# Fragments des relations

Anna Lladó

Univ. Politècnica de Catalunya
Barcelona

Journée en Hommage a

Yahya ould Hamidoune

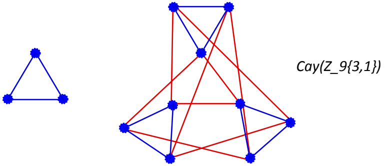
# London 1989: premier rencontre

Pourriez vous me recomender bibliographie sur
le diamètre des graphes de Cayley?

A Cayley graph $Cay(G, S)$ is quasiminimal (hierarchical) if there is an ordering $\{s_1, \ldots, s_k\}$ of $S$ such that

$$G_1 < G_2 < \cdots < G_k = G,$$

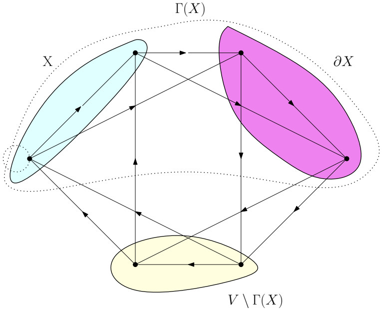
where $G_i = \langle s_1, \ldots, s_i \rangle$ and the inclusions are strict.




Le problème est trés difficile. Regarde sa connectivité. (Yahya)

## T'occupe pas des ensembles de séparation: regarde les fragments. (Yahya)

- $\Gamma \subset V \times V$ a relation (directed graph; undirected=symmetric), usually reflexive (with loops).
- $\Gamma(X)$ neighborhood of $X \subset V$.
- $\partial X = \Gamma(X) \setminus X$ boundary of $X \subset V$.




$$\kappa(\Gamma) = \min\{|\partial(X)| : 0 < |X| < |\Gamma(X)| < |V| - 1,\ X \subset G\}.$$

$X \subset G$ is a fragment if $0 < |X| < |\Gamma(X)| < |V| - 1$ and $|\partial X|$ is minimum.

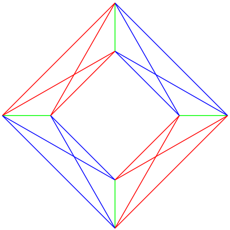
An atom is a fragment of minimum cardinality.

## Theorem (Hamidoune, L., Serra, 1992)

*The connectivity of a quasiminimal Cayley (directed) graph equals the degree except for the family of undirected graphs with*

$$s_1^2 = 1, \qquad s_2^2 = \cdots = s_k^2 = s_1.$$

*for which the connectivity is the degree minus one.*



Example of the exception: $G < \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

$s_1 = (2, 0, 0)$, $s_2 = (1, 1, 0)$, $s_3 = (1, 0, 1)$.

Wide generalization of previous results
by Godsil, by Alspach and by Baumslag for minimal and undirected.

# Barcelona 1990: Vers Vosperianity

How the minimum cuts look like?
For a maximally connected regular graph, the neighborhood of one vertex.
What else? (Yahya)

An inspiration:

## Theorem (Vosper, 1955)

*Let $p$ be a prime and $A, B \subset \mathbb{Z}_p$. If $A$ is not an arithmetic progression, then*

$$|A+B| \geq |A| + |B|$$

*unless*

$$\min\{|A|, |B|\} = 1 \quad \text{or} \quad |A+B| = p-1.$$

In graphic terms: For $\Gamma = Cay(\mathbb{Z}_p, A)$ (with $0 \in A$), $B \subset \mathbb{Z}_p$.

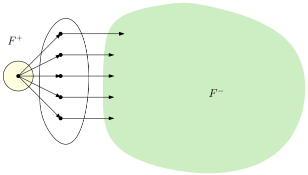
If $|B| = 1$, then $|\Gamma(B)| = |B + A| = |A|$ and $|\partial B| = |A| - 1$.

