Equivalent boundary conditions for acoustic media with exponential densities. Application to the atmosphere in helioseismology.

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Equivalent boundary conditions for acoustic media with exponential densities. Application to the atmosphere in helioseismology.

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Abstract: We present equivalent boundary conditions and asymptotic models for the solution of a transmission problem set in a domain which represents the Sun and its atmosphere. This problem models the propagation of an acoustic wave in time-harmonic regime. The specific non-standard feature of this problem lies in the presence of a small parameter δ which represents the inverse rate of the exponential decay of the density in the atmosphere. The atmosphere can be approximated by local equivalent boundary conditions for small values of this parameter. We derive rigorously equivalent conditions up to the fourth order of approximation with respect to δ for the exact solution u. The construction of equivalent conditions is based on a multiscale expansion in power series of δ for u. Numerical results illustrate the good behavior of these equivalent boundary conditions for realistic values close to those observed for the Sun. Finally we measure the boundary layer phenomenon by introducing a "skin depth" function that turns out to depend on the mean curvature of the interface between the subdomains.

Key-words: Multiscale Expansions, Equivalent Boundary Conditions, Numerical Methods for Helioseismology

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Conditions aux limites approchées pour des milieux acoustiques à densités exponentielles. Application à l'heliosismologie.

Résumé : Nous présentons des conditions aux limites équivalentes et des modèles asymptotiques en vue de la résolution de problèmes de transmission posés dans un domaine représentant le soleil et son atmosphère. Le problème modélise la propagation d'ondes acoustiques en régime harmonique. L'originalité de ce problème réside en la présence d'un petit paramètre δ qui représente l'inverse du taux de décroissance exponentielle de la masse volumique de l'atmosphère. Celle-ci peut être approchée par des conditions aux limites locales équivalentes pour des petites valeurs de ce paramètre. Nous dérivons rigoureusement des conditions équivalentes jusqu'à l'ordre 4 vis-à-vis de δ par rapport à la solution exacte u. La construction de conditions équivalentes se base sur un développement multi-échelle en puissances de δ pour u. Des résultats numériques illustrent le bon comportement de ces conditions aux limites équivalentes pour des valeurs réalistes proches de celles du Soleil. Enfin, nous quantifions un phénomène de couche limite en introduisant une fonction "épaisseur de peau" qui s'avère dépendre de la courbure moyenne de l'interface entre les deux domaines.

Mots-clés : Analyse Asymptotique, Conditions aux Limites Équivalentes, Méthodes Numériques pour l'Héliosismologie

Contents

1 Introduction

The concept of Equivalent Boundary Conditions (also called approximate, effective, or impedance conditions) is rather classical in the modeling of wave propagation phenomena. Equivalent Conditions (ECs) are introduced to reduce the computational domain, for instance for scattering problems from highly absorbing obstacles [18, 14, 13, 19, 12, 1, 10, 8, 4, 3, 7]. The main idea consists in replacing an "exact" model inside a part of the domain of interest by an approximate boundary condition. This idea is relevant when the effective condition can be readily handled for numerical computations, for instance when this condition is local [19, 2, 11].

The main application of this work concerns helioseismology. It requires a precise knowledge of the acoustic wave field both in the Sun and in the atmosphere of the Sun. We intend to work in the context of this application for which we consider that the region of interest consists of two subdomains : a first spherical region (the Sun) which is surrounded by a second region (the atmosphere) where the density is *exponentially decreasing* and behaves like $\exp(-\alpha\, r)$ (here α is a large parameter and r represents the distance to the surface of the Sun). Then the rapid decay of the density inside the atmosphere limits the penetration of the acoustic wave inside a boundary layer which occurs inside the atmosphere in a vicinity of the Sun. The characteristic size of this boundary layer is proportional to the small parameter $\delta = 1/\alpha$.

This boundary layer phenomenon is an obstacle that prevents from reaching the required accuracy for the acoustic field. It raises the difficulty of applying a discretization method (e.g. a finite element method FEM) on a mesh that combines fine cells in the second region and much larger cells in the first region. To overcome this difficulty and to solve this problem, we adopt an asymptotic method which consists in "approximating" the second region by an EC. It is then possible to solve a simpler problem resulting of the Helmholtz equation (set in the first region) with this new boundary condition. The boundary layer phenomenon which is described in this paper looks like the so-called *skin effect* phenomenon which occurs inside a highly absorbing obstacle in electromagnetism [18, 14, 15, 10, 11, 16, 8, 4]. However these boundary layer phenomena are different.

In this paper we derive ECs up to the fourth order of approximation with respect to the parameter δ. These ECs are of "Neumann–to–Dirichlet" (NtD) nature for the wave field u since a local impedance operator links the Neumann traces of the solution u and the Dirichlet traces. From NtD ECs, we deduce "Dirichlet–to–Neumann" (DtN) ECs up to the fourth order of approximation. The construction of ECs relies on a derivation of a multiscale expansion for u in power series of δ. This expansion exhibits *profile* terms (defined in the second region) rapidly decaying and which describe the boundary layer phenomenon. One difficulty to validate the equivalent conditions lies in the proof of uniform energy estimates for the exact solution. We overcome this difficulty using a compactness argument and removing a discret set of resonant frequencies which may appear in the first region. As a consequence of these estimates we develop an argument for the convergence of the multiscale expansion by proving error estimates. We prove stability results for DtN ECs up to the fourth order of approximation. The proof is based on a compactness argument. Finally we can deduce convergence results for ECs from both error estimates and stability results.

The outline of the paper proceeds as follows. In Section 2 we introduce the mathematical model, we present briefly a formal derivation of equivalent conditions and we state uniform estimates for the solution of the transmission problem. In section 3 we prove uniform estimates for the solution of the transmission problem. In section 4, we exhibit the first terms of a multiscale expansion for the solution of the transmission problem and we prove error estimates. Then, we state a hierarchy of equivalent "Neumann-to-Dirichlet" boundary conditions up to the fourth order of approximation, we derive "Dirichlet–to–Neumann" ECs. In section 5 we present and prove stability results for ECs. In section 6, we present numerical results. In Appendix A we present elements of derivations for the multiscale expansion. In Appendix B, we present additional results and applications of this expansion. We measure the boundary layer phenomenon by introducing a "skin depth" function that turns out to depend on the mean curvature of the interface and we make the comparison with the skin effect in electromagnetism.

2 The mathematical model, equivalent conditions and main results

In this section, we introduce the model problem and the framework, §2.1. We present a formal derivation of the approximate boundary conditions. Then we define the order of accuracy for equivalent conditions and we state uniform estimates for the solution of the exact problem.

2.1 The model problem. Framework

In this paper, we denote by (r, θ, ϕ) the classical spherical coordinates. Let Ω be a smooth bounded domain in $\mathbb{R}^3$, with boundary $\partial\Omega$, and $\Omega_- := \{r < r_t\} \subset \Omega$ be a ball of radius r_t, with boundary Σ. We denote by $\Omega_+ := \Omega \setminus \{r \leqslant r_t\}$ the complementary of $\overline{\Omega_-}$ in Ω, see Figure 1.

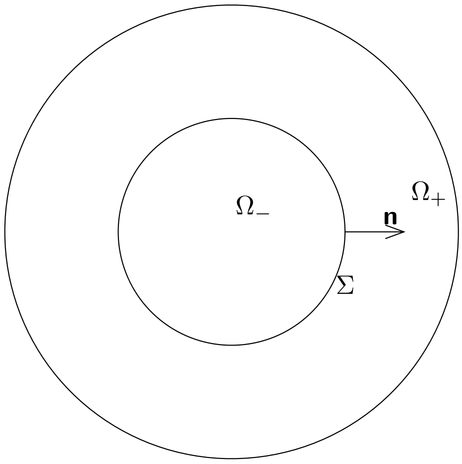


Figure 1 – A cross-section of the domain Ω and the subdomains Ω_- and Ω_+

Notation 2.1 *We denote by* u^+ *(resp.* u^-*) the restriction of any function* u *to* Ω_+ *(resp.* Ω_-*).*

In this paper we investigate the Helmholtz equation

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla \mathsf{u}\right) - \frac{\omega^2}{\rho c^2}(1 + i\underline{\nu})\mathsf{u} = f \quad \text{in} \quad \Omega\,.$$

where ρ is the density, c the celerity of acoustic waves, ω the pulsation, $\underline{\nu}$ a damping parameter. Here the density ρ satisfies a specific assumption in the domain Ω_+, that is : $\rho_+ := \rho|_{\Omega_+}$ is a radial function defined as

$$\rho_+(r) = \gamma \exp\left(-\alpha(r - r_t)\right)\,,$$

where the parameter $\alpha > 0$ is large with respect to the wave number at the surface $k = \dfrac{\omega}{c}$ and γ is a real constant. The coefficient $\underline{\nu}$ takes two different values in $\Omega_\pm$: $\underline{\nu} = (0, \nu)$ in $\Omega_- \times \Omega_+$ where $\nu \in \mathbb{R} \setminus \{0\}$ (i.e. the domain Ω_+ corresponds to a dissipative media). For the sake of simplicity, we assume that the right-hand side f (which represents an acoustic source) is a smooth function and its support does not touch the domain Ω_+. We will work under usual assumptions (regularity and positiveness) on the density and the celerity

Hypothesis 2.2 (i) *The density ρ and the celerity c are real valued smooth functions in $\overline{\Omega}$.*
(ii) *The density ρ and the celerity c are strictly positive in Ω.*

A consequence of the hypothesis 2.2 is that ρ and c are bounded with strictly positive constants:

$$\rho_{0,-} \leq \rho_- \leq \rho_{1,-}, \quad c_{0,-} \leq c_- \leq c_{1,-}, \qquad c_{0,+} \leq c_+ \leq c_{1,+}$$

We introduce the small parameter $\delta := \alpha^{-1} > 0$. Then $\mathsf{u} = (\mathsf{u}_\delta^-, \mathsf{u}_\delta^+)$ satisfies the following problem

$$\left\{\begin{array}{rcll} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_\delta^-\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_\delta^- &=& f & \text{in} \quad \Omega_- \,, \\ -\Delta \mathsf{u}_\delta^+ - \dfrac{1}{\delta}\partial_r \mathsf{u}_\delta^+ - \dfrac{\omega^2}{c^2}(1+i\nu)\mathsf{u}_\delta^+ &=& 0 & \text{in} \quad \Omega_+ \,, \\ \mathsf{u}_\delta^+ &=& \mathsf{u}_\delta^- & \text{on} \quad \Sigma \,, \\ \partial_{\mathbf{n}} \mathsf{u}_\delta^+ &=& \partial_{\mathbf{n}} \mathsf{u}_\delta^- & \text{on} \quad \Sigma \,, \\ \mathsf{u}_\delta^+ &=& 0 & \text{on} \quad \partial\Omega \,. \end{array}\right. \tag{2.1}$$

Our goal is to derive approximate models for the restriction u_δ^- of u_δ to the interior domain Ω_- when $\delta \to 0$. The limit problem (when δ tends to zero) has the following solution : $\mathsf{u}_0^+ = 0$ in the domain Ω_+ and u_0^- (in the domain Ω_-) which satisfies the problem

$$\left\{\begin{array}{lcll} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_0^-\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_0^- &=& f & \text{in} \quad \Omega_- \,, \\ \mathsf{u}_0^- &=& 0 & \text{on} \quad \Sigma \,. \end{array}\right. \tag{2.2}$$

Remark 2.1 *The limit problem* (2.2) *with $f = 0$ may exhibit eigenfrequencies ω.*

In this paper we prove stability and convergence results for approximate models (which are denoted by u_k^δ in Sec. 2.3) when $\delta \to 0$ by removing resonant frequencies which may appear in the domain Ω_-, Rem 2.1. Therefore we work under the following spectral assumption.

Hypothesis 2.3 *The angular frequency ω is non-zero and is not an eigenfrequency of the problem*

$$\left\{\begin{array}{ll} \operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}\right) + \dfrac{\omega^2}{\rho c^2}\mathsf{u} = 0 & \textit{in} \quad \Omega_- \\ \mathsf{u} = 0 & \textit{on} \quad \Sigma \,. \end{array}\right. \tag{2.3}$$

In the framework above, we prove that it is possible to replace the domain Ω_+ by approximate boundary conditions set on Σ provided that the parameter δ is small enough.

2.2 Uniform estimate

We introduce a suitable variational framework for the solution of the problem (2.1) with more general right-hand sides. This framework is useful to prove error estimate (2.18).

