THIN LAYER MODELS FOR ELECTROMAGNETISM

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ABSTRACT. We present a review on the accuracy of asymptotic models for the scattering problem of electromagnetic waves in domains with thin layer of dielectric or absorbing material. These models appear as first order approximations with respect to the size of the thin layer. The main tool is a multiscale expansion of the partial differential equations, that makes possible to replace the thin layer by approximate conditions. We present the advantages and the drawbacks of several approximations together with numerical validations and simulations. The main motivation of this work concerns the computation of electromagnetic field in biological cells. The main difficulty to compute the local electric field lies in the thinness of the membrane and in the high contrast between the electrical conductivities of the cytoplasm and of the membrane, which provides a specific behavior of the electromagnetic field at low frequencies.

CONTENTS

1. INTRODUCTION

This aim of the work is to provide a review of several asymptotic models for the scattering problem of time-harmonic electromagnetic waves in domains with thin layer of dielectric or absorbing material. These models appear as first order approximations (at the first or the second order, depending on the frequency of the harmonic wave) with respect to the size of the thin layer. Media with thin inclusions appear in many domains: geophysical applications, microwave imaging, biomedical applications, phones radiations, radar applications, non destructive testing. In this paper, the simplified configuration is mainly motivated by the computation of the electromagnetic field in a biological cell.

The electromagnetic modeling of biological cells has become extremely important since several years, in particular in the biomedical research area. In the simple model of Fear, and Stuchly or Foster and Schwan [8, 9, 10], the biological cell is composed of a conducting cytoplasm surrounded by a thin insulating membrane. When the cell is exposed to an electric field, the

1991 *Mathematics Subject Classification.* 34E05, 34E10, 58J37.

Key words and phrases. asymptotics, time-harmonic Maxwell's equations, finite element method, edge elements.

local field near the membrane may overcome physiological values. Then, complex phenomenon known as electropermeabilization (or electroporation) may occur [20]: the cell membrane is destructured and some outer molecules might be internalized inside the cell. These process hold great promises in oncology and gene therapy, particularly, to deliver drug molecules in cancer treatment.

This is the reason why several papers in the bioelectromagnetic research area deal with numerical modeling of the cell (see for instance [11, 19, 18]) and with numerical computations of the membrane voltage. Actually, the main difficulties of the calculation of the local electric field lie in the thinness of the membrane and in the high contrast between the electromagnetic properties of the cytoplasm and the membrane. More precisely, despite the electric permittivities of these two media are of the same order of magnitude, the membrane conductivity is much lower than the cytoplasm conductivity, which provides particular behavior of the electromagnetic field at low frequencies.

In previous papers [17, 15, 16, 14], Poignard *et al.* have proposed an asymptotic analysis to compute electric potential in domains with thin layer. In particular, Perrussel and Poignard have derived the asymptotic expansion of the electric potential at any order in domains with resistive thin layer [13].

More recently, we have derived an asymptotic model for the electromagnetic field in biological cell [7] at mid-frequency. All these papers are based on a Wentzel-Kramer-Brillouin (WKB) expansion of the partial differential equations, that makes possible to replace the thin layer by appropriate transmission conditions.

The aim of this paper is to present a review on the accuracy of the approximation of the time-harmonic electromagnetic field in domain with thin layer. We present the advantages and the drawbacks of different approximations (WKB expansion), generalized impedance transmission conditions (GITC) and we make the link with the work of [5]. This work is concerned essentially with numerical objectives. The theoretical pertinence of these approximations have already been shown in [7, 5].

Note that in this paper, we focus on the time-harmonic regime, despite transient field is also of great interest [3, 2].

The outline proceeds as follows. In section 2, we introduce the mathematical model. In section 3, we present the first terms of a multiscale expansion for the low-frequency and the mid-frequency cases. In section 4, we exhibit a GITC model of order 2 for the mid-frequency case and we make the link with the low-frequency case. Section 4.3 is devoted to a generic variational framework for the two cases. In section 5, we present numerical validations and simulations for different models and we end the paper with the concluding section 6.

2. THE MATHEMATICAL MODEL

2.1. **Notations.** For any orientable smooth surface without boundary $\mathcal{S}$ of $\mathbb{R}^3$, the unit normal vector $\mathbf{n}$ on $\mathcal{S}$ is outwardly oriented from the interior domain enclosed by $\mathcal{S}$ towards the outer domain.

We denote by $\vec{\mathrm{curl}}_{\mathcal{S}}$ the tangential rotational operator (which applies to functions defined on $\mathcal{S}$) and $\mathrm{curl}_{\mathcal{S}}$ the surfacic rotational operator (which applies to vector fields) [12]

$$\forall\, f \in C^\infty(\mathcal{S}), \quad \vec{\mathrm{curl}}_{\mathcal{S}} f = (\nabla_{\mathcal{S}} f) \times \mathbf{n}\ ,$$
$$\forall\, \mathbf{v} \in (C^\infty(\mathcal{S}))^3\ , \quad \mathrm{curl}_{\mathcal{S}}\, \mathbf{v} = \mathrm{div}_{\mathcal{S}}\, (\mathbf{v} \times \mathbf{n})\ ,$$

where $\nabla_{\mathcal{S}}$ and $\mathrm{div}_{\mathcal{S}}$ are respectively the tangential gradient and the surfacic divergence on $\mathcal{S}$. We denote respectively by $\mathsf{TH}^{-1/2}(\mathrm{div}_{\mathcal{S}}, \mathcal{S})$ and $\mathsf{TH}(\mathrm{curl}_{\mathcal{S}}, \mathcal{S})$ the spaces of tangent vector fields of the above operators $\mathrm{div}_{\mathcal{S}}$ and $\mathrm{curl}_{\mathcal{S}}$ [12]:

$$\mathsf{TH}^{-1/2}(\mathrm{div}_{\mathcal{S}}, \mathcal{S}) = \{\mathbf{v} \in \mathbf{H}^{-1/2}(\mathcal{S}),\, \mathrm{div}_{\mathcal{S}}\, \mathbf{v} \in \mathrm{H}^{-1/2}(\mathcal{S})\}\ ,$$
$$\mathsf{TH}(\mathrm{curl}_{\mathcal{S}}, \mathcal{S}) = \{\mathbf{v} \in \mathbf{L}^{2}(\mathcal{S}),\, \mathrm{curl}_{\mathcal{S}}\, \mathbf{v} \in \mathrm{L}^{2}(\mathcal{S})\}\ ,$$

where $\mathbf{L}^2(\mathcal{S}) = \mathrm{L}^2(\mathcal{S})^3$ and $\mathbf{H}^{-1/2}(\mathcal{S}) = \mathrm{H}^{-1/2}(\mathcal{S})^3$.

Equipped with their graph norm, $\mathsf{TH}^{-1/2}(\mathrm{div}_{\mathcal{S}}, \mathcal{S})$ and $\mathsf{TH}(\mathrm{curl}_{\mathcal{S}}, \mathcal{S})$ are Hilbert spaces.

We denote by $\mathbf{v}_{\mathsf{T}}|_{\mathcal{S}}$ the tangent component of the vector field $\mathbf{v}$ defined in a neighborhood of $\mathcal{S}$:

$$\mathbf{v}_{\mathsf{T}} = \mathbf{n} \times (\mathbf{v}|_{\mathcal{S}} \times \mathbf{n})\ ,$$

and we denote by $[\mathbf{v}]_{\mathcal{S}}$ the jump of $\mathbf{v}$ across $\mathcal{S}$:

$$[\mathbf{v}]_{\mathcal{S}} = \mathbf{v}|_{\mathcal{S}^+} - \mathbf{v}|_{\mathcal{S}^-}\ .$$

2.2. **Time-harmonic Maxwell equation in single cell.** Biological cells consist of a cytoplasm surrounded by a thin resistive layer. Throughout the paper we denote by $\mathcal{O}$ the domain of interest which is composed of the outer cell medium and the cell. Let us denote by $\mathcal{O}_{\mathrm{c}}$ the cell cytoplasm, and by $\mathcal{O}_{\mathrm{m}}^{\varepsilon}$ the cell membrane surrounding $\mathcal{O}_{\mathrm{c}}$, whose thickness is constant and denoted by ε. Assuming, without loss of generality, that the domain $\mathcal{O}_{\mathrm{c}}$ is independent of ε, the extracellular domain is then ε–dependent. We denote it by $\mathcal{O}_{\mathrm{e}}^{\varepsilon}$, in a such way that (see Figure 1) :

$$\mathcal{O} = \mathcal{O}_{\mathrm{c}} \cup \overline{\mathcal{O}_{\mathsf{m}}^{\varepsilon}} \cup \mathcal{O}_{\mathsf{e}}^{\varepsilon}.$$

The boundary of the cytoplasm is the smooth surface denoted by Γ while Γ^{ε} is the cell boundary, *i.e.* Γ^{ε} is the boundary of $\overline{\mathcal{O}_{\mathrm{c}}} \cup \mathcal{O}_{\mathrm{m}}^{\varepsilon}$.

