High-Order Optimal Edge Element for Pyramids, Prisms and Hexahedra

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Bibliography and motivation

- Nedelec's first family not optimal on non-affine hexahedra and prisms
  R.S. Falk, P. Gatto and P. Monk
  Hexahedral $H(div)$ and $H(curl)$ finite elements

- Difficult case of finite elements on pyramids
  N. Nigam, J. Phillips
  Higher-order finite elements on pyramids
  J.-L. Coulomb, F.-X. Zgainski and Y. Maréchal
  A pyramidal element to link hexahedral, prismatic and tetrahedral edge finite elements

- Is it possible to construct finite elements providing an optimal H(curl) estimate in $O(h^r)$?

Polynomial spaces

$$\mathbb{Q}_{m,n,p}(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; 0 \le i \le m,\; 0 \le j \le n,\; 0 \le k \le p\right\}$$

$$\mathbb{P}_r(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; i,j,k \ge 0,\; i+j+k \le r\right\}$$

$$\mathbb{B}_r \;=\; \mathbb{P}_r(x,y,z) \;\oplus\; \sum_{k=0}^{r-1} \mathbb{P}_k(x,y)\left(\frac{xy}{1-z}\right)^{r-k}$$

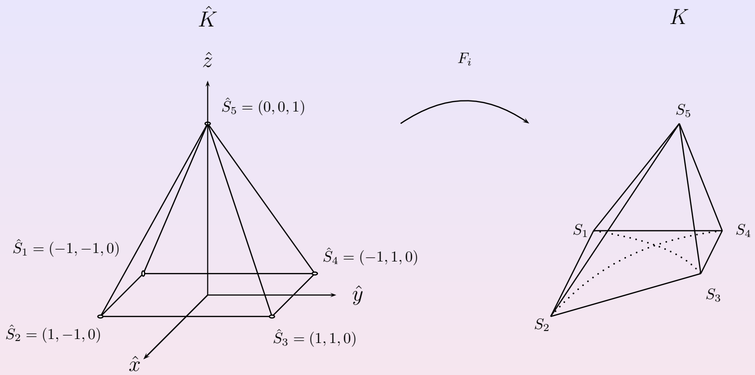
$$\widetilde{\mathbb{P}}_r(x,y,z) \;=\; Span\left\{x^i y^j z^k,\; i,j,k \ge 0,\; i+j+k = r\right\}$$

$$\mathcal{S}_r(x,y,z) \;=\; \left\{u \in \widetilde{\mathbb{P}}_r^{\,3} \text{ so that } u_1\, x + u_2\, y + u_3\, z = 0\right\}$$

$$\mathcal{R}_r(x,y,z) \;=\; \mathbb{P}^3_{r-1} \oplus \mathcal{S}_r$$

$\mathcal{R}_r(x,y,z)$: Nedelec's first family on tetrahedra

Condition of optimality



Expression of F for the pyramid :

$$F(\hat{x},\hat{y},\hat{z}) = A + B\hat{x} + C\hat{y} + D\hat{z} + \frac{\hat{x}\hat{y}}{4(1-\hat{z})}(S_1 + S_3 - S_2 - S_4)$$

F affine if the basis of the pyramid is a parallelogramm.

Condition of optimality

Finite element space :

$$V_h = \{u \in H(\mathrm{curl}, \Omega) \text{ so that } u|_K \in P_r^F(K)\}$$

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Use of Piola transform to write the space in the reference element $\hat{K}$:

$$P_r^F(K) = \{u \text{ such that } DF^* \, u \circ F \in \hat{P}_r(\hat{K})\}$$

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Condition of optimality for a given choice of $\hat{P}_r(\hat{K})$:

$$\forall K, \quad \mathcal{R}_r(x, y, z) \subset P_r^F(K)$$

This condition is sufficient to obtain optimal estimates in $O(h^r)$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Hexahedra

$$\mathbb{Q}_{r-1,r+1,r+1} \times \mathbb{Q}_{r+1,r-1,r+1} \times \mathbb{Q}_{r+1,r+1,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x},\hat{y}) \otimes \mathbb{P}_{r+1}(\hat{z})) \;\times\; (\mathbb{P}_{r+1}(\hat{x},\hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Optimal Finite Element spaces

Minimal spaces $\hat{P}_r(\hat{K})$ satisfying the condition of optimality :

