1_f(F_{\mathfrak{q}},Y_y^*(1))\hookrightarrow H^1(F_{\mathfrak{q}},Y_y^*(1))$ the canonical embeddings. Let $[\widetilde{x}^+_{\mathfrak{q}}]\in H^1_f(F_{\mathfrak{q}},V)$ be the unique element such that $g_{\mathfrak{q}}([x^+_{\mathfrak{q}}])=[x_{\mathfrak{q}}]$. From the product formula (107) it follows, that $h^{\mathrm{spl}}_{V,D}$ can be defined by

$$h^{\mathrm{spl}}_{V,D}([x],[y])=\sum_{\mathfrak{q}\in S}\ell_{\mathfrak{q}}\left([\widehat{x}_{\mathfrak{q}}]-g_{\mathfrak{q},y}\circ s_{y,\mathfrak{q}}([\widetilde{x}^+_{\mathfrak{q}}])\right),$$

where $[\widehat{x}]\in H^1_S(V)$ is an arbitrary lift of $[x]$.

2) The pairing $h^{\mathrm{spl}}_{V,D}$ is a bilinear skew-symmetric map. This can be shown directly, but follows from the interpretation of $h^{\mathrm{spl}}_{V,D}$ in terms of Nekovář's height pairing (see Proposition 6.2.3 below).

6.2. Comparision with Nekovář's height pairing

6.2.1. — We relate the pairing $h^{\mathrm{spl}}_{V,D}$ to the p-adic height pairing constructed by Nekovář in [**54**, Section 4]. First recall Nekovář's construction. If $[y]\in H^1_f(V^*(1))$, the extension (114) is crystalline at all $\mathfrak{q}\in S_p$, and therefore the sequence

$$0\to\mathbf{D}_{\mathrm{cris}}(V^*_{\mathfrak{q}}(1))\to\mathbf{D}_{\mathrm{cris}}(Y_{y,\mathfrak{q}})\to\mathbf{D}_{\mathrm{cris}}(E(0)_{\mathfrak{q}})\to 0$$

is exact. Since $\mathbf{D}_{\mathrm{cris}}(V_{\mathfrak{q}}^*(1))^{\varphi=1}=0$, we have an isomorphism of vector spaces

$$\mathbf{D}_{\mathrm{cris}}(E(0)_{\mathfrak{q}}) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(Y_{y,\mathfrak{q}})^{\varphi=1},$$

which can be extended by linearity to a map $\mathbf{D}_{\mathrm{dR}}(E(0)_{\mathfrak{q}}) \to \mathbf{D}_{\mathrm{dR}}(Y_{y,\mathfrak{q}})$. Passing to duals, we obtain a $F_{\mathfrak{q}}$-linear map $\mathbf{D}_{\mathrm{dR}}(Y_{y,\mathfrak{q}}^*(1)) \to \mathbf{D}_{\mathrm{dR}}(E(1)_{\mathfrak{q}})$ which defines a splitting $s_{\mathrm{dR},\mathfrak{q}}$ of the exact sequence

$$0 \longrightarrow \mathbf{D}_{\mathrm{dR}}(E(1)_{\mathfrak{q}}) \longrightarrow \mathbf{D}_{\mathrm{dR}}(Y_{y,\mathfrak{q}}^*(1)) \underset{\longrightarrow}{\overset{s_{\mathrm{dR},\mathfrak{q}}}{\longleftarrow}} \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \longrightarrow 0.$$

Fix a splitting $w_{\mathfrak{q}}: \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})$ of the canonical projection

$$\mathrm{pr}_{\mathrm{dR},V_{\mathfrak{q}}}: \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) \to \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}). \tag{115}$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1(F_{\mathfrak{q}},E(1)) & \longrightarrow & H^1_f(F_{\mathfrak{q}},Y_y^*(1)) & \underset{\longrightarrow}{\overset{s^w_{y,\mathfrak{q}}}{\longleftarrow}} & H^1_f(F_{\mathfrak{q}},V) & \longrightarrow 0 \\
 & & & \uparrow \exp_{Y^*_{y,\mathfrak{q}}(1)} \simeq & & \uparrow \exp_{V_{\mathfrak{q}}} \simeq & \\
 & & & \dfrac{\mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1))}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1))} & \longrightarrow & \dfrac{\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})} & \\
 & & & \uparrow \mathrm{pr}_{\mathrm{dR},Y^*_{y,\mathfrak{q}}(1)} & & \downarrow w_{\mathfrak{q}} & \\
 & & & \mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1)) & \xleftarrow{s_{\mathrm{dR},\mathfrak{q}}} & \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}). &
\end{array}$$

Then the map $s^w_{y,\mathfrak{q}}: H^1_f(F_{\mathfrak{q}},V) \to H^1_f(F_{\mathfrak{q}},Y^*_y(1))$ defined by

$$s^w_{y,\mathfrak{q}} = \exp_{Y^*_{y,\mathfrak{q}}(1)} \circ \mathrm{pr}_{\mathrm{dR},Y^*_{y,\mathfrak{q}}(1)} \circ s_{\mathrm{dR},\mathfrak{q}} \circ w_{\mathfrak{q}} \circ \exp^{-1}_{V_{\mathfrak{q}}}$$

gives a splitting of the top row of the diagram, which depends only on the choice of $w_{\mathfrak{q}}$ and $[y]$.

***Definition* (Nekovář). —** *The p-adic height pairing associated to a family $w=(w_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ of splitting $w_{\mathfrak{q}}$ of the projections (115) is defined to be the map*

$$h^{\mathrm{Hodge}}_{V,w}: H^1_f(V)\times H^1_f(V^*(1)) \to E$$

given by

$$h^{\mathrm{Hodge}}_{V,w}([x],[y]) = \sum_{\mathfrak{q}|p} \ell_{\mathfrak{q}}\left([\widehat{x}_{\mathfrak{q}}] - s^w_{y,\mathfrak{q}}([x_{\mathfrak{q}}])\right),$$

where $[\widehat{x}] \in H^1_f(Y^*_y(1))$ is a lift of $[x]\in H^1_f(V)$ and $[\widehat{x}_{\mathfrak{q}}]$ denotes its localization at $\mathfrak{q}$.

In [**54**], it is proved that $h^{\mathrm{Hodge}}_{V,w}$ is a E-bilinear map.

