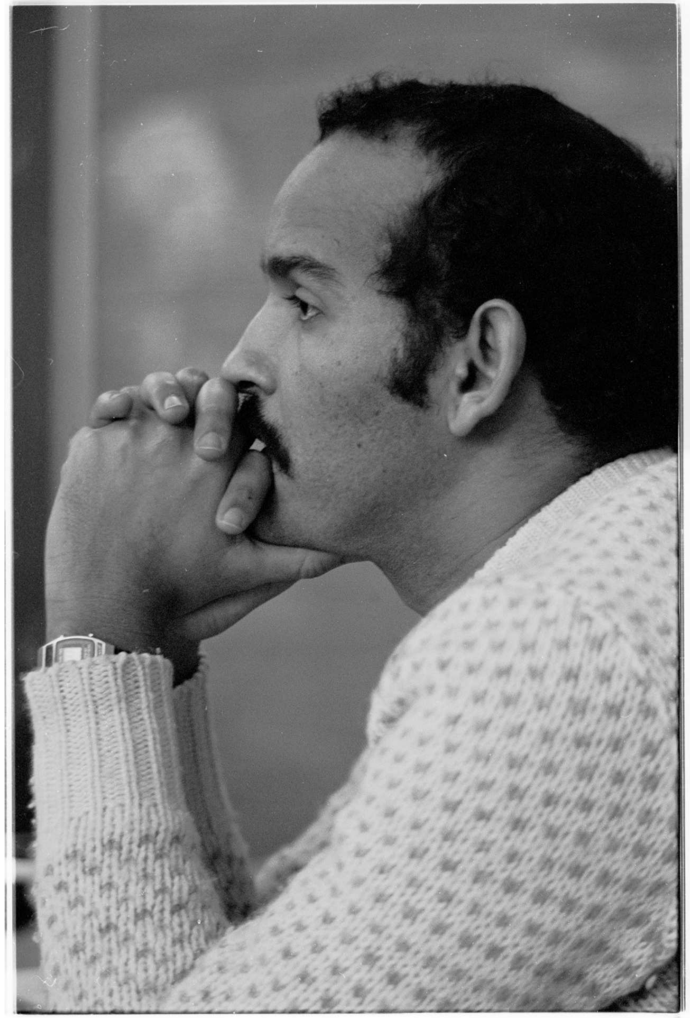
THE CACCETTA–HÄGGKVIST CONJECTURE

Adrian Bondy

Journée en hommage à Yahya Hamidoune

UPMC, Paris, March 29, 2011



Yahya Hamidoune

CAGES

(d,g)-Cage: smallest d-regular graph of girth g

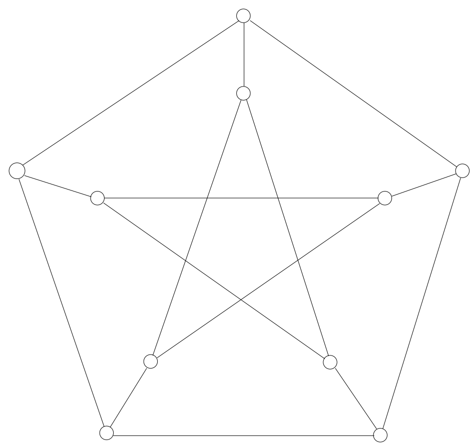
Lower bound on order of a (d,g)-cage:

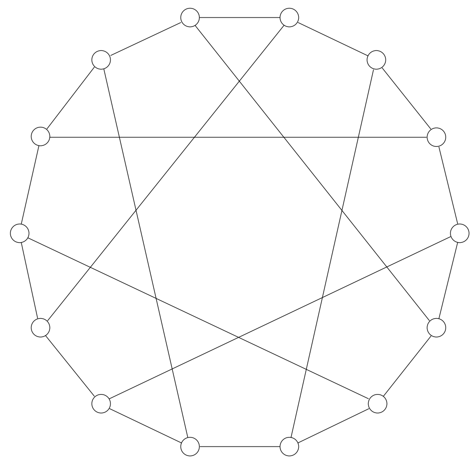
girth $g = 2r$ order $\frac{2(d-1)^r-2}{d-2}$

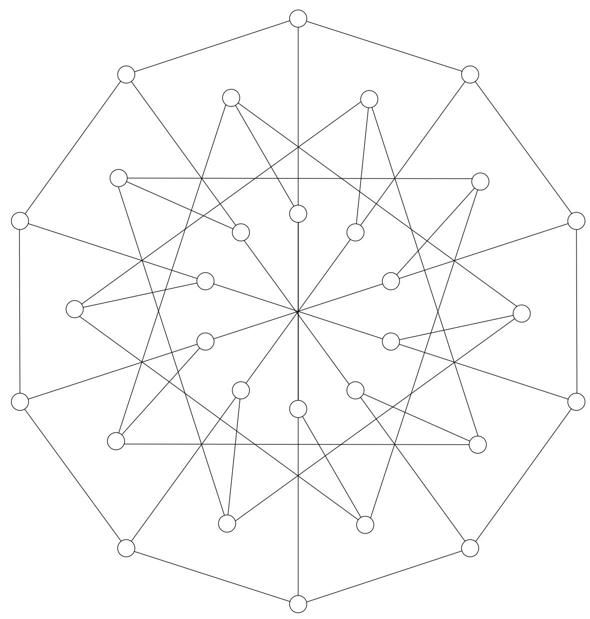
girth $g = 2r+1$ order $\frac{d(d-1)^r-2}{d-2}$

Examples with equality:

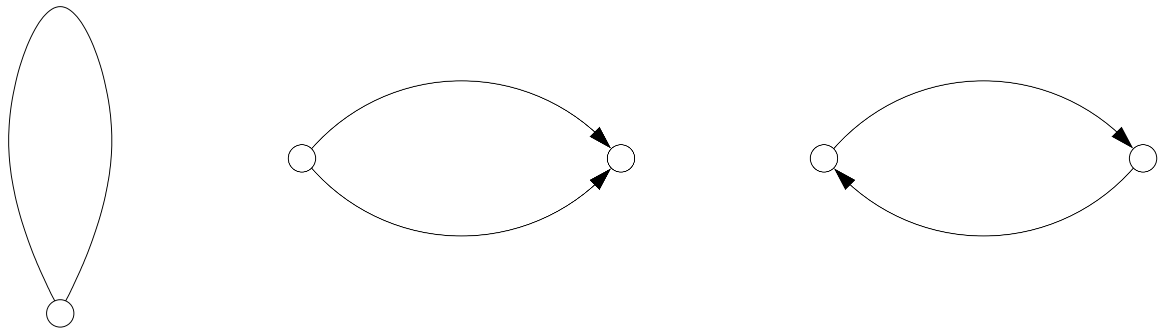
Petersen, Heawood, Coxeter-Tutte, Hoffman-Singleton . . .







Oriented Graph: no loops, parallel arcs or directed 2-cycles



We consider only oriented graphs.

DIRECTED CAGES

Directed (d, g)-cage:

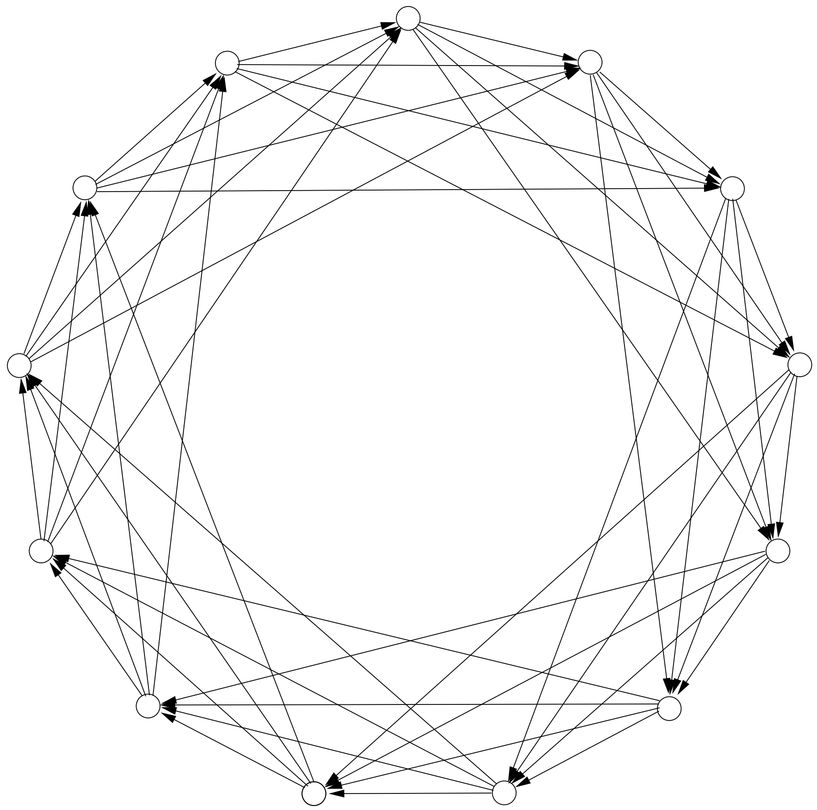
smallest d-diregular digraph of directed girth g

