Optimized transmission conditions for domain decomposition methods and Helmholtz equation. Application to higher order finite element methods.

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Model problem

Helmholtz equation

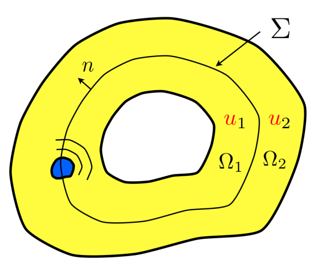
$$\begin{cases} -\rho\,\omega^2 u - \text{div}(\mu \nabla u) = 0, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial n} - ik(\omega) u = \dfrac{\partial u^{\text{inc}}}{\partial n} - ik(\omega) u^{\text{inc}}, & \text{on } \Gamma_2 \end{cases}$$

with $k(\omega)$ the wave number :

$$k(\omega) = \omega \sqrt{\frac{\rho}{\mu}}$$

and u^{inc} an incident plane wave.

Transmission conditions



Transmission conditions :

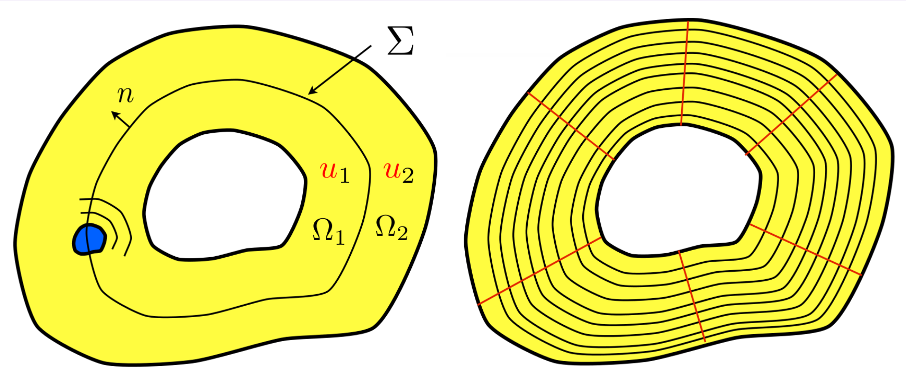
$$u_1 = u_2, \quad \mu_1 \, \partial_n u_1 = \mu_2 \, \partial_n u_2$$

Iterative DDMs : produce a sequence (u_1^n, u_2^n)

- (u_1^n, u_2^n) computed from previous iterations by solving local problems in Ω_1 and Ω_2
- $(u_1^n, u_2^n) \rightarrow (u_1, u_2)$ when n tends to the infinity

Transmission conditions

Concentric interfaces without intersection:



Equivalent transmission conditions

Transmission conditions are rewritten (coefficients ρ, μ continuous across the interface Σ)

$$\mu \left(\partial_n u_1 - z \left[k(\omega) u_1 + c T(u_1)\right]\right) = \mu \left(\partial_n u_2 - z \left[k(\omega) u_2 + c T(u_2)\right]\right)$$

$$\mu \left(\partial_n u_1 - \bar{z} \left[k(\omega) u_1 + c T(u_1)\right)\right) = \mu \left(\partial_n u_2 - \bar{z} \left[k(\omega) u_2 + c T(u_2)\right]\right)$$

Let us denote

$$B_{z,c} = \mu \left(\partial_n - z \left[k(\omega) \mathbb{I} + c T\right]\right)$$

Transmission conditions given as :

$$B_{z,c} u_1 = B_{z,c} u_2, \quad B_{\bar{z},c} u_1 = B_{\bar{z},c} u_2$$

Jacobi iterative algorithm

Sequence (u_1^n, u_2^n) obtained with Jacobi iterative algorithm:

$$\left\{ \begin{array}{ll} -\rho\,\omega^2 u_1^n - \mathrm{div}(\mu u_1^n) = 0 & \text{in } \Omega_1 \\ -\rho\,\omega^2 u_2^n - \mathrm{div}(\mu u_2^n) = 0 & \text{in } \Omega_2 \\ B_{z,c} u_1^n = B_{z,c} u_2^{n-1} & \\ B_{\bar{z},c} u_2^n = B_{\bar{z},c} u_1^{n-1} & \end{array} \right.$$