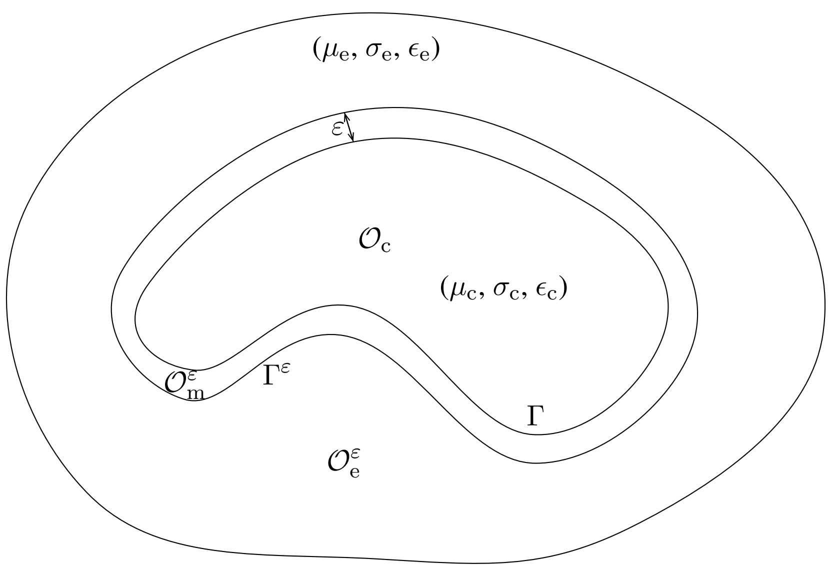

FIGURE 1. A cross-section of the domain $\mathcal{O}$ and its subdomains $\mathcal{O}_{\mathrm{c}}$, $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$, $\mathcal{O}_{\mathrm{e}}$

The electromagnetic properties of $\mathcal{O}$ are given by the following piecewise-constant functions $\underline{\mu}$, $\underline{\epsilon}$, and $\underline{\sigma}$ corresponding to the respective magnetic permeability, electrical permittivity, and conductivity of $\mathcal{O}$:

$$\underline{\mu} = \begin{cases} \mu_{\rm c}, & \text{in } \mathcal{O}_{\rm c}, \\ \mu_{\rm m}, & \text{in } \mathcal{O}_{\rm m}^{\varepsilon}, \\ \mu_{\rm e}, & \text{in } \mathcal{O}_{\rm e}^{\varepsilon}, \end{cases} \qquad \underline{\epsilon} = \begin{cases} \epsilon_{\rm c}, & \text{in } \mathcal{O}_{\rm c}, \\ \epsilon_{\rm m}, & \text{in } \mathcal{O}_{\rm m}^{\varepsilon}, \\ \epsilon_{\rm e}, & \text{in } \mathcal{O}_{\rm e}^{\varepsilon}, \end{cases} \qquad \underline{\sigma} = \begin{cases} \sigma_{\rm c}, & \text{in } \mathcal{O}_{\rm c}, \\ \sigma_{\rm m}, & \text{in } \mathcal{O}_{\rm m}^{\varepsilon}, \\ \sigma_{\rm e}, & \text{in } \mathcal{O}_{\rm e}^{\varepsilon}, \end{cases} .$$

Let us denote by $\mathbf{J}$ the time-harmonic current source and let ω be the frequency. For the sake of simplicity, we assume that $\mathbf{J}$ is smooth , supported in $\mathcal{O}_{\rm e}^{\varepsilon}$ and that it vanishes in a neighborhood of the cell membrane. Maxwell's equations link the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$, through Faraday's and Ampère's laws in $\mathcal{O}$:

$$\operatorname{curl} \mathbf{E}^{\varepsilon} - i\omega\underline{\mu}\mathbf{H}^{\varepsilon} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H}^{\varepsilon} + (i\omega\underline{\epsilon} - \underline{\sigma})\, \mathbf{E}^{\varepsilon} = \mathbf{J} \quad \text{in} \quad \mathcal{O}\,.$$

We complement this problem with a Silver-Müller boundary condition set on $\partial\mathcal{O}$.

Denoting by $\underline{\kappa}$ the complex wave number given by

$$\forall x \in \mathcal{O}, \quad \underline{\kappa}^2(x) = \omega^2 \underline{\mu}(x) \left(\underline{\epsilon}(x) + i\frac{\underline{\sigma}(x)}{\omega} \right), \quad \Im(\underline{\kappa}(x)) \geq 0,$$

Maxwell's system of first order partial differential equations can be reduced to the following second-order equation

$$\text{(1a)} \qquad \operatorname{curl}\operatorname{curl} \mathbf{E}^{\varepsilon} - \underline{\kappa}^2 \mathbf{E}^{\varepsilon} = i\omega\underline{\mu}\mathbf{J} \quad \text{in} \quad \mathcal{O}_{\rm c} \cup \mathcal{O}_{\rm m}^{\varepsilon} \cup \mathcal{O}_{\rm e}^{\varepsilon}\,,$$

with the following transmission conditions across Γ and Γ^{ε}

$$\text{(1b)} \qquad \mathbf{E}_{\rm e}^{\varepsilon} \times \mathbf{n}|_{\Gamma^{\varepsilon}} = \mathbf{E}_{\rm m}^{\varepsilon} \times \mathbf{n}|_{\Gamma^{\varepsilon}}, \quad \frac{1}{\mu_{\rm e}} \operatorname{curl} \mathbf{E}_{\rm e}^{\varepsilon} \times \mathbf{n}|_{\Gamma^{\varepsilon}} = \frac{1}{\mu_{\rm m}} \operatorname{curl} \mathbf{E}_{\rm m}^{\varepsilon} \times \mathbf{n}|_{\Gamma^{\varepsilon}},$$

$$\text{(1c)} \qquad \mathbf{E}_{\rm c}^{\varepsilon} \times \mathbf{n}|_{\Gamma} = \mathbf{E}_{\rm m}^{\varepsilon} \times \mathbf{n}|_{\Gamma}, \quad \frac{1}{\mu_{\rm c}} \operatorname{curl} \mathbf{E}_{\rm c}^{\varepsilon} \times \mathbf{n}|_{\Gamma} = \frac{1}{\mu_{\rm m}} \operatorname{curl} \mathbf{E}_{\rm m}^{\varepsilon} \times \mathbf{n}|_{\Gamma},$$

where $\mathbf{E}_{\rm e}^{\varepsilon}$, $\mathbf{E}_{\rm m}^{\varepsilon}$, $\mathbf{E}_{\rm c}^{\varepsilon}$ denote the respective restrictions of $\mathbf{E}^{\varepsilon}$ to the domains $\mathcal{O}_{\rm e}^{\varepsilon}$, $\mathcal{O}_{\mathsf{m}}^{\varepsilon}$ and $\mathcal{O}_{\rm c}$. The boundary condition writes

$$\text{(1d)} \qquad \operatorname{curl} \mathbf{E}^{\varepsilon} \times \mathbf{n} - i\kappa_{\rm e}\, \mathbf{n} \times \mathbf{E}^{\varepsilon} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}.$$

3. A MULTISCALE EXPANSION

In order to avoid the meshing of the thin membrane, it is convenient to approximate the solution to problem (1), by replacing the thin layer by appropriate conditions across the surface Γ. The idea, as presented in [7], consists in rewriting the operator $\operatorname{curl}\operatorname{curl}$ in the domain $\mathcal{O}_{\rm m}^{\varepsilon}$ in *local coordinates* $(\mathbf{x}_{\mathsf{T}}, x_3)$ [1, 7]. The variable $x_3 \in (0, \varepsilon)$ is the Euclidean distance to Γ and $\mathbf{x}_{\mathsf{T}}$ denotes tangential coordinates on Γ. Then performing the change of variable $x_3 = \varepsilon\eta$, rewriting the operator in $(\mathbf{x}_{\mathsf{T}}, \eta)$–coordinates and assuming that $\mathbf{E}^{\varepsilon}$ can be developed as a formal expansion in power series of ε, we obtain the approximation of $\mathbf{E}^{\varepsilon}$ at the desired order of accuracy. The rigorous derivation of the expansion is not in the scope of the present paper and we refer the reader to [7] for a detailed description of the calculation. Only the two first terms of the multiscale expansion are presented in this paper, namely:

$$\mathbf{E}^\varepsilon \approx \mathbf{E}^0 + \varepsilon \mathbf{E}^1, \qquad \text{in} \quad \mathcal{O}_\mathrm{c} \cup \mathcal{O}_\mathrm{e}$$
$$\mathbf{E}^\varepsilon \approx \mathbf{E}^0_\mathrm{m}\left(\mathbf{x}_\mathsf{T}, \frac{x_3}{\varepsilon}\right) + \varepsilon \mathbf{E}^1_\mathrm{m}\left(\mathbf{x}_\mathsf{T}, \frac{x_3}{\varepsilon}\right), \qquad \text{for almost any} \quad (\mathbf{x}_\mathsf{T}, x_3) \in \Gamma \times (0, \varepsilon)\ ,$$

where $\mathcal{O}_\mathrm{e}$ denotes the domain $\mathcal{O}_\mathrm{e} = \mathcal{O} \setminus \overline{\mathcal{O}_\mathrm{c}}$. For such a purpose it is convenient to introduce the electromagnetic properties of the "background" problem, *i.e* the domain $\mathcal{O}$ without the membrane:

$$\mu = \begin{cases} \mu_\mathrm{c}, & \text{in } \mathcal{O}_\mathrm{c}, \\ \mu_\mathrm{e}, & \text{in } \mathcal{O}_\mathrm{e}, \end{cases} \qquad \epsilon = \begin{cases} \epsilon_\mathrm{c}, & \text{in } \mathcal{O}_\mathrm{c}, \\ \epsilon_\mathrm{e}, & \text{in } \mathcal{O}_\mathrm{e}, \end{cases} \qquad \sigma = \begin{cases} \sigma_\mathrm{c}, & \text{in } \mathcal{O}_\mathrm{c}, \\ \sigma_\mathrm{e}, & \text{in } \mathcal{O}_\mathrm{e}, \end{cases} \quad ,$$

and we define similarly κ as

$$\kappa = \begin{cases} \kappa_\mathrm{c}, & \text{in } \mathcal{O}_\mathrm{c}, \\ \kappa_\mathrm{e}, & \text{in } \mathcal{O}_\mathrm{e}, \end{cases} \quad .$$

It is worth noting that even in the linear regime, biological cell is a complex material, which behaves differently when the frequency of the excitation changes.