6.2.2. — Now, let $D_{\mathfrak{q}}$ be a splitting submodule of $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})$. We have

$$\mathbf{D}_{\mathrm{dR}/L}(V_{\mathfrak{q}}) = D_{\mathfrak{q},L} \oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}/L}(V_{\mathfrak{q}}), \qquad D_{\mathfrak{q},L} = D_{\mathfrak{q}} \otimes_{L_0} L. \tag{116}$$

Set $D_{\mathfrak{q},F_{\mathfrak{q}}} = (D_{\mathfrak{q},L})^{G_{F_{\mathfrak{q}}}}$. Since the decomposition (116) is compatible with the Galois action, taking Galois invariants we have

$$\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) = D_{\mathfrak{q},F_{\mathfrak{q}}} \oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}).$$

This decomposition defines a splitting of the projection (115) which we will denote by $w_{D,\mathfrak{q}}$.

***Proposition 6.2.3.* —** *Let V be a p-adic representation of $G_{F,S}$ such that for each $\mathfrak{q} \in S_p$ the restriction of V on the decomposition group at $\mathfrak{q}$ is potentially semistable and satisfies condition* **S**). *Let $(D_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ be a family of splitting submodules and let $w_D = (w_{D,\mathfrak{q}})_{\mathfrak{q}\in S_p}$ be the associated system of splittings. Then*

$$h^{\mathrm{spl}}_{V,D} = h^{\mathrm{Hodge}}_{V,w_D}.$$

We need the following auxiliary result. As before, we denote by $\mathbf{D}_{\mathfrak{q}}$ the $(\varphi,\Gamma_{\mathfrak{q}})$-module associated to $D_{\mathfrak{q}}$.

***Lemma 6.2.4.* —** *The following diagram*

$$\begin{CD} \mathscr{D}_{\mathrm{dR}}(\mathbf{D}_{\mathfrak{q}}) @>{s_{\mathbf{D}_{\mathfrak{q}},y}}>> \mathscr{D}_{\mathrm{dR}}(\mathbf{D}_{\mathfrak{q},y}) \\ @VVV @VVV \\ \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) @>{s_{\mathrm{dR},\mathfrak{q}}}>> \mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1)), \end{CD}$$

where the vertical maps are induced by the canonical inclusions of corresponding $(\varphi,\Gamma_{\mathfrak{q}})$-modules and $s_{\mathbf{D}_{\mathfrak{q}},y}$ is the map induced by the splitting (82), is commutative.

Proof of the lemma. — The proof is an easy exercice and is omitted here. □

Proof of Proposition 6.2.3. — From the functoriality of the exponential map and Proposition 4.1.4 it follows that the diagram

$$\begin{CD} \mathscr{D}_{\mathrm{dR}}(\mathbf{D}_{\mathfrak{q}}) @>{\exp_{\mathbf{D}_{\mathfrak{q}}}}>> H^1_f(\mathbf{D}_{\mathfrak{q}}) \\ @V{=}VV @V{=}VV \\ \mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) @>{\exp_{V_{\mathfrak{q}}}}>> H^1_f(F_{\mathfrak{q}},V) \end{CD} \tag{117}$$

is commutative. The same holds if we replace $V_{\mathfrak{q}}$ and $\mathbf{D}_{\mathfrak{q}}$ by $Y^*_{y,\mathfrak{q}}(1)$ and $\mathbf{D}_{\mathfrak{q},y}$ respectively. Consider the diagram

$$
(118) \qquad
\begin{array}{ccc}
\mathscr{D}_{\mathrm{dR}}(\mathbf{D}_{\mathfrak{q}}) & \xrightarrow{s_{\mathbf{D}_{\mathfrak{q},y}}} & \mathscr{D}_{\mathrm{dR}}(\mathbf{D}_{\mathfrak{q},y}) \\
\downarrow & & \uparrow {\scriptstyle =} \\
\dfrac{\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}})} & & \dfrac{\mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1))}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1))}, \\
\downarrow {\scriptstyle w_{D,\mathfrak{q}}} & & \uparrow {\scriptstyle \mathrm{pr}_{\mathrm{dR},Y^*_{y,\mathfrak{q}}(1)}} \\
\mathbf{D}_{\mathrm{dR}}(V_{\mathfrak{q}}) & \xrightarrow{s_{\mathrm{dR},\mathfrak{q}}} & \mathbf{D}_{\mathrm{dR}}(Y^*_{y,\mathfrak{q}}(1)).
\end{array}
$$

From the definition of $w_{D,\mathfrak{q}}$, it follows that the composition of vertical maps in the left (resp. right) colomn is induced by the inclusion $\mathbf{D}_{\mathfrak{q}} \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$ (resp. by $\mathbf{D}_{\mathfrak{q},y} \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y^*_{y,\mathfrak{q}}(1))$) and therefore the diagram (118) is commutative by Lemma 6.2.4. From the commutativity of (117) and (118) and the definition of $s_{y,\mathfrak{q}}$ and $s^{w}_{y,\mathfrak{q}}$, it follows now that $s_{y,\mathfrak{q}} = s^{w}_{y,\mathfrak{q}}$ for all $\mathfrak{q} \in S_p$, and the proposition is proved. □

6.3. Comparision with $h^{\mathrm{norm}}_{V,D}$

6.3.1. — In this section, we compare the pairing $h^{\mathrm{spl}}_{V,D}$ with the pairing $h^{\mathrm{norm}}_{V,D}$ constructed in Chapter 5. Let V be a p-adic representation of $G_{F,S}$ that is potentially semistable at all $\mathfrak{q} \in S_p$. Fix a system $(D_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ of splitting submodules and denote by $(\mathbf{D}_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ the system of $(\varphi,\Gamma_{\mathfrak{q}})$-submodules of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_{\mathfrak{q}})$ associated to $(D_{\mathfrak{q}})_{\mathfrak{q}\in S_p}$ by Theorem 2.2.3. We will assume, that (V,D) satisfies condition **S)** of Section 6.1 and condition **N2)** of Section 5.1. Note that **S)** implies **N1)**. We also remark, that from Proposition 2.9.2 i) and the fact that the Hodge–Tate weights of $\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})/D_{\mathfrak{q}}$ and $\mathbf{D}_{\mathrm{st}/L}(V^*_{\mathfrak{q}}(1))/D^{\perp}_{\mathfrak{q}}$ are positive, it follows that, under our assumptions, **N2)** is equivalent to the following condition

N2*) For each $\mathfrak{q} \in S_p$,

$$(\mathbf{D}_{\mathrm{st}/L}(V_{\mathfrak{q}})/D_{\mathfrak{q}})^{\varphi=1,N=0,G_{L/F_{\mathfrak{q}}}} = (\mathbf{D}_{\mathrm{st}/L}(V^*_{\mathfrak{q}}(1))/D^{\perp}_{\mathfrak{q}})^{\varphi=1,N=0,G_{L/F_{\mathfrak{q}}}} = 0,$$

where L is a finite extension of $F_{\mathfrak{q}}$ such that $V_{\mathfrak{q}}$ (respectively $V^*_{\mathfrak{q}}(1)$) is semistable over L.

