Asymptotic Models for Electromagnetism Scattering :

from Strongly Absorbing Obstacles

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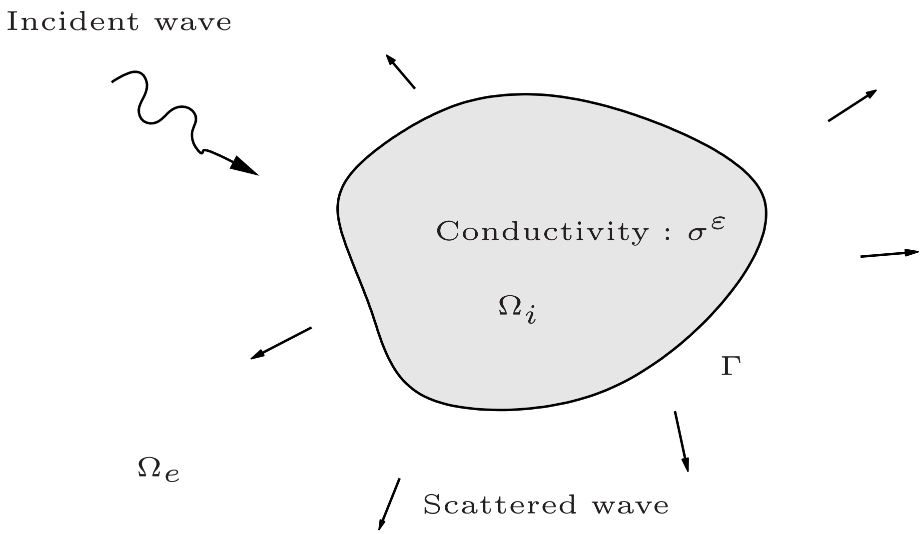
Project POEMS

Strongly Absorbing Media



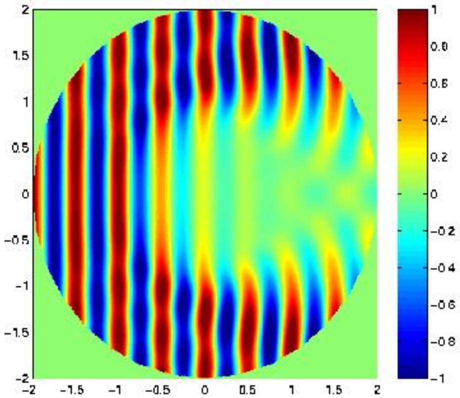
$$\sigma^\varepsilon(x) = 0, \text{ in } \Omega_e \,, \qquad\qquad \sigma^\varepsilon(x) = \sigma^\varepsilon > 0, \text{ in } \Omega_i \,.$$

We are interested in describing the solution behavior for large σ^ε :

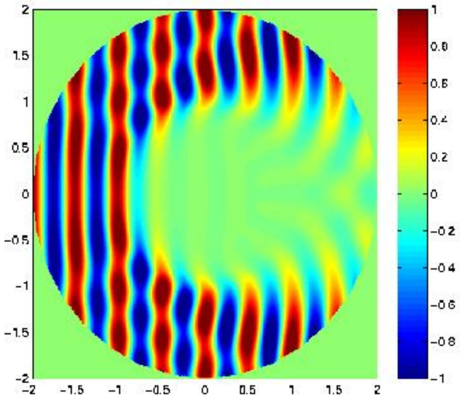
$$\varepsilon = \sqrt{1/(\omega\sigma^\varepsilon)} \Longleftrightarrow \sigma^\varepsilon = 1/(\omega\, \varepsilon^2) \,.$$

$\varepsilon \approx$ the width of the penetrable region in Ω_i (skin depth).

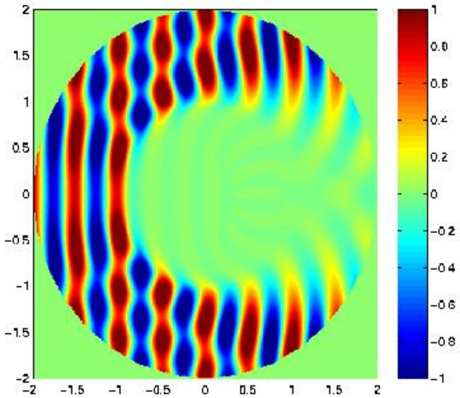
Numerical illustration of Skin layer effect