Behzad-Chartrand-Wall Conjecture 1970

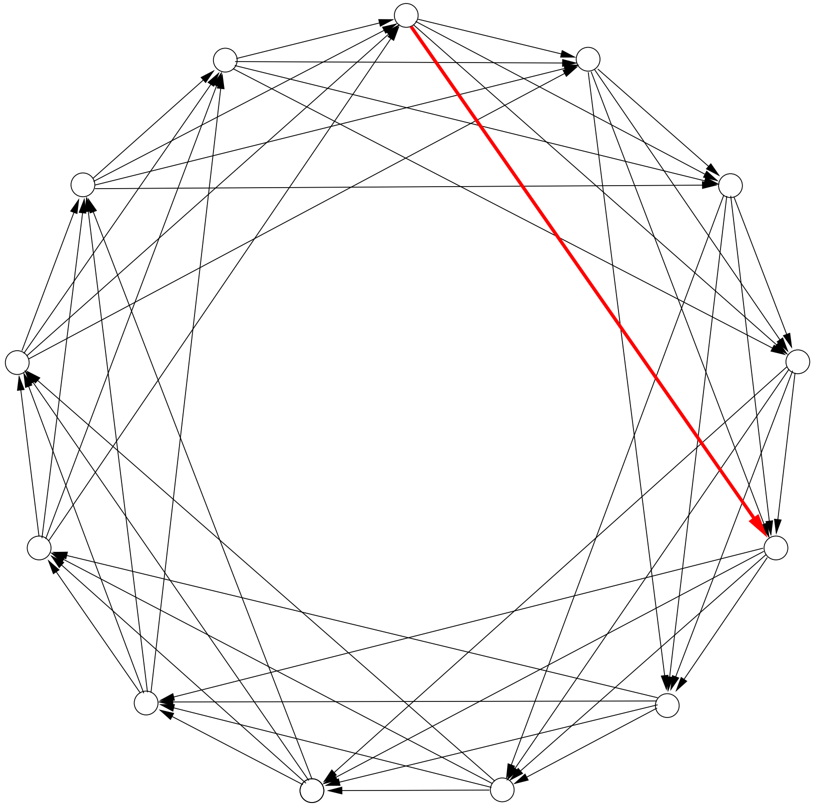
For all $d \geq 1$ and $g \geq 2$, the order of a directed (d, g)-cage is $d(g-1)+1$

Example:

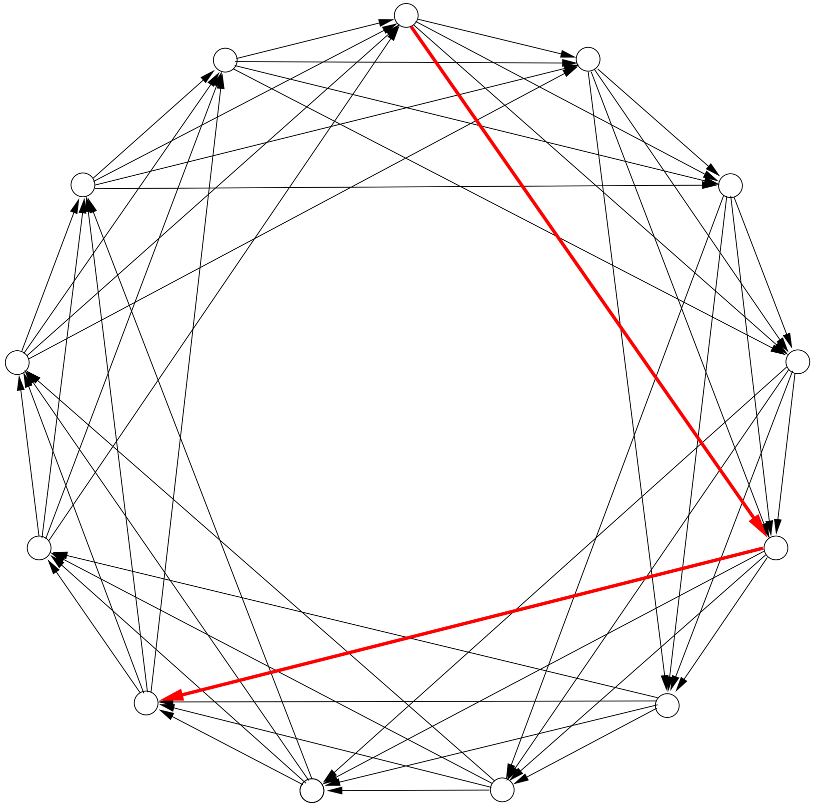
the dth power of a directed cycle of length $d(g-1)+1$



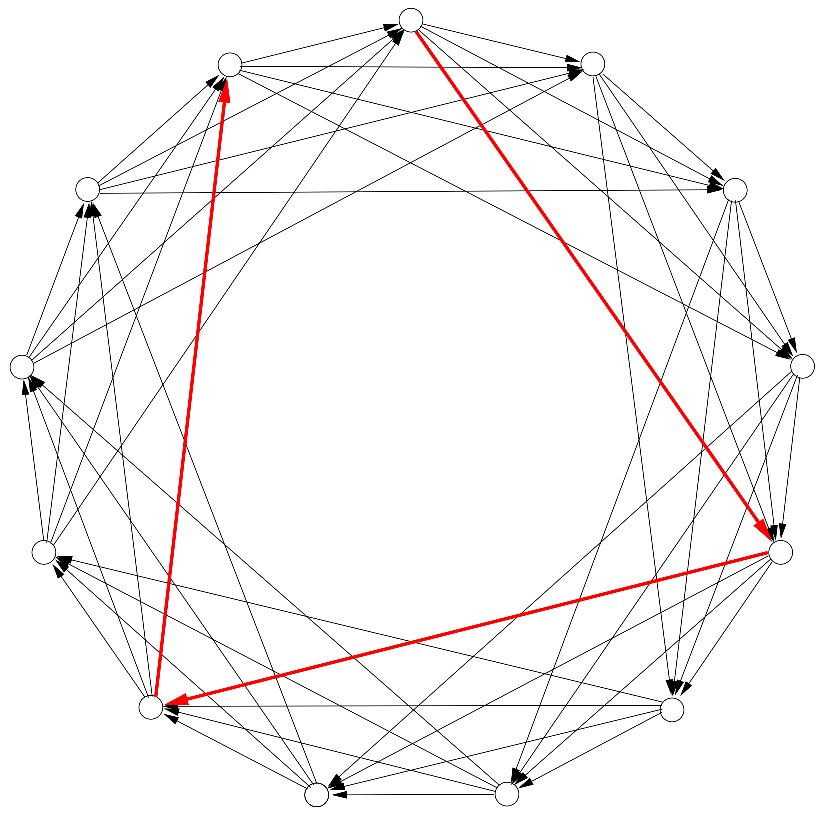
Conjectured directed (4, 4)-cage



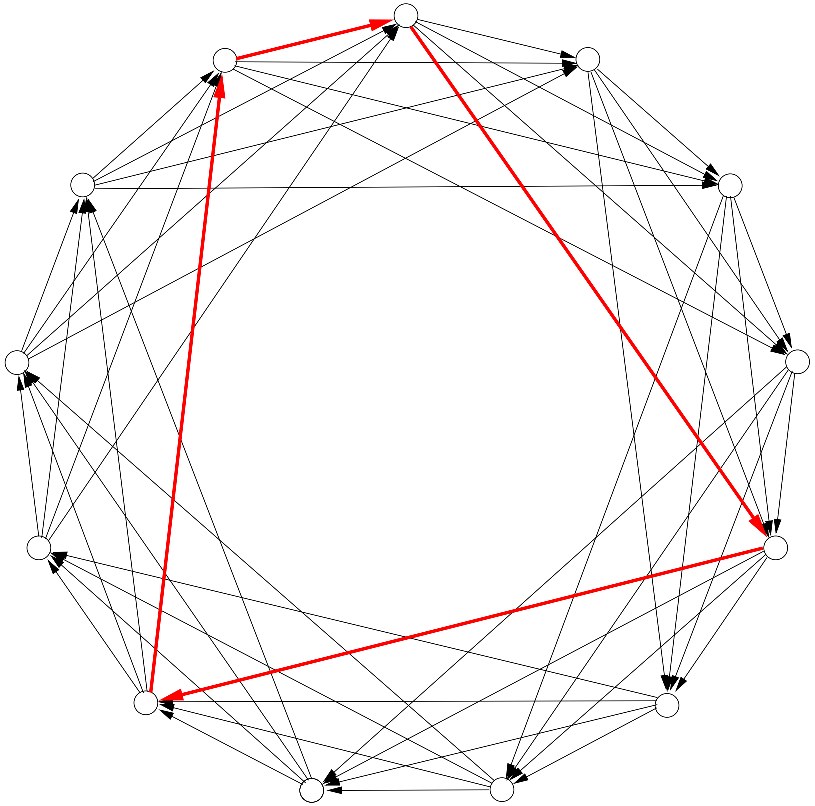
Conjectured directed $(4,4)$-cage



CONJECTURED DIRECTED $(4,4)$-CAGE



Conjectured directed $(4, 4)$-cage

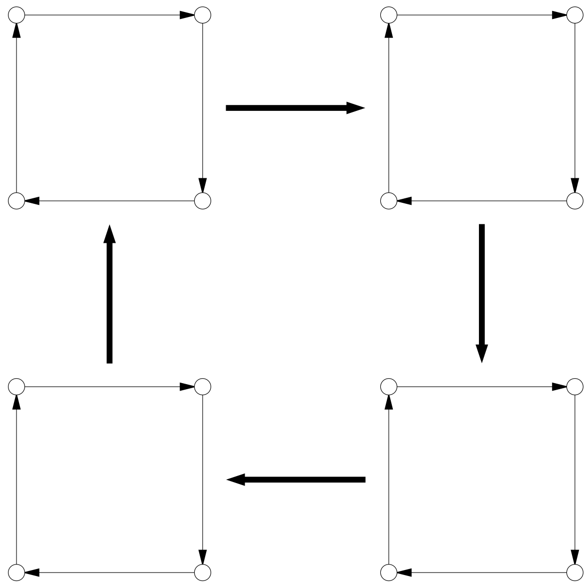


Conjectured directed $(4,4)$-cage

COMPOSITIONS

The Behzad-Chartrand-Wall Conjecture would imply that *directed cages of girth g are closed under composition*.