Pyramids

$$\mathbb{B}^3_{r-1} \oplus \left\{ \frac{\hat{x}^p \hat{y}^p}{(1-\hat{z})^{p+2}} \begin{bmatrix} \hat{y}(1-\hat{z}) \\ \hat{x}(1-\hat{z}) \\ \hat{x}\hat{y} \end{bmatrix}, \quad 0 \le p \le r-1 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^m \hat{y}^{n+2}}{(1-\hat{z})^{m+2}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \frac{\hat{x}^{n+2} \hat{y}^m}{(1-\hat{z})^{m+2}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \quad 0 \le m \le n \le r-2 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^p \hat{y}^q}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} (1-\hat{z}) \\ 0 \\ \hat{x} \end{bmatrix}, \frac{\hat{x}^q \hat{y}^p}{(1-\hat{z})^{p+q+1-r}} \begin{bmatrix} 0 \\ (1-\hat{z}) \\ \hat{y} \end{bmatrix}, \quad \begin{matrix} 0 \le p \le r-1 \\ 0 \le q \le r+1 \end{matrix} \right\}$$

Optimal Finite Element spaces

When expressed on the cube $[-1,1]^3$, it is more friendly :

Pyramids (expressed on the cube)

$$\left(\mathbb{B}_{r-1} \circ T(\tilde{x},\tilde{y},\tilde{z})\right)^3 \oplus \left\{ \tilde{x}^p \tilde{y}^p (1-\tilde{z})^p \begin{bmatrix} \tilde{y} \\ \tilde{x} \\ \tilde{x}\tilde{y} \end{bmatrix}, \quad 0 \leq p \leq r-1 \right\}$$

$$\oplus \left\{ \tilde{x}^m \tilde{y}^{n+2} (1-\tilde{z})^{n+1} \begin{bmatrix} 1 \\ 0 \\ \tilde{x} \end{bmatrix}, \ \tilde{x}^{n+2} \tilde{y}^m (1-\tilde{z})^{n+1} \begin{bmatrix} 0 \\ 1 \\ \tilde{y} \end{bmatrix}, \quad 0 \leq m \leq n \leq r-2 \right\}$$

$$\oplus \left\{ \tilde{x}^p \tilde{y}^q (1-\tilde{z})^r \begin{bmatrix} 1 \\ 0 \\ \tilde{x} \end{bmatrix}, \ \tilde{x}^q \tilde{y}^p (1-\tilde{z})^r \begin{bmatrix} 0 \\ 1 \\ \tilde{y} \end{bmatrix}, \quad \begin{matrix} 0 \leq p \leq r-1 \\ 0 \leq q \leq r+1 \end{matrix} \right\}$$

Nedelec's first family

Nedelec's first family $\hat{P}_r^1(\hat{K})$:

Hexahedra

$$\mathbb{Q}_{r-1,r,r} \times \mathbb{Q}_{r,r-1,r} \times \mathbb{Q}_{r,r,r-1}$$

Prisms

$$(\mathcal{R}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_r(\hat{z})) \ \times \ (\mathbb{P}_r(\hat{x}, \hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Pyramids

Same expression with $0 \leq p \leq r-1, \ 0 \leq q \leq r$
This finite element space is new.

H(curl) conformity

Tangential restrictions on triangular faces :

$$\mathcal{R}_r(x, y)$$

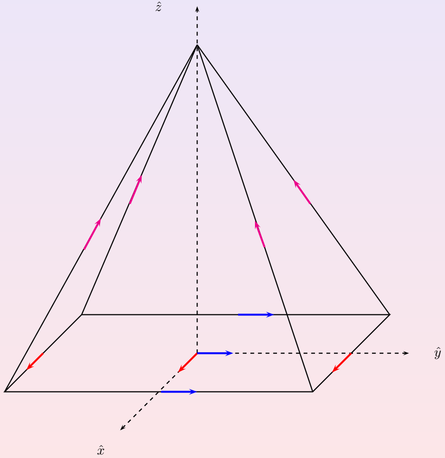
Tangential restrictions on quadrilateral faces :

$$\mathbb{Q}_{r-1,r+1}(x, y) \times \mathbb{Q}_{r+1,r-1}(x, y)$$

Nodal basis functions

$(\hat{M}_i)$: position of degrees of freeedom

$\hat{t}_i$ the associated direction



Nodal basis functions

$(\hat{M}_i)$: position of degrees of freeedom
$\hat{t}_i$ the associated direction
$(\hat{\psi}_i)$ a basis of the finite element space $\hat{P}_r$