Weak solutions

For given data $(\mathsf{f}_-, \mathsf{f}_+, \mathsf{g})$ we consider the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_\delta^-\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_\delta^- = \mathsf{f}_- & \text{in} \quad \Omega_- \\ -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_\delta^+\right) - \dfrac{\omega^2}{\rho c^2}(1+i\nu)\mathsf{u}_\delta^+ = \mathsf{f}_+ & \text{in} \quad \Omega_+ \\ \mathsf{u}_\delta^+ = \mathsf{u}_\delta^- & \text{on} \quad \Sigma \\ \partial_{\mathbf{n}}\mathsf{u}_\delta^+ = \partial_{\mathbf{n}}\mathsf{u}_\delta^- + \mathsf{g} & \text{on} \quad \Sigma \\ \mathsf{u}_\delta^+ = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.4}$$

Hereafter, we explicit a weak formulation of the problem (2.4). The variational problem writes : Find $\mathsf{u}_\delta \in V = \mathrm{H}_0^1(\Omega)$ such that

$$\forall \mathsf{v} \in V, \quad a_\delta\left(\mathsf{u}_\delta, \mathsf{v}\right) = \langle F, \mathsf{v}\rangle_{V',V} \;, \tag{2.5}$$

where the sesquilinear form a_δ is defined as

$$a_\delta\left(\mathsf{u}, \mathsf{v}\right) = \int_{\Omega_\pm} \frac{1}{\rho_\pm}\nabla \mathsf{u}^\pm \cdot \nabla \bar{\mathsf{v}}^\pm \,\mathrm{d}\mathbf{x} - \omega^2 \int_{\Omega_-} \frac{1}{\rho_- c_-^2}\mathsf{u}^-\bar{\mathsf{v}}^- \,\mathrm{d}\mathbf{x} - \omega^2(1+i\nu)\int_{\Omega_+} \frac{1}{\rho_+ c_+^2}\mathsf{u}^+\bar{\mathsf{v}}^+ \,\mathrm{d}\mathbf{x}\,,$$

(here the data ρ_+ depends on the parameter δ: $\rho_+(r) = \gamma \exp\left(-\delta^{-1}(r - r_t)\right)$) and the right-hand side F is defined as

$$\langle F, \mathsf{v}\rangle_{V',V} = -\int_\Omega \mathsf{f}\bar{\mathsf{v}}\,\mathrm{d}\mathbf{x} - \int_\Sigma \mathsf{g}\bar{\mathsf{v}}\,\mathrm{d}\sigma\,.$$

We assume that the data (f, g) satisfies the regularity assumption

$$\mathsf{f} \in \mathrm{L}^2(\Omega) \quad \text{and} \quad \mathsf{g} \in \mathrm{L}^2(\Sigma)\,. \tag{2.6}$$

Statement of uniform estimate

In the framework of Sec. 2.1 we prove δ-uniform a priori estimates for the solution of problem (2.5). The following theorem is the main result in this section.

Theorem 2.4 *Under Hypothesis 2.2-2.3, there is a constant $C > 0$ such that for all $\delta > 0$ the problem (2.5) with data (f, g) satisfying (2.6) has a unique solution $\mathsf{u}_\delta \in V$ which satisfies the uniform estimates*

$$\|\mathsf{u}_\delta^-\|_{1,\Omega_-} + \|\tfrac{1}{\sqrt{\rho_+}}\mathsf{u}_\delta^+\|_{0,\Omega_+} + \|\tfrac{1}{\sqrt{\rho_+}}\nabla\mathsf{u}_\delta^+\|_{0,\Omega_+} \leqslant C(\|\mathsf{f}\|_{0,\Omega} + \|\mathsf{g}\|_{0,\Sigma})\,. \tag{2.7}$$

This result is proved in Sec. 3. As an application of uniform estimate (2.7), we develop in Sec. 4.2 an argument for the convergence of the asymptotic expansion introduced in Sec. 2.3.

2.3 Formal derivation of equivalent conditions

In this sub-section, the main results of the paper are presented : equivalent boundary conditions (ECs) up to the fourth order of approximation. In the framework above, it is possible to derive high-order boundary conditions set on the interface Σ, when the parameter δ is small enough. We shall derive for all $k \in \{0, 1, 2, 3\}$ an EC on Σ which is associated with the problem (2.1) and satisfied by u_δ^k, i.e. u_δ^k solves the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_\delta^k\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_\delta^k & = \quad f \quad \text{in} \quad \Omega_-\,, \\ \mathsf{u}_\delta^k + \mathsf{D}_{k,\delta}(\partial_{\mathbf{n}}\mathsf{u}_\delta^k) & = \quad 0 \quad \text{on} \quad \Sigma\,. \end{cases} \tag{2.8}$$

Here $\mathsf{D}_{k,\delta}$ is a surface differential operator acting on functions defined on Σ and which depends on δ. In this section, we first present a multiscale expansion ansatz for the solution. Then Equivalent Conditions (ECs) up to the fourth order are derived for the solution of the exact problem. These conditions are of "Neumann–to–Dirichlet" (NtD) nature. Then we introduce "Dirichlet–to–Neumann" (DtN) conditions.

First step : a multiscale expansion

The first step consists in deriving a multiscale expansion for the solution $(\mathsf{u}_\delta^-, \mathsf{u}_\delta^+)$ of the model problem (2.1): it possesses an asymptotic expansion in power series of the small parameter δ

$$\mathsf{u}_\delta^-(\mathbf{x}) = \mathsf{u}_0^-(\mathbf{x}) + \delta \mathsf{u}_1^-(\mathbf{x}) + \delta^2 \mathsf{u}_2^-(\mathbf{x}) + \delta^3 \mathsf{u}_3^-(\mathbf{x}) + \cdots \quad \text{in} \quad \Omega_-\,,$$
$$\mathsf{u}_\delta^+(\mathbf{x}) = \mathsf{u}_0^+(\mathbf{x};\delta) + \delta \mathsf{u}_1^+(\mathbf{x};\delta) + \delta^2 \mathsf{u}_2^+(\mathbf{x};\delta) + \delta^3 \mathsf{u}_3^+(\mathbf{x};\delta) + \cdots \quad \text{in} \quad \Omega_+\,,$$
$$\text{with} \quad \mathsf{u}_j^+(\mathbf{x};\delta) = \chi(r)\mathfrak{U}_j(\theta,\phi,\frac{r-r_t}{\delta})\,.$$

The term $\mathfrak{U}_j(\theta,\phi,S)$ is a "*profile*" defined on $\Sigma\times\mathbb{R}^+$, and (r,θ,ϕ) are the classical spherical coordinates. The function $\mathbf{x}\mapsto\chi(r)$ is a smooth cut-off with support in $\overline{\Omega_+}$ and equal to 1 in a tubular neighborhood of Σ. This cut-off function is needed to satisfy Dirichlet condition $u=0$ on the external boundary of Ω. Each *profile* $\mathfrak{U}_j$ tends to 0 when $S\to+\infty$. We make explicit the first asymptotics $(\mathsf{u}_j^-,\mathfrak{U}_j)$ for $j = 0,1,2,3$ in Section 4.1.

Second step : construction of Neumann-to-Dirichlet (NtD) equivalent conditions of order $k+1$

The second step consists in identifying for all $k\in\{0,1,2,3\}$ a simpler problem satisfied, up to a residual term in $\mathcal{O}(\delta^{k+1})$, by the truncated expansion

$$\mathsf{u}_{k,\delta}^- := \mathsf{u}_0^- + \delta\mathsf{u}_1^- + \cdots + \delta^k \mathsf{u}_k^-$$

We define u_k^δ as the solution of the simpler problem:

$$\left\{\begin{array}{lcl} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla\mathsf{u}_k^\delta\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_k^\delta & = & f \quad \text{in} \quad \Omega_-\,,\\ \mathsf{u}_k^\delta + \mathsf{D}_{k,\delta}(\partial_{\mathbf{n}}\mathsf{u}_k^\delta) & = & 0 \quad \text{on} \quad \Sigma\,. \end{array}\right.$$

This simpler problem is the problem set in Ω_- with the equivalent boundary conditions (EC) set on Σ. Here f is the right-hand side of problem (2.1) and $\mathsf{D}_{k,\delta}$ is a differential operator acting on functions defined on the surface Σ and which depends on δ:

$$\mathsf{D}_{0,\delta} = 0\,, \tag{2.9}$$

$$\mathsf{D}_{1,\delta} = \delta\,\mathbb{I}\,, \tag{2.10}$$

$$\mathsf{D}_{2,\delta} = \delta\left(1-\frac{2\delta}{r_t}\right)\mathbb{I}\,, \tag{2.11}$$

$$\mathsf{D}_{3,\delta} = \delta\left(1-\frac{2\delta}{r_t}+\delta^2\left\{\frac{6}{r_t^2}+\frac{\omega^2}{c_0^2}(1+i\nu)+\Delta_\Sigma\right\}\right)\,. \tag{2.12}$$

c_0 is the Dirichlet trace of c on Σ. Δ_Σ is the Laplace-Beltrami operator along the surface Σ. It writes in spherical coordinates (θ,ϕ) as

$$\Delta_\Sigma = \frac{1}{r_t^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{r_t^2\sin^2\theta}\partial_\phi^2\,. \tag{2.13}$$

Remark 2.2 *For all $k \in \{0, 1, 2\}$, $\mathsf{D}_{k,\delta}$ is a scalar operator whereas $\mathsf{D}_{3,\delta}$ is a second order differential operator.*

This second step and the construction of equivalent conditions are detailed in Section 4.3.

Third step : Dirichlet-to-Neumann (DtN) conditions

A Dirichlet-to-Neumann equivalent condition of order $k+1$ writes

$$\mathsf{N}_{k,\delta}(\mathsf{v}_k^\delta) + \partial_{\mathbf{n}} \mathsf{v}_k^\delta = 0 \quad \text{on} \quad \Sigma \,,$$

where $\mathsf{N}_{k,\delta}$ denotes a local approximation of the operator N_δ, and the simpler problem writes

$$\left\{ \begin{array}{rcll} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{v}_k^\delta\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{v}_k^\delta & = & f & \text{in} \quad \Omega_- \,, \\ \mathsf{N}_{k,\delta}(\mathsf{v}_k^\delta) + \partial_{\mathbf{n}} \mathsf{v}_k^\delta & = & 0 & \text{on} \quad \Sigma \,. \end{array} \right. \tag{2.14}$$

We compute in section 4.4 the expressions of $\mathsf{N}_{1,\delta}$, $\mathsf{N}_{2,\delta}$ and $\mathsf{N}_{3,\delta}$

$$\mathsf{N}_{1,\delta} = \delta^{-1}\,\mathbb{I}\,, \tag{2.15}$$

$$\mathsf{N}_{2,\delta} = \delta^{-1}\left(1 + \frac{2\delta}{r_t}\right)\mathbb{I}\,, \tag{2.16}$$

$$\mathsf{N}_{3,\delta} = \delta^{-1}\left((1 + \frac{2\delta}{r_t})\,\mathbb{I} - \delta^2\left\{\frac{2}{r_t^2} + \frac{\omega^2}{c_0^2}(1 + i\nu) + \Delta_\Sigma\right\}\right)\,. \tag{2.17}$$

For shortness of notations, we define a DtN condition of order 1 as the Dirichlet BC on Σ.

2.4 Stability and convergence of NtD Equivalent conditions

We present well-posedness and convergence results for NtD conditions, Sec. 2.4. Elements of derivation and mathematical validations for ECs are presented in Sec. 4 and Sec. 5.

Our goal in the next sections is to validate ECs set on Σ (Sec. 4.4) proving δ-estimates for $\mathsf{u}_\delta - \mathsf{u}_\delta^k$ ($k \in \{0, 1, 2, 3\}$) where u_δ^k is the solution of the approximate model (2.14) , and u_δ satisfies the problem (2.1).The functional setting for u_δ^k is described by the Hilbert space $\mathbf{V}^k$:

Notation 2.5 $\mathbf{V}^k$ *denotes the space* $\mathrm{H}_0^1(\Omega_-)$ *when* $k = 0$*;* $\mathbf{V}^k = \mathrm{H}^1(\Omega_-)$ *when* $k = 1, 2$*, and* $\mathbf{V}^3 = \{\mathsf{u} \in \mathrm{H}^1(\Omega_-) \mid \quad \mathsf{u}|_\Sigma \in \mathrm{H}^1(\Sigma)\}$

The main result of this section is the following statement, that is for all $k \in \{0, 1, 2, 3\}$ the problem (2.14) is well-posed in the space $\mathbf{V}^k$, and its solution satisfies uniform H^1 error estimates.

Theorem 2.6 *Under Hypothesis 2.2-2.3, for all $k \in \{0, 1, 2, 3\}$ there are constants $\delta_k, C_k > 0$ such that for all $\delta \in (0, \delta_k)$, the problem (2.14) with a data $f \in \mathrm{L}^2(\Omega_-)$ has a unique solution $\mathsf{u}_\delta^k \in \mathbf{V}^k$ which satisfies the uniform estimates:*

$$\|\mathsf{u}_\delta - \mathsf{u}_\delta^k\|_{1,\Omega_-} \leqslant C_k \delta^{k+1}\,. \tag{2.18}$$

The stability result for the problem (2.8) is stated in Thm. 5.1 and it is proved in Sec. 5.2. It appears nontrivial to work straightforwardly with the difference $\mathsf{u}_\delta - \mathsf{u}_\delta^k$. A usual method consists in using the truncated series $\mathsf{u}_{k,\delta}$ introduced in Sec. 4.3 as intermediate quantities [11, 17]. Then, the error analysis is split into two steps :

1. We prove a uniform estimate for the remainder $\mathsf{u}_\delta - \mathsf{u}_{k,\delta}$ in Thm. 4.1, Sec. 4.2 ,
2. The second step consists of proving a uniform estimate for the difference $\mathsf{u}_{k,\delta} - \mathsf{u}_\delta^k$

$$\|\mathsf{u}_{k,\delta} - \mathsf{u}_\delta^k\|_{1,\Omega_-} \leqslant \widetilde{C}_k \delta^{k+1} .$$

The first step is detailed in the next sections. The second step can be deduced straightforwardly from stability estimates (5.1a)-(5.1b) stated in Th. 5.1 (Sec. 5.1) since by construction the quantity $\mathsf{u}_{k,\delta} - \mathsf{u}_\delta^k$ satisfies Problem (2.14) with a right-hand side of order δ^{k+1} and which has a support on the surface Σ. Estimates (5.1a)-(5.1b) are proved in Section 5. We refer the reader to [17, Sec. 5.2]-[11] where the authors adopt the same method and use similar arguments.