EXAMPLE: $g = 4$



CONJECTURED DIRECTED $(5, 4)$-CAGE

Reformulation:

Behzad-Chartrand-Wall Conjecture 1970

Every d-diregular digraph on n vertices has a directed cycle of length at most $\lceil n/d \rceil$

VERTEX-TRANSITIVE GRAPHS

HAMIDOUNE:

The Behzad-Chartrand-Wall Conjecture is true for vertex-transitive digraphs.

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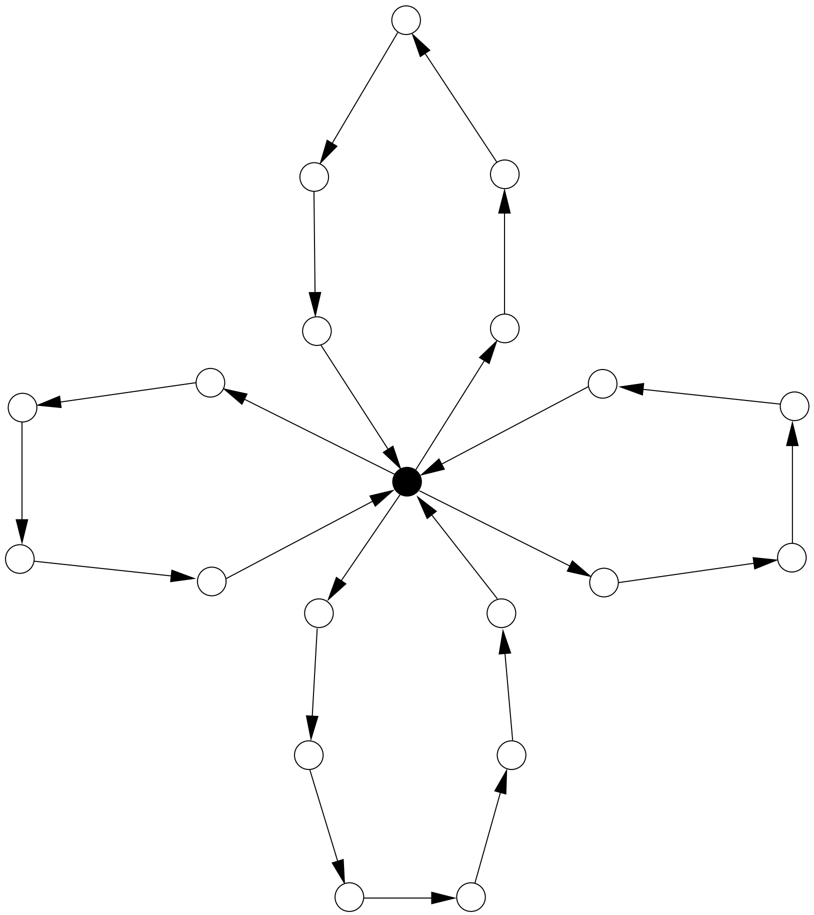
The Behzad-Chartrand-Wall Conjecture is true for vertex-transitive digraphs.

MADER:

In a d-diregular vertex-transitive digraph, there are d directed cycles $C_1, \ldots, C_d$ passing through a common vertex, any two meeting only in that vertex.

VERTEX-TRANSITIVE GRAPHS



So

$$\sum_{i=1}^{d} |V(C_i)| \leq n + d - 1$$

One of the cycles C_i is therefore of length at most

$$\frac{n + d - 1}{d} = \left\lceil \frac{n}{d} \right\rceil$$

Thus the Behzad-Chartrand-Wall Conjecture is true for vertex-transitive graphs.

VERTEX-TRANSITIVE GRAPHS

MADER:

In a d-diregular vertex-transitive digraph, there are d directed cycles $C_1, \ldots, C_d$ *passing through a common vertex, any two meeting only in that vertex.*

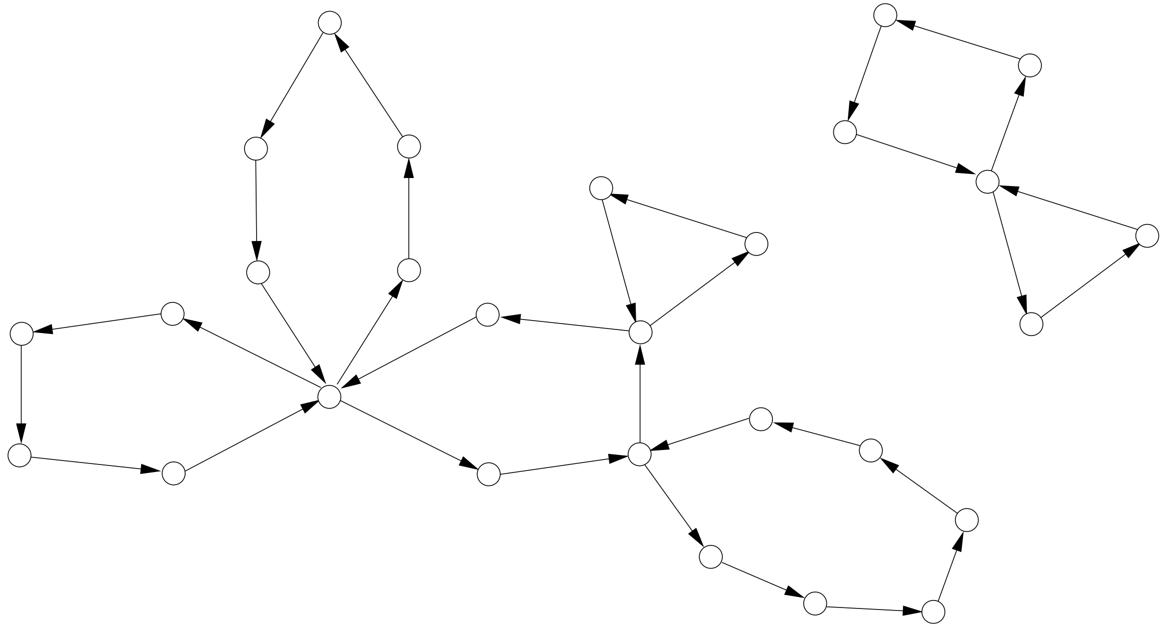
HAMIDOUNE:

Short proof of Mader's theorem.

NEARLY DISJOINT DIRECTED CYCLES

Hoáng-Reed Conjecture 1987

In a d-diregular digraph, there are d directed cycles $C_1, \ldots, C_d$ such that C_j meets $\cup_{i=1}^{j-1} C_i$ in at most one vertex, $1 < j \leq d$



FOREST OF d DIRECTED CYCLES

As before, one of these cycles would be of length at most $\lceil \frac{n}{d} \rceil$.

QUESTION: Is there a star of such cycles?

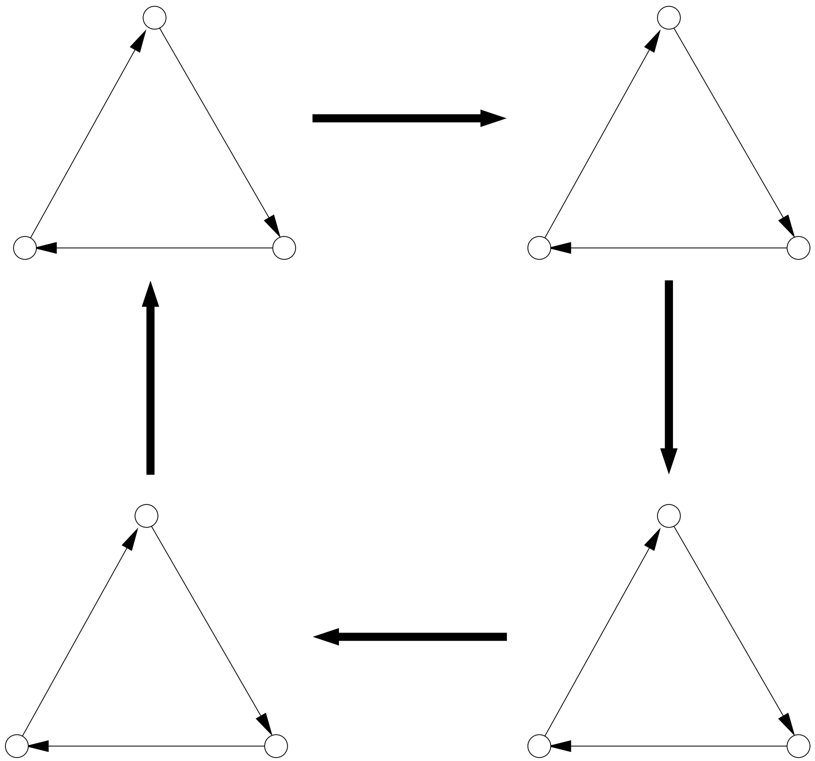
MADER: Not if $d \geq 8$

QUESTION: Is there a linear forest of such cycles?

