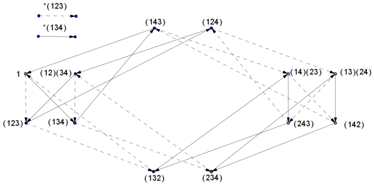
On vosperian and superconnected vertex-transitive graphs.

Susana-C. López
Universitat Politècnica de Catalunya

joint work Yahya Hamidoune and Anna Lladó.



Outline

Characterize superconnected vertex-transitive graphs

Problem

Let $\mathbf{G}$ be a $\mathbf{d}$-regular **vertex-transitive** graph with $\mathbf{d} \geq \mathbf{3}$ and connectivity $\kappa(\mathbf{G}) = \mathbf{d}$. If $\mathbf{G}$ **does not** contain $\mathbf{K_4} \setminus \mathbf{e}$ nor $\mathbf{K_{2,3}}$ then every minimum cutset of $\mathbf{G}$ **isolates** a single vertex.

Finite graph

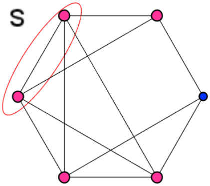
Let $\Gamma = (V, E)$ be a **graph**:

- ✓ the elements of V are the **vertices**,
- ✓ the elements of E, pairs of **distinct** vertices, are **the edges.**

The **neighbors** of $x \in V$: $\Gamma(x)$.

Given $S \subset V$,

- ✓ $\Gamma(S) = \bigcup_{x \in S} \Gamma(x)$.



Finite graph

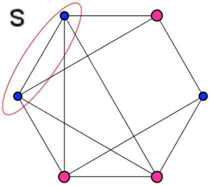
Let $\Gamma = (V, E)$ be a **graph**:

- ✓ the elements of V are the **vertices**,
- ✓ the elements of E, pairs of **distinct** vertices, are **the edges.**

The **neighbors** of $x \in V$: $\Gamma(x)$.

Given $S \subset V$,

- ✓ $\Gamma(S) = \bigcup_{x \in S} \Gamma(x)$.
- ✓ $\partial(S) = \Gamma(S) \setminus S$.



Superconnected. Vosperian

Let $\Gamma = (V, E)$ be a connected d-regular graph ($|\Gamma(x)| = d$, $\forall x \in V$),

$$|V| \geq d + 4.$$

If we remove: $\Gamma(x)$ then $\{x\}$ becomes an **isolated** vertex.

Every $S \subset V$, such that $|S| \leq d$, whose removal destroys the connectedness of Γ,

- ✓ is $\Gamma(x)$, Γ **superconnected**.

Superconnected. Vosperian

Let $\Gamma = (V, E)$ be a connected d-regular graph ($|\Gamma(x)| = d$, $\forall x \in V$),

$$|V| \geq d + 4.$$

If we remove: $\Gamma(x)$ then $\{x\}$ becomes an **isolated** vertex.

Every $S \subset V$, such that $|S| \leq d$, whose removal destroys the connectedness of Γ,

- ✓ is $\Gamma(x)$, Γ **superconnected**.
- ✓ creates exactly 2 connected components, one of size 1, Γ **vosperian**.

Superconnected. Vosperian

Let $\Gamma = (V, E)$ be a connected d-regular graph ($|\Gamma(x)| = d$, $\forall x \in V$),

$$|V| \geq d + 4.$$

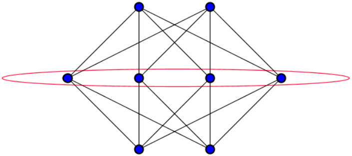
If we remove: $\Gamma(x)$ then $\{x\}$ becomes an **isolated** vertex.

Every $S \subset V$, such that $|S| \leq d$, whose removal destroys the connectedness of Γ,

- ✓ is $\Gamma(x)$, Γ **superconnected**.

  $\Uparrow$ **but** $\not\Downarrow$

- ✓ creates exactly 2 connected components, one of size 1, Γ **vosperian**.



Superconnected. Vosperian

Let $\Gamma = (V, E)$ be a connected d-regular graph ($|\Gamma(x)| = d$, $\forall x \in V$),

$$|V| \geq d + 4.$$

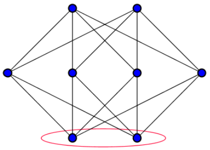
If we remove: $\Gamma(x)$ then $\{x\}$ becomes an **isolated** vertex.

Every $S \subset V$, such that $|S| \leq d$, whose removal destroys the connectedness of Γ,

- ✓ is $\Gamma(x)$, Γ **superconnected**.