The following statement is known ([**62, 10**]), but we prove it here for completeness.

Proposition 6.3.2. — *Assume that V is a p-adic representation satisfying conditions* **S)** *and* **N2*)**. *Then* *i)* $H^1_f(F_\mathfrak{q},V)=H^1_f(\mathbf{D}_\mathfrak{q})=H^1(\mathbf{D}_\mathfrak{q})$ *and* $H^1_f(F_\mathfrak{q},V^*(1))=H^1_f(\mathbf{D}^\perp_\mathfrak{q})=H^1(\mathbf{D}^\perp_\mathfrak{q})$ *for all* $\mathfrak{q}\in S_p$.

ii) $H^1_f(V)\simeq H^1(V,\mathbf{D})$ *and* $H^1_f(V^*(1))\simeq H^1(V^*(1),\mathbf{D}^\perp)$.

Proof. — i) The first statement follows from **N2)** and Proposition 4.1.4 iii).

ii) Note that by i)

$$\mathbf{R}^1\Gamma(F_\mathfrak{q},V,\mathbf{D})=\begin{cases}H^1_f(F_\mathfrak{q},V), & \text{if } \mathfrak{q}\in\Sigma_p,\\ H^1(\mathbf{D}_\mathfrak{q}), & \text{if } \mathfrak{q}\in S_p.\end{cases}$$

By definition, the group $H^1(V,\mathbf{D})$ is the kernel of the morphism

$$H^1_S(V)\bigoplus\left(\bigoplus_{\mathfrak{q}\in\Sigma_p}H^1_f(F_\mathfrak{q},V)\right)\bigoplus\left(\bigoplus_{\mathfrak{q}\in S_p}H^1(\mathbf{D}_\mathfrak{q})\right)\to\bigoplus_{\mathfrak{q}\in S}H^1(F_\mathfrak{q},V)$$

given by

$$([x],[y_\mathfrak{q}]_{\mathfrak{q}\in S})\mapsto([x_\mathfrak{q}]-g_\mathfrak{q}([y_\mathfrak{q}]))_{\mathfrak{q}\in S},\qquad [x_\mathfrak{q}]=\mathrm{res}_\mathfrak{q}([x]),$$

where $g_\mathfrak{q}$ denotes the canonical inclusion $H^1_f(F_\mathfrak{q},V)\to H^1(F_\mathfrak{q},V)$ if $\mathfrak{q}\in\Sigma_p$ and the map $H^1(\mathbf{D}_\mathfrak{q})\to H^1(F_\mathfrak{q},V)$ if $\mathfrak{q}\in S_p$. In the both cases, $g_\mathfrak{q}$ is injective and, in addition, for each $\mathfrak{q}\in S_p$ we have $H^1(\mathbf{D}_\mathfrak{q})=H^1_f(F_\mathfrak{q},V)$ by i). This implies that $H^1(V,\mathbf{D})=H^1_f(V)$. The same argument shows that $H^1(V^*(1),\mathbf{D}^\perp)=H^1_f(V^*(1))$. □

Theorem 6.3.3. — *Let V be a p-adic representation such that $V_\mathfrak{q}$ is potentially semistable for each $\mathfrak{q}\in S_p$, and let $(D_\mathfrak{q})_{\mathfrak{q}\in S_p}$ be a family of splitting submodules. Assume that (V,D) satisfies conditions* **S)** *and* **N2*)**. *Then*

$$h^{\mathrm{norm}}_{V,\mathbf{D}}=h^{\mathrm{spl}}_{V,D}.$$

where $\mathbf{D}=(\mathbf{D}_\mathfrak{q})_{\mathfrak{q}\in S_p}$ denotes the family of $(\varphi,\Gamma_\mathfrak{q})$-modules associated to $D=(D_\mathfrak{q})_{\mathfrak{q}\in S_p}$.

Proof. — First note that in our case the element $[\widetilde{x}^+_\mathfrak{q}]$, defined in Section 6.1.3, coincides with $[x^+_\mathfrak{q}]$. Comparing the definitions of $h^{\mathrm{norm}}_{V,D}$ and $h^{\mathrm{spl}}_{V,D}$ we see that it is enough to show that $\ell_\mathfrak{q}\left(\mathrm{pr}_{\mathfrak{q},y}([x^{\mathrm{Iw}}_{\mathfrak{q},y}])-s_{\mathfrak{q},y}([x^+_\mathfrak{q}])\right)=0$ for all $\mathfrak{q}\in S_p$. The splitting $s_{y,\mathfrak{q}}$ of the exact sequence

$$0\to H^1_f(F_\mathfrak{q},E(1))\to H^1_f(\mathbf{D}_{\mathfrak{q},y})\to H^1(\mathbf{D}_\mathfrak{q})\to 0$$

(see (83)) gives an isomorphism

$$H^1_{\mathrm{Iw}}(\mathbf{D}_{\mathfrak{q},y})_{\Gamma^0_\mathfrak{q}}\simeq H^1_{\mathrm{Iw}}(\mathbf{D}_\mathfrak{q})_{\Gamma^0_\mathfrak{q}}\oplus H^1_{\mathrm{Iw}}(\mathscr{R}_{F_\mathfrak{q},E}(\chi))_{\Gamma^0_\mathfrak{q}}\simeq H^1(\mathbf{D}_\mathfrak{q})\oplus H^1_{\mathrm{Iw}}(F_\mathfrak{q},E(1))_{\Gamma^0_\mathfrak{q}}.$$

Since $\pi_{\mathbf{D},\mathfrak{q}}\left(\mathrm{pr}_{\mathfrak{q},y}([x^{\mathrm{Iw}}_{\mathfrak{q},y}])-s_{\mathfrak{q},y}([x^+_\mathfrak{q}])\right)=0$, from this decomposition it follows that

$$\mathrm{pr}_{\mathfrak{q},y}([w_\mathfrak{q}])-s_{\mathfrak{q},y}([x^+_\mathfrak{q}])\in H^1_{\mathrm{Iw}}(F_\mathfrak{q},E(1))_{\Gamma^0_\mathfrak{q}}=\ker(\ell_\mathfrak{q}),$$

and the theorem is proved. □

Corollary 6.3.4.* —** *If (V,D) satisfies conditions* **S)** *and* **N2*), then*

$$h^{\mathrm{sel}}_{V,\mathbf{D},1} = h^{\mathrm{norm}}_{V,\mathbf{D}} = -h^{\mathrm{spl}}_{V,D}$$

coincide.

