High order time stepping and local time stepping for first order wave problems

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Bibliography and motivation

- Case of second-order hyperbolic problems treated by:
  Jean-Charles Gilbert and Patrick Joly, *Higher order time stepping for second order hyperbolic problems and optimal CFL conditions*,
  Julien Diaz and Marcus Grote, *Energy Conserving Explicit Local Time-Stepping for Second-Order Wave Equations*

- For first-order hyperbolic problems, second-order time scheme:
  Serge Piperno, *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems*
Model problem

First-order hyperbolic problem:

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{d} A_i(x) \frac{\partial U}{\partial x_i} = f(x, t)$$

with $A_i(x)$ symmetric matrices.

Use of Local Discontinuous formulation with centered fluxes:

$$\int_K \frac{\partial U}{\partial t} \varphi \, dx - \int_K \sum_{i=1}^{d} A_i(x) U \frac{\partial \varphi}{\partial x_i} \, dx + \int_{\partial K} \left( \sum_{i=1}^{d} A_i(x)n_i \right) \{U\} \varphi \, dx = \int_K f(x, t) \varphi \, dx$$
First-order hyperbolic problem:

\[ \frac{\partial U}{\partial t} + \sum_{i=1}^{d} A_i(x) \frac{\partial U}{\partial x_i} = f(x, t) \]

with \( A_i(x) \) symmetric matrices.

Evolution problem:

\[ \frac{dU}{dt} + A_h U = F_h \]

With conservative boundary conditions, \( A_h \) is skew-symmetric.
Modified equation approach

Leap-frog scheme:

\[ \frac{U^{n+1} - U^{n-1}}{2\Delta t} + A_h U^n = F^n_h, \]

Stability condition of this scheme:

\[ \Delta t \| A_h \|_2 \leq 1 \]

In absence of source, the exact solution is given by

\[ \frac{U^{n+1} - U^{n-1}}{2} = i \sin(i\Delta t A_h) U^n \]

Taylor expansion of the sinus provide the following scheme:

\[ \Delta t A_h + \sum_{q=1}^{m} \frac{(\Delta t A_h)^{2q+1}}{(2q + 1)!} U^n = 0. \]
Let us denote the polynomial:

\[ \tau_m(x) = x + \sum_{q=1}^{m} (-1)^q \frac{x^{2q+1}}{(2q + 1)!} \]

Stability is obtained if

\[ |\tau_m(x)| \leq 1 \iff x \in [0, \alpha_m] \]
Stability condition of modified equation

For $m$ even, \[ \alpha_m \leq \frac{3\pi}{2} \]
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Improvement of modified equation

Higher-order terms are added

\[
\frac{U^{n+1} - U^{n-1}}{2} + \left[ \sum_{q=0}^{m} \frac{(\Delta t A_h)^{2q+1}}{(2q + 1)!} \right] U^n + \left[ \sum_{q=m+1}^{r} \alpha_q (\Delta t A_h)^{2q+1} \right] U^n = 0
\]

This scheme is written under the form

\[
U^{n+1} - U^{n-1} + 2i \mathcal{T}_{2r+1}(i\Delta t A_h) U^n = 0
\]
Improvement of modified equation

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This scheme is written under the form

\[
U^{n+1} - U^{n-1} + 2i T_{2r+1}(i \Delta t A_h) U^n = 0
\]

Optimal polynomial for \( m = 0 \), and nearly optimal for \( m = 1 \)

\[
T_{2r+1}^m(x) = \frac{1}{\xi_r} T_{2r+1}^{\text{Cheb}} \left( \frac{(-1)^r \xi_r^m x}{2r + 1} \right)
\]

where \( T_{2r+1}^{\text{Cheb}} \) are Chebyshev polynomials of the first kind and

\[
\xi_0 = 1, \quad \xi_1 = \frac{2r + 1}{2\sqrt{r(r + 1)}},
\]
Improvement of modified equation