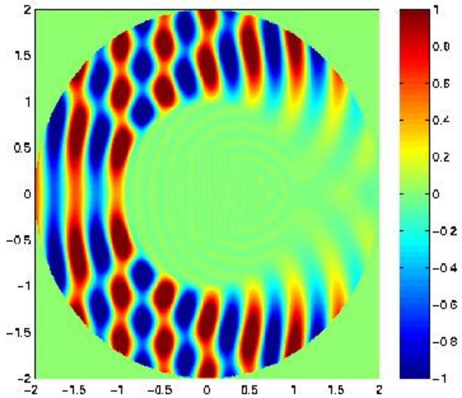
$\sigma = 45$



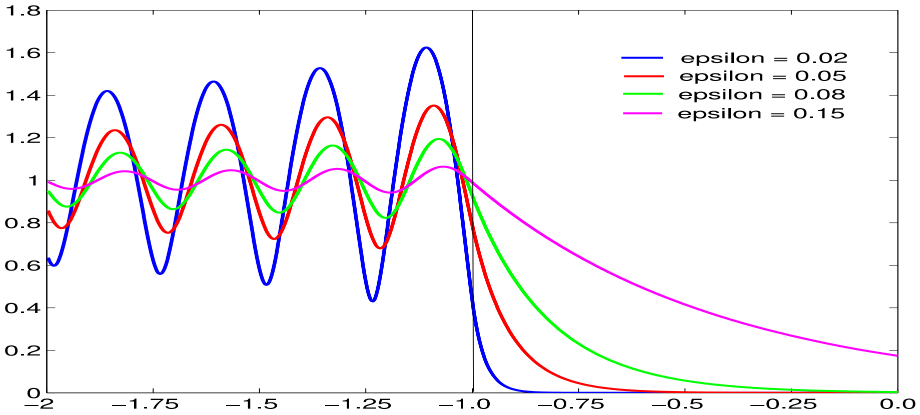
$\sigma = 156$



$\sigma = 400$

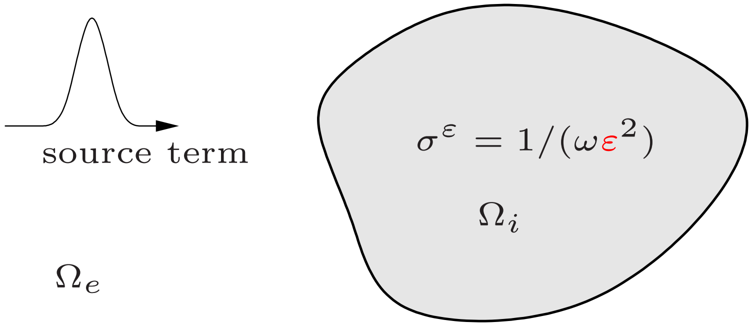


$\sigma = 2500$



Variation of $x_1 \mapsto |u(x_1, 0)|$

Theoretical Justification of Skin Effect



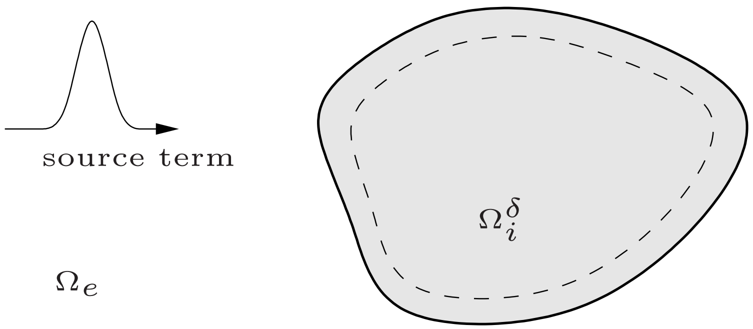
Consider a scalar wave of the form : $U^\varepsilon(x,t) = \mathcal{R}e\,(u^\varepsilon(x)\exp i\omega t)$.

$$\begin{cases} -\Delta u^\varepsilon - \omega^2 u^\varepsilon + i\omega\, \sigma^\varepsilon(x) u^\varepsilon = f\, \chi_{\Omega_e}, & \text{in } \mathbb{R}^n \\ +R.C. \end{cases}$$

Theorem 1 *There exits a constant $C > 0$ independent of ε such that*

$$\boxed{\|u^\varepsilon\|_{H^1(\Omega_i)} \leq C\,\varepsilon^{1/2}\,\|f\|_{L^2(\Omega)} \quad \textit{and} \quad \|u^\varepsilon\|_{L^2(\Omega_i)} \leq C\,\varepsilon^{3/2}\,\|f\|_{L^2(\Omega_i)}\,.}$$

Theoretical Justification of Skin Effect



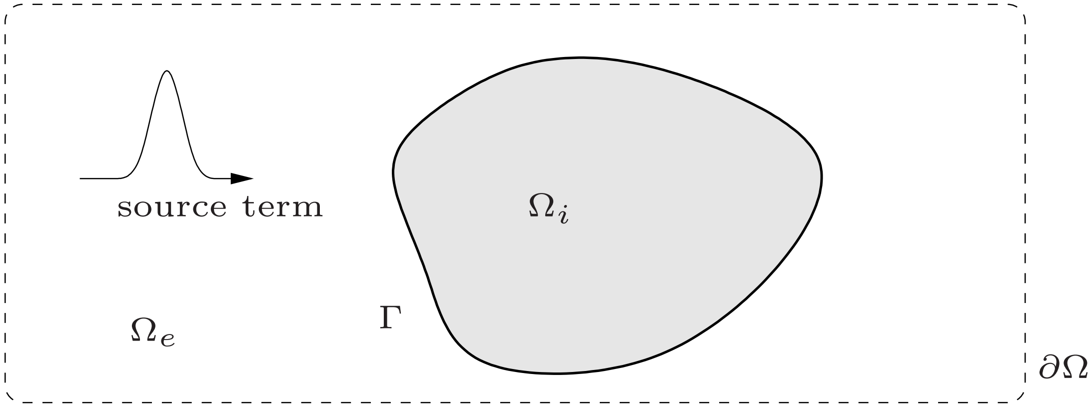
Consider a scalar wave of the form : $U^\varepsilon(x,t) = \mathcal{R}e\,(u^\varepsilon(x)\exp i\omega t)$.

$$\begin{cases} -\Delta u^\varepsilon - \omega^2 u^\varepsilon + i\omega\;\sigma^\varepsilon(x)u^\varepsilon = f\;\chi_{\Omega_e}, & \text{in } \mathbb{R}^n \\ +R.C. \end{cases}$$

Theorem 2 *Let $\delta > 0$ and define $\Omega_i^\delta = \{x \in \Omega_i; \mathrm{dist}(x, \partial\Omega_i) > \delta\}$, then there exist two positive constants C^δ and γ^δ independent of ε such that*

$$\boxed{\|u^\varepsilon\|_{H^1(\Omega_i^\delta)} \le C^\delta \exp\left(-\gamma^\delta/\varepsilon\right) \|f\|_{L^2}\,.}$$

Study of the scalar case



$$
(1) \qquad \begin{cases}
(*) \quad -\Delta u_e^\varepsilon - \omega^2 u_e^\varepsilon = f, & \text{in } \Omega_e, \\
\quad -\Delta u_i^\varepsilon - \omega^2 u_i^\varepsilon + \dfrac{i}{\varepsilon^2} u_i^\varepsilon = 0, & \text{in } \Omega_i, \\
(*) \quad u_e^\varepsilon = u_i^\varepsilon, & \text{on } \Gamma, \\
\quad \partial_n u_i^\varepsilon = \partial_n u_e^\varepsilon, & \text{on } \Gamma, \\
(*) \quad \partial_n u_e^\varepsilon + i\omega u_e^\varepsilon = 0, & \text{on } \partial\Omega
\end{cases}
$$