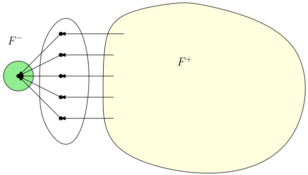
## Corollary

*If $A$ is not an arithmetic progression, the only minimum cutsets of $Cay(\mathbb{Z}_p, A)$ are the out–neighborhood or in–neighborhood of a point.*

A maximally connected regular graph $\Gamma$ is

- superconnected if the only minimal cuts are the out–neighborhood or in–neighborhood of a point.
- vosperian if the only fragments are the points or the complements of its neighborhoods. (slightly stronger)



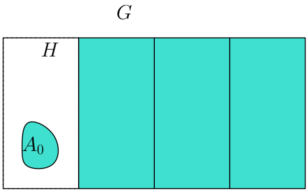



# Barcelona 1990: Vosperianity, the abelian case

## Theorem (Hamidoune, L. , Serra, 1991)

*Let $G$ be an abelian group and $A \subset G$, $|A| < |G| - 3$.*
*The graph $\Gamma = Cay(G, A)$ is vosperian unless one of the following conditions hold:*

- *There is a nontrivial subgroup $H$ which is a fragment.*
- *$A \cup \{0\}$ is an arithmetic progression.*
- *$A = (G \setminus H) \cup A_0$, $A_0 \subset H$ and $Cay(H, A_0)$ is one of the above types.*




An illustration of the third possibility in the theorem.

The proof is based on the Kemperman Structure Theorem which characterizes the critical pairs for Kneser Theorem.

# Paris 1990:
# An application to the Cacceta–Häggkvist Conjecture

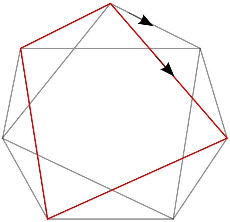
**Conjecture** (Cacceta–Häggkvist)

*The girth $g$ of a digraph $G$ with order $n$ and minimum out–degree $r$ satisfies*

$$n \geq r(g-1)+1.$$

If true, the conjecture is tight:

$$Cay(Z_n, \{1,2,\ldots,r\}), \quad n \not\equiv 0 \pmod{r}, \quad n > 2r.$$



Proved for vertex transitive graphs by Hamidoune (1981).

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**Theorem (Hamidoune, L., Serra, 1991)**

*Let $G$ be an abelian group and $\Gamma = Cay(G,S)$. Then*

$$n \geq (r+1)(g-1)-1,$$

*unless*

$\Gamma$ *is the lex product of graphs* $\Gamma_i = Cay(\mathbb{Z}_{n_i}, \{1,2,\ldots,r_i\})$.

For abelian Cayley graphs the above is the unique example.

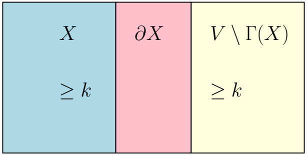
# Barcelona 1993: Beyond connectivity

What is the next size of cutsets? (Yahya)

$$\Gamma \subset V \times V$$

Connectivity: $\kappa(\Gamma) = \min\{|\partial X| : 0 < |X| < |\Gamma X| < |V| - 1,\ X \subset V\}$

$k$–connectivity: $\kappa_k(\Gamma) = \min\{|\partial X| : k < |X| < |\Gamma X| < |V| - k,\ X \subset V\}$

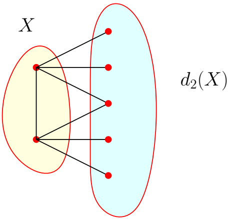


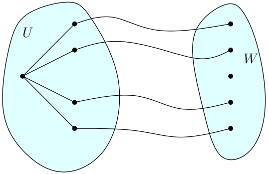
## Theorem (Hamidoune, L., Serra, 1996)

*Let* $\Gamma = Cay(G, S \cup S^{-1})$ *with* $S = \{s_1, \dots, s_m\}$ *quasiminimal. Let*

$$r = |S \cup S^{-1}|.$$

- $\Gamma$ *is Vosperian unless* $s_1^2 = 1$ *and* $s_i^2 = s_1, \ i = 2, \dots, m$
- $\kappa_2(\Gamma) = d_2(\Gamma)$ *(the minimum boundary of a set with two vertices).*
- *For every two disjoint sets* $U, W \subset G$, $U, W \neq \Gamma(x)$, *with* $|U| = |W| = r + 1$, *there are* $r + 1$ *pairwise disjoint paths from* $U$ *to* $W$. *(extended Menger.)*




Prohibited case.