3 Uniform estimate for the reference solution

In this section, we prove the Thm. 2.4, that is a uniform estimate for the solution of the reference problem. We consider the problem (2.4) at a fixed pulsation ω satisfying Hypothesis 2.3. We are going to prove the following Lemma

Lemma 3.1 *Under Hypothesis 2.2-2.3, there is a constant $C > 0$ such that for all $\delta > 0$ any solution $\mathsf{u}_\delta \in \mathrm{H}_0^1(\Omega)$ of problem (2.5) with a data $\mathsf{f} \in \mathrm{L}^2(\Omega)$ satisfies the uniform estimate with $g = 0$:*

$$\|\mathsf{u}_\delta\|_{0,\Omega} \leqslant C\|\mathsf{f}\|_{0,\Omega} \tag{3.1}$$

We conjecture that the following estimate holds when $g \neq 0$:

$$\|\mathsf{u}_\delta\|_{0,\Omega} + \|\mathsf{u}_\delta\|_{0,\Sigma} \leqslant C(\|\mathsf{f}\|_{0,\Omega} + \|\mathsf{g}\|_{0,\Sigma})$$

This Lemma is going to be proved in this section. The proof of this result involves both a compactness argument and the spectral hypothesis 2.3. Then as a consequence of estimate (3.1), we deduce estimate (2.7). Finally Thm. 2.4 is obtained as a consequence of the Fredholm alternative since the problem (2.5) is of Fredholm type.

Proof of Lemma 3.1 : Uniform estimate of u_δ

We prove this lemma by contradiction : We assume that there exists a sequence $\{\mathsf{u}_m\} \in V$, $m \in \mathbb{N}$, of solutions of problem (2.5), each solution being associated with a parameter $\delta_m > 0$ and a right-hand side $\mathsf{f}_m \in \mathrm{L}^2$:

$$\forall \mathsf{v} \in V, \quad \int_{\Omega_\pm} \frac{1}{\rho_\pm} \nabla \mathsf{u}_m^\pm \cdot \nabla \bar{\mathsf{v}}^\pm \, \mathrm{d}\mathbf{x} - \omega^2 \int_{\Omega_-} \frac{1}{\rho_- c_-^2} \mathsf{u}_m^- \bar{\mathsf{v}}^- \, \mathrm{d}\mathbf{x} \\ - \omega^2(1+i\nu) \int_{\Omega_+} \frac{1}{\rho_+ c_+^2} \mathsf{u}_m^+ \bar{\mathsf{v}}^+ \, \mathrm{d}\mathbf{x} = \int_\Omega \mathsf{f}_m \bar{\mathsf{v}} \, \mathrm{d}\mathbf{x} \, , \tag{3.2}$$

satisfying the following conditions

$$\|\mathsf{u}_m\|_{0,\Omega} = 1 \qquad \text{for all } m \in \mathbb{N}, \tag{3.3a}$$

$$\|\mathsf{f}_m\|_{0,\Omega} \to 0 \qquad \text{as } m \to \infty. \tag{3.3b}$$

Choosing tests functions $\mathsf{v} = \mathsf{u}_m$ in (3.2) and taking the imaginary part, since $\nu \neq 0$, we obtain with the help of conditions (3.3a)-(3.3b) the following convergence result :

(i) the sequence $\{\frac{1}{\sqrt{\rho_+}}\mathsf{u}_m^+\}$ converges to 0 in $\mathrm{L}^2(\Omega_+)$.

Then taking the real part, we obtain as a consequence of this convergence result the following uniform bounds on the gradients :

$$\|\nabla \mathsf{u}_m^-\|_{0,\Omega_-} + \|\frac{1}{\sqrt{\rho_+}}\nabla \mathsf{u}_m^+\|_{0,\Omega_+} \leqslant C \tag{3.4}$$

Since $\rho_+ \leqslant \rho_{0,-}$ and ρ_- is bounded, we deduce $\|\nabla \mathsf{u}_m\|_{0,\Omega} \leqslant C$ and the sequence $\{\mathsf{u}_m\}$ is bounded in H^1.

Limit of the sequence and conclusion

The domain Ω being bounded, the embedding of H^1 in L^2 is compact. Hence, since the sequence $\{\mathsf{u}_m\}$ is bounded in H^1 we can extract a subsequence of $\{\mathsf{u}_m\}$ (still denoted by $\{\mathsf{u}_m\}$) which is strongly converging in $\mathrm{L}^2(\Omega)$ and weakly converging in H^1

$$\begin{cases} \nabla \mathsf{u}_m \rightharpoonup \nabla \mathsf{u} & \text{in} \quad \mathbf{L}^2(\Omega) = (\mathrm{L}^2(\Omega))^3 \\ \mathsf{u}_m \to \mathsf{u} & \text{in} \quad \mathrm{L}^2(\Omega). \end{cases} \tag{3.5}$$

A consequence of the strong convergence in $\mathrm{L}^2(\Omega)$ and (3.3a) is that $\|\mathsf{u}\|_{0,\Omega} = 1$. Using Hypothesis 2.3, we are going to prove that $\mathsf{u} = 0$, which will contradict $\|\mathsf{u}\|_{0,\Omega} = 1$, and finally prove estimate (3.1).

Since $\|\mathsf{u}_m^+\|_{0,\Omega_+} \leqslant \sqrt{\rho_{0,-}}\,\|\frac{1}{\sqrt{\rho_+}}\mathsf{u}_m^+\|_{0,\Omega_+}$, letting $m \to +\infty$ and using *(i)* we get $\|\mathsf{u}^+\|_{0,\Omega_+} = 0$. Hence

$$\mathsf{u} = 0 \quad \text{in} \quad \Omega_+ \,. \tag{3.6}$$

In particular, (3.6) implies that $\mathsf{u}^- := \mathsf{u}|_{\Omega_-}$ belongs to the space $\mathrm{H}_0^1(\Omega_-)$.

Let $\mathsf{w} \in \mathrm{H}_0^1(\Omega_-)$. The extension w_0 of w by 0 on Ω_+ defines an element of $\mathrm{H}_0^1(\Omega)$. We can use w_0 as test function in (3.2) and we obtain

$$\int_{\Omega_-} \frac{1}{\rho_-}\,\nabla \mathsf{u}_m^- \cdot \nabla \overline{\mathsf{w}}^-\,\mathrm{d}\mathbf{x} - \omega^2 \int_{\Omega_-} \frac{1}{\rho_- c_-^2}\mathsf{u}_m^- \overline{\mathsf{w}}^-\,\mathrm{d}\mathbf{x} = \int_{\Omega} \mathsf{f}_m^- \overline{\mathsf{w}}\,\mathrm{d}\mathbf{x} \,.$$

According to (3.5) and (3.3b), taking limits as $m \to +\infty$, we deduce from the previous equalities

$$\int_{\Omega_-} \frac{1}{\rho_-}\,\nabla \mathsf{u}^- \cdot \nabla \overline{\mathsf{w}}\,\mathrm{d}\mathbf{x} - \omega^2 \int_{\Omega_-} \frac{1}{\rho_- c_-^2}\mathsf{u}^- \overline{\mathsf{w}}\,\mathrm{d}\mathbf{x} = 0 \,, \tag{3.7}$$

i.e., $\mathsf{u}^- \in \mathrm{H}_0^1(\Omega_-)$ satisfies (3.7) for all $\mathsf{w} \in \mathrm{H}_0^1(\Omega_-)$. Then integrating by parts we find

$$\begin{cases} \operatorname{div}(\dfrac{1}{\rho_-}\nabla \mathsf{u}^-) + \frac{\omega^2}{\rho_- c_-^2}\mathsf{u}^- = 0 & \text{in} \quad \Omega_- \\ \mathsf{u}^- = 0 & \text{on} \quad \Sigma \,. \end{cases} \tag{3.8}$$

By Hypothesis 2.3, we deduce

$$\mathsf{u}^- = 0 \quad \text{in} \quad \Omega_- .$$

Hence, with (3.6), we have $\mathsf{u} = 0$ in Ω, which contradicts $\|\mathsf{u}\|_{0,\Omega} = 1$ and ends the proof of Lemma 3.1.

4 Derivation and stability of equivalent conditions

It is possible to exhibit series expansions in powers of δ for u_δ in (2.1) :

$$\mathsf{u}_\delta^-(\mathbf{x}) \approx \sum_{j\geqslant 0} \delta^j \mathsf{u}_j^-(\mathbf{x}) \,, \tag{4.1}$$

$$\mathsf{u}_\delta^+(\mathbf{x}) \approx \sum_{j\geqslant 0} \delta^j \mathsf{u}_j^+(\mathbf{x};\delta) \quad \text{with} \quad \mathsf{u}_j^+(\mathbf{x};\delta) = \chi(r)\mathfrak{U}_j(\theta,\phi,\frac{r-r_t}{\delta}) \,. \tag{4.2}$$

The term $\mathfrak{U}_j(\theta,\phi,S)$ is a "*profile*" which is defined on $\Sigma\times\mathbb{R}^+$, it tends to 0 when $S\to+\infty$. The function χ is a smooth cut-off, see Sec. 2.3. Elements of derivations for this expansion are presented in detail in Appendix A. In this section, we explicit the first terms in asymptotics, §4.1. Then we construct formally equivalent conditions, §4.3. In §4.2 we validate the asymptotic expansion with estimates for the remainders.

4.1 First terms of the asymptotic expansion

Straightforward calculations lead to the first-order terms $\mathfrak{U}_j$ and u_j^-. In this section we explicit these terms when $j\leq 3$. We give elements of proof in Appendix A, Section A.3. There holds

$$\mathsf{u}_0^+ = 0\ ,\ (\mathfrak{U}_0=0).$$

Then u_0^- solves the problem

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla\mathsf{u}_0^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_0^- &=& f & \text{in} \quad \Omega_-\ ,\\ \mathsf{u}_0^- &=& 0 & \text{on} \quad \Sigma\ . \end{array}\right. \tag{4.3}$$

The next term which is determined in the asymptotics is the profile $\mathfrak{U}_1$

$$\mathfrak{U}_1(\theta,\phi,S) = -\mathrm{e}^{-S}\partial_{\mathbf{n}}\mathsf{u}_0^-\left(\mathbf{X}(r_t\theta,r_t\phi)\right)\ . \tag{4.4}$$

Here,

$$\mathbf{X}(r_t\theta,r_t\phi) = (r_t\sin\theta\cos\phi, r_t\sin\theta\sin\phi, r_t\cos\theta)\in\Sigma\ . \tag{4.5}$$

Then u_1^- solves

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla\mathsf{u}_1^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_1^- &=& 0 & \text{in} \quad \Omega_-\ ,\\ \mathsf{u}_1^- &=& -\partial_{\mathbf{n}}\mathsf{u}_0^- & \text{on} \quad \Sigma\ . \end{array}\right. \tag{4.6}$$

The next term which is determined is the profile $\mathfrak{U}_2$

$$\mathfrak{U}_2(\theta,\phi,S) = (a_2(\theta,\phi) + Sb_2(\theta,\phi))\mathrm{e}^{-S}\ , \tag{4.7}$$

where

$$a_2(\theta,\phi) = \left(-\partial_{\mathbf{n}}\mathsf{u}_1^- + \frac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^-\right)\left(\mathbf{X}(r_t\theta,r_t\phi)\right) \quad \text{and} \quad b_2(\theta,\phi) = \frac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^-\left(\mathbf{X}(r_t\theta,r_t\phi)\right)\ .$$

Then u_2^- solves the problem

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla\mathsf{u}_2^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_2^- &=& 0 & \text{in} \quad \Omega_-\ ,\\ \mathsf{u}_2^- &=& -\partial_{\mathbf{n}}\mathsf{u}_1^- + \dfrac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^- & \text{on} \quad \Sigma\ . \end{array}\right. \tag{4.8}$$

The next term which is determined is the profile $\mathfrak{U}_3$

$$\mathfrak{U}_3(\theta,\phi,S) = (a_3(\theta,\phi) + Sb_3(\theta,\phi) + S^2c_3(\theta,\phi))\mathrm{e}^{-S}\ , \tag{4.9}$$

where the functions a_3, b_3, c_3 are given by

$$\begin{cases} a_3 = -\partial_{\mathbf{n}} \mathsf{u}_2^- + \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{ \dfrac{6}{r_t^2} + \dfrac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathsf{u}_0^- \;, \\ b_3 = \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{ \dfrac{6}{r_t^2} + \dfrac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathsf{u}_0^- \;, \\ c_3 = -\dfrac{3}{r_t^2}\partial_{\mathbf{n}} \mathsf{u}_0^- \;. \end{cases}$$

Then, u_3^- solves the problem

$$\begin{cases} -\operatorname{div}(\dfrac{1}{\rho}\nabla \mathsf{u}_3^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_3^- = 0 & \text{in} \quad \Omega_- \;, \\ \mathsf{u}_3^- = -\partial_{\mathbf{n}} \mathsf{u}_2^- + \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{ \dfrac{6}{r_t^2} + \dfrac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma \right\} \partial_{\mathbf{n}} \mathsf{u}_0^- & \text{on} \quad \Sigma \;. \end{cases} \tag{4.10}$$

Existence and regularity results in Sobolev spaces for the asymptotics $\mathfrak{U}_j$ and u_j^- are stated in Prop. A.1, Sec. A.4.