Relaxation with parameter r:

$$B_{z,c} u_1^n = r B_{z,c} u_2^{n-1} + (1-r) B_{z,c} u_1^{n-1}$$

$$B_{\bar{z},c} u_2^n = r B_{\bar{z},c} u_1^{n-1} + (1-r) B_{\bar{z},c} u_2^{n-1}$$

Classical choices for operator B

Scalar Impedance

$$\boxed{z = i \quad T = 0}$$

- Després (1990), Després, Joly, Roberts (1992)
- Collino, Ghanemi, Joly (1998)

Local operators :

$$\boxed{z = i \quad T = (\mathbb{I} - \alpha_1 \Delta_\Sigma)^{-1} (\mathbb{I} - \alpha_2 \Delta_\Sigma)}$$

- Gander, Magoules, Nataf (2002), Japhet, Nataf (2002)
- J.F. Lee (2006), Antoine, Boubendir, Geuzaine (2012)

Exponential convergence

For a non-local operator T of the form

$$T = \Lambda\Lambda^*$$

where Λ is an isomorphism from $L^2(\Sigma)$ to $H^{-1/2}(\Sigma)$, there exists $\tau(r, z, c, \Lambda)$ such that

$$||u_1^n - u_1|| + ||u_2^n - u_2|| \leq C\,\tau^n$$

Optimization of parameters z, c for circular layers.

Class of non-local operators

Λ pseudo-differential operator of order $1/2$, its symbol would be

$$\hat{\Lambda} = \left|\frac{\xi}{\omega}\right|^{1/2}$$

In 2-D, an operator satisfying these properties is :

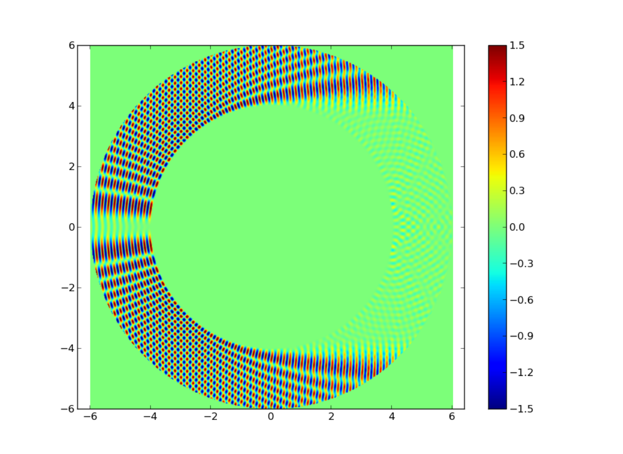
$$(\Lambda u, \varphi) = \int_\Sigma \int_\Sigma \chi(|x-y|)\sqrt{|x-y|}\,\partial_s u(x)\,\partial_s \varphi(y)\,d\sigma(x)\,d\sigma(y)$$

Justification of this form in Collino-Joly-Lecouvez (Waves 2013)

Cut-off function χ used to obtain a quasi local operator.

Comparison with Després method

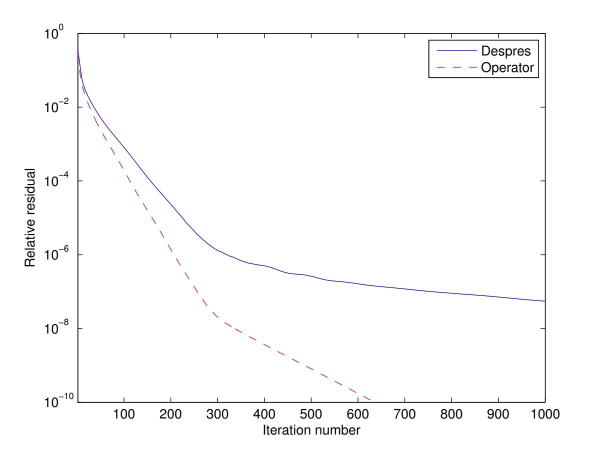
Scattering by a disc of radius $R = 4$, absorbing boundary condition set on $R = 6$, $\omega = 10\pi$