Actually, if the complex constants κ_e and κ_c are of similar order, for some frequencies we call low-frequency range, the modulus $|\kappa_\mathrm{m}/\kappa_\mathrm{e}|^2$ is small and of order similar to ε, while for higher frequencies, called mid-frequency range, κ_m, κ_e (and thus κ_c) are of the same order. We refer to [15] for more details. Therefore the asymptotic expansion has to take this feature into account.

3.1. **The mid-frequency case.** In the mid-frequency range, the cell is a soft contrast material. The results of [7] hold straightforwardly.

Two first orders of the asymptotic expansion. The first term $\mathbf{E}^0$ of the expansion satisfies the problem without the layer:

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^0 - \kappa_\mathrm{e}^2 \mathbf{E}^0 = i\omega\mu_\mathrm{e}\mathbf{J}\ , \text{ in } \mathcal{O}_\mathrm{e}, \tag{2a}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^0 - \kappa_\mathrm{c}^2 \mathbf{E}^0 = 0\ , \text{ in } \mathcal{O}_\mathrm{c}, \tag{2b}$$

with the transmission conditions:

$$\left[\mathbf{E}^0 \times \mathbf{n}\right]_\Gamma = 0, \quad \frac{1}{\mu_\mathrm{e}} \operatorname{curl}\mathbf{E}^0 \times \mathbf{n}|_{\Gamma+} = \frac{1}{\mu_\mathrm{c}} \operatorname{curl}\mathbf{E}^0 \times \mathbf{n}|_{\Gamma-}, \tag{2c}$$

and the Silver-Müller condition

$$\operatorname{curl}\mathbf{E}^0 \times \mathbf{n} - i\kappa_\mathrm{e}\,\mathbf{n} \times \mathbf{E}^0 \times \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}. \tag{2d}$$

The influence of the thin layer appears in the problem satisfied by the second term $\mathbf{E}^1$. According to [7], it is necessary to introduce the following tangent operators $\mathbf{T}$ and $\mathbf{S}$ on Γ^1:

$$\text{(3)}\quad \mathbf{T}(\mathbf{E}) = \left(\frac{\mu_\mathrm{m}}{\kappa_\mathrm{m}^2} - \frac{\mu_\mathrm{e}}{\kappa_\mathrm{e}^2}\right)\mathbf{n}\times \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\big(\frac{1}{\mu_\mathrm{c}}\operatorname{curl}\mathbf{E})_\mathsf{T}|_{\Gamma^-} + (\mu_\mathrm{m}-\mu_\mathrm{e})\frac{1}{\mu_\mathrm{c}}\left(\operatorname{curl}\mathbf{E}\times\mathbf{n}\right)|_{\Gamma^-},$$

$$\text{(4)}\quad \mathbf{S}(\mathbf{E}) = -\left(\frac{\kappa_\mathrm{m}^2}{\mu_\mathrm{m}} - \frac{\kappa_\mathrm{e}^2}{\mu_\mathrm{e}}\right)(\mathbf{n}\times\mathbf{E}\times\mathbf{n})|_{\Gamma^+} + \left(\frac{1}{\mu_\mathrm{m}} - \frac{1}{\mu_\mathrm{e}}\right)\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma(\mathbf{n}\times\mathbf{E}\times\mathbf{n})|_{\Gamma^+}\,.$$

such as $\mathbf{E}^1$ satisfies

$$\text{(5a)}\qquad \operatorname{curl}\operatorname{curl}\mathbf{E}^1 - \kappa^2\mathbf{E}^1 = 0\ ,\ \text{in } \mathcal{O}_\mathrm{c}\cup\mathcal{O}_\mathrm{e},$$

$$\text{(5b)}\qquad \operatorname{curl}\mathbf{E}^1\times\mathbf{n} - i\kappa_\mathrm{e}\,\mathbf{n}\times\mathbf{E}^1\times\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O},$$

with the following transmission conditions on Γ

$$\text{(5c)}\qquad \mathbf{n}\times\mathbf{E}^1|_{\Gamma^+}\times\mathbf{n} = \mathbf{n}\times\mathbf{E}^1|_{\Gamma^-}\times\mathbf{n} + \mathbf{T}(\mathbf{E}^0)\ ,$$

$$\text{(5d)}\qquad \frac{1}{\mu_\mathrm{e}}\left(\operatorname{curl}\mathbf{E}^1\times\mathbf{n}\right)|_{\Gamma^+} = \frac{1}{\mu_\mathrm{c}}\left(\operatorname{curl}\mathbf{E}^1\times\mathbf{n}\right)|_{\Gamma^-} + \mathbf{S}(\mathbf{E}^0)\ .$$

3.2. **The low-frequency case.** For low frequencies, the ratio $|\kappa_\mathrm{m}/\kappa_\mathrm{e}|^2$ is of order ε, and thus the derivation of the expansion changes [6]. It is convenient to introduce the complex $\widetilde{\kappa}_\mathrm{m}$ such as

$$\kappa_\mathrm{m}^2 = \varepsilon\widetilde{\kappa}_\mathrm{m}^2,$$

where $\widetilde{\kappa}_\mathrm{m}$ is such that its imaginary and real parts have the same sign as these of κ_m.

Performing the same reasoning as done in [7] and assuming that $\widetilde{\kappa}_\mathrm{m}$ is independent of ε, we find that at low frequency, the membrane influence appears at the zeroth order term, meaning that the membrane influence should not be neglected [6].

$$\text{(6a)}\qquad \operatorname{curl}\operatorname{curl}\mathbf{E}^0 - \kappa_\mathrm{e}^2\mathbf{E}^0 = i\omega\mu_\mathrm{e}\mathbf{J}, \quad \text{in } \mathcal{O}_\mathrm{e},$$

$$\text{(6b)}\qquad \operatorname{curl}\operatorname{curl}\mathbf{E}^0 - \kappa_\mathrm{c}^2\mathbf{E}^0 = 0, \quad \text{in } \mathcal{O}_\mathrm{c},$$

$$\text{(6c)}\qquad \operatorname{curl}\mathbf{E}^0\times\mathbf{n} - i\kappa_\mathrm{e}\mathbf{n}\times\mathbf{E}^0\times\mathbf{n} = 0, \quad \text{on } \partial\mathcal{O},$$

with the following transmission conditions set on Γ

$$\text{(6d)}\qquad \frac{1}{\mu_\mathrm{e}}\operatorname{curl}\mathbf{E}^0|_{\Gamma^+}\times\mathbf{n} = \frac{1}{\mu_\mathrm{c}}\operatorname{curl}\mathbf{E}^0|_{\Gamma^-}\times\mathbf{n},$$

$$\text{(6e)}\qquad \left[\mathbf{n}\times\mathbf{E}^0\right]_\Gamma = \frac{\mu_\mathrm{m}}{\widetilde{\kappa}_\mathrm{m}^2}\frac{\kappa_\mathrm{c}^2}{\mu_\mathrm{c}}\nabla_\Gamma\left(\mathbf{E}^0|_{\Gamma^-}\cdot\mathbf{n}\right).$$

Transmission condition (6e) can be rewritten in terms of the tangent components of the magnetic field. Actually, since $\mathbf{J}$ vanishes in a neighborhood of the membrane, one has on Γ the following condition:

$$\left(\operatorname{curl}\operatorname{curl}\mathbf{E}^0\right)|_{\Gamma^-}\cdot\mathbf{n} - \kappa_\mathrm{c}^2\mathbf{E}^0|_{\Gamma^-}\cdot\mathbf{n} = 0,$$

from which we infer

$$\frac{\kappa_\mathrm{c}^2}{\mu_\mathrm{c}}\nabla_\Gamma\left(\mathbf{E}^0|_{\Gamma^-}\cdot\mathbf{n}\right) = \vec{\operatorname{curl}}_\Gamma\operatorname{curl}_\Gamma\left(\frac{1}{\mu_\mathrm{c}}\operatorname{curl}\mathbf{E}^0|_{\Gamma^-}\times\mathbf{n}\right).$$

[1] In the reference [7, Eq. (18)], there is a sign error in front of the term $\left(\frac{\kappa_\mathrm{m}^2}{\mu_\mathrm{m}} - \frac{\kappa_\mathrm{e}^2}{\mu_\mathrm{e}}\right)(\mathbf{n}\times\mathbf{E}\times\mathbf{n})|_{\Gamma^+}$ which appears in the expression of $\mathbf{S}(\mathbf{E})$.

Therefore, transmission condition (6e) can be rewritten into

$$\left[\mathbf{n} \times \mathbf{E}^0\right]_\Gamma = -\frac{\mu_\mathrm{m}}{\widetilde{\kappa}_\mathrm{m}^2} \vec{\mathrm{curl}}_\Gamma \, \mathrm{curl}_\Gamma \left(\frac{1}{\mu_\mathrm{c}} \, \mathrm{curl}\, \mathbf{E}^0|_{\Gamma^-}\right)_\mathsf{T} . \tag{7}$$

Note that unlike the mid-frequency case, the zeroth order term satisfies a non standard problem which links the jump of the electric field $\mathbf{E}^0$ to the tangential gradient of its normal component. Existence and uniqueness for such a problem is non trivial and will be discussed in the section 4.2.