4.2 Estimates for the remainders

The validation of the asymptotic expansion (4.1)-(4.2) consists in proving estimates for remainders r_δ^N defined in Ω as

$$r_\delta^N = \mathsf{u}_\delta - \sum_{n=0}^{N} \delta^n \, \mathsf{u}_n \tag{4.11}$$

The convergence result is the following statement.

Theorem 4.1 *Under Hypothesis 2.2-2.3, the solution u_δ of problem (2.1) has a two-scale expansion which can be written in the form (4.1)-(4.2), with $\mathsf{u}_n^- \in \mathrm{H}^1(\Omega_-)$ and $\mathfrak{U}_n \in \mathrm{H}^1\left(\Sigma \times \mathbb{R}^+\right)$. For each $N \in \mathbb{N}$, the remainders r_δ^N satisfy*

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \delta^{-1/2}\|\tfrac{1}{\sqrt{\rho_+}} r_\delta^{N,+}\|_{0,\Omega_+} + \delta^{1/2}\|\tfrac{1}{\sqrt{\rho_+}}\nabla r_\delta^{N,+}\|_{0,\Omega_+} \leqslant C_N \delta^{N+1} \tag{4.12}$$

with a constant C_N independent of δ.

PROOF. The error estimate (4.12) is obtained through an evaluation of the right-hand sides when the Helmholtz operator is applied to the remainder r_δ^N. By construction, the remainder is solving the problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla r_\delta^{N,-}\right) - \dfrac{\omega^2}{\rho c^2} r_\delta^{N,-} = 0 & \text{in} \quad \Omega_- \\ -\operatorname{div}\left(\dfrac{1}{\rho}\nabla r_\delta^{N,+}\right) - \dfrac{\omega^2}{\rho c^2}(1+i\nu) r_\delta^{N,+} = \mathsf{f}_{N,\delta} & \text{in} \quad \Omega_+ \\ r_\delta^{N,+} = r_\delta^{N,-} & \text{on} \quad \Sigma \\ \partial_{\mathbf{n}} r_\delta^{N,+} = \partial_{\mathbf{n}} r_\delta^{N,-} + \mathsf{g}_{N,\delta} & \text{on} \quad \Sigma \\ r_\delta^N = 0 & \text{on} \quad \partial\Omega \;, \end{cases} \tag{4.13}$$

which is nothing but the boundary value problem (2.4) with right-hand sides $(\mathsf{f}_-, \mathsf{f}_+, \mathsf{g}) = (0, \mathsf{f}_{N,\delta}, \mathsf{g}_{N,\delta})$. The right hand sides $\mathsf{g}_{N,\delta}$ and $\mathsf{f}_{N,\delta}$ are residuals which are roughly of the order δ^N. By construction of the expansion, we obtain :

$$\mathsf{g}_{N,\delta} = \delta^N \partial_{\mathbf{n}} \mathsf{u}_N^- \quad \text{on} \quad \Sigma \tag{4.14}$$

and

$$\mathsf{f}_{0,\delta} = 0 \quad \text{and for all } N \geqslant 1, \quad \mathsf{f}_{N,\delta} = -\chi\rho_+^{-1}\delta^{N-1}\left(\mathrm{A}_0\mathfrak{U}_{N+1}(\theta, \phi, \frac{r - r_t}{\delta}) + \mathcal{O}(\delta)\right) = \mathcal{O}(\delta^{N-1}) \tag{4.15}$$

where the operator A_0 is defined in appendix (A.2a). The terms with derivatives of χ should appear with higher powers of δ (see [5, Eq. (5.6)] and [16, Prop. 7.4, Eq. $(7.17)_2$]). We can apply Theorem 2.4 to the remainder r_δ^N :

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \|\tfrac{1}{\sqrt{\rho_+}} r_\delta^{N,+}\|_{0,\Omega_+} + \|\tfrac{1}{\sqrt{\rho_+}} \nabla r_\delta^{N,+}\|_{0,\Omega_+} \leqslant C_N \left(\|\mathsf{f}_{N,\delta}\|_{0,\Omega_+} + \|\mathsf{g}_{N,\delta}\|_{0,\Sigma}\right) \tag{4.16}$$

We deduce from (4.14) and (4.15) the estimates

$$\|\mathsf{g}_{N,\delta}\|_{0,\Sigma} \leqslant C\delta^N \quad \text{and} \quad \|\mathsf{f}_{N,\delta}\|_{0,\Omega_+} \leqslant C\delta^{N-1/2}$$

since $\mathrm{A}_0\mathfrak{U}_{N+1} \in \mathrm{H}^1(\Sigma \times \mathbb{R}^+)$. Here C may depend on N. Combining this estimate with (4.16), we infer

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \|\tfrac{1}{\sqrt{\rho_+}} r_\delta^{N,+}\|_{0,\Omega_+} + \|\tfrac{1}{\sqrt{\rho_+}} \nabla r_\delta^{N,+}\|_{0,\Omega_+} \leqslant C_N \delta^{N-1/2}$$

we therefore have:

$$\|r_\delta^{N,-}\|_{1,\Omega_-} + \delta^{-1/2}\|\tfrac{1}{\sqrt{\rho_+}} r_\delta^{N,+}\|_{0,\Omega_+} + \delta^{1/2}\|\tfrac{1}{\sqrt{\rho_+}} \nabla r_\delta^{N,+}\|_{0,\Omega_+} \leqslant C\delta^{N-1}$$

In order to prove optimal estimates for r_δ^N, we apply the previous estimates to r_δ^{N+2}. Using the formula

$$r_\delta^{N+2} = r_\delta^N + \delta^{N+1}\mathsf{u}_{N+1} + \delta^{N+2}\mathsf{u}_{N+2}$$

we finally obtain the wanted estimates since we have for any $n \in \mathbb{N}$

$$\|\mathsf{u}_n^-\|_{1,\Omega_-} + \delta^{-1/2}\|\tfrac{1}{\sqrt{\rho_+}} \mathsf{u}_n^+\|_{0,\Omega_+} + \delta^{1/2}\|\tfrac{1}{\sqrt{\rho_+}} \nabla \mathsf{u}_n^+\|_{0,\Omega_+} \leqslant C\,.$$

■

4.3 Construction of Neumann to Dirichlet (NtD) equivalent conditions

In this section, we derive formally equivalent boundary conditions (Sec. 2.3).

Order 1

Since u_0^- solves the problem (4.3), the condition of order 1 is the homogeneous Dirichlet boundary condition, see (2.9)

$$\mathsf{u}_0 = 0 \quad \text{on} \quad \Sigma$$

Order 2

According to (4.3)-(4.6), the truncated expansion $\mathsf{u}^-_{1,\delta} := \mathsf{u}^-_0 + \delta\mathsf{u}^-_1$ solves the Helmholtz equation $-\operatorname{div}(\frac{1}{\rho}\nabla\mathsf{u}^-_{1,\delta}) - \frac{\omega^2}{\rho c^2}\mathsf{u}^-_{1,\delta} = f$ in the domain Ω_- together with the Robin boundary condition set on Σ

$$\mathsf{u}^-_{1,\delta} + \delta\partial_{\mathbf{n}}\mathsf{u}^-_{1,\delta} = \delta^2\partial_{\mathbf{n}}\mathsf{u}^-_1 \quad \text{on} \quad \Sigma$$

Neglecting the term of order 2 in the previous right-hand side, we infer the equivalent boundary condition, see (2.10)

$$\mathsf{u}^\delta_1 + \delta\partial_{\mathbf{n}}\mathsf{u}^\delta_1 = 0 \quad \text{on} \quad \Sigma$$

Order 3

According to (4.3)-(4.6)-(4.8), the truncated expansion $\mathsf{u}^-_{2,\delta} := \mathsf{u}^-_0 + \delta\mathsf{u}^-_1 + \delta^2\mathsf{u}^-_2$ solves the Helmholtz equation in Ω_- together with the boundary condition set on Σ

$$\mathsf{u}^-_{2,\delta} + \delta\left(1 - \frac{2}{r_t}\delta\right)\partial_{\mathbf{n}}\mathsf{u}^-_{2,\delta} = -\frac{2}{r_t}\delta^3\partial_{\mathbf{n}}\mathsf{u}^-_1 + \delta^3\left(1 - \frac{2}{r_t}\delta\right)\partial_{\mathbf{n}}\mathsf{u}^-_2 \quad \text{on} \quad \Sigma$$

We neglect the terms of order 3 in the previous right-hand side and we infer the equivalent boundary condition of order 3, see (2.11)

$$\mathsf{u}^\delta_2 + \delta\left(1 - \frac{2\delta}{r_t}\right)\partial_{\mathbf{n}}\mathsf{u}^\delta_2 = 0 \quad \text{on} \quad \Sigma$$

Order 4

According to (4.3)-(4.6)-(4.8)-(4.10), the truncated expansion $\mathsf{u}^-_{3,\delta} := \mathsf{u}^-_0 + \delta\mathsf{u}^-_1 + \delta^2\mathsf{u}^-_2 + \delta^3\mathsf{u}^-_3$ solves the Helmholtz equation in Ω_- together with a boundary condition set on Σ

$$\mathsf{u}^-_{3,\delta} + \delta\left(1 - \frac{2\delta}{r_t} + \delta^2\left\{\frac{6}{r_t^2} + \frac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma\right\}\right)\partial_{\mathbf{n}}\mathsf{u}^-_{3,\delta} = \mathcal{O}(\delta^4)$$

We neglect the terms of order 4 in the previous right-hand side to infer the equivalent boundary condition of order 4, see (2.12)

$$\mathsf{u}^\delta_3 + \delta\left(1 - \frac{2\delta}{r_t} + \delta^2\left\{\frac{6}{r_t^2} + \frac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma\right\}\right)\partial_{\mathbf{n}}\mathsf{u}^\delta_3 = 0$$

4.4 Dirichlet-to-Neumann (DtN) conditions

We introduce the operator $\mathsf{N}_\delta = (\mathsf{D}_\delta)^{-1}$ where the operator D_δ is the exact Neumann-to-Dirichlet operator that characterizes u^+_δ. Then the exact boundary condition for u^-_δ can be rewritten as

$$\mathsf{N}_\delta\mathsf{u}^-_\delta + \partial_{\mathbf{n}}\mathsf{u}^-_\delta = 0 \quad \text{on} \quad \Sigma\,.$$

A Dirichlet-to-Neumann equivalent condition of order $k+1$ writes

$$\mathsf{N}_{k,\delta}(\mathsf{v}^\delta_k) + \partial_{\mathbf{n}}\mathsf{v}^\delta_k = 0 \quad \text{on} \quad \Sigma\,,$$

where $\mathsf{N}_{k,\delta}$ denotes a local approximation of the operator N_δ, and the simpler problem writes

$$\left\{\begin{array}{lcl} -\operatorname{div}\left(\frac{1}{\rho}\nabla\mathsf{v}^\delta_k\right) - \frac{\omega^2}{\rho c^2}\mathsf{v}^\delta_k & = & f \quad \text{in} \quad \Omega_-\,, \\ \mathsf{N}_{k,\delta}(\mathsf{v}^\delta_k) + \partial_{\mathbf{n}}\mathsf{v}^\delta_k & = & 0 \quad \text{on} \quad \Sigma\,. \end{array}\right. \tag{4.17}$$

Remark 4.1 *There holds $\mathsf{N}_{1,\delta} = \delta^{-1}\,\mathbb{I}$, i.e the DtN condition of order 2 coincides with the NtD condition of order 2 (2.10).*

The expression of $\mathsf{N}_{k,\delta}$, for any $k \geqslant 2$, can be obtained by a formal Taylor expansion of $(\mathsf{D}_{k,\delta})^{-1}$ such that

$$\mathsf{D}_{k,\delta} = (\mathsf{N}_{k,\delta})^{-1} + \mathcal{O}(\delta^{k+1})\,.$$

Straightforward calculi lead to the following expansions

$$(\mathsf{D}_{2,\delta})^{-1} = \delta^{-1}\left(1 + \frac{2\delta}{r_t}\right)\mathbb{I} + \mathcal{O}(\delta)\,,$$

$$(\mathsf{D}_{3,\delta})^{-1} = \delta^{-1}\left((1 + \frac{2\delta}{r_t})\,\mathbb{I} - \delta^2\left\{\frac{2}{r_t^2} + \frac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma\right\}\right) + \mathcal{O}(\delta^2)\,.$$

We infer the expressions of $\mathsf{N}_{2,\delta}$ and $\mathsf{N}_{3,\delta}$

$$\mathsf{N}_{2,\delta} = \delta^{-1}\left(1 + \frac{2\delta}{r_t}\right)\mathbb{I}\,,$$

$$\mathsf{N}_{3,\delta} = \delta^{-1}\left((1 + \frac{2\delta}{r_t})\,\mathbb{I} - \delta^2\left\{\frac{2}{r_t^2} + \frac{\omega^2}{c_0^2}(1+i\nu) + \Delta_\Sigma\right\}\right)\,.$$

For shortness of notations, we define a DtN condition of order 1 as the Dirichlet BC on Σ (2.9).