Exact Impedance Boundary Conditions

Exact non local Neumann-to-Dirichlet (DtN) b.c.

$$(2) \qquad u_e^\varepsilon + \mathcal{D}^\varepsilon \partial_n u_e^\varepsilon = 0, \qquad \text{on } \Gamma,$$

where $\mathcal{D}^\varepsilon : H^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma)$ is the boundary operator defined by :

$$\mathcal{D}^\varepsilon \varphi = u_i^\varepsilon,$$

where u_i^ε is the unique solution of the interior boundary value problem :

$$(3) \qquad \begin{cases} -\Delta u_i^\varepsilon - \omega^2 u_i^\varepsilon + \dfrac{i}{\varepsilon^2} u_i^\varepsilon = 0 & \text{in } \Omega_i, \\ -\partial_n u_i^\varepsilon = \varphi & \text{on } \Gamma. \end{cases}$$

$$(4) \qquad \forall\, \varphi \in H^{-\frac{1}{2}}(\Gamma), \quad \mathcal{I}m \, \langle\, \mathcal{D}^\varepsilon \varphi, \varphi \rangle_\Gamma \;=\; -\frac{1}{\varepsilon^2} \int_{\Omega_i} |u_i^\varepsilon(\varphi)|^2 \, dx \leq 0.$$

Skin-effect : as $\varepsilon \to 0$, this operator becomes “almost local” (even differential)

Neumann-to-Dirichlet GIBC's

DEFINITION :

• A *NtD GIBC* is a boundary condition of the type

$$u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n u^{\varepsilon,k} = 0, \qquad \text{on } \Gamma, \tag{5}$$

where $\mathcal{D}^{\varepsilon,k}$ is a (suitable) local approximation of $\mathcal{D}^{\varepsilon}$.

•• The GIBC is of order k if the solution $u^{\varepsilon,k}$ to

$$\begin{cases} -\Delta u^{\varepsilon,k} - \omega^2 u^{\varepsilon,k} = f & \text{in } \Omega_e, \\ u^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\partial_n u^{\varepsilon,k} = 0, & \text{on } \Gamma, \\ \partial_n u^{\varepsilon,k} + i\omega u^{\varepsilon,k} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

is an approximation of u^{ε} up to $O(\varepsilon^{k+1})$ error.

Expression of NtD GIBC's

setting : $\sqrt{i} := \frac{\sqrt{2}}{2}(1+i)$,

$$\textbf{For k} = \mathbf{0}, \qquad \mathcal{D}^{\varepsilon,0} = 0,$$

$$\textbf{For k} = \mathbf{1}, \qquad \mathcal{D}^{\varepsilon,1} = \frac{\varepsilon}{\sqrt{i}},$$

$$\textbf{For k} = \mathbf{2}, \qquad \mathcal{D}^{\varepsilon,2} = \frac{\varepsilon}{\sqrt{i}} + i\mathcal{H}\varepsilon^2,$$

$$\textbf{For k} = \mathbf{3}, \qquad \mathcal{D}^{\varepsilon,3} = \frac{\varepsilon}{\sqrt{i}} + i\mathcal{H}\varepsilon^2 - \frac{\sqrt{i}\ \varepsilon^3}{2}\left(3\mathcal{H}^2 - G + \omega^2 + \Delta_\Gamma\right),$$

Theorem 3 *Let $k = 0, 1, 2$ or 3, then, for sufficiently small ε, the boundary value problem (6) has a unique solution $u^{\varepsilon,k} \in H^1(\Omega_e)$. Moreover, there exists a constant C_k, independent of ε, such that*

$$\|u_e^\varepsilon - u^{\varepsilon,k}\|_{H^1(\Omega_e)} \leq C_k\ \varepsilon^{k+1}. \tag{7}$$

Dirichlet-to-Neumann GIBC's

Exact DtN b.c. : we introduce $\mathcal{N}^\varepsilon := (\mathcal{D}^\varepsilon)^{-1}$,

$$\partial_n u_e^\varepsilon + \mathcal{N}^\varepsilon u_e^\varepsilon = 0, \quad \text{on } \Gamma, \tag{8}$$

DtN $GIBC$ is of the form : $\boxed{\partial_n u^{\varepsilon,k} + \mathcal{N}^{\varepsilon,k} u^{\varepsilon,k} = 0, \quad \text{on } \Gamma,}$

where $\mathcal{N}^{\varepsilon,k}$ denotes some local approximation of $\mathcal{N}^\varepsilon$.

"Natural conditions" do not satisfy in general : $\mathcal{D}^{\varepsilon,k} = \left(\mathcal{N}^{\varepsilon,k}\right)^{-1}$

However, they formally satisfy : $\boxed{\mathcal{D}^{\varepsilon,k} = \left(\mathcal{N}^{\varepsilon,k}\right)^{-1} + O(\varepsilon^{k+1}).}$

$$\textbf{For } \mathbf{k = 2}, \qquad \mathcal{N}^{\varepsilon,2} = \frac{\sqrt{i}}{\varepsilon} + \mathcal{H},$$

$$\textbf{For } \mathbf{k = 3}, \qquad \mathcal{N}^{\varepsilon,3} = \frac{\sqrt{i}}{\varepsilon} + \mathcal{H} - \frac{\varepsilon}{2\sqrt{i}}(\Delta_\Gamma + \mathcal{H}^2 - G + \omega^2).$$

Theorem 4 *Theorem 3 holds for DtN GIBCs.*

Some history on GIBC's...

Impedance Boundary Conditions : the original idea goes to Leontovich (1930) where he replaced the earth by a boundary condition.

Rytov (1940) : asymptotic analysis of the problem (formal based on high frequency asymptotics ansatz)

Senior-Volakis (1995)

Antoine-Barucq-Vernhet (2001) : microlocal expansions of pseudo-differential operators

Haddar-Joly-Nguyen (2005) : we revisited this "old problem" having in mind three goals :

- Try to build GIBC using "standard" scaling techniques as in the case of thin coatings
- Give rigorous mathematical justification of GIBC's order
- Consider the case of corners (under study...)