Remark 3.1 (Link with the quasistatic potential). Note that equation (6) is the extension to the electric field of the steady-state potential approximation as given by Perrussel and Poignard [13].

Actually, the quasi-static approximation consists in assuming that the solution to (6) derives from a potential, *i.e.* that $\mathbf{E}^0 = -\nabla U$. Then we deduce the following partial differential equation for U:

$$\begin{aligned}
&\kappa_\mathrm{e}^2 \Delta U = i\omega\mu_\mathrm{e} \nabla \cdot \mathbf{J}, \quad \text{in } \mathcal{O}_\mathrm{e} \, , \\
&\kappa_\mathrm{c}^2 \Delta U = 0, \quad \text{in } \mathcal{O}_\mathrm{c}, \\
&i\kappa_\mathrm{e} \partial_\mathbf{n} U = 0, \quad \text{on } \partial\mathcal{O} \, .
\end{aligned}$$

The continuity of $\mu^{-1}\kappa^2 \mathbf{E}^0 \cdot \mathbf{n}$ across Γ and transmission condition (6e) lead to the following transmission conditions

$$\begin{aligned}
&\kappa_\mathrm{e}^2 \partial_\mathbf{n} U|_{\Gamma^+} = \kappa_\mathrm{c}^2 \partial_\mathbf{n} U|_{\Gamma^-}, \\
&\frac{\widetilde{\kappa}_\mathrm{m}^2}{\mu_\mathrm{m}} [U]_\Gamma = \frac{\kappa_\mathrm{c}^2}{\mu_\mathrm{c}} \partial_\mathbf{n} U|_{\Gamma^-},
\end{aligned}$$

which is exactly the first-order approximate condition for the quasistatic potential as given by [13]. Note that the above problem is simpler than problem (6) since it has a straighforward variational formulation as shown in [13].

3.3. **Influence of the position of the fictive boundary and of weighted average of the traces on the expansion.** In the above sections 3.1–3.2, we have chosen to write the condition on the boundary Γ of the inner domain $\mathcal{O}_\mathrm{c}$ but this is an arbitrary convention. Sometimes it might be interesting to place the fictive surface on which the transmission conditions hold between the boundary of the inner domain and the surface Γ^ε. Actually, for any $\beta \in [0, 1]$ we can define the family of surfaces that are *parallel* to Γ by

$$\Gamma_\beta = \{\mathbf{x}_\mathsf{T} + \beta\varepsilon\mathbf{n}(\mathbf{x}_\mathsf{T}), \quad \mathbf{x}_\mathsf{T} \in \Gamma\} \, .$$

In addition, in the definition of $\mathbf{S}$ and $\mathbf{T}$, the surface Γ^- is involved but here again it is a convention, and a weighted average between Γ^+ and Γ^- could have been chosen.

In order to study numerically the influence of such conventions on the convergence rate, for any $\alpha \in [0, 1]$, and for any vector field $\mathbf{v}$ defined in a neighborhood of Γ, let $\langle \mathbf{v}|_\Gamma \rangle_\alpha$ be defined by

$$\langle \mathbf{v}|_\Gamma \rangle_\alpha = \alpha \, \mathbf{v}|_{\Gamma^+} + (1-\alpha) \, \mathbf{v}|_{\Gamma^-} .$$

We now define the operators $\mathbf{T}_{\alpha,\beta}$ and $\mathbf{S}_{\alpha,\beta}$ as

$$\mathbf{T}_{\alpha,\beta}(\mathbf{E}) = A_\beta\, \mathbf{n} \times \vec{\operatorname{curl}}_{\Gamma_\beta} \operatorname{curl}_{\Gamma_\beta} \langle \left(\frac{1}{\mu} \operatorname{curl} \mathbf{E}\right)_{\mathsf{T}} |_{\Gamma_\beta} \rangle_\alpha + B_\beta \, \langle \left(\frac{1}{\mu} \operatorname{curl} \mathbf{E} \times \mathbf{n}\right) |_{\Gamma_\beta} \rangle_\alpha, \tag{8a}$$

$$\mathbf{S}_{\alpha,\beta}(\mathbf{E}) = C_\beta\, \vec{\operatorname{curl}}_{\Gamma_\beta} \operatorname{curl}_{\Gamma_\beta} \langle \mathbf{E}_{\mathsf{T}} |_{\Gamma_\beta} \rangle_\alpha - D_\beta \, \langle \mathbf{E}_{\mathsf{T}} |_{\Gamma_\beta} \rangle_\alpha, \tag{8b}$$

where

$$A_\beta = \frac{\mu_{\mathrm{m}}}{\kappa_{\mathrm{m}}^2} - \beta \frac{\mu_{\mathrm{c}}}{\kappa_{\mathrm{c}}^2} - (1-\beta)\frac{\mu_{\mathrm{e}}}{\kappa_{\mathrm{e}}^2}, \tag{9a}$$

$$B_\beta = \mu_{\mathrm{m}} - \beta\mu_{\mathrm{c}} - (1-\beta)\mu_{\mathrm{e}}, \tag{9b}$$

$$C_\beta = \frac{1}{\mu_{\mathrm{m}}} - \frac{\beta}{\mu_{\mathrm{c}}} - \frac{1-\beta}{\mu_{\mathrm{e}}}, \tag{9c}$$

$$D_\beta = \frac{\kappa_{\mathrm{m}}^2}{\mu_{\mathrm{m}}} - \beta \frac{\kappa_{\mathrm{c}}^2}{\mu_{\mathrm{c}}} - (1-\beta)\frac{\kappa_{\mathrm{e}}^2}{\mu_{\mathrm{e}}}. \tag{9d}$$

With such notations, for any $(\alpha, \beta) \in [0,1]^2$, approximate transmission conditions (5c)–(5d) of the mid-frequency case have to be replaced on Γ_β by

$$\mathbf{n} \times \mathbf{E}^1|_{\Gamma_\beta^+} \times \mathbf{n} = \mathbf{n} \times \mathbf{E}^1|_{\Gamma_\beta^-} \times \mathbf{n} + \mathbf{T}_{\alpha,\beta}(\mathbf{E}^0)\,,$$

$$\frac{1}{\mu_{\mathrm{e}}}\left(\operatorname{curl} \mathbf{E}^1 \times \mathbf{n}\right)|_{\Gamma_\beta^+} = \frac{1}{\mu_{\mathrm{c}}}\left(\operatorname{curl} \mathbf{E}^1 \times \mathbf{n}\right)|_{\Gamma_\beta^-} + \mathbf{S}_{\alpha,\beta}(\mathbf{E}^0)\,,$$

while transmission condition (7) of the low frequency case has to be replaced on Γ_β by

$$\left[\mathbf{n} \times \mathbf{E}^0\right]_{\Gamma_\beta} = -\frac{\mu_{\mathrm{m}}}{\widetilde{\kappa}_{\mathrm{m}}^2} \vec{\operatorname{curl}}_{\Gamma_\beta} \operatorname{curl}_{\Gamma_\beta} \left(\langle \frac{1}{\mu} \operatorname{curl} \mathbf{E}^0 |_{\Gamma_\beta} \rangle_\alpha \right)_{\mathsf{T}}.$$

4. Generalized Impedance Transmission Conditions

As seen in section 3.1, the computation of the approximate field requires to solve two similar problems which are independent of ε: one for $\mathbf{E}^0$ and one for $\mathbf{E}^1$. The advantage of such approach lies in the parametric study of the problem: if one is interested in several values of the parameter ε, one just has to compute $\mathbf{E}^0$ and $\mathbf{E}^1$ once, and then it remains to recover the final approximation $\mathbf{E}^\varepsilon$ for the desired values of ε with the simple operation:

$$\mathbf{E}^\varepsilon \approx \mathbf{E}^0 + \varepsilon \mathbf{E}^1.$$

However if the membrane thickness is well-known, it could be interesting to solve only one problem. For such approach, the idea is to write the problem satisfied by $\mathbf{E}_1^\varepsilon = \mathbf{E}^0 + \varepsilon\mathbf{E}^1$:

$$\operatorname{curl}\operatorname{curl} \mathbf{E}_1^\varepsilon - \kappa_{\mathrm{e}}^2 \mathbf{E}_1^\varepsilon = i\omega\mu_{\mathrm{e}} \mathbf{J}\,, \text{ in } \mathcal{O}_{\mathrm{e}}^\beta\,, \tag{10a}$$

$$\operatorname{curl}\operatorname{curl} \mathbf{E}_1^\varepsilon - \kappa_{\mathrm{c}}^2 \mathbf{E}_1^\varepsilon = 0\,, \text{ in } \mathcal{O}_{\mathrm{c}}, \tag{10b}$$

$$\operatorname{curl} \mathbf{E}_1^\varepsilon \times \mathbf{n} - i\kappa_{\mathrm{e}}\, \mathbf{n} \times \mathbf{E}_1^\varepsilon \times \mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O}, \tag{10c}$$

with the following transmission conditions on Γ_β

(10d) $$\mathbf{n}\times \mathbf{E}_1^\varepsilon|_{\Gamma_\beta^+}\times \mathbf{n} = \mathbf{n}\times \mathbf{E}_1^\varepsilon|_{\Gamma_\beta^-}\times \mathbf{n} + \varepsilon \mathbf{T}_{\alpha,\beta}(\mathbf{E}^0),$$

(10e) $$\frac{1}{\mu_\mathrm{e}}\left(\mathrm{curl}\,\mathbf{E}_1^\varepsilon \times \mathbf{n}\right)|_{\Gamma_\beta^+} = \frac{1}{\mu_\mathrm{c}}\left(\mathrm{curl}\,\mathbf{E}_1^\varepsilon \times \mathbf{n}\right)|_{\Gamma_\beta^-} + \varepsilon \mathbf{S}_{\alpha,\beta}(\mathbf{E}^0).$$

Here, $\mathcal{O}_\mathrm{e}^\beta = \{\mathbf{x}\in\mathcal{O}_\mathrm{e} \mid \mathrm{dist}(\mathbf{x},\Gamma) > 2\beta\varepsilon\}$.