Optimal polynomial for $m = 0$, and nearly optimal for $m = 1$

$$T^m_{2r+1}(x) = \frac{1}{\xi_r} T^{Cheb}_{2r+1}\left(\frac{(-1)^r \xi^m_r x}{(2r + 1)}\right)$$

where $T^{Cheb}_{2r+1}$ are Chebyshev polynomials of the first kind and

$$\xi^0_r = 1, \quad \xi^1_r = \frac{2r + 1}{2\sqrt{r(r + 1)}},$$

Stability condition:

$$\Delta t \| A_h \|_2 \leq \frac{2r + 1}{\xi^m_r}$$

with

$$\xi^1_r = 1 + O\left(\frac{1}{r^2}\right)$$
Use of Horner algorithm leads to numerical instabilities for large values of $r$

Use of Chebyshev recurrence leads to stable algorithms:

\begin{align*}
Q_0 &= U^n \\
Q_1 &= \frac{\xi_r}{2r + 1} \Delta t A_h U^n \\
Q_n &= \frac{2\xi_r}{2r + 1} \Delta t A_h Q_{n-1} + Q_{n-2} \\
&\quad \vdots \\
U^{n+1} &= U^{n-1} - \frac{2}{\xi_r} Q_{2r+1}
\end{align*}
Two-level time stepping

Computational domain split into a “fine region” and a “coarse region”

\( Ph \): projector onto the fine region

\[
\begin{align*}
\frac{U^{n+1} - U^{n-1}}{2} + \left[ \sum_{q=0}^{m} \frac{(\Delta t A_h)^{2q+1}}{(2q + 1)!} \right] U^n \\
+ \left[ \sum_{q=m+1}^{r} \alpha_q (\Delta t A_h P_h)^{2q} \right] A_h U^n = 0,
\end{align*}
\]

Presence of \( Ph \) \(\Rightarrow\) terms of second sum are computed only on the “fine region”

Skew-symmetry of the matrices \( A_h P_h A_h \cdots A_h P_h A_h \) \(\Rightarrow\) stability of this scheme
Stable two-level algorithm

For $m = 0$, it is equivalent to the following scheme (obtained by reproducing the strategy of Diaz and Grote):

$$
\begin{align*}
  w_h &= A_h (I - P_h) U^n \\
  Q_0 &= U^n \\
  Q_1 &= -\frac{\Delta t}{2r + 1} (w_h + A_h P_h Q_0) \\
  \text{For } k = 1, 2r \\
  Q_{k+1} &= Q_{k-1} - \frac{2\Delta t}{2r + 1} (A_h P_h Q_k + w \delta_k \text{ even}) \\
  \text{End For} \\
  U^{n+1} &= U^{n-1} + 2Q_{2r+1}
\end{align*}
$$

Stable algorithm even for large values of $r$
Multilevel algorithm

Domain split into hierarchical subdomains

\[ \Omega = \bigcup \Omega_i = \bigcup K_e \]

with

\[ \Omega \supset \Omega_1 \supset \Omega_2 \cdots \supset \Omega_r \]
Multilevel algorithm

For each element, a nominal time step is computed

$$\Delta t_e = \frac{2r + 1}{\xi_r^m \| P_e A_h P_e \|_2}$$
by considering only direct neighbors of each element
Global time step $\Delta t$ is chosen by the user, then a level $i$ is affected to each element with respect to the rule:

$$\text{if } \Delta t_e \leq \frac{\xi_i m \Delta t}{2i + 1}, \text{ then } K_e \in \Omega_i.$$
Multilevel algorithm

We consider the following time scheme

\[
\frac{U^{n+1} - U^{n-1}}{2\Delta t} + A_h U^n + \Delta t^2 A_h P_1 A_h P_1 A_h U^n \\
+ \Delta t^4 A_h P_1 A_h P_2 A_h P_2 A_h P_1 A_h U^n \\
+ \Delta t^6 A_h P_1 A_h P_2 A_h P_3 A_h P_3 A_h P_2 A_h P_1 A_h U^n + \cdots = 0
\]

where \( P_k \) are diagonal matrices:

\[
P_k = \begin{pmatrix}
\beta_0^k \\
\vdots \\
\beta_1^k \\
\vdots \\
\beta_r^k
\end{pmatrix}
\]

with

\[
\beta_m^k = 0, \ \forall m < k
\]
Multilevel algorithm

If we write the expansion of optimal polynomial $\tau_{opt}^k(X)$ as:

$$\tau_{opt}^k(X) = X + \gamma_1^k X^3 + \gamma_2^k X^5 + \cdots + \gamma_k^k X^{2k+1}$$