Proof. — This follows from Theorems 5.2.2 and 6.3.3. □

CHAPTER 7

p-ADIC HEIGHT PAIRINGS IV: EXTENDED SELMER GROUPS

7.1. Extended Selmer groups

7.1.1. — Let $F=\mathbf{Q}$. Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p. We fix a splitting submodule D_p of V_p which we will denote simply by D. In Section 4.3, we associated to D a canonical filtration $\left(F_i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)\right)_{-2\leqslant i\leqslant 2}$. Recall that $F_0\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)=\mathbf{D}$, where $\mathbf{D}$ is the $(\varphi,\Gamma_{\mathbf{Q}_p})$-module associated to D. We maintain the notation of Section 4.3 and set $\mathbf{M}_0=\mathbf{D}/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)$, $\mathbf{M}_1=F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)/\mathbf{D}$ and $\mathbf{W}=F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)/F_{-1}\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)$. The exact sequence

$$0\to\mathbf{M}_0\to\mathbf{W}\to\mathbf{M}_1\to 0$$

induces the coboundary map $\delta_0\,:\,H^0(\mathbf{M}_1)\to H^1(\mathbf{M}_0)$. Note that if V satisfies conditions **N1-2)** of Section 5.1 we have $\mathbf{M}_0=\mathbf{M}_1=0$. We first describe the structure of the Selmer group $H^1(V,\mathbf{D})$. Recall the following conditions introduced in Section 4.3

F1a) $H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p))=H^0(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p^*(1)))/F_1\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p^*(1)))=0.$

F2a) The composed map

$$\delta_{0,c}\,:\,H^0(\mathbf{M}_1)\xrightarrow{\delta_0}H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_c}H^1_c(\mathbf{M}_0),$$

where the second arrow denotes the canonical projection on $H^1_c(\mathbf{M}_0)$, is an isomorphism.

Let $\rho_{\mathbf{D},f}$ and$\rho_{\mathbf{D},c}$ denote the composed maps

$$\rho_{\mathbf{D},f}\,:\,H^1(\mathbf{D})\to H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_f}H^1_f(\mathbf{M}_0),\qquad (119)$$
$$\rho_{\mathbf{D},c}\,:\,H^1(\mathbf{D})\to H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_c}H^1_c(\mathbf{M}_0).$$

Note that $H^0(\mathbf{M}_1)=H^0(\mathbf{D}')$, where $\mathbf{D}'=\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_p)/\mathbf{D}$.

Proposition 7.1.2. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ which is potentially semistable at p. Assume that the restriction of V on the decomposition group at p satisfies conditions* **F1a)** *and* **F2a)**. *Then*

i) There exists an exact sequence

$$0 \to H^0(\mathbf{D}') \xrightarrow{\partial_0} H^1(V,\mathbf{D}) \to H^1_f(V) \to 0. \tag{120}$$

ii) The map

$$\begin{aligned} \mathrm{spl}^c_{V,\mathbf{D}} : \quad & H^1(V,\mathbf{D}) \to H^0(\mathbf{D}'), \\ & \left[(x,(x^+_{\mathfrak{q}}),(\lambda_{\mathfrak{q}}))\right] \mapsto \delta_{0,c}^{-1} \circ \rho_{\mathbf{D},c}\left([x^+_p]\right) \end{aligned}$$

defines a canonical splitting of (120).

Proof. — The first statement follows directly from the definition of Selmer complexes and the exact sequence (99). See also [**10**, Proposition 11]. The second statement follows immediately from the definition of $\mathrm{spl}_{V,\mathbf{D}}$. □

Definition. — *If the data (V,D) satisfy conditions* **F1a)** *and* **F2a)**, *we call $H^1(V,\mathbf{D})$ the extended Selmer group associated to (V,D).*

From Proposition 7.1.2 it follows that we have a decomposition

$$H^1(V,\mathbf{D}) \simeq H^1_f(V) \oplus H^0(\mathbf{D}'),$$

and we denote by

$$\mathfrak{s}^c_{V,\mathbf{D}} : H^1_f(V) \to H^1(V,\mathbf{D})$$

the injection induced by this splitting.

If, in addition, (V,D) satisfies **F2b)**, we have another natural splitting of (120), namely

$$\begin{aligned} \mathrm{spl}^f_{V,\mathbf{D}} : \quad & H^1(V,\mathbf{D}) \to H^0(\mathbf{D}'), \\ & \left[(x,(x^+_{\mathfrak{q}}),(\lambda_{\mathfrak{q}}))\right] \mapsto \delta_{0,f}^{-1} \circ \rho_{\mathbf{D},f}\left([x^+_p]\right), \end{aligned}$$

and we denote by

$$\mathfrak{s}^f_{V,\mathbf{D}} : H^1_f(V) \to H^1(V,\mathbf{D})$$

the resulting injection.

7.2. Comparision with $h^{\mathrm{spl}}_{V,D}$

7.2.1. — Assume that, in addition to **F1a)** and **F2a)**, (V,D) satisfies condition

F2b) The map

$$\delta_{0,f}\,:\, H^0(\mathbf{M}_1)\xrightarrow{\delta_0} H^1(\mathbf{M}_0)\xrightarrow{\mathrm{pr}_f} H^1_f(\mathbf{M}_0),$$

where the second arrow denotes the canonical projection on $H^1_f(\mathbf{M}_0)$, is an isomorphism (see Section 4.3).

Define a bilinear map

$$\langle\ ,\ \rangle_{\mathbf{D},f}\,:\, H^1_f(\mathbf{M}_0)\times H^1_f(\mathbf{M}_1^*(\chi_p))\to E$$

as the composition

$$\begin{array}{ccccc} H^1_f(\mathbf{M}_0)\times H^1_f(\mathbf{M}_1^*(\chi_p)) & \xrightarrow{(\delta_{0,f}^{-1},\mathrm{id})} & H^0(\mathbf{M}_1)\times H^1_f(\mathbf{M}_1^*(\chi_p)) & \xrightarrow{\cup} & H^1(\mathscr{R}_{\mathbf{Q}_p,E}(\chi_p)) \\ & & & & \big\downarrow{\scriptstyle \ell_{\mathbf{Q}_p}} \\ & & & & E. \end{array}$$

Lemma 7.2.2. — *For all $x\in H^1_f(\mathbf{M}_0)$ and $y\in H^1_f(\mathbf{M}_1^*(\chi_p))$ we have*

$$\langle x,y\rangle_{\mathbf{D},f}=-[i^{-1}_{\mathbf{M}_1^*(\chi_p),f}(y),\delta_{0,f}^{-1}(x)]_{\mathbf{M}_1},$$

where $[\ ,\ ]_{\mathbf{M}_0}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1^*(\chi_p))\times\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1)\to E$ denotes the canonical duality and $i_{\mathbf{M}_1^*(\chi_p),f}\,:\,\mathscr{D}_{\mathrm{cris}}(\mathbf{M}_1^*(\chi_p))\to H^1_f(\mathbf{M}_1^*(\chi_p))$ is the isomorphism constructed in Proposition 2.9.4.