MADER: NO

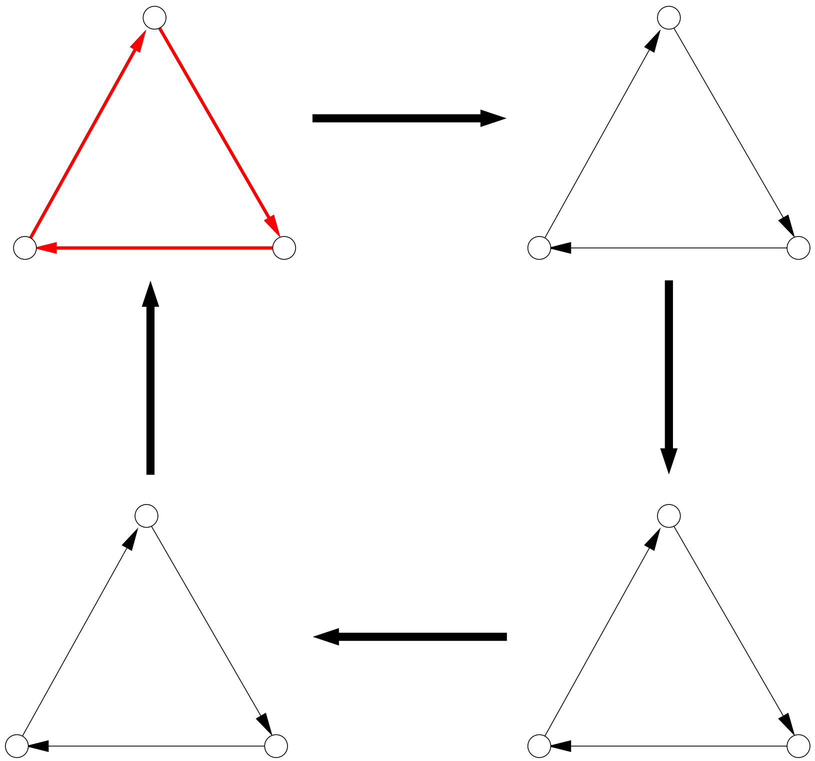
The composition $\overrightarrow{C_d}[\overrightarrow{C_{d-1}}]$ *is* d*-regular (in fact, vertex-transitive) but contains no path of* d *directed cycles*, each cycle (but the first) meeting the preceding one in exactly one vertex.

$d = 4$



NO PATH OF FOUR DIRECTED CYCLES

$d = 4$



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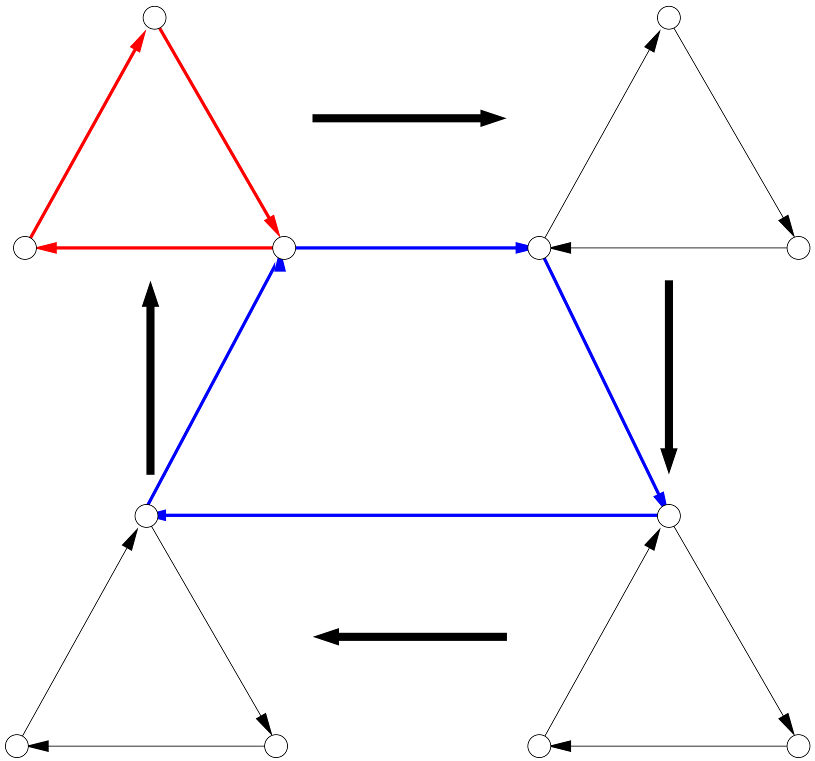


NO PATH OF FOUR DIRECTED CYCLES

PRESCRIBED MINIMUM OUTDEGREE

Caccetta-Häggkvist Conjecture 1978

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $\lceil n/d \rceil$

Caccetta and Häggkvist: $d = 2$

Hamidoune: $d = 3$

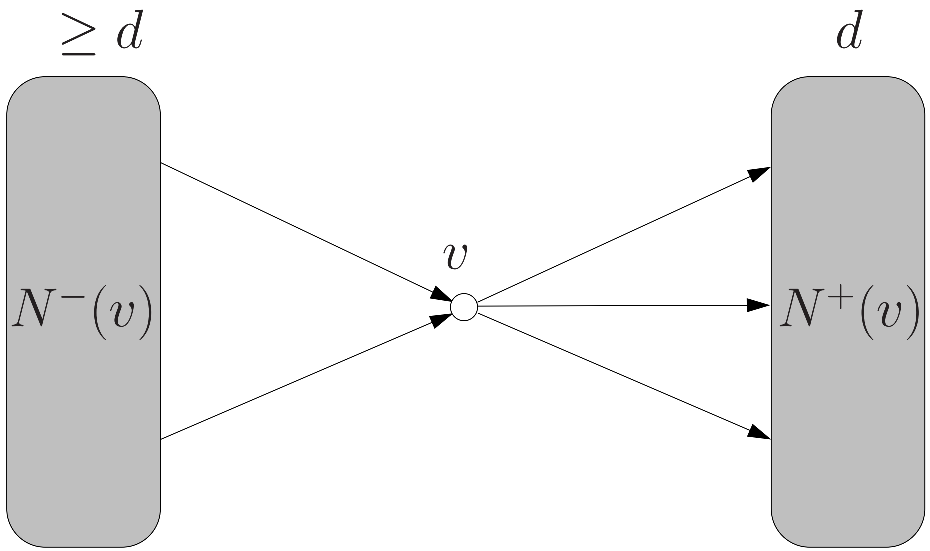
Hoáng and Reed: $d = 4, 5$

Shen: $d \leq \sqrt{n/2}$

Chvátal and Szemerédi:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $2n/d$.

Proof by Induction on n:



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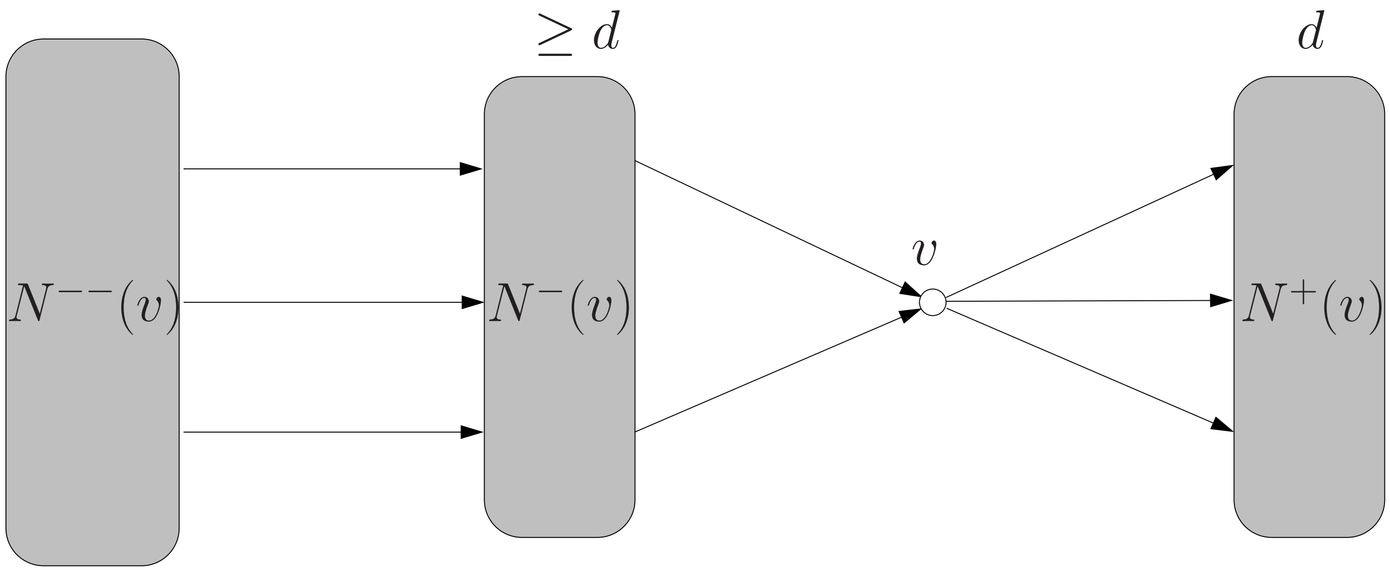