5 Analysis of DtN Equivalent Conditions

The main result of this section is Theorem 5.1 in §5.1 which proves the stability of equivalent conditions.

5.1 Main result for DtN equivalent conditions (ECs)

In this section we address the issue of stability results for ECs. We focus on DtN equivalent condition which are defined for $k = 1, 2, 3$ (2.14). For shortness of notations we define for $k = 0$ the DtN EC of order 1 as the Dirichlet BC on Σ. We consider the problem (2.14) at a fixed non-zero frequency ω satisfying Hypothesis 2.3. The functional setting for u_δ^k is described by the Hilbert space $\mathbf{V}^k$ (see Notation 2.5). The main result of this section is the following statement, that is for all $k \in \{0, 1, 2, 3\}$ the problem (2.14) is well-posed in the space $\mathbf{V}^k$, and its solution satisfies uniform H^1 estimates.

Theorem 5.1 *Under Hypothesis 2.2-2.3, for all $k \in \{0, 1, 2, 3\}$ there are constants $\delta_k, C_k > 0$ such that for all $\delta \in (0, \delta_k)$, the problem (2.14) with a data $f \in \mathrm{L}^2(\Omega_-)$ has a unique solution $\mathsf{u}_\delta^k \in \mathbf{V}^k$ which satisfies the uniform estimates:*

$$\|\mathsf{u}_\delta^k\|_{1,\Omega_-} \leqslant C_k \|f\|_{0,\Omega_-} \quad \textit{for all} \quad k \in \{0,1,2\}\ , \tag{5.1a}$$

$$\|\mathsf{u}_\delta^3\|_{1,\Omega_-} + \sqrt{\delta}\|\nabla_\Sigma \mathsf{u}_\delta^3\|_{0,\Sigma} + \sqrt{\delta}^{-1}\|\mathsf{u}_\delta^3\|_{0,\Sigma} \leqslant C_3\|f\|_{0,\Omega_-}\ . \tag{5.1b}$$

The key for the proof of Thm. 5.1 is the following Lemma.

Lemma 5.2 *Under Hypothesis 2.2-2.3, for all $k \in \{0, 1, 2, 3\}$ there exists constants $\delta_k, C_k > 0$ such that for all $\delta \in (0, \delta_k)$, any solution $\mathsf{u}_\delta^k \in \mathbf{V}^k$ of problem (2.14) with a data $f \in \mathrm{L}^2(\Omega_-)$ satisfies the uniform estimate:*

$$\|\mathsf{u}_\delta^k\|_{0,\Omega_-} \leqslant C_k\|f\|_{0,\Omega_-} \tag{5.2}$$

Remark 5.1 *For $k = 0$, the Theorem 5.1 and the Lemma 5.2 hold for all $\delta > 0$ since the EC is nothing but the Dirichlet BC on Σ. For $k = 1, 2, 3$, one proves uniform estimate when δ is small enough. The proof is based on a compactness argument.*

The Lemma 5.2 is proved in Section 5.2. As a consequence of this Lemma, any solution of the problem (2.14) satisfies uniform H^1-estimates (5.1a), (5.1b), respectively when $k = 0, 1, 2, 3$. Then, the proof of the Thm. 5.1 is obtained as a consequence of the Fredholm alternative since the problem (2.14) is of Fredholm type.

In this section, one first proves the Lemma 5.2, i.e. uniform L^2-estimate (5.2) for any solution of problem (2.14), Sec. 5.2. For $k = 0$, the proof of Lemma 5.2 is a consequence of the Hypothesis 2.3 together with the Fredholm alternative. We focus on the proof of Lemma 5.2 for $k = 3$ since the proofs when $k = 0, 1, 2$ are simpler. Hence we consider the problem (here $\mathsf{u} = \mathsf{u}_\delta^3$)

$$\begin{cases} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}) - \frac{\omega^2}{\rho c^2}\mathsf{u} &= f \quad \text{in} \quad \Omega_- \,, \\ \partial_{\mathbf{n}}\mathsf{u} + \mathsf{N}_\delta \mathsf{u} &= 0 \quad \text{on} \quad \Sigma \,, \end{cases} \tag{5.3}$$

where the operator N_δ is defined as

$$\mathsf{N}_\delta = \delta^{-1}\mathcal{J}_\delta \mathbb{I} - \delta \Delta_\Sigma$$

Here, $\mathcal{J}_\delta$ is a scalar defined as

$$\mathcal{J}_\delta = \left((1 + \frac{2\delta}{r_t}) - \delta^2 \left\{ \frac{2}{r_t^2} + \frac{\omega^2}{c_0^2}(1 + i\nu) \right\} \right) .$$

To prepare for the proof, we introduce the variational formulation for u. If $\mathsf{u} \in \mathbf{V}^3$ is a solution of (5.3), then it satisfies for all $\mathsf{v} \in \mathbf{V}^3$:

$$\int_{\Omega_-} \left(\frac{1}{\rho}\nabla \mathsf{u} \cdot \nabla \bar{\mathsf{v}} - \frac{\omega^2}{\rho c^2}\mathsf{u}\bar{\mathsf{v}} \right) \,\mathrm{d}\mathbf{x} + \frac{\delta}{\rho(r_t)} \int_\Sigma \nabla_\Sigma \mathsf{u} \cdot \nabla_\Sigma \bar{\mathsf{v}} \,\mathrm{d}\sigma + \frac{\delta^{-1}}{\rho(r_t)} \mathcal{J}_\delta \int_\Sigma \mathsf{u}\, \bar{\mathsf{v}} \,\mathrm{d}\sigma = \int_{\Omega_-} f \bar{\mathsf{v}} \,\mathrm{d}\mathbf{x} \,. \tag{5.4}$$

5.2 Proof of Lemma 5.2

Proof by contradiction: We assume that there is a sequence $(\mathsf{u}_m) \in \mathbf{V}^3$, $m \in \mathbb{N}$, of solutions of the problem (5.3) associated with a parameter δ_m and a right-hand side $f_m \in \mathrm{L}^2(\Omega_-)$:

$$-\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_m) - \frac{\omega^2}{\rho c^2}\mathsf{u}_m = f_m \quad \text{in} \quad \Omega_- \,, \tag{5.5a}$$

$$\partial_{\mathbf{n}}\mathsf{u}_m + \delta_m^{-1}\mathcal{J}_m \mathsf{u}_m - \delta_m \Delta_\Sigma \mathsf{u}_m = 0 \quad \text{on} \quad \Sigma \,, \tag{5.5b}$$

(with $\mathcal{J}_m := \mathcal{J}_{\delta_m}$) satisfying the following conditions

$$\delta_m \to 0 \qquad \text{as } m \to \infty \,, \tag{5.6a}$$

$$\|\mathsf{u}_m\|_{0,\Omega_-} = 1 \qquad \text{for all } m \in \mathbb{N} \,, \tag{5.6b}$$

$$\|f_m\|_{0,\Omega_-} \to 0 \qquad \text{as } m \to \infty \,. \tag{5.6c}$$

5.2.1 Estimates of the sequence $\{\mathsf{u}_m\}$

We first prove that the sequence $\{\mathsf{u}_m\}$ is bounded in $\mathrm{H}^1(\Omega_-)$ (not in $\mathbf{V}^3$). We particularize the weak formulation (5.4) for the sequence $\{\mathsf{u}_m\}$: For all $\mathsf{v} \in \mathbf{V}^3$:

$$\int_{\Omega_-} \left(\frac{1}{\rho}\nabla \mathsf{u}_m \cdot \nabla \bar{\mathsf{v}} - \tfrac{\omega^2}{\rho c^2}\mathsf{u}_m \bar{\mathsf{v}}\right)\,\mathrm{d}\mathbf{x} + \tfrac{\delta_m}{\rho(r_t)} \int_\Sigma \nabla_\Sigma \mathsf{u}_m \cdot \nabla_\Sigma \bar{\mathsf{v}}\,\mathrm{d}\sigma + \tfrac{\delta_m^{-1}}{\rho(r_t)} \mathcal{J}_m \int_\Sigma \mathsf{u}_m\,\bar{\mathsf{v}}\,\mathrm{d}\sigma = \int_{\Omega_-} f_m \bar{\mathsf{v}}\,\mathrm{d}\mathbf{x}\,. \tag{5.7}$$

Then choosing $\mathsf{v} = \mathsf{u}_m$ in (5.7) and taking the real part, we obtain with the help of condition (5.6b) the uniform bound (since $\dfrac{1}{\rho c} \in \mathrm{L}^\infty(\Omega_-)$)

$$\int_{\Omega_-} \frac{1}{\rho_-}\|\nabla \mathsf{u}_m\|^2\,\mathrm{d}\mathbf{x} + \tfrac{\delta_m}{\rho(r_t)}\|\nabla_\Sigma \mathsf{u}_m\|^2_{0,\Sigma} + \tfrac{\delta_m^{-1}}{\rho(r_t)}\Re(\mathcal{J}_m)\|\mathsf{u}_m\|^2_{0,\Sigma} \leqslant \omega^2 \|\frac{1}{\rho c}\|_\infty + \|f_m\|_{0,\Omega_-}\,.$$

We infer that the sequence $\{\mathsf{u}_m\}$, resp. $\{\sqrt{\delta_m}\nabla_\Sigma \mathsf{u}_m\}$, is bounded in $\mathrm{H}^1(\Omega_-)$ (since $\dfrac{1}{\rho_-} \geqslant D_0 > 0$), resp. in $\mathrm{L}^2(\Sigma)$; and since $\mathcal{J}_m$ tends to 1, the sequence $\{\delta_m^{-1/2}\mathsf{u}_m\}$ is bounded in $\mathrm{L}^2(\Sigma)$:

$$\|\mathsf{u}_m\|_{1,\Omega_-} \leqslant C\,, \tag{5.8a}$$
$$(\delta_m)^{\frac{1}{2}}\|\nabla_\Sigma \mathsf{u}_m\|_{0,\Sigma} \leqslant C\,, \tag{5.8b}$$
$$(\delta_m)^{-\frac{1}{2}}\|\mathsf{u}_m\|_{0,\Sigma} \leqslant C\,. \tag{5.8c}$$

Another consequence of (5.8a) is that the sequence $\{\mathsf{u}_m\}$ is bounded in $\mathrm{H}^{\frac{1}{2}}(\Sigma)$.

$$\|\mathsf{u}_m\|_{\frac{1}{2},\Sigma} \leqslant C\,. \tag{5.9}$$

5.2.2 Limit of the sequence and conclusion

The domain Ω_- being bounded, the embedding of $\mathrm{H}^1(\Omega_-)$ in $\mathrm{L}^2(\Omega_-)$ is compact. Hence as a consequence of (5.8a) we can extract a subsequence of $\{\mathsf{u}_m\}$ (still denoted by $\{\mathsf{u}_m\}$) which is converging in $\mathrm{L}^2(\Omega_-)$ and we can assume that the sequence $\{\nabla\mathsf{u}_m\}$ is weakly converging in $\mathbf{L}^2(\Omega_-)$. As a consequence of (5.9), up to the extraction of a subsequence, we can also assume that the sequence $\{\mathsf{u}_m\}$ is strongly converging in $\mathrm{L}^2(\Sigma)$: We deduce that there is $\mathsf{u} \in \mathrm{L}^2(\Omega_-)$ such that

$$\mathsf{u}_m \to \mathsf{u} \quad \text{in} \quad \mathrm{L}^2(\Omega_-) \tag{5.10a}$$
$$\nabla\mathsf{u}_m \rightharpoonup \nabla\mathsf{u} \quad \text{in} \quad \mathbf{L}^2(\Omega_-) \tag{5.10b}$$
$$\mathsf{u}_m \to \mathsf{u} \quad \text{in} \quad \mathrm{L}^2(\Sigma)\,. \tag{5.10c}$$

A consequence of the strong convergence result in $\mathrm{L}^2(\Omega_-)$ (5.10a) and (5.6b) is that $\|\mathsf{u}\|_{0,\Omega_-} = 1$. Furthermore, as a consequence of (5.10c) and (5.8c) one can deduce that $\mathsf{u} = 0$ on Σ.