Proof. — Recall that for each $z\in H^1(\mathscr{R}_{\mathbf{Q}_p,E}(\chi_p))$ we have $\mathrm{inv}_p(w_p\cup z)=\ell_p(z)$, where $w_p=(0,\log\chi_p(\gamma_{\mathbf{Q}_p}))$. Therefore, using Proposition 2.9.4, we obtain

$$\begin{aligned}\langle x,y\rangle_{\mathbf{D},f}&=\ell_{\mathbf{Q}_p}(\delta_{0,f}^{-1}(x)\cup y)=\mathrm{inv}_p(w_p\cup\delta_{0,f}^{-1}(x)\cup y)=\\&=-\mathrm{inv}_p(i_{\mathbf{M}_1,c}(\delta_{0,f}^{-1}(x))\cup y))=\\&=-\mathrm{inv}_p(i_{\mathbf{M}_1,c}(\delta_{0,f}^{-1}(x))\cup i_{\mathbf{M}_1^*(\chi_p),f}\circ i^{-1}_{\mathbf{M}_1^*(\chi_p),f}(y))=\\&=-[i^{-1}_{\mathbf{M}_1^*(\chi_p),f}(y),\delta_{0,f}^{-1}(x)]_{\mathbf{M}_1}.\end{aligned}$$

□

7.2.3. — Assume that (V,D) satisfies conditions **F1a-b)** and **F2a-b)**. Then condition **S)** holds by Proposition 4.3.13 iv) and the height pairing $h^{\mathrm{spl}}_{V,D}$ is defined.

Theorem 7.2.4. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p and satisfies conditions* **F1a-b)** *and* **F2a-b)***. Then for all $[x^{\mathrm{sel}}]=[(x,(x_{\mathrm{q}}^+),(\lambda_{\mathrm{q}}))]\in H^1(V,\mathbf{D})$ and $[y^{\mathrm{sel}}]=[(y,(y_{\mathrm{q}}^+),(\mu_{\mathrm{q}}))]\in H^1(V^*(1),\mathbf{D}^\perp)$ we have*

$$h^{\mathrm{sel}}_{V,\mathbf{D}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])=-h^{\mathrm{spl}}_{V,D}([x],[y])+\left\langle\rho_{\mathbf{D},f}([x_p^+]),\rho_{\mathbf{D}^\perp,f}([y_p^+])\right\rangle_{\mathbf{D},f},$$

where the map $\rho_{\mathbf{D},f}$ and $\rho_{\mathbf{D}^\perp,f}$ are defined in (119).

Proof. — The proof is the same as that of [**56**, Theorem 11.4.6] with some modifications. Recall that we have a split exact sequence

$$0 \longrightarrow H^0(\mathbf{D}') \longrightarrow H^1(V,\mathbf{D}) \underset{\longrightarrow}{\overset{\mathfrak{s}_{V,\mathbf{D}}}{\longleftarrow}} H^1_f(V) \longrightarrow 0.$$

Let $[x^{\mathrm{sel}}] = [(x,(x^+_{\mathfrak{q}}),(\lambda_{\mathfrak{q}}))] \in H^1(V,\mathbf{D})$. Then $\mathfrak{s}_{V,\mathbf{D}}([x^{\mathrm{sel}}]) = \left[(x,(\widetilde{x}^+_{\mathfrak{q}}),(\widetilde{\lambda}_{\mathfrak{q}}))\right]$, where

$$\widetilde{x}^+_p = x^+_p - \partial_0 \circ \left(\delta^{-1}_{0,c} \circ \rho_{\mathbf{D},c}\left([x^+_p]\right)\right). \tag{121}$$

Since $H^0(\mathbf{M}_0)=0$, $H^2(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p))=0$ and $H^1_f(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)) = H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p))$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)) & \longrightarrow & H^1_f(\mathbf{D}) & \longrightarrow & H^1_f(\mathbf{M}_0) & \longrightarrow & 0\\
 & & \Big\downarrow = & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)) & \longrightarrow & H^1(\mathbf{D}) & \longrightarrow & H^1(\mathbf{M}_0) & \longrightarrow & 0.
\end{array}$$

The image of $[\widetilde{x}^+_p] \in H^1(\mathbf{D})$ in $H^1(\mathbf{M}_0)$ is equal to

$$\begin{aligned}
&\rho_{\mathbf{D},f}([x^+_p]) + \rho_{\mathbf{D},c}([x^+_p]) - \partial_0 \circ \left(\delta^{-1}_{0,c} \circ \rho_{\mathbf{D},c}\left([x^+_p]\right)\right) = \\
&= \rho_{\mathbf{D},f}([x^+_p]) - \delta_{0,f}\left(\delta^{-1}_{0,c} \circ \rho_{\mathbf{D},c}\left([x^+_p]\right)\right) \in H^1_f(\mathbf{M}_0),
\end{aligned}$$

and therefore $[\widetilde{x}^+_p] \in H^1_f(\mathbf{D})$. Consider the following diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \Big\downarrow & & \Big\downarrow & & \\
 & & & & H^0(\mathbf{D}') & \xrightarrow{=} & H^0(\mathbf{D}') & & \\
 & & & & \Big\downarrow \partial_0 & & \Big\downarrow \partial_0 & & \\
0 & \longrightarrow & H^1(\mathbf{Q}_p,E(1)) & \longrightarrow & H^1(\mathbf{D}_y) & \underset{\pi_{\mathbf{D}}}{\overset{s_{y,p}}{\rightleftarrows}} & H^1(\mathbf{D}) & \longrightarrow & 0\\
 & & \Big\downarrow = & & \Big\downarrow g_{p,y} & & \Big\downarrow g_p & & \\
0 & \longrightarrow & H^1(\mathbf{Q}_p,E(1)) & \longrightarrow & H^1(\mathbf{Q}_p,Y^*_y(1)) & \xrightarrow{\pi_p} & H^1(\mathbf{Q}_p,V), & &
\end{array} \tag{122}$$

where $s_{y,p}$ is the canonical splitting constructed in Section 4.2. Recall that by Proposition 4.3.13 iii), $\mathrm{Im}(g_p) = H^1_f(\mathbf{Q}_p,V)$. Let $[\widehat{x}] \in H^1_f(Y^*_y(1))$ be any lift of $[x]$ and let $[\widehat{x}_p] \in H^1(\mathbf{Q}_p,Y^*_y(1))$ denote its localization at p. Then by definition, we have

$$h^{\mathrm{spl}}_{V,D}([x],[y]) = \ell_p\left([\widehat{x}_p] - g_{p,y} \circ s_{y,p}([\widetilde{x}^+_p])\right).$$