Vandermonde matrix :

$$VDM_{i,j} = \hat{\psi}_i(\hat{M}_j) \cdot \hat{t}_j$$

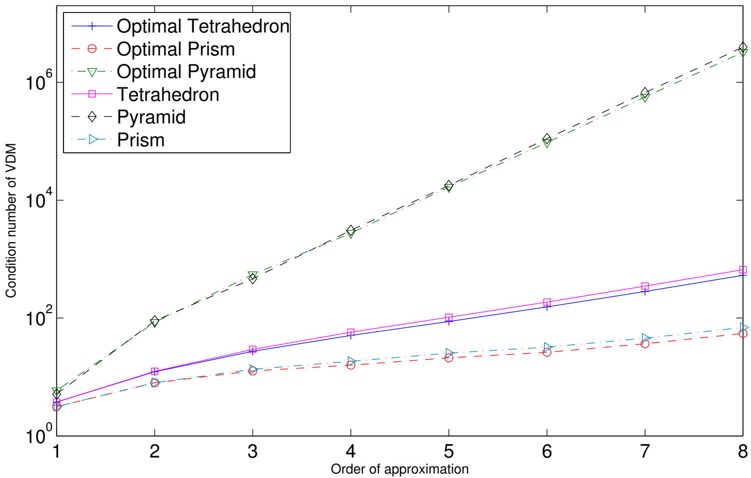
The basis function $\hat{\varphi}_i$ associated with the point $\hat{M}_i$:

$$\hat{\varphi}_i = \sum_j (VDM^{-1})_{i,j} \, \hat{\psi}_j.$$

Nodal basis functions

Vandermonde matrix :

$$VDM_{i,j} = \hat{\psi}_i(\hat{M}_j) \cdot \hat{t}_j$$



Hierarchical basis functions for the pyramid

Parameters β_i are associated with triangular faces
Parameters λ_i are associated with vertices of the pyramid

$$\begin{cases} \beta_1 = \dfrac{1 - \hat{x} - \hat{z}}{2} \\ \beta_2 = \dfrac{1 - \hat{y} - \hat{z}}{2} \\ \beta_3 = \dfrac{1 + \hat{x} - \hat{z}}{2} \\ \beta_4 = \dfrac{1 + \hat{y} - \hat{z}}{2} \end{cases} \qquad \begin{cases} \lambda_1 = \dfrac{\beta_1 \, \beta_2}{1 - \hat{z}} \\ \lambda_2 = \dfrac{\beta_2 \, \beta_3}{1 - \hat{z}} \\ \lambda_3 = \dfrac{\beta_3 \, \beta_4}{1 - \hat{z}} \\ \lambda_4 = \dfrac{\beta_4 \, \beta_1}{1 - \hat{z}} \\ \lambda_5 = \hat{z} \end{cases}$$

Hierarchical basis functions for the pyramid

γ_i are parametrizations of vertical edges
γ_i are parametrizations of horizontal edges

$$\left\{ \begin{array}{l} \gamma_1 = \dfrac{2\hat{z} + \hat{x} + \hat{y}}{2} \\[2ex] \gamma_2 = \dfrac{2\hat{z} - \hat{x} + \hat{y}}{2} \\[2ex] \gamma_3 = \dfrac{2\hat{z} - \hat{x} - \hat{y}}{2} \\[2ex] \gamma_4 = \dfrac{2\hat{z} + \hat{x} - \hat{y}}{2} \end{array} \right.$$

$$\left\{ \begin{array}{l} \delta_1 = \delta_3 = \hat{x} \\[2ex] \delta_2 = \delta_4 = \hat{y} \end{array} \right.$$

Hierarchical basis functions for the pyramid

Use of Jacobi polynomials $P_i^{\alpha,\beta}(x)$ orthogonal with respect to weight $(1-x)^\alpha(1+x)^\beta$

For two horizontal edges :

$$(\lambda_1 \nabla(\lambda_2+\lambda_3) - \lambda_2 \nabla(\lambda_1+\lambda_4))\, P_i^{0,0}(\delta_1),\ 0 \le i \le r-1$$

$$(\lambda_1 \nabla(\lambda_3+\lambda_4) - \lambda_4 \nabla(\lambda_1+\lambda_2))\, P_i^{0,0}(\delta_2),\ 0 \le i \le r-1$$