Then, remarking that $\varepsilon\mathbf{S}_{\alpha,\beta}(\mathbf{E}^0)$ and $\varepsilon\mathbf{S}_{\alpha,\beta}(\mathbf{E}_1^\varepsilon)$ differ from a term in ε^2 (and similarly for $\varepsilon\mathbf{T}_{\alpha,\beta}(\mathbf{E}_1^\varepsilon)$), the final field $\mathbf{E}_{[1]}^\varepsilon$, which approximates $\mathbf{E}^\varepsilon$ at the order 2, is obtained by solving only one problem:

(11a) $$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon - \kappa_\mathrm{e}^2\mathbf{E}_{[1]}^\varepsilon = i\omega\mu_\mathrm{e}\mathbf{J}\ ,\ \text{in } \mathcal{O}_\mathrm{e}^\beta\ ,$$

(11b) $$\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon - \kappa_\mathrm{c}^2\mathbf{E}_{[1]}^\varepsilon = 0\ ,\ \text{in } \mathcal{O}_\mathrm{c},$$

(11c) $$\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon\times\mathbf{n} - i\kappa_\mathrm{e}\,\mathbf{n}\times\mathbf{E}_{[1]}^\varepsilon\times\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{O},$$

with the following transmission conditions on Γ_β, called generalized impedance transmission conditions (GITC) of order 2:

(11d) $$\mathbf{n}\times \mathbf{E}_{[1]}^\varepsilon|_{\Gamma_\beta^+}\times \mathbf{n} = \mathbf{n}\times \mathbf{E}_{[1]}^\varepsilon|_{\Gamma_\beta^-}\times \mathbf{n} + \varepsilon \mathbf{T}_{\alpha,\beta}(\mathbf{E}_{[1]}^\varepsilon),$$

(11e) $$\frac{1}{\mu_\mathrm{e}}\Big(\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon \times \mathbf{n}\Big)|_{\Gamma_\beta^+} = \frac{1}{\mu_\mathrm{c}}\Big(\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon \times \mathbf{n}\Big)|_{\Gamma_\beta^-} + \varepsilon \mathbf{S}_{\alpha,\beta}(\mathbf{E}_{[1]}^\varepsilon).$$

4.1. **An equivalent Lagrangian formulation.** In order to solve equations (11) and also equations (6), we provide an equivalent Lagrangian formulation. We introduce the Lagrangian multiplier λ defined as

$$\lambda = \langle \frac{1}{\mu}(\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon|_{\Gamma_\beta})_\mathsf{T}\rangle_\alpha.$$

Then the GITCs for the mid-frequency case write:

$$\Big[\mathbf{n}\times\mathbf{E}_{[1]}^\varepsilon\Big]_{\Gamma_\beta} = \varepsilon\left(-A_\beta\,\vec{\mathrm{curl}}_{\Gamma_\beta}\mathrm{curl}_{\Gamma_\beta}\lambda\ +\ B_\beta\lambda\right)$$

$$\left[\frac{\mathbf{n}}{\mu}\times\mathrm{curl}\,\mathbf{E}_{[1]}^\varepsilon\right]_{\Gamma_\beta} = \varepsilon\left(-C_\beta\,\vec{\mathrm{curl}}_{\Gamma_\beta}\mathrm{curl}_{\Gamma_\beta}\Big\langle(\mathbf{E}_{[1]}^\varepsilon)_\mathsf{T}\Big\rangle_{1-\alpha}\ +\ D_\beta\Big\langle(\mathbf{E}_{[1]}^\varepsilon)_\mathsf{T}\Big\rangle_{1-\alpha}\right),$$

where constants $A_\beta, B_\beta, C_\beta, D_\beta$ are defined in (9).

Delourme *et al.* have derived in [5, Eqs. (1), (5)] a model for periodic oscillating thin layer that can be explicited in our simpler case of homogeneous thin layer. After calculations, it falls that the model of Delourme *et al.* corresponds to our GITC of order 2 in the specific symmetric case $(\alpha,\beta) = (1/2, 1/2)$.

Note that under the assumption $\Im(A_\beta)\neq 0$, the operator $\mathcal{G}$ defined from $\mathsf{T}\mathrm{H}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\beta},\Gamma_\beta)$ onto $\mathsf{T}\mathrm{H}(\mathrm{curl}_{\Gamma_\beta},\Gamma_\beta)$ by

(12)

for any $g\in\mathsf{T}\mathrm{H}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\beta},\Gamma_\beta)$, $\quad\mathcal{G}(g)=\lambda,$

where λ satisfies

$$A_\beta\,\vec{\mathrm{curl}}_{\Gamma_\beta}\mathrm{curl}_{\Gamma_\beta}\lambda -\ B_\beta\lambda = g \quad \text{on} \quad \Gamma_\beta\ ,$$

is well defined and invertible. Therefore the result of Delourme *et al.* can be applied straightforwardly: in the framework of section 4.3.1, existence and uniqueness of the solution $\mathbf{E}^{\varepsilon}_{[1]} \in \mathbf{V}_{\alpha,\beta}$ to (11) hold.

4.2. **Link with the low-frequency case.** The above GITC should provide an approximation of $\mathbf{E}^{\varepsilon}$ of order $O(\varepsilon^2)$ under the framework of mid-frequency. However, observe that if we now replace κ_{m}^2 by $\varepsilon\widetilde{\kappa}_{\mathrm{m}}^2$, meaning that if we look at the low-frequency case, and if we drop the terms of order $O(\varepsilon)$ we observe that the term $\varepsilon \mathbf{S}_{\alpha,\beta}(\mathbf{E}^{\varepsilon}_{[1]})$ has to be dropped off, while $\varepsilon \mathbf{T}_{\alpha,\beta}(\mathbf{E}^{\varepsilon}_{[1]})$ should be identified with

$$-\frac{\mu_{\mathrm{m}}}{\widetilde{\kappa}_{\mathrm{m}}^2} \vec{\mathrm{curl}}_{\Gamma_\beta} \mathrm{curl}_{\Gamma_\beta} \left(\langle \frac{1}{\mu} \mathrm{curl}\, \mathbf{E}_{[0]} |_{\Gamma_\beta} \rangle_\alpha \right)_{\mathsf{T}} \times \mathbf{n},$$

where $\mathbf{E}_{[0]}$ satisfies the same problem as $\mathbf{E}^0$, given by (6), thanks to (7).

Therefore, from low to mid frequency, the GITC (11) provides an approximation of $\mathbf{E}^{\varepsilon}$. The order of approximation should be $O(\varepsilon)$ at low frequency (*i.e.* for $|\kappa_{\mathrm{m}}/\kappa_{\mathrm{e}}|^2 = O(\varepsilon)$) and $O(\varepsilon^2)$ at mid frequency, where $|\kappa_{\mathrm{m}}/\kappa_{\mathrm{e}}| \sim 1$.

Note that in this case, (6) also falls into the above framework by changing A_β, B_β, C_β and D_β into

$$A_\beta^R = \frac{1}{\varepsilon} \frac{\mu_{\mathrm{m}}}{\widetilde{\kappa}_{\mathrm{m}}^2}, \qquad B_\beta^R = C_\beta^R = D_\beta^R = 0.$$

However since $B_\beta = 0$, the operator $\mathcal{G}$ defined by (12) is no longer invertible from $\mathsf{T}\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ onto $\mathsf{T}\mathrm{H}(\mathrm{curl}_\Gamma, \Gamma)$. Note however that it is invertible from the space

$$\mathsf{T}\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma, 0) = \left\{ \mathbf{v} \in \mathsf{T}\mathrm{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \mid \mathrm{div}_\Gamma\, \mathbf{v} = 0 \right\},$$

onto the space

$$\mathsf{T}\mathrm{H}(\mathrm{curl}_\Gamma, \Gamma, 0) = \left\{ \mathbf{v} \in \mathsf{T}\mathrm{H}(\mathrm{curl}_\Gamma, \Gamma) \mid \mathrm{div}_\Gamma\, \mathbf{v} = 0 \right\},$$

and thus ad hoc modifications of the results of Delourme *et al.* would lead to similar existence and uniqueness results in the framework of section 4.3.2 .

4.3. **Generic variational formulations.** Let write now the variational fomulations for the two cases.