# Paris 1996: A test case: torsion–free groups

Non abelian groups should have stronger connectivity.
Boundary constraints introduce technical problems if the group is finite: try infinite. (Yahya)

**Theorem (Hamidoune, L., Serra, 1998)**

*Let $A, B$ be finite subsets of a torsion–free group $G$. Then*

$$|AB| \geq |A| + |B| + 1,$$

*unless, $A$ and $B$ are almost arithemtic progressions with common difference in a cyclic subgroup, up to transllation.*

Almost progressions: ● ○ ● ● ● ●

This extends previous results

- Kemperman (1956): $|AB| \geq |A| + |B| - 1$ (trivial for $\mathbb{Z}$)
- Brailovsky–Freiman (1990): $|AB| \geq |A| + |B|$ unless
  (i) $A, B$ arithmetic progressions or
  (ii) $\min\{|A|, |B|\} = 1$
  (easy for $\mathbb{Z}$).

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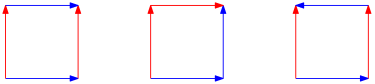
In the nonabelian case the groups which may prevent better bounds are identified as being two generated with short relations: $x^2 = y^2$ or $xyx = x$.



$xy = yx$ $\quad$ $x^2 = y^2$ $\quad$ $xyx = y$

# Barcelona 1998: Edge connectivity

Edge–connectivity is easier to deal with.
We should obtain stronger results. (Yahya)

t-edge-connectivity: of $G = (V, E)$

$$\lambda_t(G) = \min\{e(X) : \ t \leq |X| \leq |V| - t, \ X \subset V\}.$$

**Theorem (Hamidoune, L., Serra, Tindell, 1999)**

*Let $G$ be a connected vertex transitive graph with degree $r$.*
*For all $t \leq (r+3)/3$,*

$$\lambda_t(G) \geq (r - t + 2)(r - r(A)),$$

*where $r(A)$ is the degree of a $t$–atom. In particular, if $(r+3)/3 < |X| < n/2$,*

$$e(X) \geq (4/9)r^2 + (2/3)r,$$

*unless $G$ has a dense subgraph $H$ and $G/H$ is arc-transitive.*

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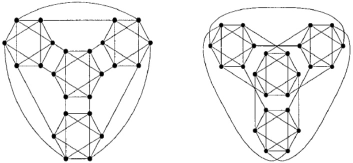
- Edge neighborhoods quadratic in $r$ occur in
  - graphs with prime order,
  - graphs with girth larger than $t$
  - arc–transitive graphs.



Examples where the alternative of the Theorem holds.

# References


- On the Connectivity of hierarquical Cayley graphs, *Discrete Applied Math.* 37 (1992) 275–280.
- Vosperian and Superconnected Abelian Cayley graphs, *Graphs and Combinatorics* 7 (1991) 143–152.
- Minimum order of Loop Networks of given degree and girth *Graphs and Combinatorics* 11 (1995) 131–138.
- Small cuts in quasiminimal Cayley graphs, *Discrete Math.* 159 (1996) 131–142.
- An isoperimetric problem in Cayley graphs, *Theory of Computing Systems* 32 (1999) 507–516.
- Sets with small product in Torsion–Free groups, *Combinatorica* 18 (1998) 587–595.
- Nontrivial edge–cuts in vertex transitive graphs, *SIAM J. Discrete Math.* 13 (2000) 139–144.

# The framework

The above results use

- The isoperimetric method of Yahya.
- They are based on the analysis of fragments (sets with minimum neighborhood) and atoms (minimum fragments).
- The study of atoms reduce the problems of connectivity to locate small sets with small boundary.
- Further applications to problems in additive combinatorics were developed in successive fragments of our relationship.

Yahya was my master et mon ami.

Merci

J'espère que son esprit, qui viens du dessert, nous acompagne.