For a vertical edge :

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\, P_i^{0,0}(\gamma_1), \quad 0 \le i \le r-1$$

Hierarchical basis functions for the pyramid

For the base:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \beta_4\, P_i^{0,0}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_j^{1,1}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1))\ \beta_3\, P_j^{1,1}(\frac{\beta_3 - \beta_1}{1 - \hat{z}}) P_i^{0,0}(\frac{\beta_4 - \beta_2}{1 - \hat{z}})(1 - \hat{z})^{\max(i,j)-1}$$

$$0 \le i, j \le r - 1$$

For a triangular face:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \lambda_5\, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\ \beta_3\, P_i^{0,0}(\delta_1) P_j^{0,0}(\gamma_1)$$

$$0 \le i + j \le r - 2$$

Hierarchical basis functions for the pyramid

For interior functions:

$$(\lambda_1 \nabla(\lambda_2 + \lambda_3) - \lambda_2 \nabla(\lambda_1 + \lambda_4))\ \beta_4\, \lambda_5\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla(\lambda_3 + \lambda_4) - \lambda_4 \nabla(\lambda_2 + \lambda_1))\ \beta_3\, \lambda_5\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$(\lambda_1 \nabla \lambda_5 - \lambda_5 \nabla \lambda_1)\ \beta_3\, \beta_4\, P_{ijk}(\hat{x}, \hat{y}, \hat{z})$$

$$0 \leq i, j \leq r - 2,$$
$$0 \leq k \leq r - 2 - max(i, j)$$

$$P_{ijk}(\hat{x}, \hat{y}, \hat{z}) = P_i^{0,0}\left(\frac{\beta_3 - \beta_1}{1 - \hat{z}}\right) P_j^{0,0}\left(\frac{\beta_4 - \beta_2}{1 - \hat{z}}\right) P_k^{2\max(i,j)+2,0}(2\hat{z} - 1)(1 - \hat{z})^{\max(i,j)-1}$$

Comparison with other pyramidal edge elements

- $\hat{P}_r^1$ is the same space as proposed by Coulomb et al, Graglia and Gheorma, Gradinaru and Hiptmair, Doucet et al, Nigam and Phillips for $r = 1$.
- First space proposed by Nigam and Phillips contains more degrees of freedom than $\hat{P}_r^1$ while providing the same order of convergence
- Second space proposed by Nigam and Phillips contains $r(r-1)$ less degrees of freedom but is not consistent for non-affine pyramids
- Basis functions of Coulomb et al, Graglia and Gheorma for $r = 2$ induce spurious modes and are providing only first-order convergence even for affine pyramids

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \text{curl}(\text{curl} E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \text{curl} E \cdot \text{curl} \varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Dispersion analysis

Maxwell's equations

$$-\omega^2 E + \text{curl}(\text{curl} E) = f$$

$$-\omega^2 \int_\Omega E \cdot \varphi_i + \text{curl} E \cdot \text{curl} \varphi_i \, dx = \int_\Omega f \cdot \varphi_i \, dx$$

$$-\omega^2 M_h E + K_h E = F_h$$

Research of eigenvalues (ω, E) with quasi-periodic conditions

$$E(\vec{x} + \vec{h}) = exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

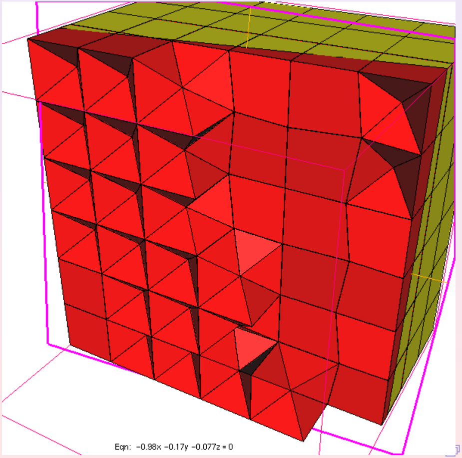
Dispersion error = $\dfrac{\omega - ||\vec{k}||}{\omega}$