4.3.1. *The mid-frequency case.* We introduce a common variational framework for the GITC model (11). The functional spaces associated with $\mathbf{E}^{\varepsilon}_{[1]}$ and λ are $\mathbf{V}_{\alpha,\beta}$ and $\mathsf{W}_\beta = \mathsf{T}\mathrm{H}(\mathrm{curl}_{\Gamma_\beta}, \Gamma_\beta)$ respectively, defined as

$$\begin{aligned} \mathbf{V}_{\alpha,\beta} = &\Big\{ \mathbf{E} \in \mathbf{L}^2(\mathcal{O}_{\mathrm{c}} \cup \mathcal{O}_{\mathrm{e}}^\beta),\ \mathrm{curl}\, \mathbf{E}_{\mathrm{c}} \in \mathbf{L}^2(\mathcal{O}_{\mathrm{c}}),\ \mathrm{curl}\, \mathbf{E}_{\mathrm{e}} \in \mathbf{L}^2(\mathcal{O}_{\mathrm{e}}^\beta), \\ &\big\langle \mathbf{E}_{\mathsf{T}} |_{\Gamma_\beta} \big\rangle_{1-\alpha} \in \mathsf{T}\mathrm{H}(\mathrm{curl}_{\Gamma_\beta}, \Gamma_\beta),\ \mathbf{E} \times \mathbf{n} \in \mathbf{L}^2(\partial\mathcal{O}) \Big\}, \end{aligned} \tag{13}$$

$$\mathsf{W}_\beta = \mathsf{T}\mathrm{H}(\mathrm{curl}_{\Gamma_\beta}, \Gamma_\beta). \tag{14}$$

Note that the functional spaces $\mathbf{V}_{\alpha,\beta}$ and W_β depend on ε since the surface Γ_β depends on ε. Then, the variational formulation for the GITC (11) writes :

Find $(\mathbf{E}^{\varepsilon}_{[1]}, \lambda) \in \mathbf{V}_{\alpha,\beta} \times \mathsf{W}_\beta$ such that for all $(\mathbf{U}, \xi) \in \mathbf{V}_{\alpha,\beta} \times \mathsf{W}_\beta$

$$
\begin{aligned}
&\int_{\mathcal{O}_c\cup\mathcal{O}_e^\beta}\frac{1}{\mu}\operatorname{curl}\mathbf{E}^\varepsilon_{[1]}\cdot\operatorname{curl}\overline{\mathbf{U}}\,\mathrm{d}\mathbf{x}-\int_{\mathcal{O}_c\cup\mathcal{O}_e^\beta}\frac{\kappa^2}{\mu}\mathbf{E}^\varepsilon_{[1]}\cdot\overline{\mathbf{U}}\,\mathrm{d}\mathbf{x}\\
&-i\kappa_e\int_{\partial O}\mathbf{E}^\varepsilon_{[1]}\times\mathbf{n}\cdot\overline{\mathbf{U}}\times\mathbf{n}\,\mathrm{d}s-\int_{\Gamma_\beta}\mathbf{n}\times\lambda\cdot\left[\,\overline{\mathbf{U}_\mathsf{T}}\,\right]\,\mathrm{d}s\\
&+\varepsilon\int_{\Gamma_\beta}C_\beta\,\operatorname{curl}_{\Gamma_\beta}\left\langle(\mathbf{E}^\varepsilon_{[1]})_\mathsf{T}\right\rangle_{1-\alpha}\operatorname{curl}_{\Gamma_\beta}\left\langle\overline{\mathbf{U}_\mathsf{T}}\right\rangle_{1-\alpha}\,\mathrm{d}s\\
&-\varepsilon\int_{\Gamma_\beta}D_\beta\left\langle(\mathbf{E}^\varepsilon_{[1]})_\mathsf{T}\right\rangle_{1-\alpha}\cdot\left\langle\overline{\mathbf{U}_\mathsf{T}}\right\rangle_{1-\alpha}\,\mathrm{d}s\\
&=\;i\omega\int_{\mathcal{O}_e}\mathbf{J}\cdot\overline{\mathbf{U}_e}\,\mathrm{d}\mathbf{x}\,,
\end{aligned}
\tag{15a}
$$

and

$$
\int_{\Gamma_\beta}\left[\mathbf{n}\times\mathbf{E}^\varepsilon_{[1]}\right]\cdot\overline{\xi}\,\mathrm{d}s+\varepsilon\int_{\Gamma_\beta}A_\beta\operatorname{curl}_{\Gamma_\beta}\lambda\operatorname{curl}_{\Gamma_\beta}\overline{\xi}\,\mathrm{d}s-\varepsilon\int_{\Gamma_\beta}B_\beta\lambda\cdot\overline{\xi}\,\mathrm{d}s\;=\;0\,.
\tag{15b}
$$

4.3.2. *Variational formulations for the low-frequency case.* For the low-frequency case we define the functional spaces $\mathbf{V}_{\alpha,\beta,0}$ and $\mathsf{W}_{\beta,0}$ similarly to $\mathbf{V}_{\alpha,\beta}$ and W_β by

$$
\begin{aligned}
\mathbf{V}_{\alpha,\beta,0}=&\{\mathbf{E}\in\mathbf{L}^2(\mathcal{O}_c\cup\mathcal{O}_e^\beta),\,\operatorname{curl}\mathbf{E}_c\in\mathbf{L}^2(\mathcal{O}_c),\,\operatorname{curl}\mathbf{E}_e\in\mathbf{L}^2(\mathcal{O}_e^\beta),\\
&\left\langle\mathbf{E}_\mathsf{T}|_{\Gamma_\beta}\right\rangle_{1-\alpha}\in\mathsf{TH}(\operatorname{curl}_{\Gamma_\beta},\Gamma_\beta,0),\quad\mathbf{E}\times\mathbf{n}\in\mathbf{L}^2(\partial\mathcal{O})\}\,,\\
\mathsf{W}_{\beta,0}=&\;\mathsf{TH}(\operatorname{curl}_{\Gamma_\beta},\Gamma_\beta,0)\,.
\end{aligned}
\tag{16}
$$

The Lagrangian variational formulation for model (6) writes :
Find $(\mathbf{E}_{[0]},\lambda)\in\mathbf{V}_{\alpha,\beta,0}\times\mathsf{W}_{\beta,0}$ such that for any test-vector field $\mathbf{U}\in\mathbf{V}_{\alpha,\beta,0}$:

$$
\begin{aligned}
&\int_{\mathcal{O}_c\cup\mathcal{O}_e^\beta}\frac{1}{\mu}\operatorname{curl}\mathbf{E}_{[0]}\cdot\operatorname{curl}\overline{\mathbf{U}}\,\mathrm{d}\mathbf{x}-\int_{\mathcal{O}_c\cup\mathcal{O}_e^\beta}\frac{\kappa^2}{\mu}\mathbf{E}_{[0]}\cdot\overline{\mathbf{U}}\,\mathrm{d}\mathbf{x}-i\kappa_e\int_{\partial O}\mathbf{E}_{[0]}\times\mathbf{n}\cdot\overline{\mathbf{U}}\times\mathbf{n}\,ds\\
&-\int_{\Gamma_\beta}\mathbf{n}\times\lambda\cdot\left[\,\overline{\mathbf{U}_\mathsf{T}}\,\right]\,\mathrm{d}s\;=\;i\omega\int_{\mathcal{O}_e}\mathbf{J}\cdot\overline{\mathbf{U}_e}\,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{17a}
$$

where λ satisfies the variational formulation for any smooth enough test-function $\xi\in\mathsf{W}_{\beta,0}$:

$$
-\int_{\Gamma_\beta}\left[\mathbf{n}\times\mathbf{E}_{[0]}\right]\cdot\overline{\xi}\,\mathrm{d}s-\frac{\mu_m}{\widetilde{\kappa}_m^2}\int_{\Gamma_\beta}\operatorname{curl}_{\Gamma_\beta}\lambda\operatorname{curl}_{\Gamma_\beta}\overline{\xi}\,\mathrm{d}s\;=\;0.
\tag{17b}
$$

4.3.3. *Uniqueness.* One easily has the following uniqueness result for the above variational formulations.

Lemma 4.1 (Uniqueness result)**.** *Assume that σ is strictly positive in $\mathcal{O}$ and suppose that a solution $(\mathbf{E},\lambda)$ to the variational formulation* (17) *(resp. to* (15)*) exists. Then it is necessarily unique.*

Proof. We prove the results for (17) when $\beta = 0$, the proof of (15) is exactly the same, *mutatis mutandis*.

If a solution $(\mathbf{E}, \lambda)$ exists with $\mathbf{J}$ equal to zero, then necessarily, by taking as test-couple $(\mathbf{U}, \xi) = (\mathbf{E}, \lambda)$, and using the conjugate form of (17b), one infers

$$\int_{\mathcal{O}_c \cup \mathcal{O}_e} \frac{1}{\mu} \left|\operatorname{curl} \mathbf{E}_{[0]}\right|^2 dx - \omega^2 \int_{\mathcal{O}_c \cup \mathcal{O}_e} (\epsilon + i\frac{\sigma}{\omega}) |\mathbf{E}_{[0]}|^2 dx - i\kappa_e \int_{\partial O} |\mathbf{E}_{[0]} \times \mathbf{n}|^2 \, ds$$
$$- \frac{\mu_m}{|\widetilde{\kappa}_m^2|^2} \widetilde{\kappa}_m^2 \int_\Gamma |\operatorname{curl}_\Gamma \lambda|^2 \, ds \;=\; 0$$

Taking the imaginary part of the above equality, one infers that $\mathbf{E}_{[0]}$ and $\operatorname{curl}_\Gamma \lambda$ vanish. Then, since

$$\operatorname{curl}_\Gamma \lambda = 0 \quad \text{and} \quad \operatorname{div}_\Gamma \lambda = 0,$$

and due to the assumption on Γ, we infer that λ identically vanishes, hence the uniqueness. □