The diagram (122) shows that there exists a unique element $[\widehat{x}^+_p] \in H^1(\mathbf{D}_y)$ such that $g_{p,y}\left([\widehat{x}^+_p]\right) = [\widehat{x}_p]$ and $\pi_{\mathbf{D}}\left([\widehat{x}^+_p]\right) = [x^+_p]$. Therefore, there exists a lift $[\widehat{x}^{\mathrm{sel}}]$ of $[x^{\mathrm{sel}}]$ of the form $\left[\widehat{x}^{\mathrm{sel}}\right] = [(\widehat{x}, \widehat{x}^+_{\mathrm{q}}, \widehat{\lambda}_{\mathrm{q}})]$. Recall that

$$\beta_{Y^*_y(1),\mathbf{D}_y}\left([\widehat{x}^{\mathrm{sel}}]\right) = (-z \cup \widehat{x}, (-w_{\mathrm{q}} \cup \widehat{x}^+_{\mathrm{q}}), (z_{\mathrm{q}} \cup \widehat{\lambda}_{\mathrm{q}})) \in S^2(Y^*_y(1), \mathbf{D}_y),$$

where z, w_{q} and z_{q} are defined in (112). Set

$$[t_p] = -\delta^{-1}_{0,f} \circ \rho_{\mathbf{D},f}([x^+_p]) \in H^0(\mathbf{M}_1) = H^0(\mathbf{D}'). \tag{123}$$

Then

$$\rho_{\mathbf{D},f}([\widetilde{x}^+_p]) + \rho_{\mathbf{D},f}(\partial_0([t_p])) = \rho_{\mathbf{D},f}([\widetilde{x}^+_p]) + \delta_{0,f}([t_p]) = 0.$$

Thus, the image of $[\widetilde{x}^+_p] + \partial_0([t_p])$ under the projection $H^1(\mathbf{D}) \to H^1(\mathbf{M}_0)$ lies in $H^1_c(\mathbf{M}_0)$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)) & \longrightarrow & H^1(\mathbf{D}) & \longrightarrow & H^1(\mathbf{M}_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cup w_p} & & \downarrow{\scriptstyle \cup w_p} & & \downarrow{\scriptstyle \cup w_p} & & \\
0 & \longrightarrow & H^2(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)) & \longrightarrow & H^2(\mathbf{D}) & \longrightarrow & H^2(\mathbf{M}_0) & \longrightarrow & 0 \\
 & & \| & & & & & & \\
 & & \{0\} & & & & & &
\end{array}$$

By Lemma 2.9.5, $H^1_c(\mathbf{M}_0) = \ker\left(\cup w_p : H^1(\mathbf{M}_0) \to H^2(\mathbf{M}_0)\right)$, and we have

$$[w_p] \cup \left([\widetilde{x}^+_p] + \partial_0([t_p])\right) = 0 \quad \text{in } H^2(\mathbf{D}).$$

Set $[\xi_p] = s_{y,p}\left([\widetilde{x}^+_p]\right) + \partial_0([t_p]) \in H^1(\mathbf{D}_y)$. Then

$$\beta_{\mathbf{D}_y}([\xi_p]) = -[w_p] \cup [\xi_p] = 0.$$

Now we can use Lemma 5.2.3 and write

$$\beta_{Y^*_y(1),\mathbf{D}_y}\left([\widehat{x}^{\mathrm{sel}}]\right) = [(\widehat{a}, (\widehat{b}_{\mathrm{q}}), (\widehat{c}_{\mathrm{q}}))],$$

where

$$\widehat{b}_p = w_p \cup (\xi_p - \widehat{x}^+_p) = w_p \cup \left(s_{y,p}\left(\widetilde{x}^+_p\right) - \widehat{x}^+_p\right) + w_p \cup \partial_0(t_p).$$

Let $\alpha_y \in C^0(G_{\mathbf{Q},S}, Y_y)$ be an element that maps to $1 \in C^0(G_{\mathbf{Q},S}, E) = E$ and satisfies $d\alpha_y = y$. The first formula of Lemma 5.2.4 reads

$$h^{\mathrm{sel}}_{V,\mathbf{D}}([x^{\mathrm{sel}}], [y^{\mathrm{sel}}]) = \mathrm{inv}_{\mathbf{Q}_p}\left([g_{p,y}(\widehat{b}_p) \cup f^\perp_p(\alpha_y) + g_p(b_p) \cup \mu_p]\right). \tag{124}$$

Set $u_p = s_{y,p}(\widetilde{x}_p^+) - \widehat{x}_p^+$. Then $u_p \in C^1_{\varphi,\gamma_p}(E(1))$ and $u_p \cup \alpha_y = u_p$. Thus

$$(125)\quad [g_{p,y}(\widehat{b}_p) \cup f_p^{\perp}(\alpha_y) + g_p(b_p) \cup \mu_p] = \\ = [w_p \cup u_p] + [g_{p,y}(w_p \cup \partial_0(t_p)) \cup f_p^{\perp}(\alpha_y) + g_p(b_p) \cup \mu_p].$$

By (113), we have

$$\mathrm{inv}_{\mathbf{Q}_p}[w_p \cup u_p] = \ell_{\mathbf{Q}_p}[u_p] = -h^{\mathrm{spl}}_{V,D}([x],[y]),$$

and from (124–125) we get

$$(126)\quad h^{\mathrm{sel}}_{V,\mathbf{D}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}]) = \\ = -h^{\mathrm{spl}}_{V,D}([x],[y]) + \mathrm{inv}_{\mathbf{Q}_p}\left([g_{p,y}(w_p \cup \partial_0(t_p)) \cup f_p^{\perp}(\alpha_y) + g_p(b_p) \cup \mu_p]\right).$$

We compute the second term on the right hand side of this formula. Since $g_{p,y}(\partial_0([t_p])) = 0$, there exists $\widetilde{t}_p \in \mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y^*(1)_p)$ such that $\widetilde{t}_p \mapsto t_p$ under the projection $\mathbf{D}^{\dagger}_{\mathrm{rig}}(Y_y^*(1)_p) \to \mathbf{D}'_y$ and we can assume that

$$\partial_0(t_p) = d_0(\widetilde{t}_p) = ((\varphi-1)(\widetilde{t}_p), (\gamma_p-1)(\widetilde{t}_p)).$$

Therefore

$$g_{p,y}(w_p \cup \partial_0(t_p)) = z_p \cup g_{p,y}(d_0(\widetilde{t}_p)) \in K^2_p(V) \subset K^2_p(Y_y^*(1)),$$
$$g_p(b_p) \cup \mu_p = z_p \cup g_p(d\widetilde{t}_p) \cup \mu_p \in K^2_p(V) \subset K^2_p(Y_y^*(1)).$$