Dispersion analysis

Research of eigenvalues (ω, E) with quasi-periodic conditions

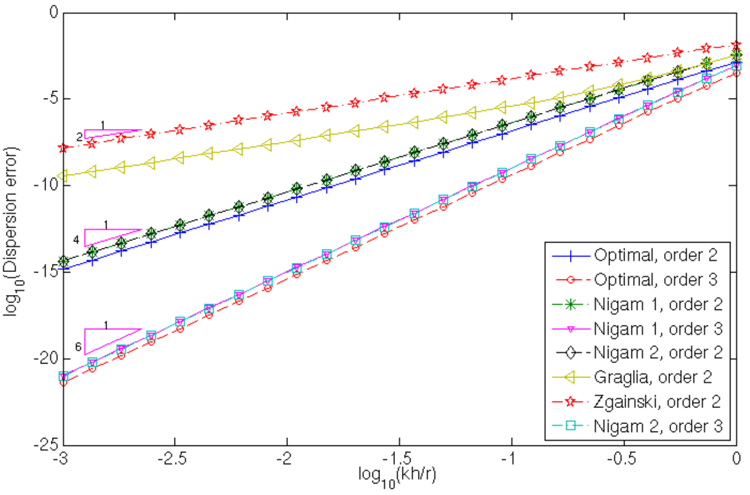
$$E(\vec{x} + \vec{h}) \; = \; exp^{i\vec{k}\cdot\vec{h}} E(\vec{x})$$

Dispersion analysis on purely pyramidal mesh

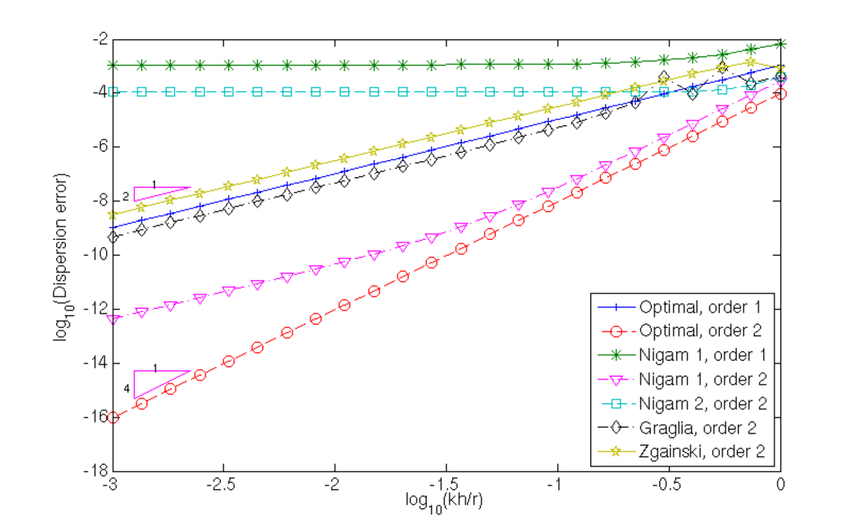


Dispersion analysis

Dispersion on affine pyramids

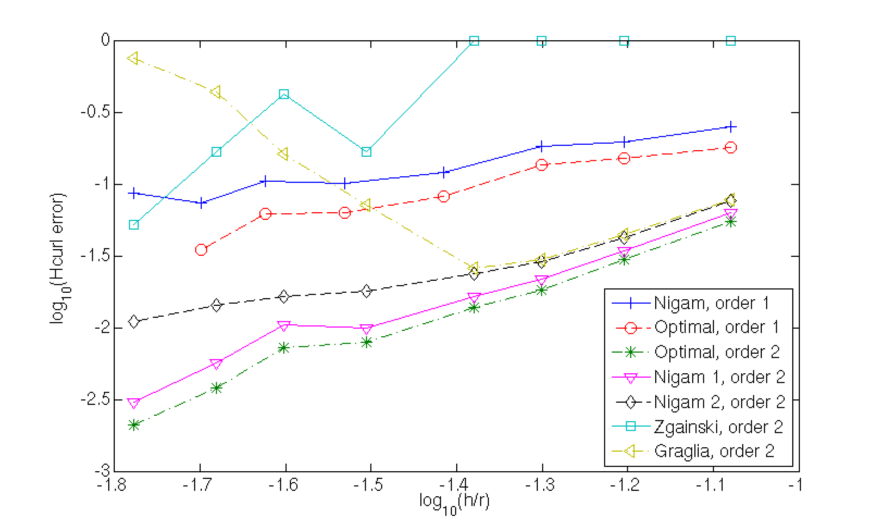


Dispersion on non-affine pyramids



Convergence for the cube

Gaussian source inside a cube and non-affine pyramids :