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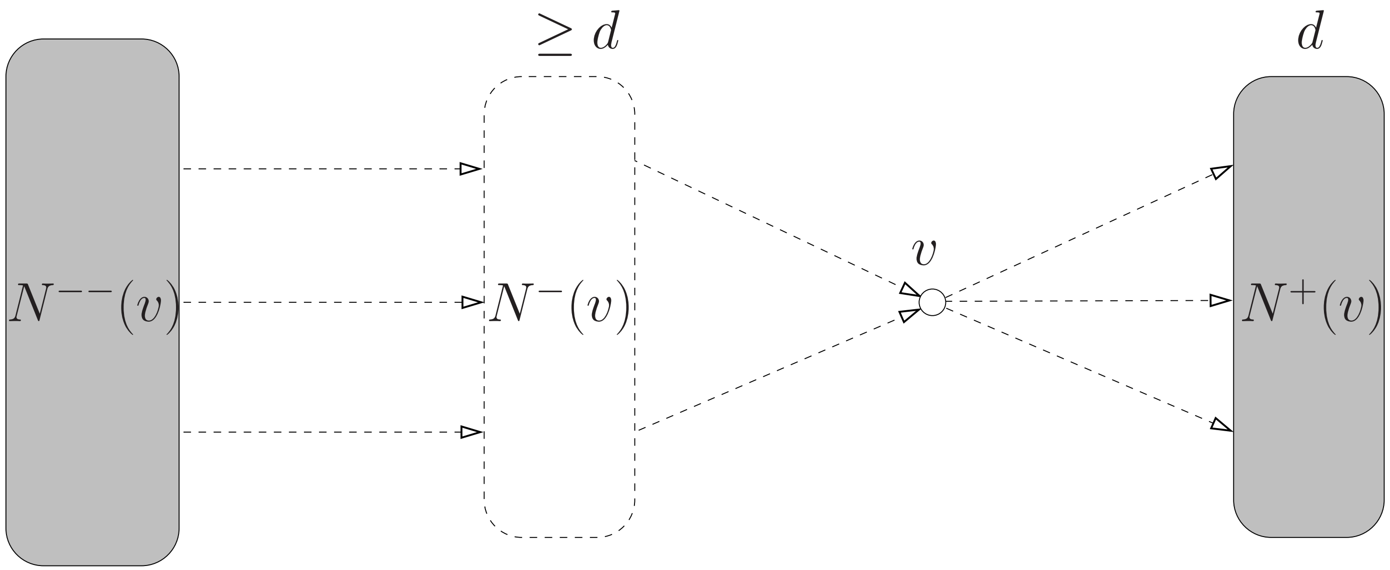

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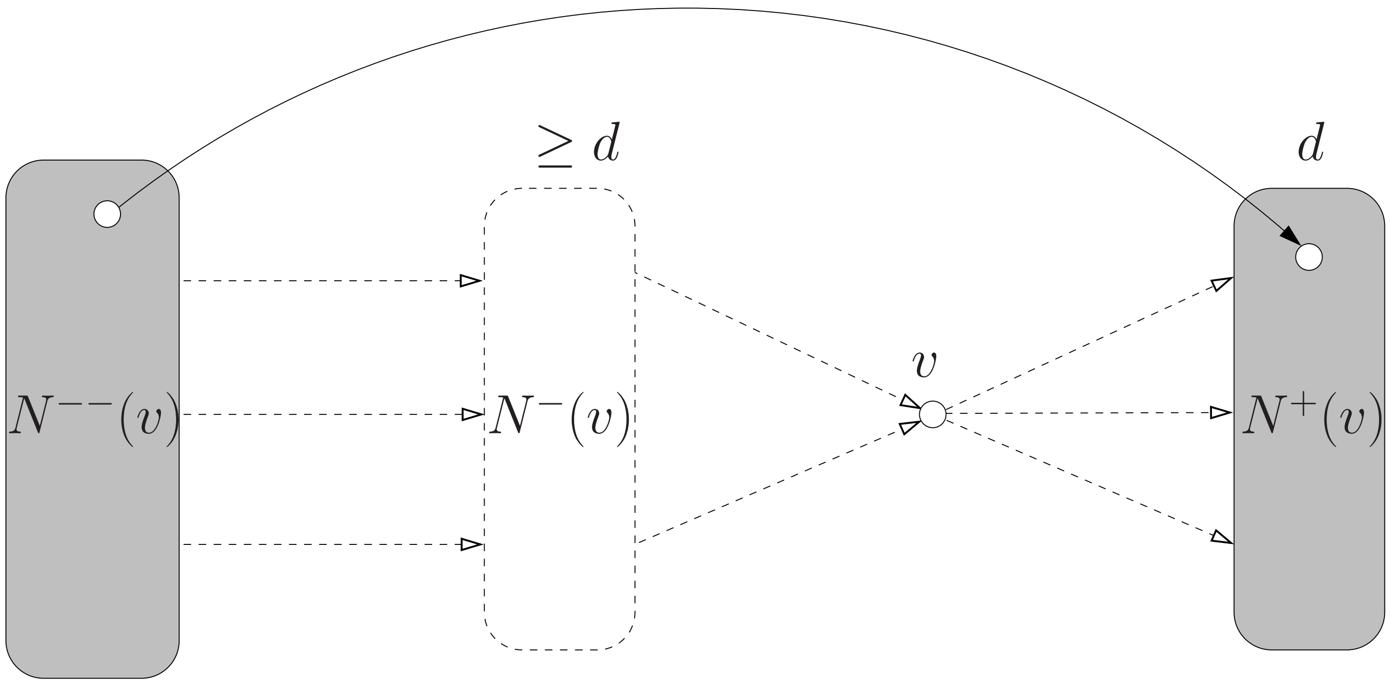

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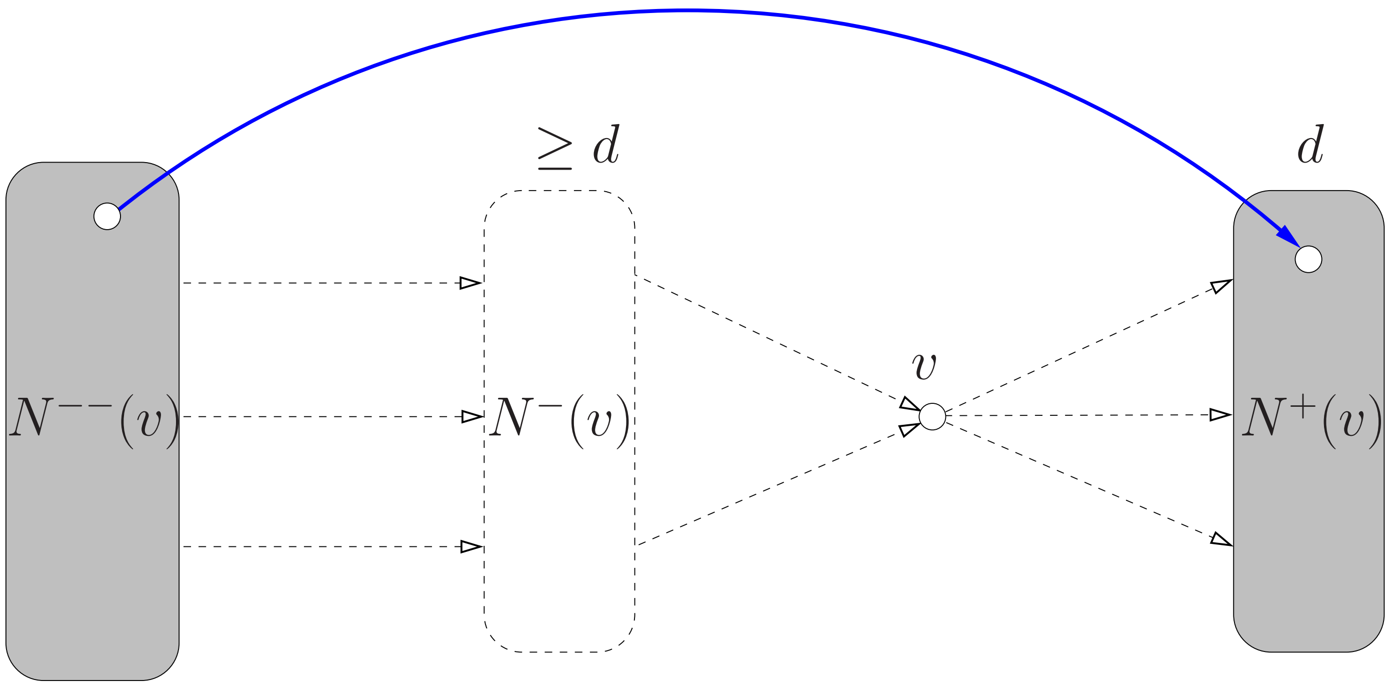