Using Hypothesis 2.3, we are going to prove that $\mathsf{u} = 0$ in Ω_-, which will contradict $\|\mathsf{u}\|_{0,\Omega_-} = 1$ and finally prove estimate (5.2). Let $\mathsf{v} \in \mathcal{C}_c^\infty(\Omega_-)$, it defines an element of $\mathbf{V}^3$. Hence we can use v as test function in (5.7). Since $\mathsf{v} = 0$ on Σ and $\nabla_\Sigma \mathsf{v} = 0$ on Σ we infer

$$\int_{\Omega_-} \left(\frac{1}{\rho}\nabla \mathsf{u}_m \cdot \nabla \bar{\mathsf{v}} - \tfrac{\omega^2}{\rho c^2}\mathsf{u}_m \bar{\mathsf{v}}\right)\,\mathrm{d}\mathbf{x} = \int_{\Omega_-} f_m \bar{\mathsf{v}}\,\mathrm{d}\mathbf{x}\,. \tag{5.11}$$

As a consequence of (5.10c), taking limits as $m \to \infty$, we can deduce from the previous equality and convergence results (5.10a)-(5.10b)-(5.11) that $\mathsf{u} \in \mathrm{H}^1_0(\Omega_-)$ satisfies for all $\mathsf{v} \in \mathcal{C}_c^\infty(\Omega_-)$

$$\int_{\Omega_-} \left(\frac{1}{\rho}\nabla \mathsf{u} \cdot \nabla \bar{\mathsf{v}} - \frac{\omega^2}{\rho c^2}\mathsf{u}\bar{\mathsf{v}}\right) \, \mathrm{d}\mathbf{x} = 0 \,.$$

Then integrating by parts we find that u satisfies the problem

$$\begin{cases} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}) - \frac{\omega^2}{\rho c^2}\mathsf{u} = 0 & \text{in} \quad \Omega_- \\ \mathsf{u} = 0 & \text{on} \quad \Sigma \,. \end{cases}$$

According to Hypothesis 2.3, we infer

$$\mathsf{u} = 0 \quad \text{in} \quad \Omega_-$$

which contradicts $\|\mathsf{u}\|_{0,\Omega_-} = 1$ and ends the proof of Lemma 5.2.

6 Numerical illustration

6.1 Numerical method

We solve Helmholtz equation in a radial configuration (r, θ, ϕ), where the data only depends on the radius r. Details and validations of the numerical method described in this section are given in [6]. The interval $[0, r_t]$ is subdivided into sub-intervals:

$$[0, r_t] = \cup [x_i, x_{i+1}]$$

One-dimensional finite elements will be used in r-coordinate, while spherical harmonics will be used in θ, ϕ. Since spherical harmonics are orthonormal and diagonalize the laplacian, we will obtain a decoupled sequence of 1-D problems to solve. The 1-D finite element space is equal to

$$V_h \;=\; \left\{ u \in H^1(\Omega) \text{ such that } u|_{[x_i, x_i+1]} \in \mathbb{P}_r \right\}$$

where $\mathbb{P}_r$ is the space of polynomials of degree lower or equal to r, r being the order of the approximation. The solution u is then searched under the form

$$u(r, \theta, \phi) = \sum_{j=0}^{N_h} \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} u_j^{\ell,m} \, \varphi_j(r) \, Y_\ell^m(\theta, \phi)$$

where φ_i are basis functions generating the finite element space V_h of dimension N_h, and Y_ℓ^m spherical harmonics given as

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\phi}$$

where P_ℓ^m are the associated Legendre polynomials. L is the maximal degree of spherical harmonics used in the expression of u. The variational formulation solved by $u^{\ell,m}$ is given as

$$\begin{aligned} -\omega^2 \int_0^{r_t} \frac{1}{\rho_0 c_0^2} r^2 \, u^{\ell,m} \varphi_i \, dr + \int_0^{r_t} \frac{r^2}{\rho_0} \frac{\partial u^{\ell,m}}{\partial r} \frac{\partial \varphi_i}{\partial r} \, dr \\ + \ell(\ell+1) \int_0^{r_t} \frac{1}{\rho_0} u^{\ell,m} \varphi_i \, dr - \left[\frac{1}{\rho_0} r^2 \frac{\partial u^{\ell,m}}{\partial r} \varphi_i \right]_0^{r_t} = f_i^{\ell,m} \end{aligned}$$

where
$$f_i^{\ell,m} = \int_0^{r_t} \int_0^{\pi} \int_0^{2\pi} r^2 \, f(r,\theta,\phi) \bar{Y}_\ell^m(\theta,\phi) \, \sin\theta \, d\phi \, d\theta \, \varphi_i \, dr$$

The boundary term in square brackets of the variational formulation is replaced by the correct term depending on the boundary condition imposed at $r = r_t$. For the discretization of the finite element space V_h, tenth order polynomials ($r = 10$) are used. Gauss-Lobatto points are used both as interpolation and quadrature points. The source $f_i^{\ell,m}$ is computed with Gauss-Legendre integration formulae.

6.2 Radial source and solution

We consider an artificial data of a star of interior radius equal to 1 with constant celerity and piecewise exponential density. More precisely,

$$\rho_0(r) = \begin{cases} D_1 \, e^{(r-1)/\delta_{in}} & \text{if } r < 1 \\ D_1 \, e^{-(r-1)/\delta} & \text{if } r \geq 1 \end{cases} \tag{6.1}$$

$$D_1 = 2 \times 10^{-4} \, \text{kg/m}^3, \qquad c_0 = 8 \times 10^3 \, \text{m/s}, \quad \delta_{in} = +\infty \tag{6.2}$$

$$\omega = \sqrt{1 + \frac{2i\gamma}{\omega_0}} \, \omega_0, \quad \gamma = \frac{\omega_0}{100} \tag{6.3}$$

The parameters values have been chosen of the same order of magnitude than the Sun's values after adimensionalization with respect to the radius, in order to face the same numerical difficulties. $\delta_{in} = +\infty$ means that the interior density is constant. Two typical frequencies are chosen : $\omega_0 = 2\pi f_0$ with $f_0 = 7 \times 10^4$ and 7×10^5 Hz (which corresponds for the Sun, before the adimensionalisation procedure to $f_0^{sun} = 0.1$ mHz and 1 mHz). They are related to the interior wavelengths $\lambda = c_0/f_0$: 1.14×10^{-1} m and 1.14×10^{-2} m. A radial gaussian source is located at the radius $r_0 = 0.8$ m :

$$f(r) \;=\; e^{-\alpha(r-r_0)^2}$$

As a result the computations are performed for a single mode $\ell = 0$. The parameter α is chosen such that

$$f(r_0 + 0.02) = 10^{-6}$$

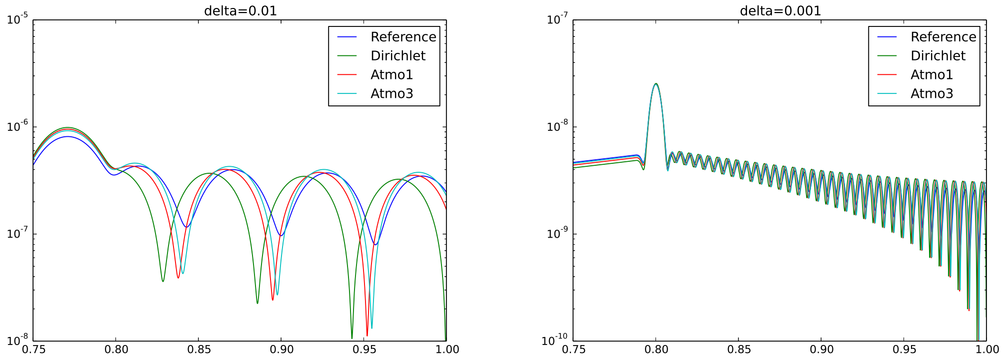
We choose to evaluate the performances of our equivalent atmosphere conditions for values of δ related to the wavelength of the considered problem. For $f_0^{sun} = 0.1$ mHz, $\delta \in [5 \cdot 10^{-5}, \, 10^{-2}]$ m. For $f_0^{sun} = 1$ mHz, $\delta \in [5 \cdot 10^{-6}, \, 10^{-3}]$ m.

6.3 Reference solutions

For each value of $\delta = \dfrac{1}{\alpha}$, a reference solution is obtained by solving the wave equation on the domain $[0, r_t]$. An exact Dirichlet-to-Neumann operator is computed by solving the following problem:

$$\begin{cases} r^2 w'' + 2rw' + \alpha r^2 w' + \left[k_\infty^2 w^2 - \ell(\ell+1)\right] w = 0 \text{ for } r \geq r_t \\ w(r_t) = 1 \end{cases} \tag{6.4}$$

for each ℓ, where
$$k_\infty^2 = \frac{\omega^2}{c_0^2}$$



(a) 0.1 mHz

(b) 1 mHz

Figure 1: Modulus of the reference solution and solutions using atmosphere boundary conditions.

The value of $\dfrac{\partial w}{\partial r}(r_t)$ will be the exact impedance Z_{exact}, and we impose

$$\frac{\partial u}{\partial r}(r_t) = Z_{\text{exact}} u(r_t)$$

as a boundary condition to compute the reference solution. A plane wave analysis provides two complex wavelengths:

$$2k^{\pm} = i\alpha \pm \sqrt{4k_\infty^2 - \alpha^2} \tag{6.5}$$

The right-going wave will oscillate (for r large enough) as $e^{ik^+ r}$ whereas a left-going wave will oscillate as $e^{ik^- r}$. The problem (6.4) is solved in the domain $[r_t, R_{\text{max}}]$ where R_{max} is chosen such that

$$|e^{i(k^+ - k^-)(R_{\text{max}} - r_t)}| = 10^{-16}$$

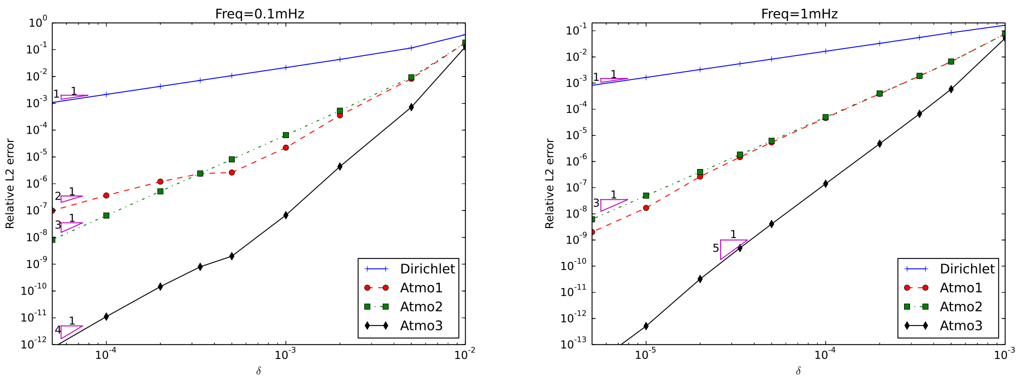
since the left term of this relation corresponds to the amplitude of the reflected wave. This exponential is always decreasing as soon as γ is positive. The problem (6.4) is solved by searching w as

$$w = v \, \exp(ik^+ r)$$

The advantage of this approach is that less degrees of freedom are needed to discretize v (since the phase is removed), and no underflow occurs.

6.4 Equivalent Atmosphere Boundary Conditions

For each value of δ, each of the equivalent atmosphere boundary conditions associated with the problem (2.14) is imposed at $r = r_t$, leading to four approximate numerical solutions (called Dirichlet, Atmo1, Atmo2 and Atmo3). Dirichlet corresponds to condition (2.9), Atmo1 to condition (2.15), Atmo2 to the condition (2.16) and Atmo3 to condition (2.17). In Fig 1 are displayed the obtained solutions. The relative L^2 errors with the reference solution on the inside interval $[0, 1]$ as δ tends to zero are displayed in Fig. 2. For the low frequency case ($f_0^{\text{sun}} = 0.1$ mHz), a preasymptotic regime is observed with a convergence in $O(\delta^3)$ for Atmo1 and Atmo2 and $O(\delta^5)$ for Atmo3, before observing the theoretical orders of convergence. For the high frequency case ($f_0^{\text{sun}} = 1$ mHz), the preasymptotic regime extends to the explored range of δ.



(a) 0.1 mHz (b) 1 mHz

Figure 2: Relative L^2 errors between the reference solution and solutions using atmosphere boundary conditions.