Thus,

$$(127)\quad \begin{aligned} &\mathrm{inv}_{\mathbf{Q}_p}\left([g_{p,y}(w_p \cup \partial_0(t_p)) \cup f_p^{\perp}(\alpha_y) + g_p(b_p) \cup \mu_p]\right) = \\ &= \mathrm{inv}_{\mathbf{Q}_p}\left(z_p \cup g_{p,y}(d\widetilde{t}_p) \cup f_p^{\perp}(\alpha_y) + z_p \cup g_p(\pi_p(d\widetilde{t}_p)) \cup \mu_p]\right) = \\ &= -\mathrm{inv}_{\mathbf{Q}_p}\left([z_p \cup g_{p,y}(\widetilde{t}_p) \cup d f_p^{\perp}(\alpha_y) + z_p \cup g_p(\widetilde{t}_p) \cup d\mu_p]\right) = \\ &= -\mathrm{inv}_{\mathbf{Q}_p}\left([z_p \cup \widetilde{t}_p \cup (f_p^{\perp}(y) + d\mu_p)]\right) = \\ &= -\mathrm{inv}_p\left([z_p \cup \widetilde{t}_p \cup g_p(y_p^+)]\right) = \\ &= -\mathrm{inv}_p\left([w_p \cup t_p \cup y_p^+]\right) = \\ &= -\ell_{\mathbf{Q}_p}\left([t_p \cup y_p^+]\right). \end{aligned}$$

Now we remark that $\ell_{\mathbf{Q}_p}\left([t_p \cup y_p^+]\right) = \ell_{\mathbf{Q}_p}\left([t_p \cup \rho_{\mathbf{D}^{\perp},f}(y_p^+])\right)$ and, taking into account (123), we have

$$(128)\quad \ell_{\mathbf{Q}_p}\left([t_p \cup y_p^+]\right) = -\ell_{\mathbf{Q}_p}\left(\delta_{0,f}^{-1} \circ \rho_{\mathbf{D},f}([x_p^+]) \cup \rho_{\mathbf{D},f}([y_p^+])\right) = \\ = -\left\langle \rho_{\mathbf{D},f}([x_p^+]), \rho_{\mathbf{D}^{\perp},f}([y_p^+])\right\rangle_{\mathbf{D},f}.$$

The theorem follows from (126–128). □

Corollary 7.2.5. — *Under conditions of Theorem 7.2.4, for all* $[x] \in H^1_f(V)$ *and* $[y] \in H^1_f(V^*(1))$ *we have*

$$h^{\mathrm{spl}}_{V,D}([x],[y]) = -h^{\mathrm{sel}}_{V,\mathbf{D}}(\mathfrak{s}^f_{V,\mathbf{D}}([x]),\mathfrak{s}^f_{V^*(1),\mathbf{D}^\perp}([y])).$$

Proof. — Set $[(x,(x^+_{\mathrm{q}}),\lambda_{\mathrm{q}})] = \mathfrak{s}^f_{V,\mathbf{D}}([x])$. Then $\rho_{\mathbf{D},f}([x^+_p]) = 0$ and the formula follows from Theorem 7.2.4. □

7.3. The pairing $h^{\mathrm{norm}}_{V,D}$ for extended Selmer groups

7.3.1. — Recall condition **F3)** introduced in Section 4.3

F3) For all $i \in \mathbf{Z}$

$$\mathscr{D}_{\mathrm{pst}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V_p)/F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V_p))^{\varphi=p^i} = \mathscr{D}_{\mathrm{pst}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V_p))^{\varphi=p^i} = 0.$$

Clearly, **F3)** implies **F1a-b)**. In this section, we generalize the construction of the height pairing $h^{\mathrm{norm}}_{V,D}$ to the case when V satisfies conditions **F3)**, **F2a)** and **F2b)**.

Let $[y] \in H^1_f(V^*(1))$ and let Y_y denote the associated extention (114). As before, we denote by $\mathbf{D}_y$ the inverse image of $\mathbf{D}$ in $\mathbf{D}^\dagger_{\mathrm{rig}}(Y^*_y(1)_p)$. Since the representation V_p satisfies condition **S)**, the exact sequence (82) have a canonical splitting $s_{\mathbf{D},y}$.

In the diagram (109), the maps g_v and $g_{v,y}$ are no more injective and we replace it by the diagram (122). Let $[x] \in H^1_f(V)$ and let $\mathfrak{s}_{V,\mathbf{D}}([x]) = [(x,(\widetilde{x}^+_{\mathrm{q}}),(\widetilde{\lambda}_{\mathrm{q}}))]$. Then $[\widetilde{x}^+_p]$ is the unique element of $H^1_f(\mathbf{D})$ such that $g_p([\widetilde{x}^+_p]) = [x_p]$. Its explicit form is given by (121), but we do not use it here. Let

$$[\widehat{x}] \in \ker\left(H^1_S(Y^*_y(1)) \to \frac{H^1(\mathbf{Q}_{\mathrm{q}},Y^*_y(1))}{H^1_f(\mathbf{Q}_{\mathrm{q}},Y^*_y(1))}\right)$$

be an arbitrary lift of $[x]$. (Note that by Lemma 6.1.2, we can even take $[\widehat{x}] \in H^1_f(Y^*_y(1))$.) As easy diagram chase (already used in the proof of Theorem 7.2.4) shows there exists a unique $[\widehat{x}^+_p] \in H^1(\mathbf{D}_y)$ such that $g_{p,y}([\widehat{x}^+_p]) = \mathrm{res}_p([\widehat{x}])$ in $H^1(\mathbf{Q}_p,Y^*_y(1))$ and $\pi_{\mathbf{D}}([\widehat{x}^+_p]) = [\widetilde{x}^+_p]$.