Formal Construction of GIBC

We introduce for sufficiently small δ (independent of ε)

$$\Omega_i^\delta = \{x \in \Omega_i \ ; \ \mathrm{dist}(x, \partial\Omega_i) < \delta\},$$

$$\left.\begin{array}{ccc} \Omega_i^\delta & \longleftrightarrow & \Gamma \times (0, \delta) \\ x & \longleftrightarrow & (x_\Gamma, \nu) \end{array}\right\} : \qquad x = x_\Gamma + \nu\, n$$

Ansatz

$$u_e^\varepsilon(x) = u_e^0(x) + \varepsilon u_e^1(x) + \varepsilon^2 u_e^2(x) + \cdots, \qquad \text{for } x \in \Omega_e,$$

$$u_i^\varepsilon(x) = u_i^0(x_\Gamma, \nu/\varepsilon) + \varepsilon u_i^1(x_\Gamma, \nu/\varepsilon) + \varepsilon^2 u_i^2(x_\Gamma, \nu/\varepsilon) + \cdots, \qquad \text{for } x \in \Omega_i^\delta$$

$u_i^\ell(x_\Gamma, \eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ are such that

$$\lim_{\eta\to\infty} u_i^\ell(x_\Gamma, \eta) = 0 \quad \text{for a.e. } x_\Gamma \in \Gamma.$$

Construction of GIBC's

The $GIBC$ of order k is obtained by considering the truncated expansion :

$$\widetilde{u}_e^{\varepsilon,k} := \sum_{\ell=0}^{k} \varepsilon^\ell u_e^\ell, \tag{9}$$

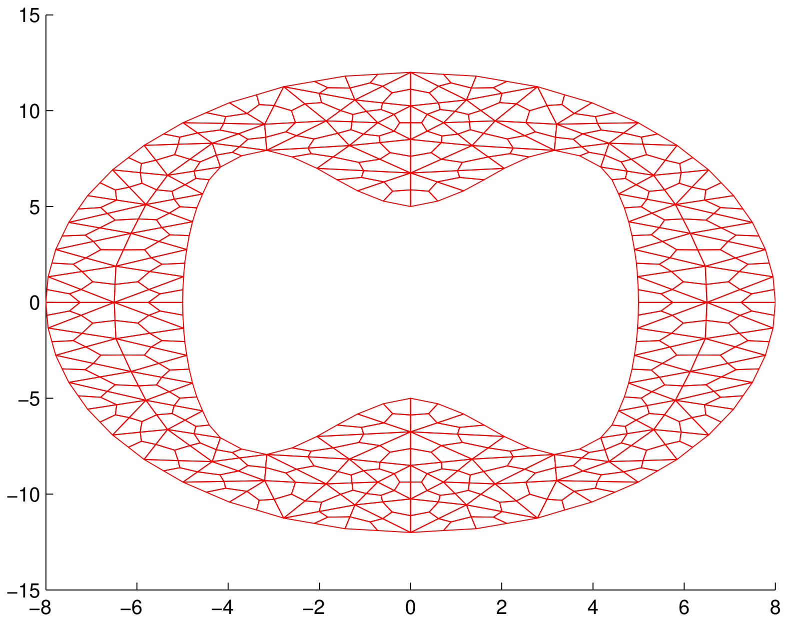
Substituting $u_i^\ell(x_\Gamma, 0)$ by their expression leads to a boundary condition of the form

$$\widetilde{u}_e^{\varepsilon,k} + \mathcal{D}^{\varepsilon,k}\, \partial_n \widetilde{u}_e^{\varepsilon,k} = \varepsilon^{k+1}\, g_k^\varepsilon \quad \text{on } \Gamma, \quad \text{with } g_k^\varepsilon = O(1),$$

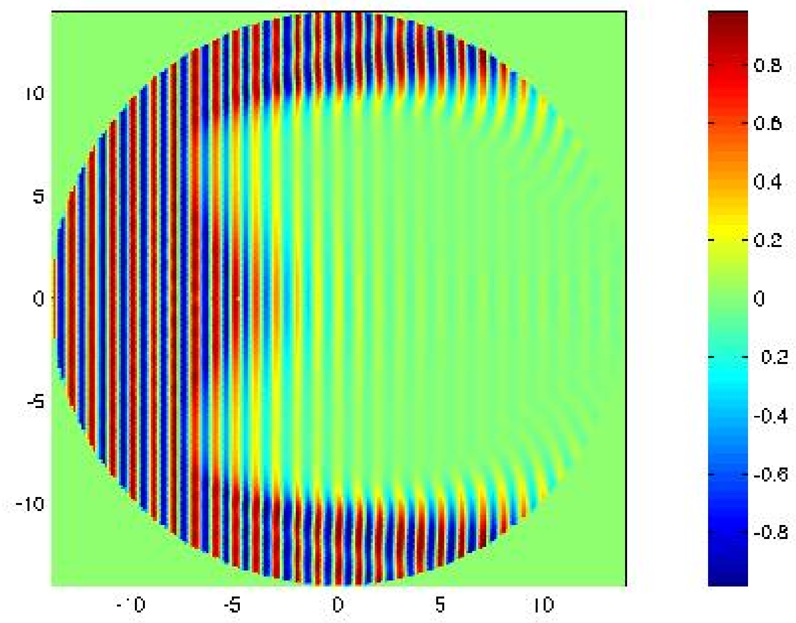
The GIBC of order k that defines $u^{\varepsilon,k}$ is then obtained by neglecting the right hand side.