Coefficients $\beta_k^m$ are chosen to coincide with these polynomials for each level

For $k = 1, r$

For $m = 1, k-1$

$$\beta_k^m = 0$$

End For

For $m = k, r$

$$\beta_k^m = \sqrt{\gamma_k^m}$$

For $n = 1, k-1$

$$\beta_k^m = \beta_k^m / \beta_n^m$$

End For

End For

End For
Use of Horner algorithm:

\[
Q_0 = \Delta t A_h U^n
\]
\[
Q_1 = \Delta t A_h P_1 Q_0
\]
\[
Q_2 = \Delta t A_h P_2 Q_1
\]
\[
\ldots
\]
\[
Q_r = \Delta t A_h P_r Q_{r-1}
\]
\[
Q_{r-1} = Q_{r-1} + \Delta t A_h P_r Q_r
\]
\[
Q_{r-2} = Q_{r-2} + \Delta t A_h P_{r-1} Q_{r-1}
\]
\[
\ldots
\]
\[
Q_0 = Q_0 + \Delta t A_h P_1 Q_1
\]
\[
U^{n+1} = U^{n-1} - 2Q_0
\]

unstable due to round-off errors when \( r \geq 14 \).
We consider wave equation

\[ A_i = \begin{pmatrix} 0 & e_i^* \\ e_i & 0 \end{pmatrix} \]

and Neumann boundary conditions so that \( A_h \) is skew-symmetric.
2-D numerical results

Box pierced with two small holes

![Image of a box pierced with two small holes]
2-D numerical results

Solution obtained for $t = 2$
2-D numerical results

Solution obtained for $t = 4$
2-D numerical results

Solution obtained for $t = 6$
Solution obtained for $t = 8$
2-D numerical results

\[ \Delta t_{\text{max}} = 0.01036, \quad \Delta t_{\text{min}} = 0.000737 \]

Ratio \[ \frac{\Delta t_{\text{max}}}{\Delta t_{\text{min}}} = 14.1 \]

Computational time with optimized fourth order (\( \Delta t = 0.005 \)): 767s
2-D numerical results

\[ \Delta t_{\text{max}} = 0.01036, \quad \Delta t_{\text{min}} = 0.000737 \]

Ratio \( \frac{\Delta t_{\text{max}}}{\Delta t_{\text{min}}} = 14.1 \)

Computational time with optimized fourth order (\( \Delta t = 0.005 \)): 767s

Fourth-order local time stepping with the following repartition:

<table>
<thead>
<tr>
<th>Level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>Number of elements</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>4</td>
</tr>
</tbody>
</table>

\( L^2 \) error for \( t = 10 \): 7.78e-6

Computational time (\( \Delta t = 0.01 \)): 177s
3-D numerical results

Scattering by a satellite
3-D numerical results

Mesh used for the simulations
3-D numerical results

Solution obtained for $t = 0.1$
3-D numerical results

Solution obtained for \( t = 0.2 \)
3-D numerical results

Solution obtained for $t = 0.3$
3-D numerical results

Solution obtained for $t = 0.4$
3-D numerical results

Solution obtained for $t = 0.5$
3-D numerical results

$\Delta t_{max} = 1.177e^{-3}$, $\Delta t_{min} = 1.442e^{-5}$

Ratio $\frac{\Delta t_{max}}{\Delta t_{min}} = 81.6$

Computational time with standard leap frog ($\Delta t = 1e^{-5}$): 63.4h
3-D numerical results

\[ \Delta t_{\text{max}} = 1.177e^{-3}, \quad \Delta t_{\text{min}} = 1.442e^{-5} \]

Ratio \[ \frac{\Delta t_{\text{max}}}{\Delta t_{\text{min}}} = 81.6 \]

Computational time with standard leap frog (\( \Delta t = 1e^{-5} \)): 63.4h

Second-order local time stepping with the following repartition:

<table>
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<th>Level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</thead>
<tbody>
<tr>
<td>Number of elements</td>
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<td>7629</td>
<td>867</td>
<td>35</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\( L^2 \) error for \( t = 0.5 \): 2.31e-3

Computational time (\( \Delta t = 2.5e^{-4} \)): 9.48h