6.5 Axisymmetric source and solutions

In a more realistic setting, we consider smooth approximations of ρ^{sun}, c^{sun} obtained by fitting eighth-order B-splines with the data given by the Standard Solar Model [9] (c and $\log \rho$ are fitted). The approximation of ρ^{sun} is plotted in figure 3, $R_\odot$ is the radius of the Sun. The density is therefore given

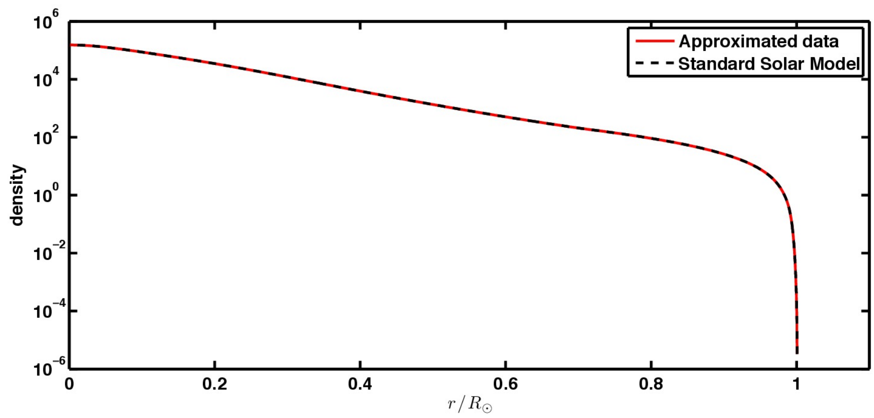


Figure 3: Exact value and approximated value of ρ^{sun} versus the relative radius $r/R_\odot$.

as:

$$\rho_0(r) = \begin{cases} \rho^{\text{sun}} & \text{if } r/R_\odot \leq 1.0007126 \\ D_1\, e^{-(r/R_\odot - 1)/\delta} & \text{otherwise} \end{cases} \tag{6.6}$$

$$D_1 \approx 3.292 \times 10^{-6}\,\text{kg/m}^3, \tag{6.7}$$

$$\omega = \sqrt{1 + \frac{2i\gamma}{\omega_0}}\,\omega_0, \quad \gamma = \frac{\omega_0}{100} \tag{6.8}$$

The velocity is given as

$$c_0(r) = \begin{cases} c^{\text{sun}} & \text{if } r/R_\odot \leq 1.0007126 \\ C_1 & \text{otherwise} \end{cases} \tag{6.9}$$

$$C_1 \approx 6.865 \times 10^3\,\text{m/s} \tag{6.10}$$

The source is a 3-D gaussian:

$$f(x) = \exp(-\alpha ||x - x_0||^2)$$

where $x_0 \approx (0, 0, 0.99)$ and α chosen such that

$$\exp(-0.007\,\alpha) = 10^{-6}.$$

The computation is done with spherical harmonics with $\ell \leq 100$ for a frequency $f_0^{\text{sun}} = 3$ mHz. The reference solution is displayed in figure 4 on a plane Oxz for a given value of δ. In figure 5(a), the

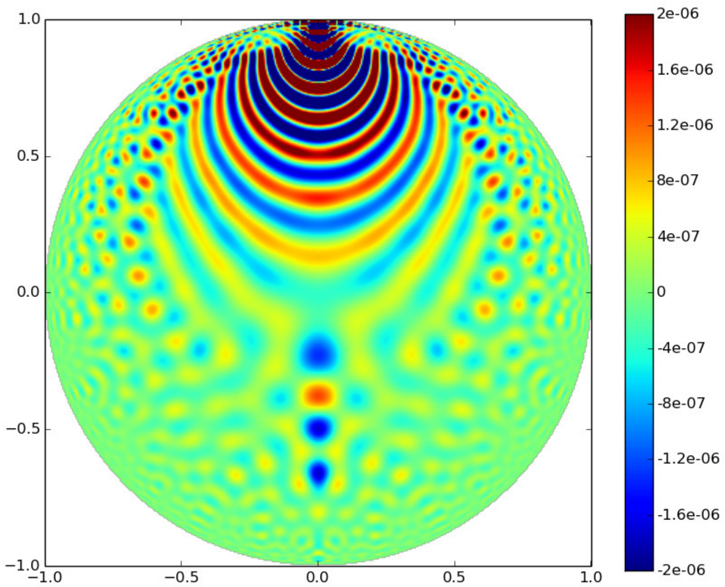
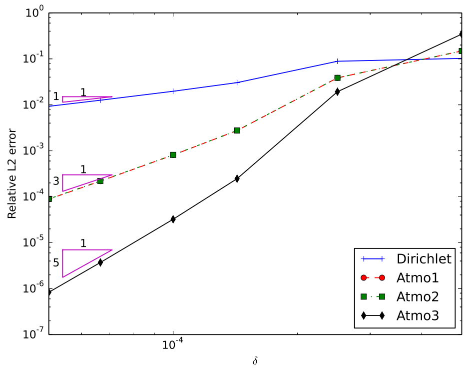
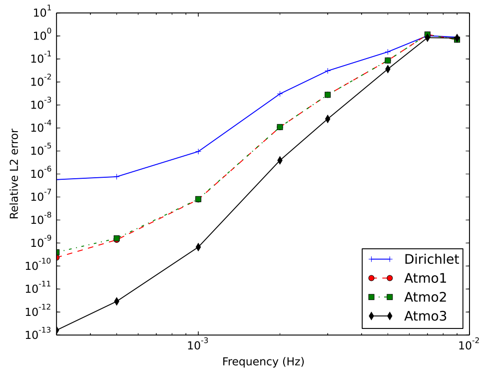


Figure 4: Real part of $\mathrm{u}/\sqrt{\rho}$ in the plane Oxz for $\dfrac{1}{\delta} = 7000$ and $f_0^{\text{sun}} = 3$ mHz and the parameters of the Sun.



(a) Relative L^2 error between exact solution and approximate solution obtained with different boundary conditions versus δ for parameters of the Sun and $f_0^{\text{sun}} = 3$ mHz.



(b) Relative L^2 error between exact solution and approximate solution obtained with different boundary conditions versus f_0^{sun} for parameters of the Sun and $\delta = 1/7000$.

Figure 5: Convergence in the realistic context of the Sun.

convergence of the different equivalent boundary conditions is represented versus δ. For this frequency and these parameters, we observe a convergence in $O(\delta^3)$ for equivalent boundary conditions Atmo1 and Atmo2, and in $O(\delta^5)$ for Atmo3. We think that we are in a preasymptotic regime. In figure 5(b), the different equivalent boundary conditions are compared for different frequencies from 0.3 mHz until 9 mHz. The condition Atmo3 is quite efficient for low frequencies. We observe that all the conditions (Dirichlet, Atmo1, Atmo2 and Atmo3) are no longer accurate for high frequencies ($f_0^{\text{Sun}} \geq 5$ mHz). It is expected since for this range of frequencies, propagative modes exist inside the atmosphere and they are no longer restricted to a small tubular neighborhood of the atmosphere. Future works will address the derivation of accurate equivalent boundary conditions in high-frequency regime.

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A Elements of derivations for the multiscale expansion and equivalent conditions

The aim of this appendix is to present elements of derivations for the multiscale expansion and equivalent conditions. We make the following ansatz

$$\begin{aligned} \mathsf{u}_\delta^- &\sim \sum_{j\geqslant 0} \delta^j \ \mathsf{u}_j^-(\mathbf{x}) \\ \mathsf{u}_\delta^+ &\sim \sum_{j\geqslant 0} \delta^j \ \mathsf{u}_j^+(\mathbf{x};\delta) \quad \text{with} \quad \mathsf{u}_j^+(\mathbf{x};\delta) = \chi(r)\mathfrak{U}_j(\theta,\phi,\frac{r-r_t}{\delta}) \end{aligned}$$

for the solution of the problem (2.1). We present all the details for the calculi of the first terms $\mathsf{u}_j^\pm$ ($j = 0, \cdots, 3$). As a consequence, we exhibit formally equivalent boundary conditions up to the fourth order.

A.1 Expansion of the operators in power series of δ

To derive the formal expansion of u_δ^+ we perform the scaling

$$S = \frac{r - r_t}{\delta} \tag{A.1}$$

in the equations set in Ω_+ and Σ in order to make appear the small parameter δ in these equations. Then it is possible to expand the Helmholtz operator in power series of δ with coefficient intrinsic operators :

$$\Delta + \frac{1}{\delta}\partial_r + \frac{\omega^2}{c^2}(1+i\nu)\mathbb{I} = \delta^{-2}\left(\sum_{n=0}^{N-1}\delta^n \mathrm{A}_n + \delta^N \mathrm{R}_{N,\delta}\right)(\partial_S,\partial_\theta,\partial_\phi;S,\theta,\phi) \quad \text{for all } N\in\mathbb{N}.$$

The remainder $\mathrm{R}_{N,\delta}$ has smooth coefficients in (S,θ,ϕ) which are bounded in δ. Since the Laplace operator Δ writes in spherical coordinates (r,θ,ϕ) as

$$\Delta = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$$

we observe that

$$\mathrm{A}_0 = \partial_S^2 + \partial_S \,, \tag{A.2a}$$

$$\mathrm{A}_1 = \frac{2}{r_t}\partial_S \,, \tag{A.2b}$$

$$\mathrm{A}_2 = \Delta_\Sigma - \frac{2}{r_t^2} S \partial_S + \frac{\omega^2}{c_0^2}(1 + i\nu) \,, \tag{A.2c}$$

where Δ_Σ is the Laplace-Beltrami operator along the surface Σ, see (2.13). Similarly the normal derivative writes

$$\partial_{\mathbf{n}}(\partial_S; \delta) = \frac{1}{\delta}\partial_S$$

on the interface Σ. Define v_δ by

$$\mathrm{v}_\delta(\theta, \phi, S) = \mathsf{u}_\delta^+(\mathbf{x}) \quad \text{where} \quad \mathbf{x} = \mathbf{X}(r_t\theta, r_t\phi) + s\mathbf{n}\left(\mathbf{X}(r_t\theta, r_t\phi)\right) \,,$$

and $\mathbf{X}(r_t\theta, r_t\phi)$ is defined by (4.5).

Remark A.1 *The coordinate system $(r_t\theta, r_t\phi, s)$ with $s = r - r_t$ is a "normal coordinate system" to the surface Σ on the manifold Ω_+ [4].*

Then, after the scaling $s \mapsto S = \dfrac{s}{\delta}$ (where $s = r - r_t$), the problem (2.1) writes

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_\delta^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_\delta^- & = & f & \text{in} \quad \Omega_- \,, \\ \mathsf{u}_\delta^- & = & \mathrm{v}_\delta & \text{on} \quad \Sigma \,, \end{array}\right. \tag{A.3}$$

and

$$\left\{\begin{array}{lcll} (\partial_S^2 + \partial_S)\mathrm{v}_\delta + \sum\limits_{n\geqslant 1} \delta^n \mathrm{A}_n \mathrm{v}_\delta & = & 0 & \text{in} \quad \Sigma \times (0, +\infty), \\ \partial_S \mathrm{v}_\delta & = & \delta \partial_{\mathbf{n}} \mathsf{u}_\delta^- & \text{on} \quad \Sigma \times \{0\} \,. \end{array}\right. \tag{A.4}$$

Moreover v_δ tends to 0 when S tends to ∞ since u_δ^+ tends to 0 when r tends to ∞.

A.2 Equations for the terms $\mathfrak{U}_n$ and u_n^-

We insert the Ansatz

$$\mathsf{u}_\delta^- \sim \sum_{n\geqslant 0} \mathsf{u}_n^-(\mathbf{x})\,\delta^n \quad \text{and} \quad \mathrm{v}_\delta \sim \sum_{n\geqslant 0} \mathfrak{U}_n(\theta, \phi, \frac{s}{\delta})\,\delta^n$$

with $\mathfrak{U}_j(\cdot, S) \longrightarrow 0$ as $S \to \infty$, in equations (A.3) and (A.4), and we perform the identification of terms with the same power in δ. Then the terms $\mathfrak{U}_n$ and u_n^- satisfy the following family of problems coupled by their conditions on the interface Σ (corresponding to $S = 0$):

$$\left\{\begin{array}{lcll} -\partial_S^2\mathfrak{U}_n - \partial_S\mathfrak{U}_n & = & \sum\limits_{p=1}^{n} \mathrm{A}_p \mathfrak{U}_{n-p} & \text{in} \quad \Sigma \times (0, +\infty) \,, \\ \partial_S \mathfrak{U}_n & = & \partial_{\mathbf{n}} \mathsf{u}_{n-1}^- & \text{on} \quad \Sigma \,, \end{array}\right. \tag{A.5}$$

and

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_n^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_n^- & = & f\delta_n^0 & \text{in} \quad \Omega_- \,, \\ \mathsf{u}_n^- & = & \mathfrak{U}_n & \text{on} \quad \Sigma \,. \end{array}\right. \tag{A.6}$$

In (A.5), we use the convention $\mathsf{u}_{-1}^- = 0$ and in (A.6) δ_n^0 denotes the Kronecker symbol. Hereafter, we make explicit the first asymptotics $\mathfrak{U}_n$ and u_n^- for $n = 0, 1, 2, 3$ by induction.