Numerical validation

Peanut C^1 geometry



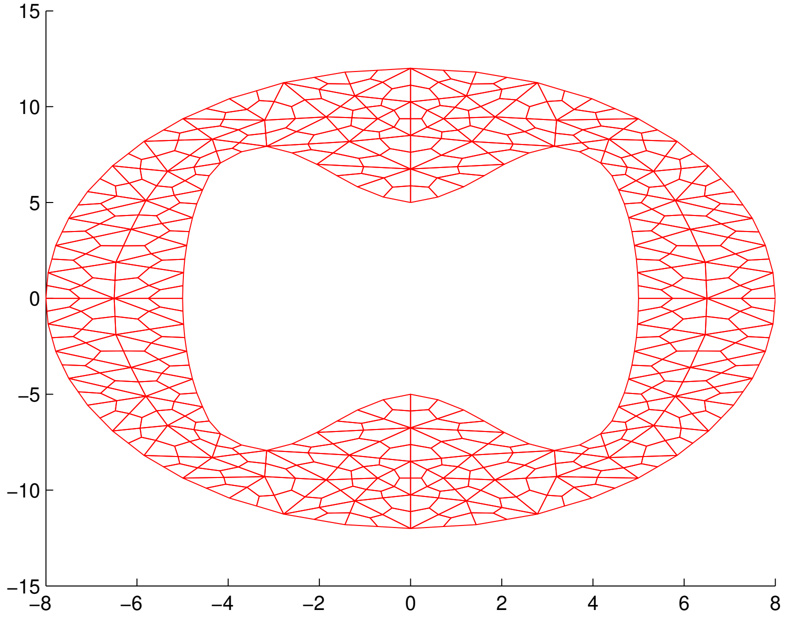
Used mesh



$\omega = 2\pi$

Numerical validation

Peanut C^1 geometry



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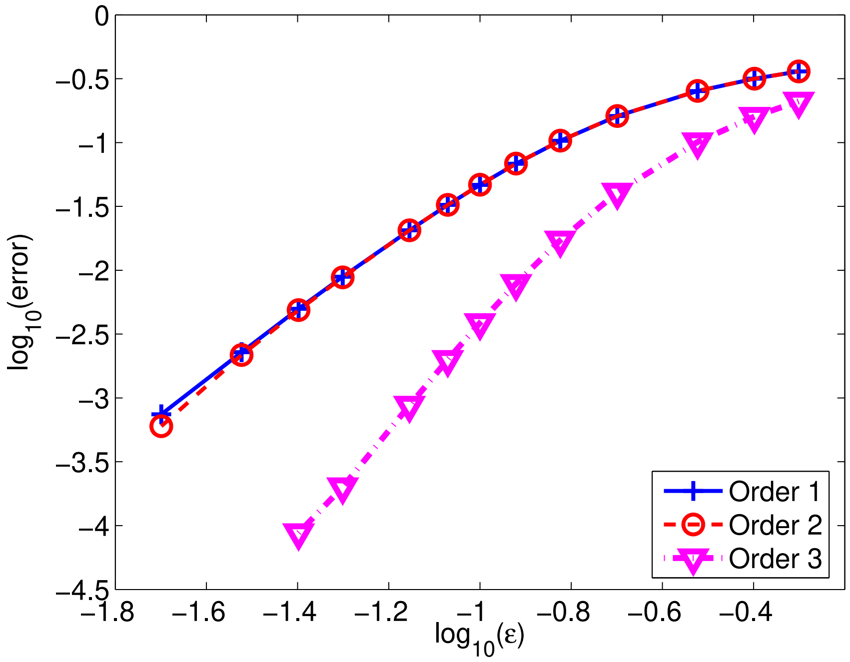

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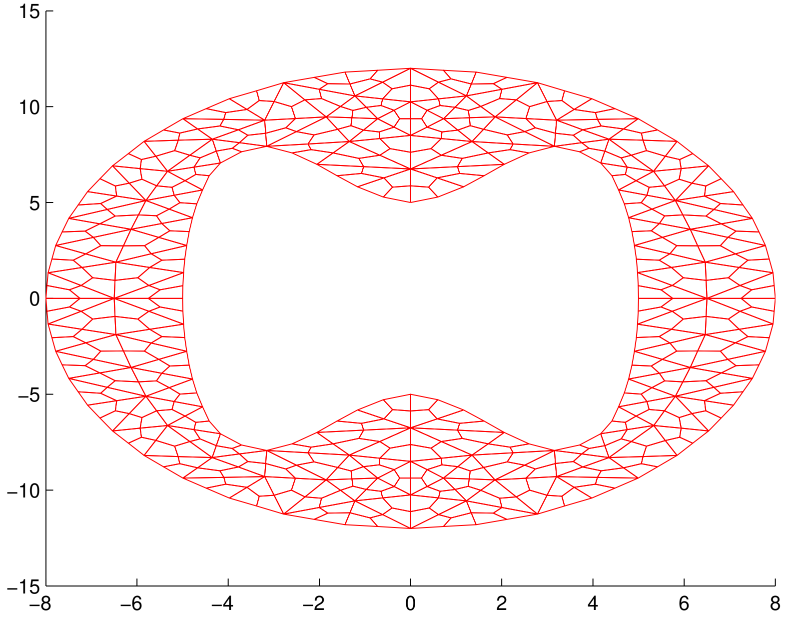