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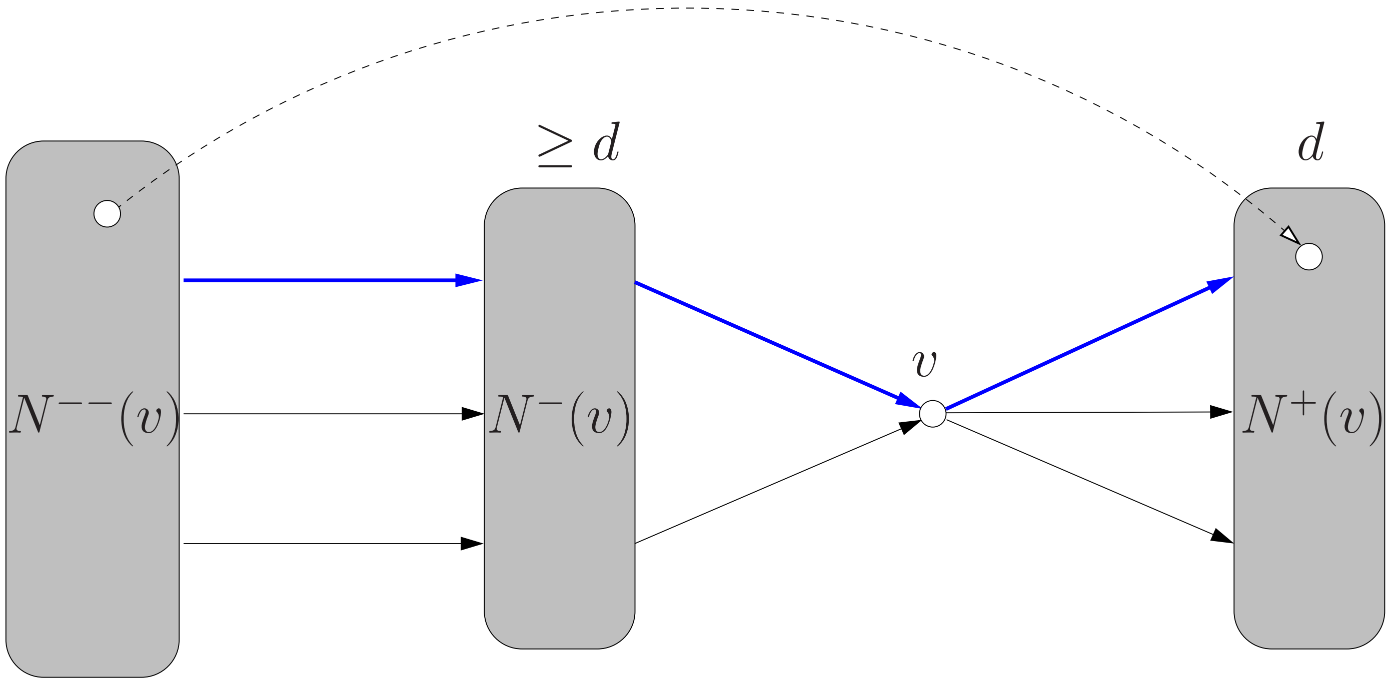
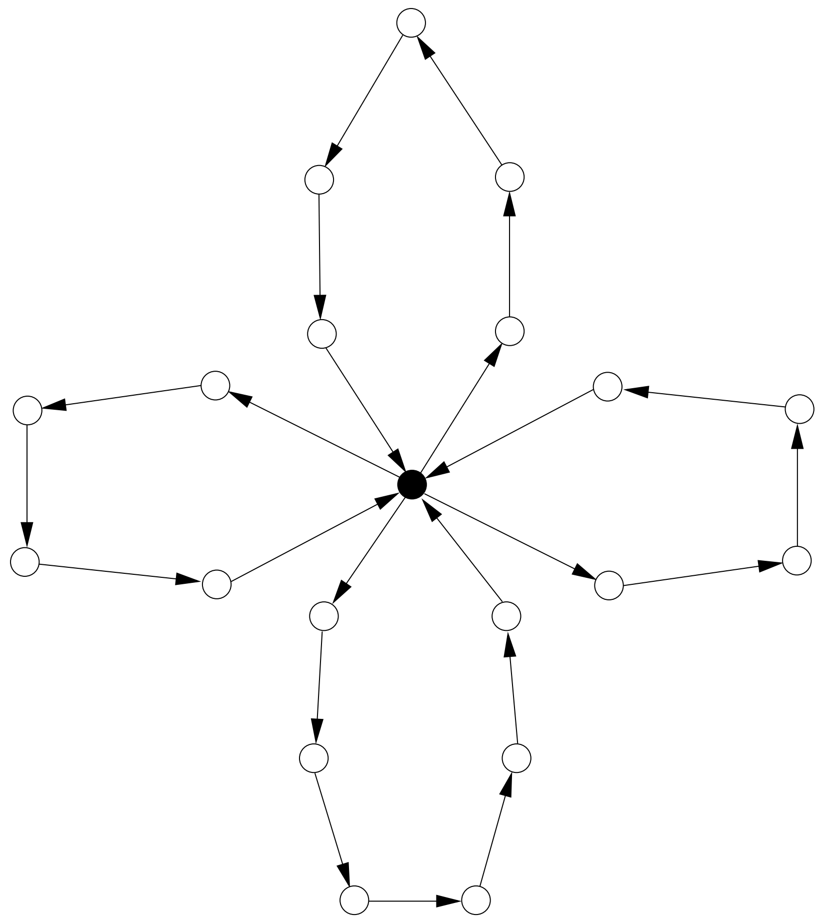




CHVÁTAL AND SZEMERÉDI:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $(n/d) + 2500$

SHEN:

Every digraph on n vertices with minimum outdegree d has a directed cycle of length at most $(n/d) + 73$

WHAT DOES THIS SAY WHEN $d = \lceil n/3 \rceil$?

Every digraph on n vertices with minimum outdegree $\lceil n/3 \rceil$ *has a directed cycle of length at most* 76

BUT THE BOUND IN THE CACCETTA–HÄGGKVIST CONJECTURE IS $\lceil n/d \rceil = 3$!

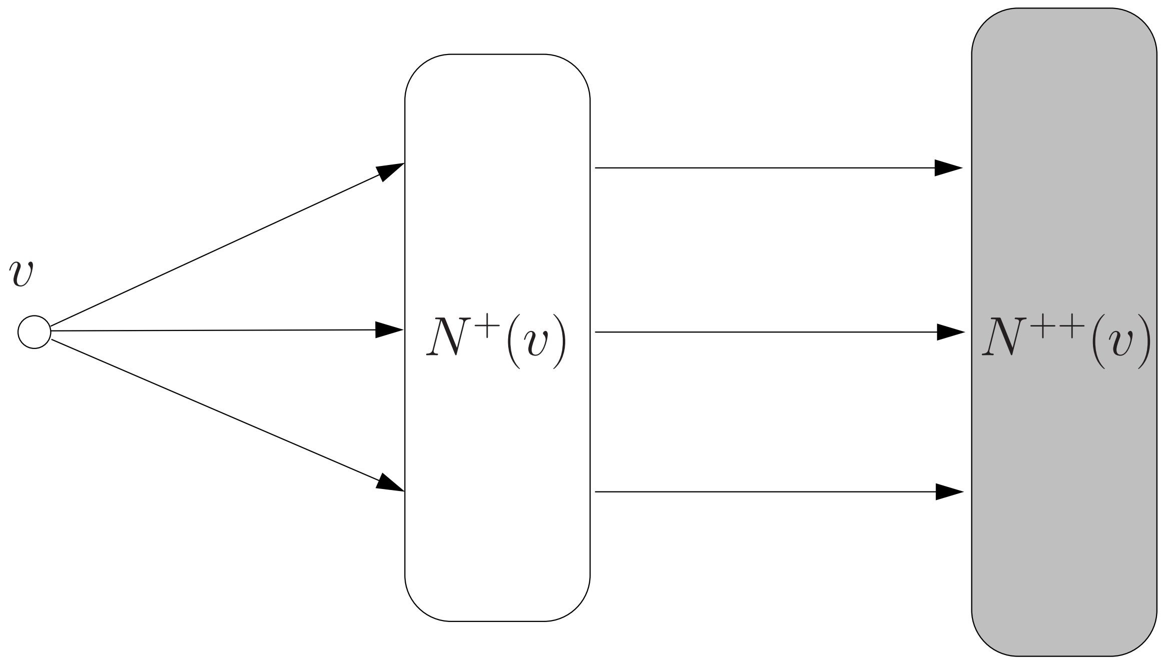
Caccetta-Häggkvist Conjecture for triangles