We have the following diagram which can be seen as an analog of the diagram (108) in our situation

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\mathscr{H}(\Gamma^0_{\mathbf{Q}_p}) \otimes_{\Lambda_{E,\mathbf{Q}_p}} H^1_{\mathrm{Iw}}(\mathbf{Q}_p, E(1)) & \longrightarrow & H^1(\mathbf{Q}_p, E(1)) & \xrightarrow{\ell_{\mathbf{Q}_p}} & E \\
\downarrow & & \downarrow & & \\
H^1_{\mathrm{Iw}}(\mathbf{D}_y) & \xrightarrow{\mathrm{pr}_{\mathbf{D},y}} & H^1(\mathbf{D}_y) & \xleftarrow{\partial_0} & H^0(\mathbf{D}') \\
\downarrow{\scriptstyle \pi^{\mathrm{Iw}}_{\mathbf{D}}} & & \downarrow{\scriptstyle \pi_{\mathbf{D}}} & & \downarrow{\scriptstyle =} \\
H^1_{\mathrm{Iw}}(\mathbf{D}) & \xrightarrow{\mathrm{pr}_{\mathbf{D}}} & H^1(\mathbf{D}) & \xleftarrow{\partial_0} & H^0(\mathbf{D}') \\
\downarrow & & \downarrow & & \\
0 & & 0. & &
\end{array}$$

From Proposition 4.3.15 it follows that there exist a unique $[t_p] \in H^0(\mathbf{D}')$ (explicitly given by (123) and $\left[x^{\mathrm{Iw}}_{p,y}\right] \in H^1_{\mathrm{Iw}}(\mathbf{D}_y)$ such that

$$[\widehat{x}^+_p] + \partial_0([t_p]) = \mathrm{pr}_{\mathbf{D}} \circ \pi^{\mathrm{Iw}}_{\mathbf{D}} \left(\left[x^{\mathrm{Iw}}_{p,y}\right]\right).$$

Set

$$[u_p] = [\widehat{x}^+_p] + \partial_0([t_p]) - \mathrm{pr}_{\mathbf{D},y} \left(\left[x^{\mathrm{Iw}}_{p,y}\right]\right). \tag{129}$$

Then $[u_p] \in H^1(\mathbf{Q}_p, E(1))$.

Definition. — *Let V be a p-adic representation that is potentially semistable at p and satisfies conditions* **F2a-b)** *and* **F3)**. *We define the height pairing*

$$h^{\mathrm{norm}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

by

$$h^{\mathrm{norm}}_{V,D}([x],[y]) = \ell_{\mathbf{Q}_p}([u_p]).$$

It is easy to see that $h^{\mathrm{norm}}_{V,D}([x],[y])$ does not depend on the choice of the lift $\left[x^{\mathrm{Iw}}_{p,y}\right]$. The following result generalizes [**56**, Theorem 11.4.6].

Theorem 7.3.2. — *Let V be a p-adic representation of $G_{\mathbf{Q},S}$ that is potentially semistable at p and satisfies conditions* **F2a-b)** *and* **F3)**. *Then*

i) $h^{\mathrm{norm}}_{V,D} = h^{\mathrm{spl}}_{V,D}$;

ii) For all $[x^{\mathrm{sel}}]=[(x,(x^+_{\mathfrak q}),(\lambda_{\mathfrak q}))]\in H^1(V,\mathbf{D})$ *and* $[y^{\mathrm{sel}}]=[(y,(y^+_{\mathfrak q}),(\mu_{\mathfrak q}))]\in H^1(V^*(1),\mathbf{D}^{\perp})$ *we have*

$$h^{\mathrm{sel}}_{V,\mathbf{D}}([x^{\mathrm{sel}}],[y^{\mathrm{sel}}])=-h^{\mathrm{norm}}_{V,D}([x],[y])+\big\langle \rho_{\mathbf{D},f}([x^+_p]),\rho_{\mathbf{D}^{\perp},f}([y^+_p])\big\rangle_{\mathbf{D},f}.$$

Proof. — i) Recall that in the definition of $h^{\mathrm{norm}}_{V,D}$ we can take $[\widehat{x}]\in H^1_f(Y^*_y(1))$. Comparing the definitions of $h^{\mathrm{norm}}_{V,D}$ and $h^{\mathrm{spl}}_{V,D}$, we see that it is enough to prove that

$$[u_p]-\big(s_{y,p}([\widehat{x}_p]-[\widetilde{x}^+_p])\big)\in\ker(\ell_{\mathbf{Q}_p}),$$

where $[u_p]$ is defined by (129) and $s_{y,p}$ denotes the splitting (83). Since the restriction of $g_{p,y}$ on $H^1(\mathbf{Q}_p,E(1))$ is the identity map, we have

$$[u_p]=g_{p,y}([u_p])=[\widehat{x}_p]-g_{p,y}([x^{\mathrm{Iw}}_{p,y}]),$$

and it is enough to check that

$$g_{p,y}([x^{\mathrm{Iw}}_{p,y}])-g_{p,y}\circ s_{y,p}([\widetilde{x}^+_p])\in\ker(\ell_{\mathbf{Q}_p}). \tag{130}$$

First remark that the canonical splitting (82) induces splittings $s^{\mathrm{Iw}}_{p,y}$ and $s_{p,y}$ in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathrm{Iw}}(\mathscr{R}_{\mathbf{Q}_p,E}(\chi)) & \longrightarrow & H^1_{\mathrm{Iw}}(\mathbf{D}_y) & \overset{s^{\mathrm{Iw}}_{p,y}}{\rightleftarrows} & H^1_{\mathrm{Iw}}(\mathbf{D}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathbf{D},y}} & & \downarrow{\scriptstyle \mathrm{pr}_{\mathbf{D}}} & & \\
0 & \longrightarrow & H^1(\mathbf{Q}_p,E(1)) & \longrightarrow & H^1(\mathbf{D}_y) & \overset{s_{p,y}}{\rightleftarrows} & H^1(\mathbf{D}) & \longrightarrow & 0.
\end{array}$$

Write $[x^{\mathrm{Iw}}_{p,y}]$ in the form

$$[x^{\mathrm{Iw}}_{p,y}]=s^{\mathrm{Iw}}_{p,y}(a^{\mathrm{Iw}})+b^{\mathrm{Iw}},\qquad a^{\mathrm{Iw}}\in H^1_{\mathrm{Iw}}(\mathbf{D}),\quad b^{\mathrm{Iw}}\in H^1_{\mathrm{Iw}}(\mathscr{R}_{\mathbf{Q}_p,E}(\chi_p)).$$

By the definition of $[x^{\mathrm{Iw}}_{p,y}]$, we have

$$\mathrm{pr}_{\mathbf{D},y}([x^{\mathrm{Iw}}_{p,y}])=s_{y,p}(a)+b,$$

where $b\in\ker(\ell_{\mathbf{Q}_p})=H^1(\mathbf{Q}_p,E(1))_{\Gamma^0_{\mathbf{Q}_p}}$ and

$$a=\partial([t_p])+s_{p,y}([\widetilde{x}^+_p])\in H^1(\mathbf{D}).$$

Since $g_{p,y}(s_{y,p}(\partial_0([t_p]))=0$, we have

$$g_{p,y}(\mathrm{pr}_{\mathbf{D},y}([x^{\mathrm{Iw}}_{p,y}]))=b+g_{p,y}(s_{y,p}(a))=b+g_{p,y}(s_{y,p}([\widetilde{x}^+_p])),$$

and (130) is checked.

ii) The second statement follows from i) and Theorem 7.2.4. □

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