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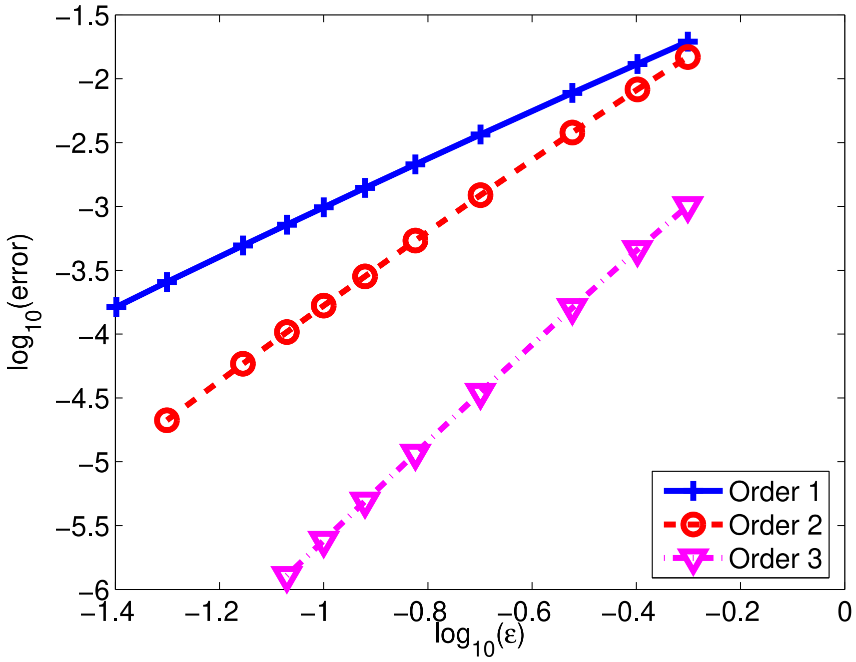
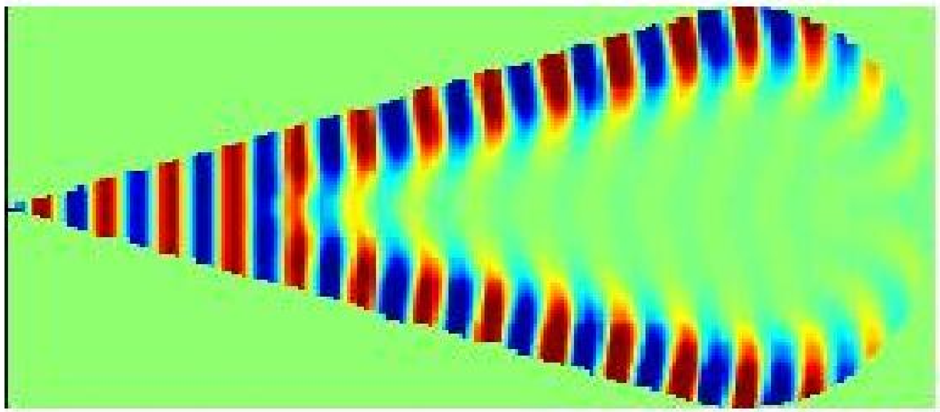


$\omega = 0.2\pi$

The case of corners

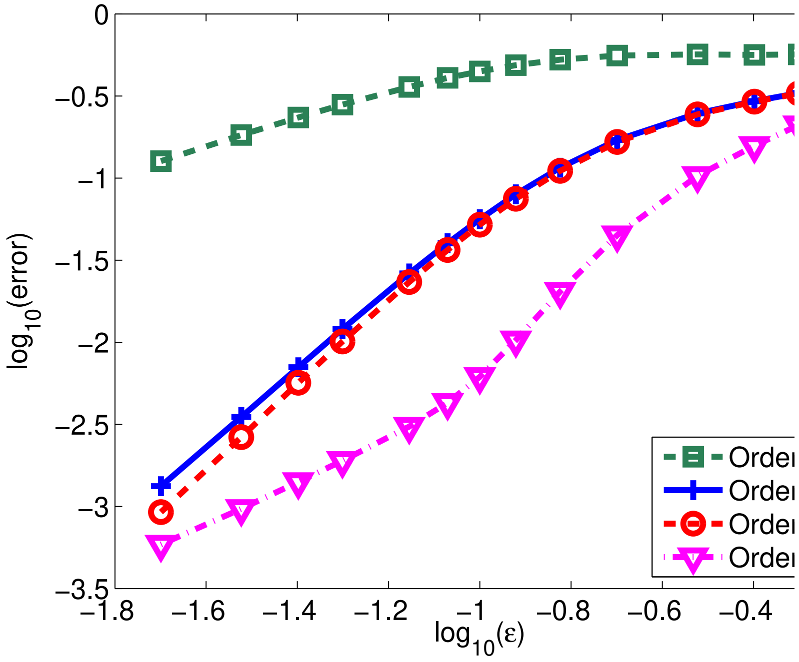
The diedron-disk geometry : $\omega = 2\pi$

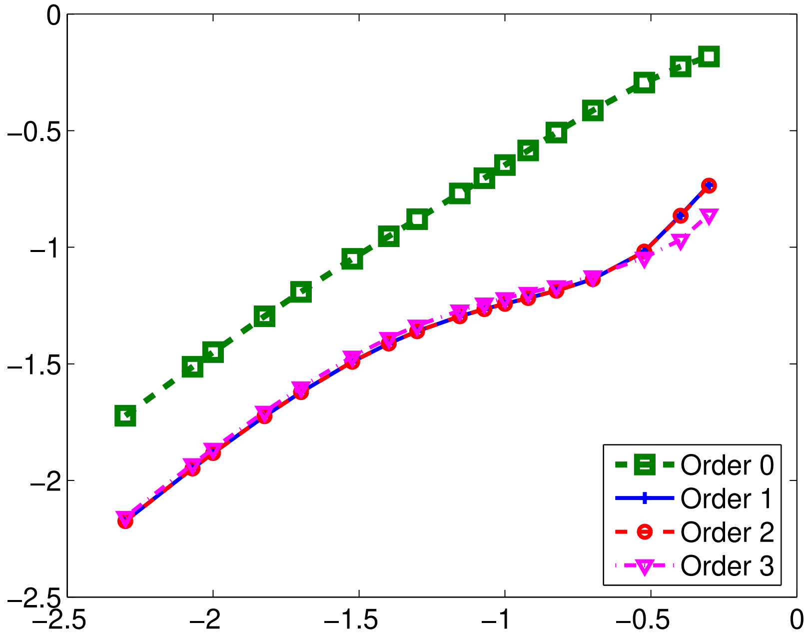


Left going incident wave

The case of corners

The diedron-disk geometry : $\omega = 2\pi$





Right going incident wave

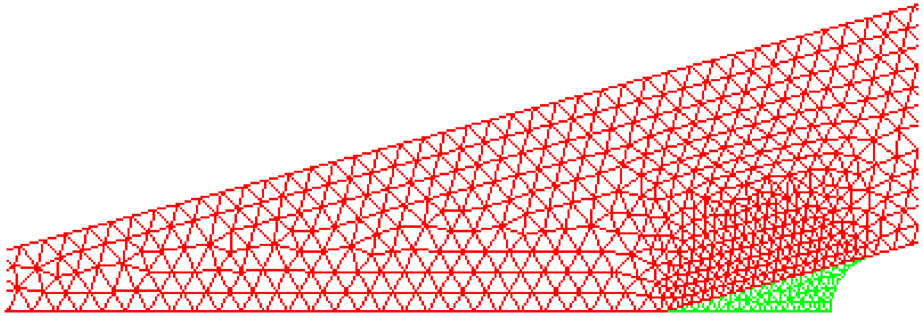
Left going incident wave

Problem : the ansatz is not correct at the corner region (phd thesis of G. Vial).