Every digraph on n vertices with minimum outdegree $\lceil n/3 \rceil$ has a directed triangle

SECOND NEIGHBOURHOODS

Seymour's Second Neighbourhood Conjecture 1990

Every digraph (without directed 2-cycles) has a vertex with at least as many second outneighbours as first outneighbours

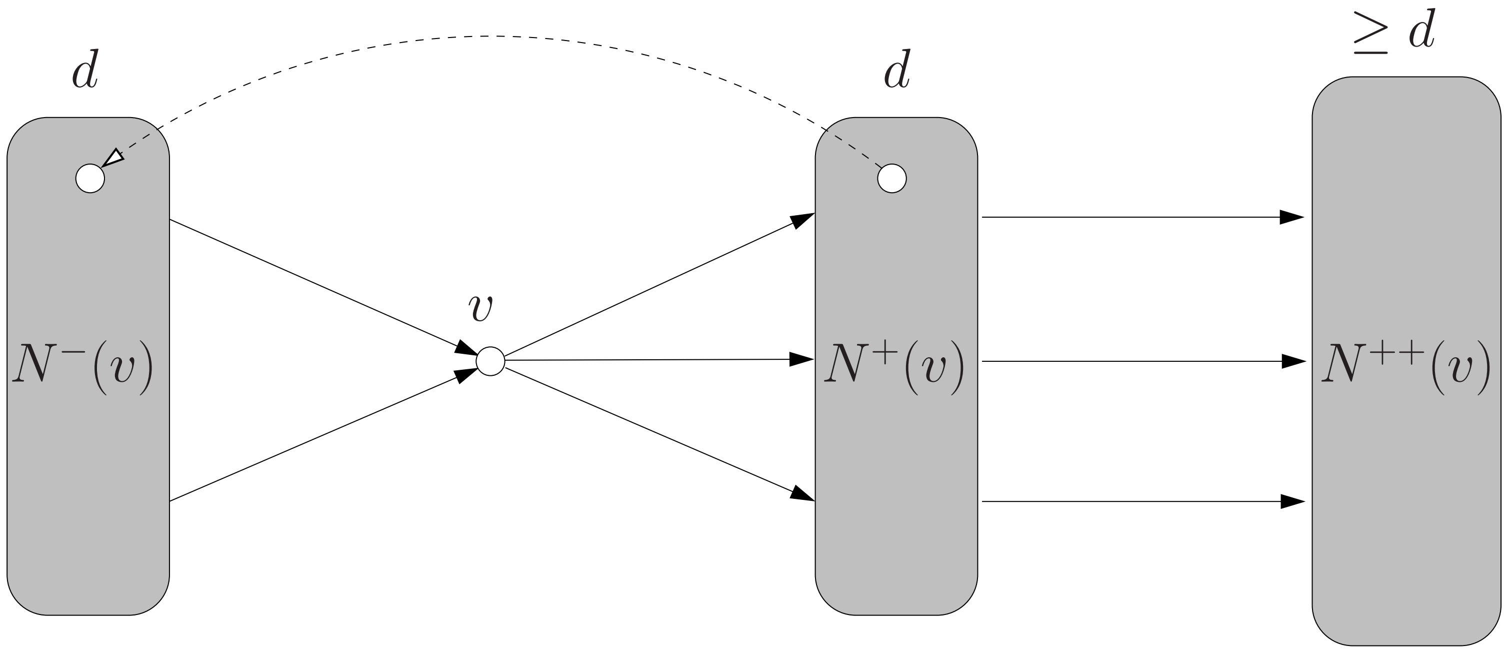


v
$N^+(v)$
$N^{++}(v)$

The Second Neighbourhood Conjecture implies the triangle case

$$d = \left\lceil \frac{n}{3} \right\rceil$$

of the Behzad-Chartrand-Wall Conjecture



If there is no directed triangle:

$$n \geq 3d + 1$$

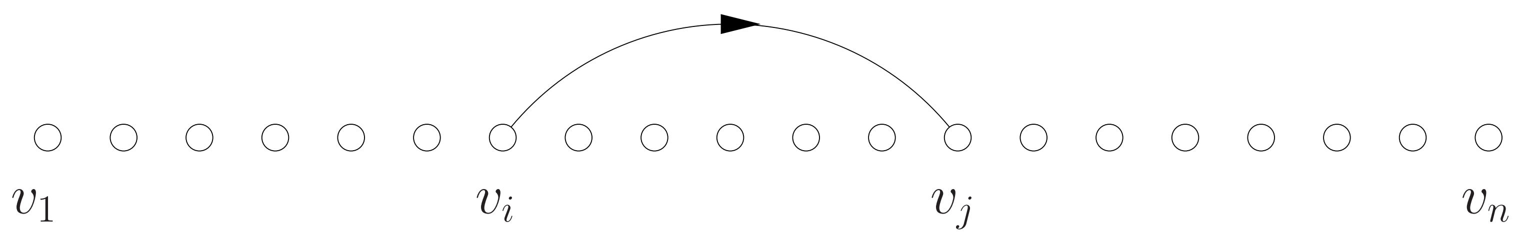
FISHER:

The Second Neighbourhood Conjecture is true for tournaments.

Proof by HAVET AND THOMASSÉ using median orders.

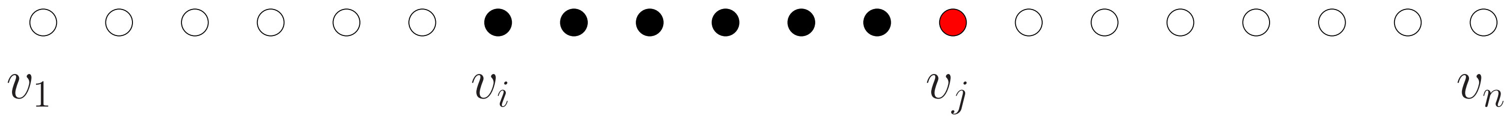
Median order: *linear order* $v_1, v_2, \ldots, v_n$ *maximizing*

$$|\{(v_i, v_j) : i < j\}|$$


v_1
v_i
v_j
v_n

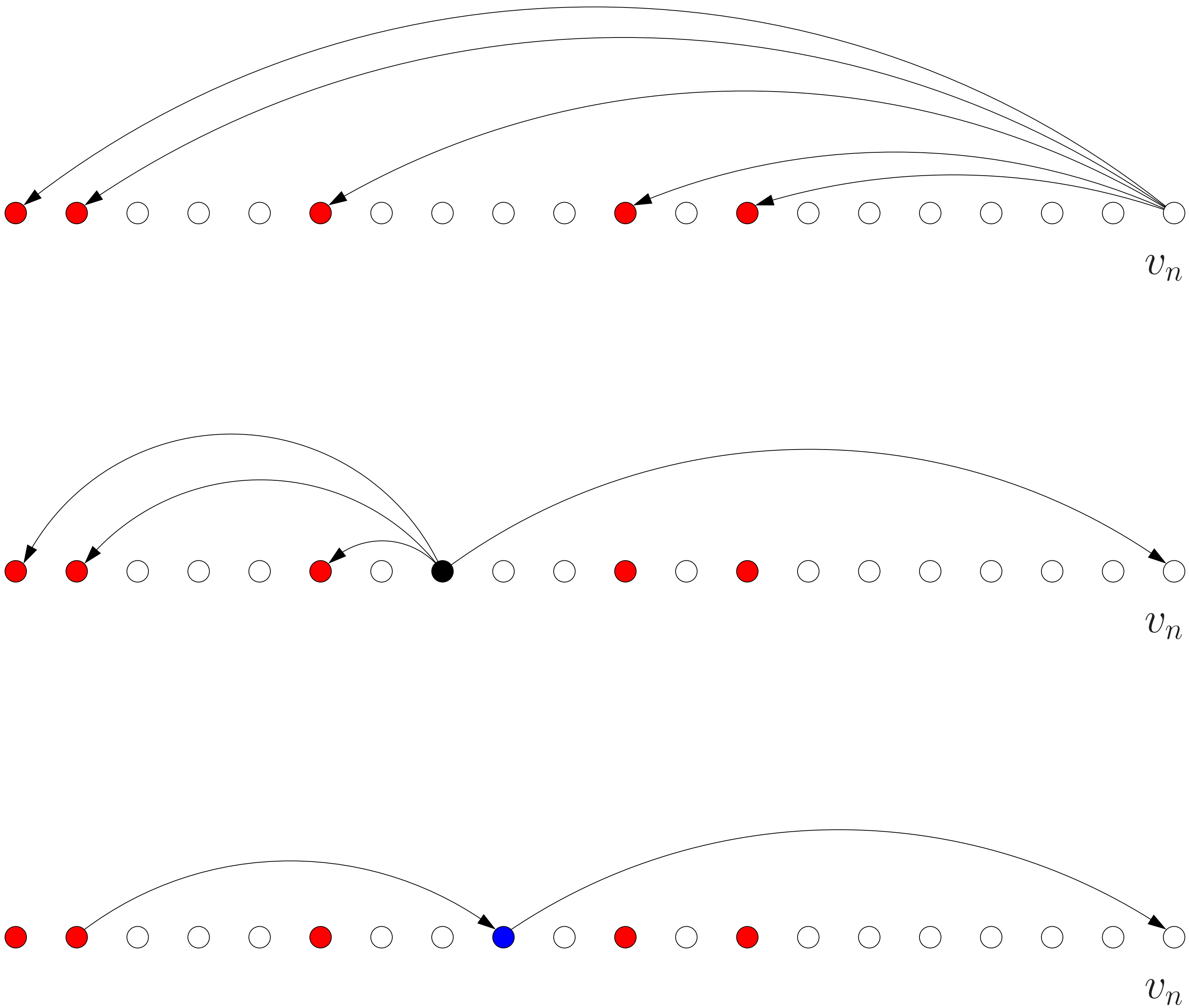
Property of median orders:

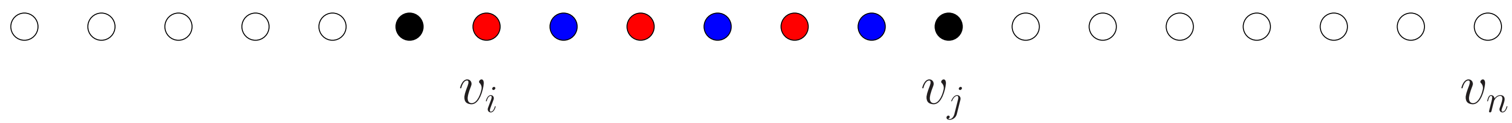
For any $i < j$, vertex v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \ldots, v_{j-1}$.

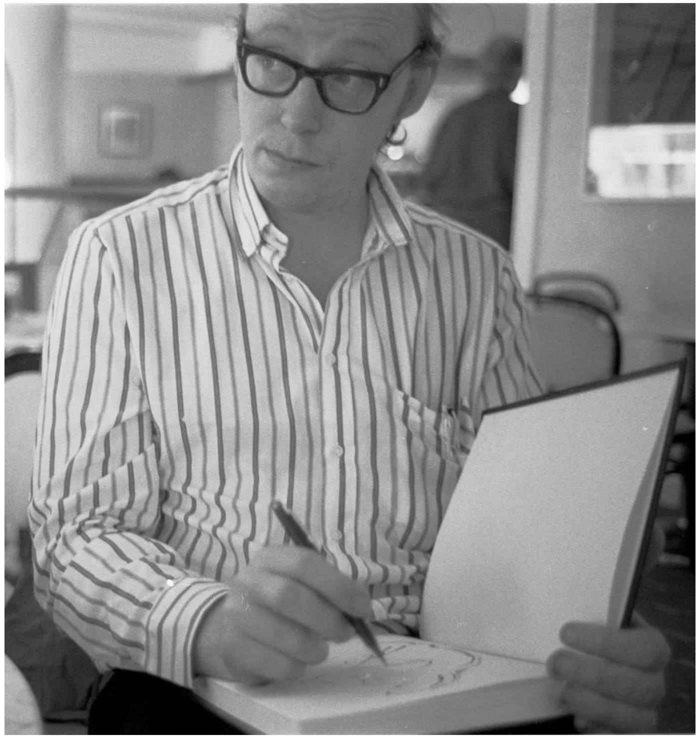


If not, move v_j before v_i.

Claim: $|N^{++}(v_n)| \geq |N^{+}(v_n)|$

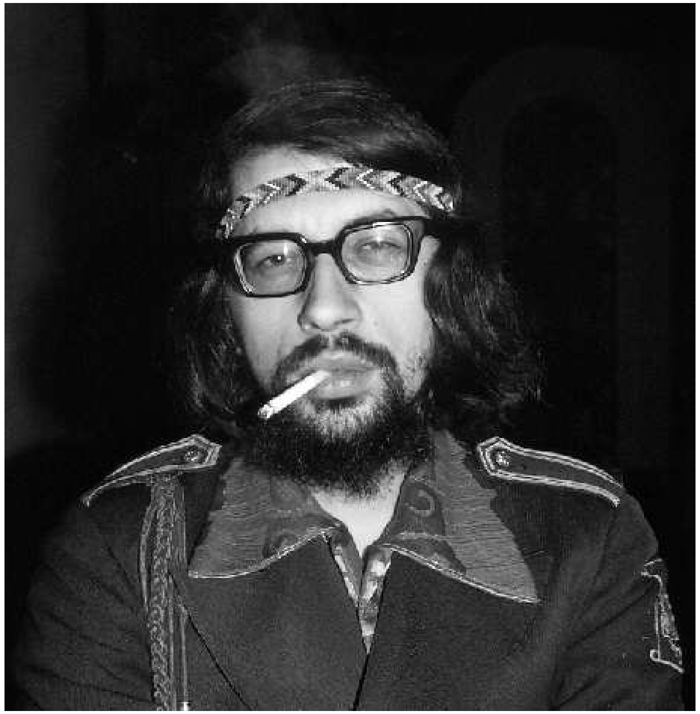

v_n
v_n
v_n


v_i
v_j
v_n

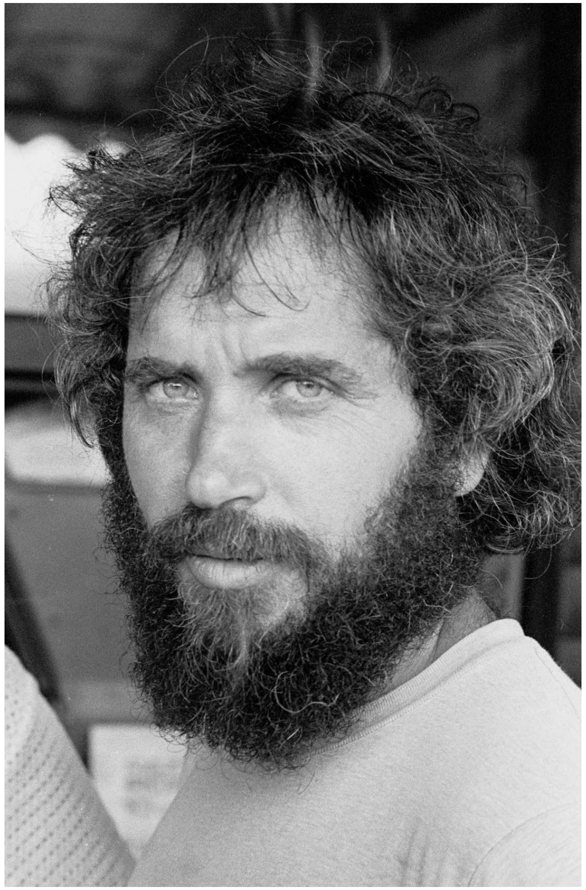


Roland Häggkvist

Paul Seymour

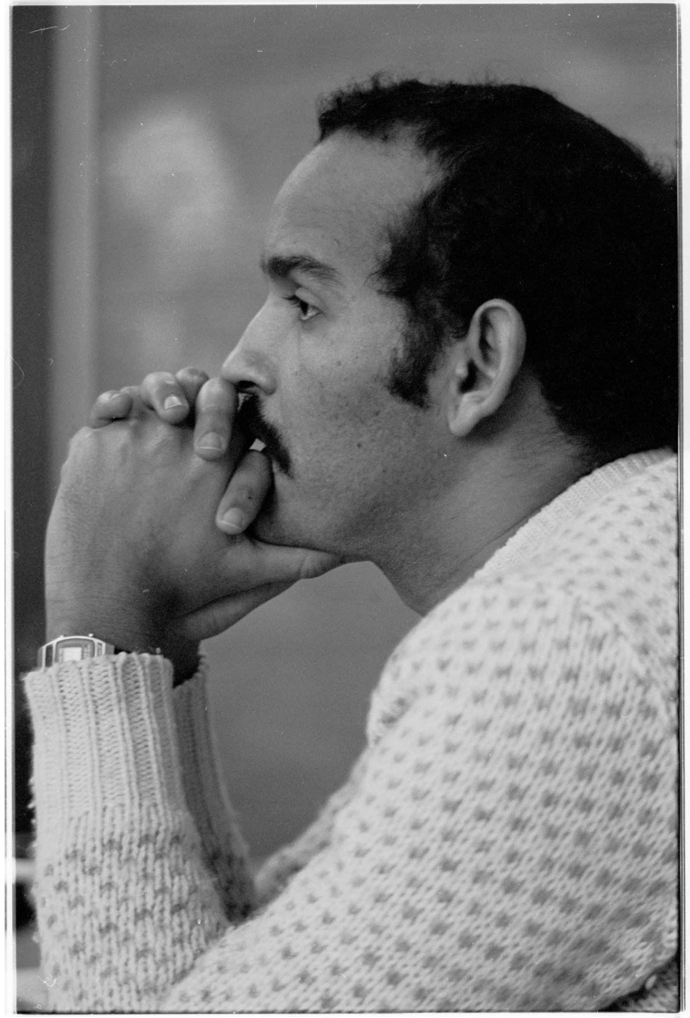


Vašek Chvátal



Endre Szemerédi

Stephan Thomassé



Yahya Hamidoune

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