Subdomains are concentric discs

Comparison with Després method

Evolution of the residual for Després method (z=i, T=0) and optimized method with $\mathbb{P}_4$ finite elements (Jacobi algorithm).



$\Rightarrow$ Geometrical convergence for the optimized method.

Influence of cut-off function

Number of iterations versus the radius of the cut-off function ($\mathbb{Q}_4$)

Radius	Jacobi	Gmres(50)
$\frac{\lambda}{8}$	497	144
$\frac{\lambda}{4}$	365	122
$\frac{\lambda}{2}$	342	114
λ	403	120
$2\,\lambda$	398	120
$4\,\lambda$	398	120

λ is here the wavelength.

Influence of the mesh size

Number of iterations versus the number of dofs per wavelength.

N	Jacobi	Gmres(50)
2	723	370
4	394	113
8	398	120
16	401	118
32	401	114
64	401	110

Influence of the number of subdomains

Number of iterations versus the number of subdomains ($\mathbb{Q}_4$).

Number of sub-domains	Jacobi	Gmres(50)
2	398	120
4	456	193
8	1322	435
16	3776	962

Computations performed with the same coefficients z, c. choosing different coefficients might reduce substantially the number of iterations

Influence of the frequency

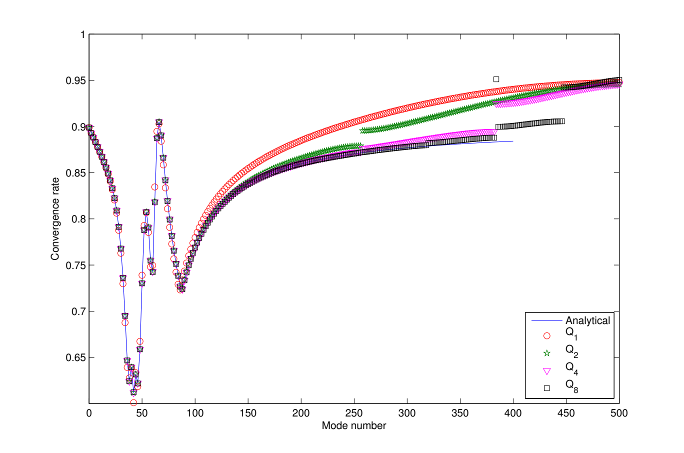
Number of iterations versus the pulsation ω ($\mathbb{Q}_4$ with eight degrees of freedom per wavelength)

ω	Jacobi	Gmres(50)
π	150	62
2π	202	72
4π	266	86
6π	280	99
10π	398	120
20π	515	149
40π	746	189

$\Rightarrow$ The influence of the frequency is rather mild on this case.

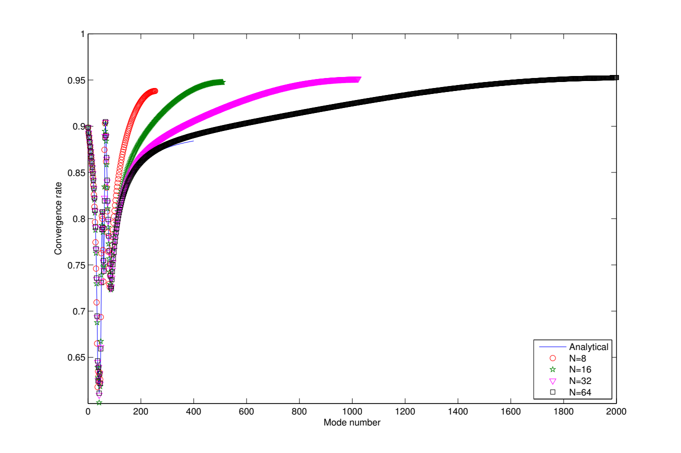
Comparison with analytical rate

Rate of convergence can be computed analytically with modes $e^{im\theta}$



For high values of *m*, the rate differ because of discretization error.
$\Rightarrow$ Maximal rate of about 0.95 instead of 0.9 for the analytical computation

Comparison with analytical rate



Final rate of 0.95 rather independant of the number of points N per wavelength.