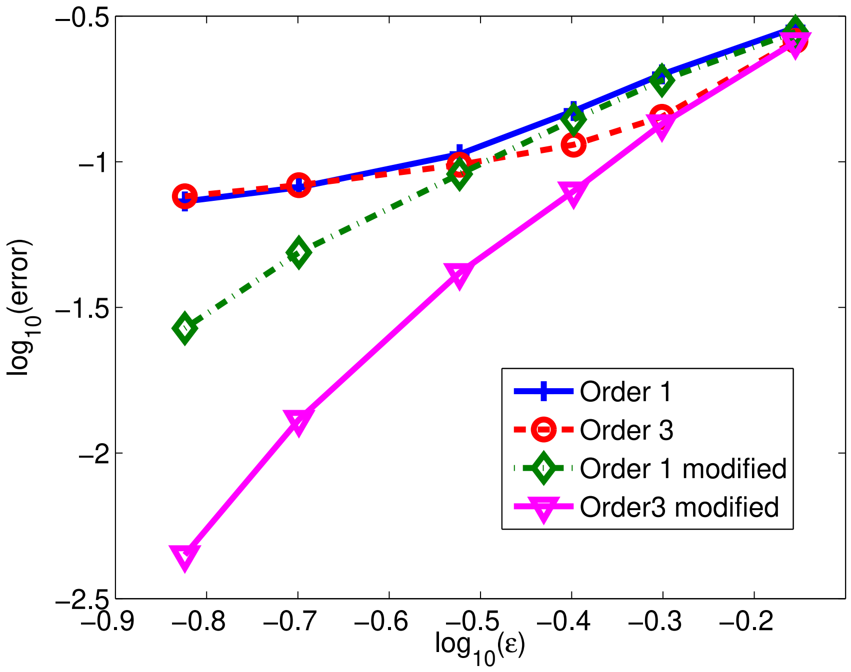
Open problem : find a suitable corner condition

Numerical treatment of corners

Idea : Let us mesh the absorbing media near the corner



Mesh of the diedron-disk's tip



Error according to ε for modified order 1 and order

Problem : the mesh has to be refined near the corner

Numerical Difficulty : singular point at the transition dielectric-empty

The Maxwell equations

$$\boxed{E^{\varepsilon,k} \times n \; + \; \omega \, \mathcal{D}^{\varepsilon,k}(H_T^{\varepsilon,k}) = 0, \quad \text{on } \Gamma,}$$

For k = 0, $\mathcal{D}^{\varepsilon,0} = 0,$

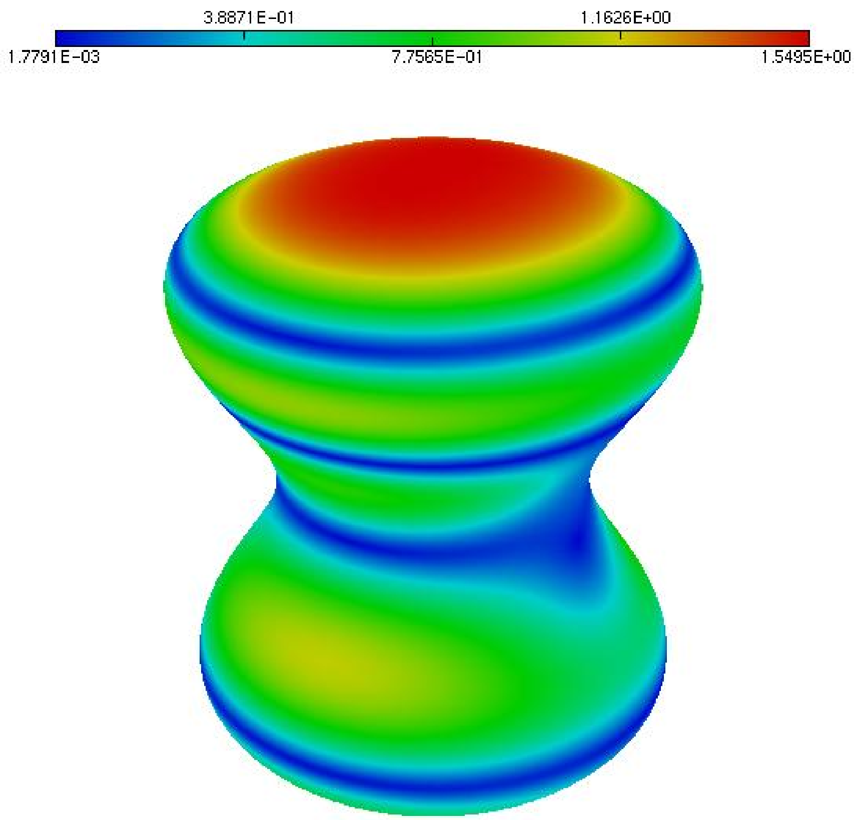
For k = 1, $\mathcal{D}^{\varepsilon,1} = \varepsilon\sqrt{i},$