5. Numerical Simulations

5.1. **Validation.** The variational formulation (15) is discretized with high-order hexahedral edge elements of the first kind (see [4] for more details). Here, a fifth-order approximation and curved elements will be used throughout all the experiments. When using edge elements, the following surface finite element matrix (φ_i being a function test associated with the boundary Γ_β)

$$M_{i,j} \;=\; \int_{\Gamma_\beta} \mathbf{n} \times \varphi_j \cdot \bar{\varphi}_i$$

is singular, so that the global finite element matrix will be non-invertible as soon as D_β (9) is equal to 0. In order to ensure the invertibility of the global matrix, we add to D_β a penalty term :

$$D_\beta \leftarrow D_\beta + \varepsilon\delta$$

δ is chosen equal to 10^{-7} in all the numerical experiments. We think that by choosing more carefully the finite element space for λ, for instance a divergence-free finite element space (such that $\operatorname{div}\lambda = 0$ is ensured by construction) like the space $\mathrm{TH}(\operatorname{curl}_\Gamma, \Gamma, 0)$, the surface finite element matrix M may become invertible, and the penalty term could be dropped. The following models will be compared :

- Symmetric model (variational formulation (15) with $\beta = 0.5$)
- GITC model (variational formulation (15) with $\beta = 0$)
- Resistive model (variational formulation (17) with $\beta = 0$)
- $\mathbf{E}_1^\varepsilon$ model (solving problem (10) with $\beta = 0$)

The models are validated for the scattering of a sphere with the following parameters

$$\begin{array}{lll} \epsilon_m \;=\; 3.5, & \mu_m \;=\; 2.0, & \sigma_m \;=\; 0.05 \\ \epsilon_c \;=\; 2.0, & \mu_c \;=\; 1.5, & \sigma_c \;=\; 0.02 \\ \epsilon_e \;=\; 1.0, & \mu_e \;=\; 1.0, & \sigma_e \;=\; 0.0 \end{array}$$

The source is imposed via the Silver-Müller condition ($\mathbf{J} = 0$):

$$\operatorname{curl} \mathbf{E} \times \mathbf{n} - i\,\kappa_e\, \mathbf{n} \times \mathbf{E} \times \mathbf{n} = \operatorname{curl} \mathbf{E}^{\text{inc}} \times \mathbf{n} - i\,\kappa_e\, \mathbf{n} \times \mathbf{E}^{\text{inc}} \times \mathbf{n}$$

where the incident plane wave is equal to

$$\mathbf{E}^{\text{inc}} \;=\; e^{i\kappa_e x} \mathbf{e}_z \,.$$

The wave vector is oriented along $\mathbf{e}_z$, the polarization along $\mathbf{e}_x$. The real part of the solution is displayed in Fig. 2 for $\varepsilon = 0.04$. As it can be observed in Fig. 3, the choice $\alpha = 0.5$ gives

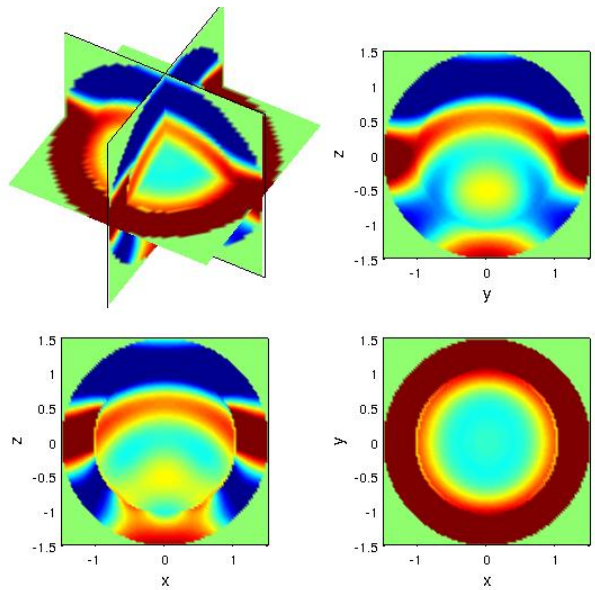

FIGURE 2. Real part of the total field for the sphere. $\varepsilon = 0.04$

more accurate results than other values of α for this model case. The GITC model and symmetric model both provide a convergence in $O(\varepsilon^2)$, the symmetric model being more accurate for this case.

5.2. **Biological cell.** We study the case of a biological cell, assumed to be a sphere whose radius is equal to 10 μm with the following parameters

$$\begin{array}{lll} \epsilon_m = 10.0, & \mu_{\mathrm{m}} = 1.0, & \sigma_m = 5.0e-7 \\ \epsilon_c = 80.0, & \mu_{\mathrm{c}} = 1.0, & \sigma_c = 1.0 \\ \epsilon_{\mathrm{e}} = 80.0, & \mu_{\mathrm{e}} = 1.0, & \sigma_e = 0.5 \end{array}$$

The source is here a Gaussian distribution:

$$\mathbf{J} = \beta_J \, exp(-\alpha_J \, r^2) \mathbf{e}_x$$

α_J and β_J are two constants. The outer boundary is a sphere of radius 20 μm, and the center of the gaussian is located at $(0, 0, 15)$. The radius of the membrane is taken equal to 10 μm, the thickness of the membrane is taken equal to 0.01μm. When the frequency is decreasing, we can observe that the solution becomes discontinuous across the interface. In Fig. 4, the solution is displayed along the axis Oz. For a frequency equal to 100 Mhz, the solution is almost continuous, whereas for a frequency equal to 10 Mhz, the solution presents an important discontinuity through the membrane. The 3-D solution is displayed for the low frequency case in Fig. 5. The symmetric model provides a solution that fits perfectly to the exact solution. In Fig. 6, the different models are compared versus the frequency. It can be observed that the symmetric model provides better results than other models except for the highest frequency. However, the relative numerical error

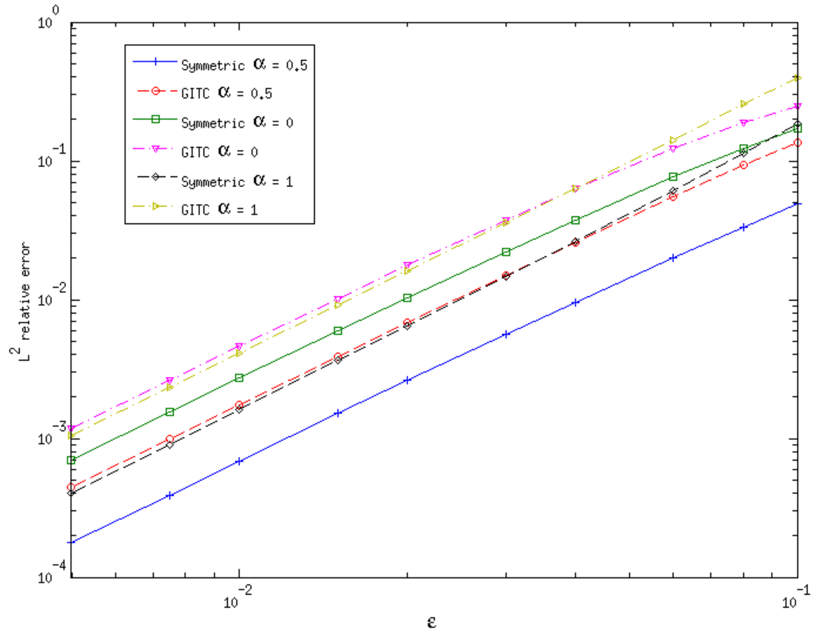

FIGURE 3. Relative L^2 error versus thickness ε for the symmetric model and GITC for different values of α

is estimated to be around 10^{-4}, and when the frequency is lower than 1 Thz, all the models (except the $\mathbf{E}_1^\varepsilon$ model) are providing an error close to the numerical error. We think that this error is mainly numerical, and not due to the model. For low-frequencies, the numerical method used is not very robust, that is the reason why the L^2 error increases a little bit for the lowest frequencies. The numerical method is rather appropriate for mid-frequency cases than for low-frequency cases.

We see also a locking phenomenon for the $\mathbf{E}_1^\varepsilon$, since the constant of convergence is worsening when the frequency is decreasing, whereas the other models seem robust.

The symmetric model and $\mathbf{E}_1^\varepsilon$ and GITC provide a second-order convergence for high-frequency, as it can be observed in Fig. 7. whereas the resistive model is only a first-order model. The model $\mathbf{E}_1^\varepsilon$ is also clearly less accurate than GITC.

For low-frequency cases, all the models provide a first-order convergence (see Fig. 8), but because of the locking phenomenon in the $\mathbf{E}_1^\varepsilon$ model, this model seems to not converge. A convergence should be observed for a very small thickness.

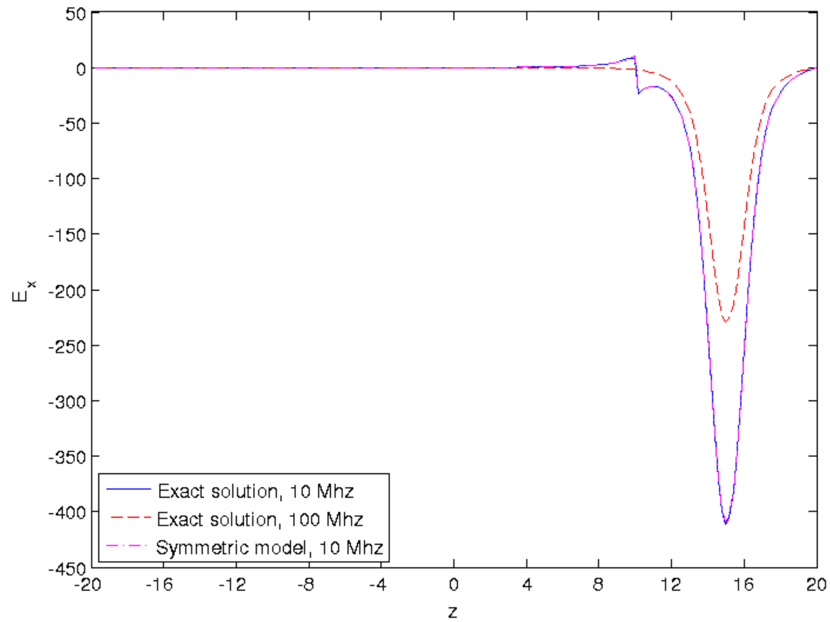

FIGURE 4. Exact solution for the biological cell for a frequency equal to 10 and 100 Mhz, and solution obtained with the symmetric model. Real part of E_x.