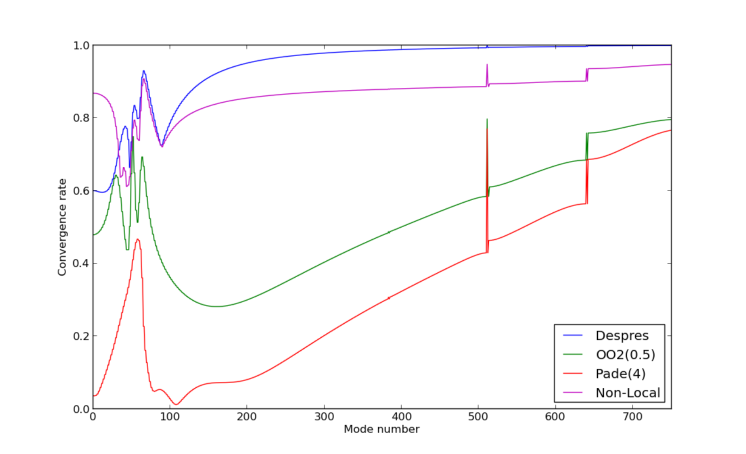
Comparison with other transmission conditions

Comparison with the following transmission conditions

- Després operator : Després (1990)
- OO2 : Optimized second-order operator Gander, Magoules, Nataf (2002)
- Padé(N) : Square root operator approximated by Padé expansion, Antoine, Boubendir, Geuzaine (2012)
- Non-local : our approach with $T = \Lambda\Lambda^*$

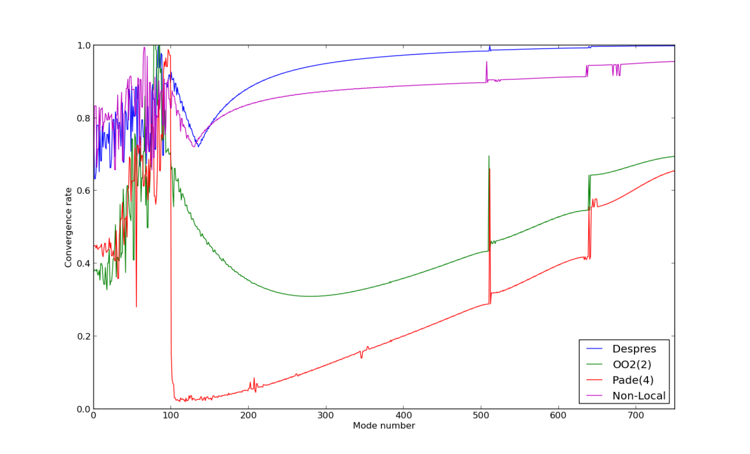
Comparison with other transmission conditions

Rate of convergence for the homogeneous disk



Comparison with other transmission conditions

Rate of convergence for a dielectric square ($\rho = 2.25, \quad \mu = 1$)



Comparison with other transmission conditions

Number of iterations with Gmres(50) for the dielectric square to reach a relative residual lower than 10^{-8} ($\mathbb{Q}_6$)

Frequency	0.01	0.1	1.0	4.0
Després	128	48	38	247
OO2	26	29	40	200
Padé(4)	94	18	34	160
Non-local	28	33	46	194

Prospects

- Adjonction of local operators in T
- Implement and test 3-D cases
- Case of intersecting interfaces
- Extension to 3-D Maxwell's equations