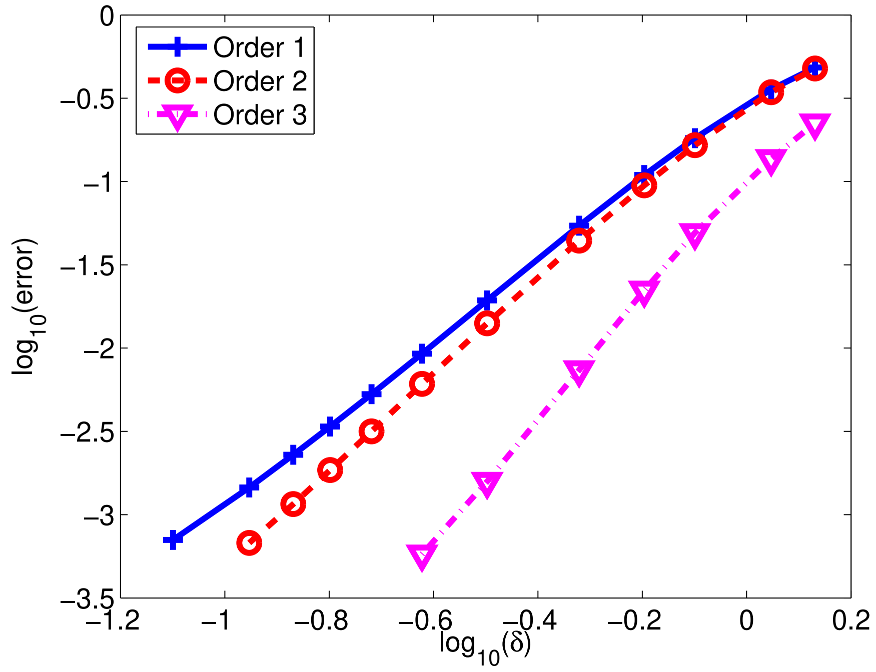
For k = 2, $\mathcal{D}^{\varepsilon,2} = \varepsilon\sqrt{i} + \varepsilon^2(\mathcal{H} - \mathcal{C}),$

For k = 3, $\mathcal{D}^{\varepsilon,3} = \varepsilon\sqrt{i} + \varepsilon^2(\mathcal{H} - \mathcal{C}) + \dfrac{\varepsilon^3}{2\sqrt{i}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\right)$

Numerical validation

Peanut C^1 geometry



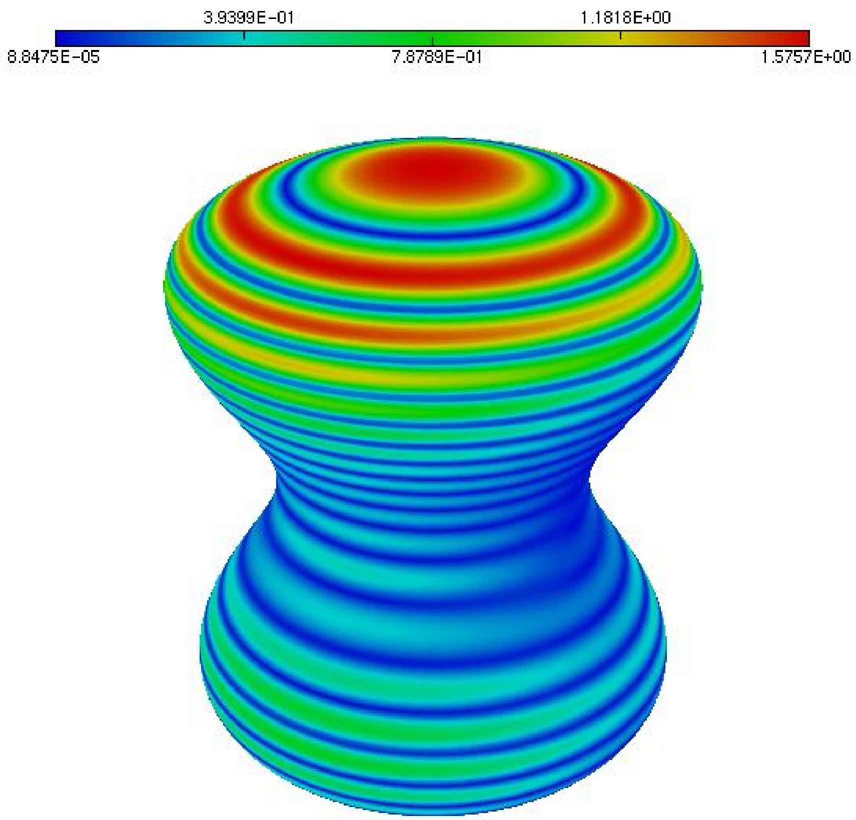


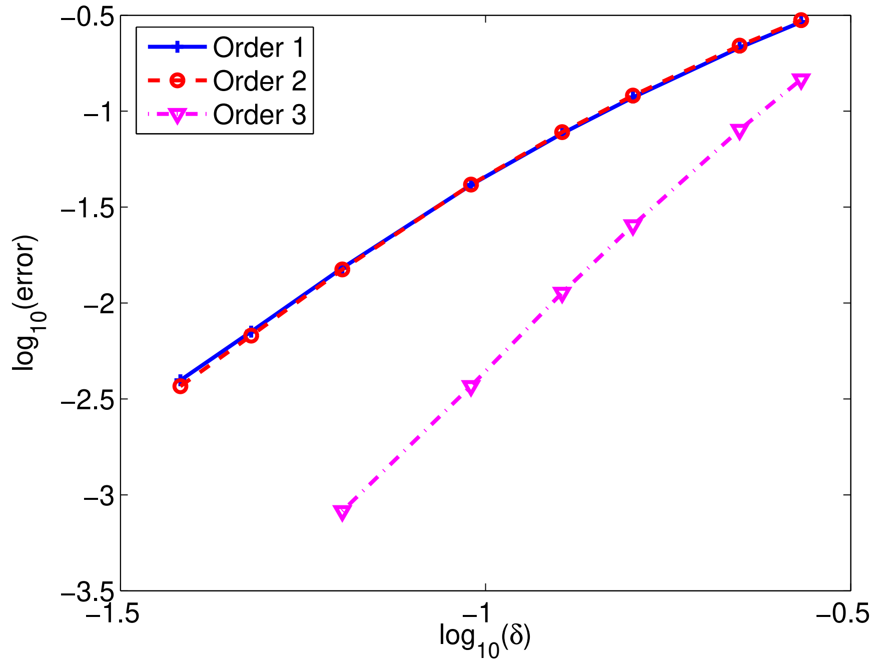
$\omega = 0.2\pi$

Solution for $\varepsilon = 0.1$

Numerical validation

Peanut C^1 geometry





$\omega = \pi$

Solution for $\varepsilon = 0.1$