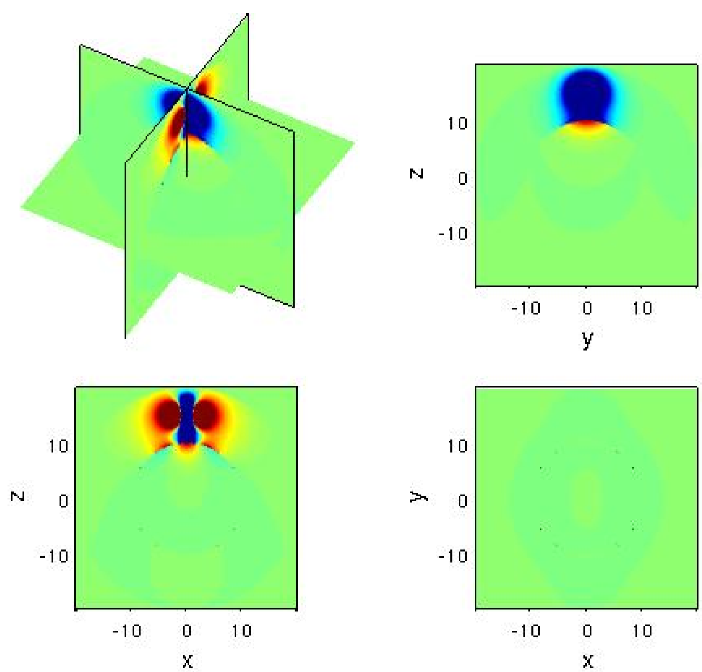

FIGURE 5. Exact solution for 10 Mhz. Real part of E_x.

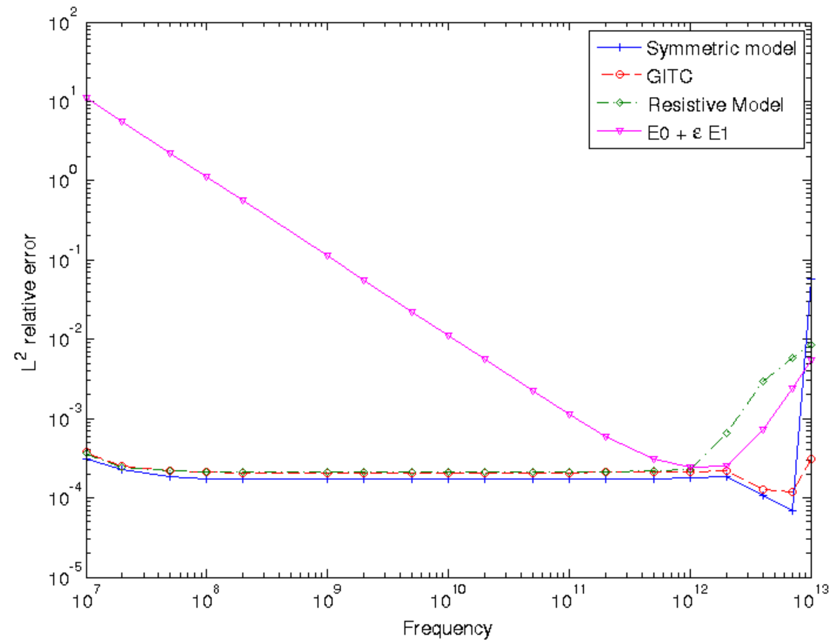

FIGURE 6. Relative L^2 error of the different models versus the frequency (in Hz).

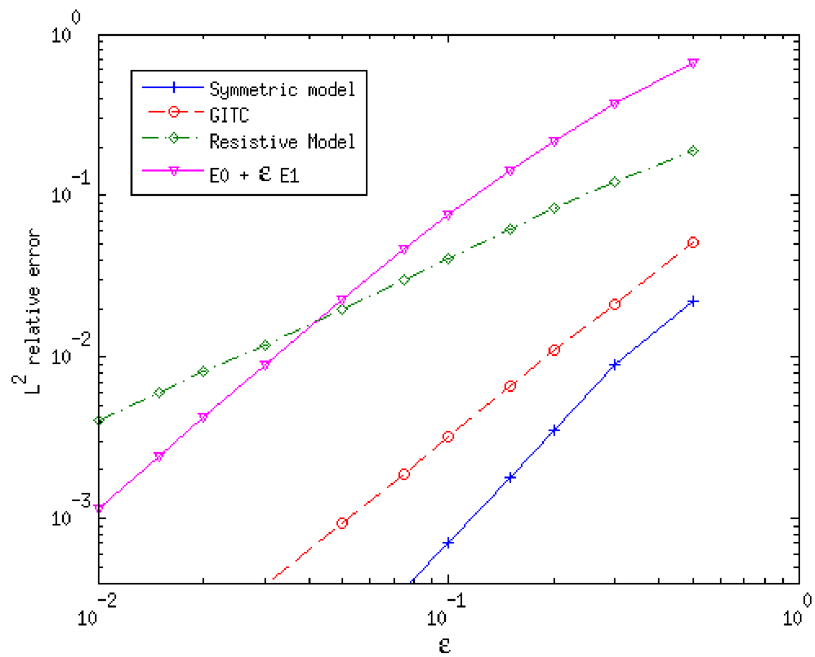

FIGURE 7. Relative L^2 error of the different models versus the thickness (in μm). Frequency equal to 500 GHz.

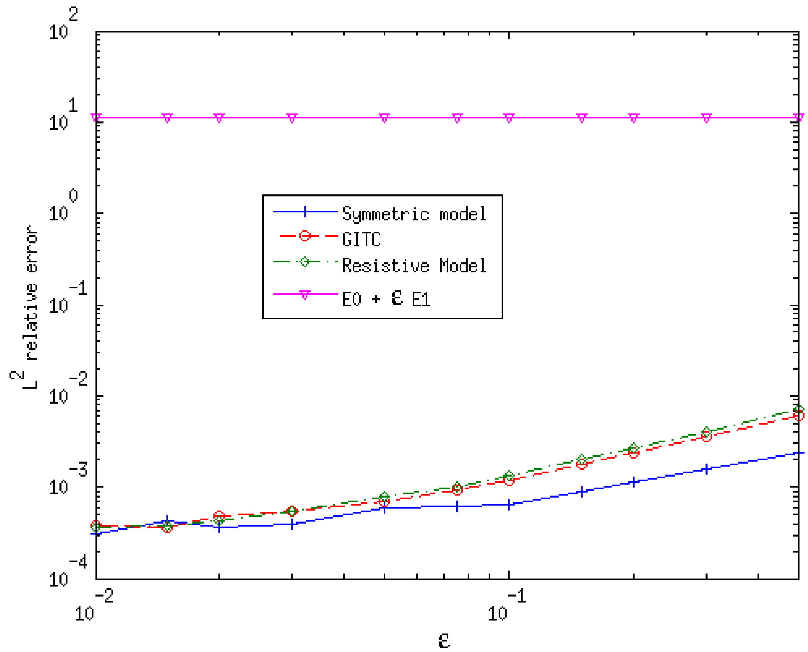

FIGURE 8. Relative L^2 error of the different models versus the thickness (in μm). Frequency equal to 10 MHz.

6. Conclusion

In this paper we have studied numerically the electromagnetic field in a domain with thin layer, which presents two different behaviors: at low frequency, when the ohmic currents are dominant, the domain is high contrasted with a very resistive thin membrane, while at mid frequencies, for which the currents of displacement are prominent the material is soft contrasted. This is the main feature of biological cells.

To avoid the meshing of the thin membrane, four approximations of the electromagnetic field have been used:

- Symmetric model (variational formulation (15) with $\beta = 0.5$)
- GITC model (variational formulation (15) with $\beta = 0$)
- Resistive model (variational formulation (17) with $\beta = 0$)
- E_1^ε model (solving problem (10) with $\beta = 0$)

Each approximation has its own advantages and drawbacks. For instance, if one wants to study the influence of the frequency on the electric field in the cell, the use of the GITC model (or the symmetric model) is more powerful since it provides a quite good approximation of the field whatever the frequency is (of order 2 at mid-frequency and of order 1 at low frequency).

However, if one wants to study the influence of the membrane thickness on the electric field by changing the value of ε, it is more efficient to use the Resistive model for the low frequency case and the E_1^ε model for mid-frequencies. Actually, the coefficients E^0 and E^1 do not depend on ε and they have to be computed only once: the change of ε appearing in the sum

$$\mathsf{E}_1^\varepsilon = \mathsf{E}^0 + \varepsilon \mathsf{E}^1,$$

whereas the GITC model or the Symmetric model involve ε and thus a change in the value of ε makes it necessary to compute the electric field again.

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