A.3 First terms of the asymptotics

$$\left\{\begin{array}{lcll} -\partial_S^2 \mathfrak{U}_0(\cdot,S) - \partial_S \mathfrak{U}_0(\cdot,S) &=& 0 & \text{for} \quad S\in(0,+\infty)\,, \\ \partial_S \mathfrak{U}_0(\cdot,0) &=& 0\,. & \end{array}\right. \tag{A.7}$$

The unique solution of (A.7) such that $\mathfrak{U}_0 \to 0$ when $S\to\infty$ is

$$\mathfrak{U}_0 = 0\,.$$

Hence, according to (A.5) for $n=0$, u_0^- solves the problem

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_0^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_0^- &=& f & \text{in} \quad \Omega_-\,, \\ \mathsf{u}_0^- &=& 0 & \text{on} \quad \Sigma\,. \end{array}\right. \tag{A.8}$$

The next term which is determined in the asymptotics is the profile $\mathfrak{U}_1$. According to (A.5) for $n=1$ and since $\mathfrak{U}_0=0$, $\mathfrak{U}_1$ solves the ODE

$$\left\{\begin{array}{lcll} -\partial_S^2 \mathfrak{U}_1(\cdot,S) - \partial_S \mathfrak{U}_1(\cdot,S) &=& 0 & \text{for} \quad S\in(0,+\infty)\,, \\ \partial_S \mathfrak{U}_1(\cdot,0) &=& \partial_{\mathbf{n}}\mathsf{u}_0^-\,. & \end{array}\right. \tag{A.9}$$

The unique solution of (A.9) such that $\mathfrak{U}_1 \to 0$ when $S\to\infty$ is

$$\mathfrak{U}_1(\theta,\phi,S) = -\mathrm{e}^{-S}\partial_{\mathbf{n}}\mathsf{u}_0^-\left(\mathbf{X}(r_t\theta,r_t\phi)\right)\,. \tag{A.10}$$

Remind that $\mathbf{X}(r_t\theta,r_t\phi)\in\Sigma$ is defined by (4.5). Hence, according to (A.6) for $n=1$, u_1^- satisfies the problem

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_1^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_1^- &=& 0 & \text{in} \quad \Omega_-\,, \\ \mathsf{u}_1^- &=& -\partial_{\mathbf{n}}\mathsf{u}_0^- & \text{on} \quad \Sigma\,. \end{array}\right. \tag{A.11}$$

The next term which is determined is the profile $\mathfrak{U}_2$. According to (A.5) for $n=2$ and since $\mathfrak{U}_0=0$, $\mathfrak{U}_2$ solves the ODE

$$\left\{\begin{array}{lcll} -\partial_S^2 \mathfrak{U}_2(\cdot,S) - \partial_S \mathfrak{U}_2(\cdot,S) &=& \frac{2}{r_t}\partial_S\mathfrak{U}_1(\cdot,S) & \text{for} \quad S\in(0,+\infty)\,, \\ \partial_S \mathfrak{U}_2(\cdot,0) &=& \partial_{\mathbf{n}}\mathsf{u}_1^-\,. & \end{array}\right. \tag{A.12}$$

According to (A.10), the unique solution of (A.12) such that $\mathfrak{U}_2\to 0$ when $S\to\infty$ is (see (4.7))

$$\mathfrak{U}_2(\theta,\phi,S) = \left(a_2(\theta,\phi) + S b_2(\theta,\phi)\right)\mathrm{e}^{-S}\,, \tag{A.13}$$

where

$$a_2(\theta,\phi) = \left(-\partial_{\mathbf{n}}\mathsf{u}_1^- + \frac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^-\right)\left(\mathbf{X}(r_t\theta,r_t\phi)\right) \quad\text{and}\quad b_2(\theta,\phi) = \frac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^-\left(\mathbf{X}(r_t\theta,r_t\phi)\right)\,.$$

Hence, according to (A.6) for $n=2$, u_2^+ solves the problem (see (4.8))

$$\left\{\begin{array}{lcll} -\operatorname{div}(\frac{1}{\rho}\nabla \mathsf{u}_2^-) - \frac{\omega^2}{\rho c^2}\mathsf{u}_2^- &=& 0 & \text{in} \quad \Omega_-\,, \\ \mathsf{u}_2^- &=& -\partial_{\mathbf{n}}\mathsf{u}_1^- + \frac{2}{r_t}\partial_{\mathbf{n}}\mathsf{u}_0^- & \text{on} \quad \Sigma\,. \end{array}\right. \tag{A.14}$$

The next term which is determined is the profile $\mathfrak{U}_3$. According to (A.5) for $n = 3$ and since $\mathfrak{U}_0 = 0$, $\mathfrak{U}_3$ solves the ODE for $S \in (0, +\infty)$

$$\left\{\begin{array}{lcl} -\partial_S^2 \mathfrak{U}_3(\cdot, S) - \partial_S \mathfrak{U}_3(\cdot, S) & = & \dfrac{2}{r_t}\partial_S \mathfrak{U}_2(\cdot, S) + \mathrm{A}_2 \mathfrak{U}_1(\cdot, S) \quad \text{for } S \in (0, +\infty) \\ \partial_S \mathfrak{U}_3(\cdot, 0) & = & \partial_{\mathbf{n}} \mathsf{u}_2^- \end{array}\right. \tag{A.15}$$

where the operator A_2 is given by (A.2c). According to (A.10)-(A.13), the right hand side in (A.15) writes

$$\left(\frac{2}{r_t}\partial_S \mathfrak{U}_2 + \mathrm{A}_2 \mathfrak{U}_1\right)(\cdot, S) = \left(\frac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left(\frac{\omega^2}{c_0^2}(1 + i\nu) + \Delta_\Sigma + \frac{6}{r_t^2} S\right)\partial_{\mathbf{n}} \mathsf{u}_0^-\right) \mathrm{e}^{-S} \,,$$

since (remind that the operator Δ_Σ is given by (2.13))

$$\Delta_\Sigma \left(\partial_{\mathbf{n}} \mathsf{u}_0^- (\mathbf{X}(r_t\theta, r_t\phi))\right) = \Delta_\Sigma \left(\partial_{\mathbf{n}} \mathsf{u}_0^-\right)(\mathbf{X}(r_t\theta, r_t\phi)) \,.$$

Then the unique solution of (A.15) such that $\mathfrak{U}_3 \to 0$ when $S \to \infty$ is (see (4.9))

$$\mathfrak{U}_3(\theta, \phi, S) = (a_3(\theta, \phi) + S b_3(\theta, \phi) + S^2 c_3(\theta, \phi))\mathrm{e}^{-S} \,, \tag{A.16}$$

where the functions a_3, b_3, c_3 are given by

$$\left\{\begin{aligned} a_3 &= -\partial_{\mathbf{n}} \mathsf{u}_2^- + \frac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{\frac{6}{r_t^2} + \frac{\omega^2}{c_0^2}(1 + i\nu) + \Delta_\Sigma\right\}\partial_{\mathbf{n}} \mathsf{u}_0^- \,, \\ b_3 &= \frac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{\frac{6}{r_t^2} + \frac{\omega^2}{c_0^2}(1 + i\nu) + \Delta_\Sigma\right\}\partial_{\mathbf{n}} \mathsf{u}_0^- \,, \\ c_3 &= -\frac{3}{r_t^2}\partial_{\mathbf{n}} \mathsf{u}_0^- \,. \end{aligned}\right.$$

Then, according to (A.6) for $n = 3$, u_3^- solves the problem (see (4.10))

$$\left\{\begin{array}{lll} -\operatorname{div}\left(\dfrac{1}{\rho}\nabla \mathsf{u}_3^-\right) - \dfrac{\omega^2}{\rho c^2}\mathsf{u}_3^- = 0 & \text{in} & \Omega_- \,, \\ \mathsf{u}_3^- = -\partial_{\mathbf{n}} \mathsf{u}_2^- + \dfrac{2}{r_t}\partial_{\mathbf{n}} \mathsf{u}_1^- - \left\{\dfrac{6}{r_t^2} + \dfrac{\omega^2}{c_0^2}(1 + i\nu) + \Delta_\Sigma\right\}\partial_{\mathbf{n}} \mathsf{u}_0^- & \text{on} & \Sigma \,. \end{array}\right. \tag{A.17}$$

A.4 Existence and regularity of the asymptotics

We claim without proving the next proposition which ensures existence and regularity results for the asymptotics.

Proposition A.1 *Let $k \in \mathbb{N}$. We assume that $f \in \mathrm{H}^k(\Omega_-)$. Then for all $n \in \{0, 1, \cdots, k+1\}$ there exists asymptotics $\mathfrak{U}_n$ and u_n^- satisfying*

- $\mathfrak{U}_0 = 0$ *and* (A.8)*, when* $n = 0$*,*
- (A.10) *and* (A.11)*, when* $n = 1$*,*
- (A.13) *and* (A.14)*, when* $n = 2$*,*
- (A.16) *and* (A.17)*, when* $n = 3$*,*

- (A.5) *and* (A.6)*, when* $n \geqslant 4$.

There holds

$$\mathfrak{U}_n \in \mathrm{H}^{k+\frac{3}{2}-n}(\Sigma \times \mathbb{R}^+) \quad \textit{and} \quad \mathsf{u}_n^- \in \mathrm{H}^{k+2-n}(\Omega_-) \,.$$

There exists functions a_n^l*,* $l \in \{1, \cdots, n\}$*, defined on* Σ *such that*

$$\mathfrak{U}_n(\theta, \phi, S) = (a_n^1(\theta,\phi) + S a_n^2(\theta,\phi) + \cdots + S^{n-1} a_n^n(\theta,\phi))\,\mathrm{e}^{-S}\,.$$

The regularity result for the asymptotics is obtained as a consequence of classical elliptic regularity results in Sobolev spaces.

B A measure of the boundary layer phenomenon. Comparison with the skin effect in electromagnetism

As another application of the multiscale expansion, we make the link in this section between the parameter δ and the "pointwise energy" of the solution of the problem (2.1) across the boundary layer. We measure this boundary layer phenomenon by introducing a "skin depth" function that turns out to depend on the mean curvature of the boundary of the domain Ω_-. An asymptotic expansion when $\delta \to 0$ for this function shows the influence of the geometry of the interface : the skin depth is larger for small values of δ when the mean curvature of the interface Σ is smaller. We refer also to the works [8, 4] where the authors measure similarly the skin effect phenomenon by introducing a suitable skin depth function.

Introduction of a skin depth function

For a data f of the problem (2.1), let us define $\mathrm{u}_\delta(\theta,\phi,s) = \mathsf{u}_\delta^+(\mathbf{x})$ for all $s \geqslant 0$ and $(\theta,\phi) \in \Sigma$ (i.e. $\mathbf{x} \in \overline{\Omega_+}$).

Definition B.1 *Let* f *be a smooth data of problem* (2.1) *such that for all* $(\theta,\phi) \in \Sigma$*,* $\mathrm{u}_\delta(\theta,\phi,0) \neq 0$*. The skin depth is the length* $\mathcal{L}(\delta;\theta,\phi)$ *defined on* Σ *and taking the smallest value such that*

$$|\mathrm{u}_\delta(\theta,\phi;\mathcal{L}(\delta;\theta,\phi))| = |\mathrm{u}_\delta(\theta,\phi,0)|\mathrm{e}^{-1}\,. \tag{B.1}$$

Thus this length $\mathcal{L}(\delta;\theta,\phi)$ is the distance from the interface where the field has decreased of a fixed rate e. It depends on δ, of each point $(\theta,\phi) \in \Sigma$ and a priori on the data f. However it is possible to exhibit the asymptotic behavior of the skin depth function $\mathcal{L}$ for small values of δ independently of both (θ,ϕ) and the data f.

Asymptotic behavior of the skin depth function when $\delta \to 0$

Theorem B.1 *Let* f *be a smooth data of problem* (2.1)*. We assume that* $\partial_{\mathbf{n}}\mathsf{u}_0^-\big(\mathbf{X}(r_t\theta, r_t\phi)\big) \neq 0$ *on* Σ*. Then the skin depth function has the following behavior for small values of* δ

$$\mathcal{L}(\delta;\theta,\phi) \approx \delta\left(1 - \frac{2}{r_t}\delta + \mathcal{O}(\delta^2)\right)\,, \quad \delta \to 0\,. \tag{B.2}$$

PROOF. The proof of this theorem is based on the formula

$$|\mathrm{u}_\delta(\theta,\phi;s)| = |\mathrm{u}_\delta(\theta,\phi;0)|\mathrm{m}(\theta,\phi;s)\,\mathrm{e}^{-\dfrac{s}{\delta}} \tag{B.3}$$

with

$$\mathrm{m}(\theta, \phi; s) = 1 - \frac{2}{r_t} s + \mathcal{O}\left((\delta + s)^2\right) .$$

This formula (B.3) is a consequence of equations (4.4) and (4.7). According to Definition B.1, the skin depth function $\mathcal{L}(\delta; .)$ satisfies

$$|\mathrm{u}_\delta(\theta, \phi; \mathcal{L}(\delta; \theta, \phi))| = |\mathrm{u}_\delta(\theta, \phi; 0)| \, \mathrm{e}^{-1} . \tag{B.4}$$

Hence,

$$\mathrm{m}(.; \mathcal{L}(\delta; .)) \, \mathrm{e}^{-\mathcal{L}(\delta; .)/\delta} = \mathrm{e}^{-1} .$$

Performing an asymptotic expansion when δ tends to zero, we infer the asymptotic behavior of the skin depth function (B.2). ■

Comparison with the skin effect in electromagnetism

According to [4, Th. 3.2] the skin depth function $\mathcal{L}_0(\sigma; y_\alpha)$ associated with the skin effect phenomenon has the following behavior for large conductivity σ,

$$\mathcal{L}_0(\sigma; y_\alpha) \approx \ell(\sigma) \left(1 - \mathcal{H}(y_\alpha)\ell(\sigma) + \mathcal{O}(\sigma^{-1})\right) , \quad \sigma \to \infty . \tag{B.5}$$

Here y_α, $\alpha = 1, 2$ are tangential coordinates on Σ, $\mathcal{H}$ is the mean curvature of the surface Σ and $\ell(\sigma) = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$ is the classical skin depth parameter which corresponds to δ in this paper. Hence, the δ-asymptotic behavior of the skin depth function (B.2) coincides only up to the first order of approximation with the behavior of the skin depth function $\mathcal{L}_0(\sigma; y_\alpha)$ characterizing the skin effect phenomenon since the mean curvature of the sphere $\Sigma = \{r = r_t\}$ is $\mathcal{H} = \frac{1}{r_t}$. Similarly the equivalent boundary conditions associated with these two boundary layer phenomena coincide only up to the second order of approximation, (2.9)-(2.10).

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