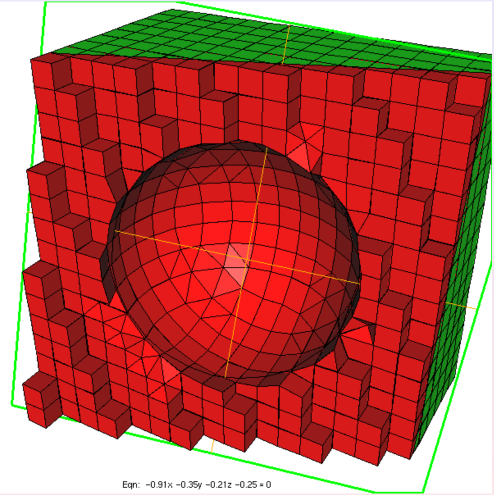


Convergence for the sphere

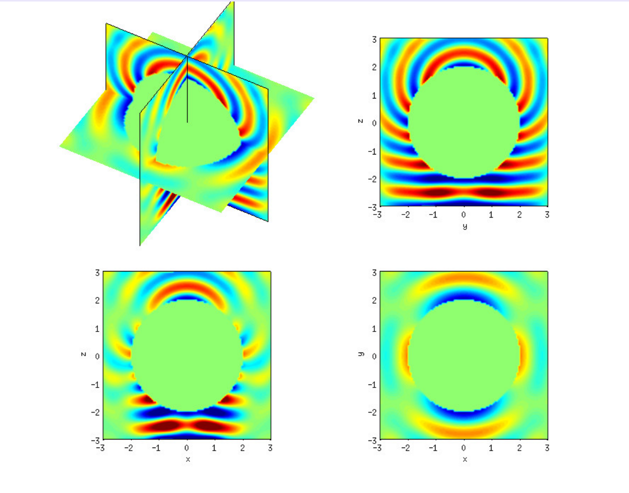
Scattering of a perfectly conducting object

$$\begin{cases} -\omega^2 \varepsilon\, E + \mathrm{curl}(\frac{1}{\mu}\, \mathrm{curl}\, E) \;=\; f \text{ in } \Omega \\ E \times n = -E^{\mathrm{inc}} \times n \text{ on } \Gamma \\ \mathrm{curl} E \times n \;=\; i\, k (n \times E) \times n \text{ on } \Sigma \end{cases}$$

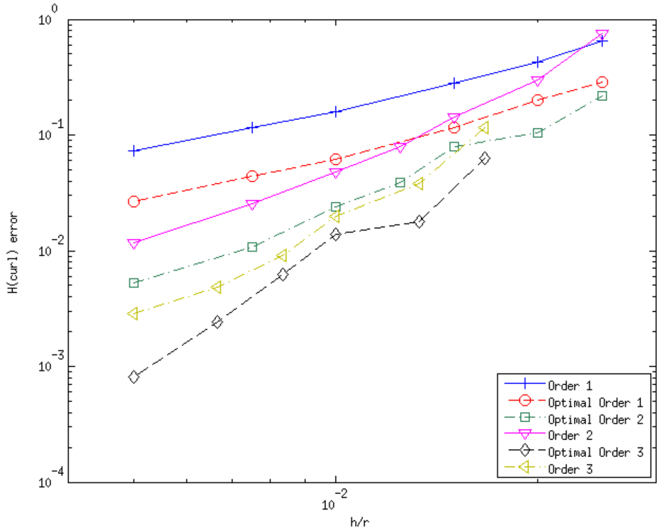
Convergence for the sphere



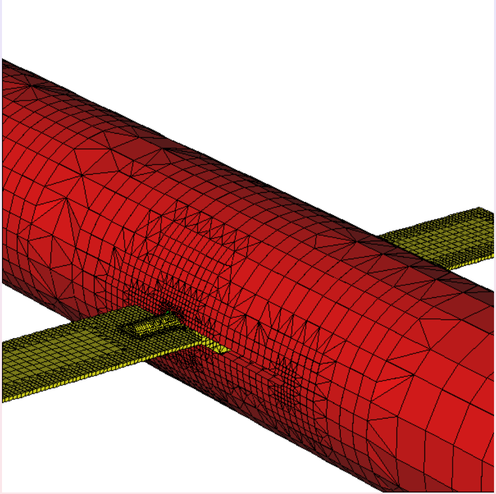
Convergence for the sphere



Convergence for the sphere

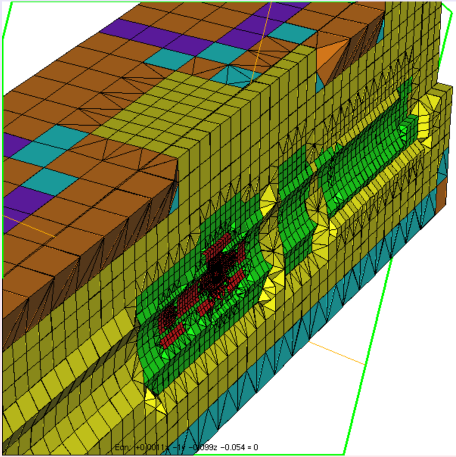


Scattering by a satellite

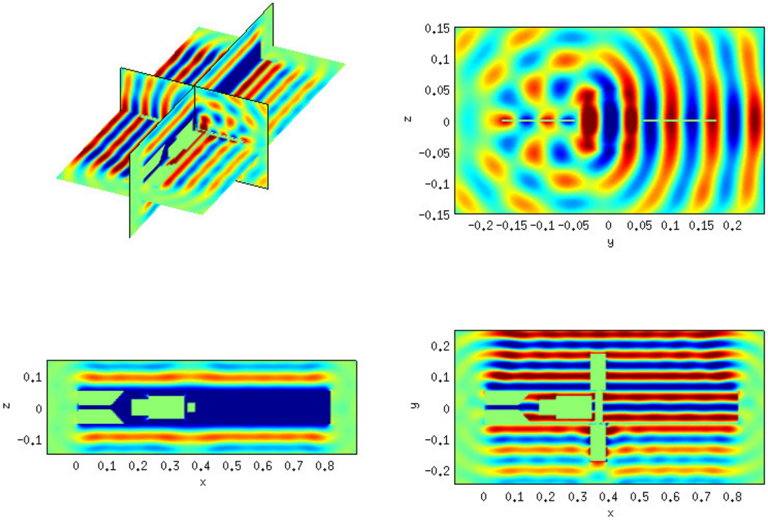


Scattering by a satellite

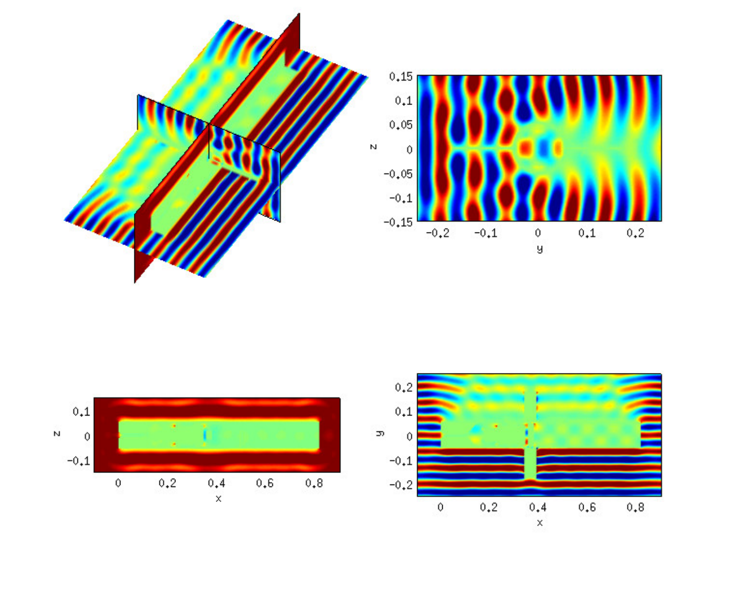
Each color is associated with an order of approximation



Scattering by a satellite



Scattering by a satellite



Scattering by a satellite

The mesh contains 35006 tetrahedra, 50390 hexahedra (40 659 affine hexahedra), 48865 pyramids (40 508 affine pyramids), 4582 wedges. We use $\hat{P}_r^1$ and there are 2 570 034 dofs.

END