  $\Uparrow$ **but** $\not\Downarrow$

- ✓ creates exactly **2** connected components, one of size **1**, Γ **vosperian**.



a **twin** pair

What we do in vertex-transitive graphs:

- ✓ We characterize **vt** graphs that remain connected after failure of a vertex and its neighbors.

  $\Downarrow$ **vosperian vt** graphs (in terms of the automorphism group)

What we do in vertex-transitive graphs:

- ✓ We characterize **vt** graphs that remain connected after failure of a vertex and its neighbors.

  $\Downarrow$ **vosperian vt** graphs (in terms of the automorphism group)

- ✓ We prove that in a **vt** graph **without twin** pairs. If every small cutset consist of some vertex neighbor, then $\mathbf{V} \setminus (\{\mathbf{x}\} \cup \mathbf{\Gamma}(\mathbf{x}))$ is connected.

  $\Downarrow$ **superconnected vt** graphs (without twins) are **vosperian**

What we do in vertex-transitive graphs:

- ✓ We characterize **vt** graphs that remain connected after failure of a vertex and its neighbors.

  ⇓ **vosperian vt** graphs (in terms of the automorphism group)

- ✓ We prove that in a **vt** graph **without twin** pairs. If every small cutset consist of some vertex neighbor, then $\mathbf{V} \setminus (\{\mathbf{x}\} \cup \mathbf{\Gamma}(\mathbf{x}))$ is connected.

  ⇓ **superconnected vt** graphs (without twins) are **vosperian**

- ✓ We characterize **Cayley graphs** that remain connected after failure of a vertex and its neighbors.

  ⇓ **vosperian Cayley** graphs (in terms of the generating set)

  ⇓ **superconnected Cayley** graphs (an aperiodic generating set)

Antecedents

- ✓ 1991, Hamidoune-Lladó-Serra, characterization of vosperian and superconnected abelian Cayley digraphs (recursive).
- ✓ 1997, Hamidoune, characterization of vosperian and superconnected abelian Cayley digraphs (non-recursive).
- ✓ 2001, Wang-Meng, characterization of hyperconnected and superconnected vertex-transitive graphs of degree 3.
- ✓ 2000(2008), Tian-Meng, characterization of hyperconnected and superconnected vertex-transitive graphs of degree 4,5(6).
- ✓ 2003, Meng, characterization of hyperconnected and superconnected vertex and edge transitive graphs.

Antecedents

- ✓ 1991, Hamidoune-Lladó-Serra, characterization of vosperian and superconnected abelian Cayley digraphs (recursive).
- ✓ 1997, Hamidoune, characterization of vosperian and superconnected abelian Cayley digraphs (non-recursive).
- ✓ 2001, Wang-Meng, characterization of hyperconnected and superconnected vertex-transitive graphs of degree 3.
- ✓ 2000(2008), Tian-Meng, characterization of hyperconnected and superconnected vertex-transitive graphs of degree 4,5(6).
- ✓ 2003, Meng, characterization of hyperconnected and superconnected vertex and edge transitive graphs.

$\Downarrow$

isoperimetric method

The isoperimetric method: terminology

Let $\Gamma = (V, E)$ be a graph and $X \subset V$:

- ✓ the **boundary**: $\partial_\Gamma(X) = \Gamma(X) \setminus X$,
- ✓ the **exterior**: $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$.

The isoperimetric method: terminology

Let $\Gamma = (V, E)$ be a graph and $X \subset V$:

- ✓ the **boundary**: $\partial_\Gamma(X) = \Gamma(X) \setminus X$,
- ✓ the **exterior**: $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$.

If $\min\{|X|, |\nabla(X)|\} \geq k$, for some $X \subset V$: Γ is **k-separable**.

$$\kappa_k(\Gamma) = \min\{|\partial X| : X \subset V, \min\{|X|, |\nabla(X)|\} \geq k\}$$

The isoperimetric method: terminology

Let $\Gamma = (V, E)$ be a graph and $X \subset V$:

- ✓ the **boundary**: $\partial_\Gamma(X) = \Gamma(X) \setminus X$,
- ✓ the **exterior**: $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$.

If $\min\{|X|, |\nabla(X)|\} \geq k$, for some $X \subset V$: Γ is k-**separable**.

$$\kappa_k(\Gamma) = \min\{|\partial X| : X \subset V, \min\{|X|, |\nabla(X)|\} \geq k\}$$

In particular,

- $\kappa_1(\Gamma) = \kappa$ is the **connectivity**
- $\kappa_2(\Gamma) = \min\{|\partial(X)| : X \subset V, \min\{|X|, |\nabla(X)|\} \geq 2\}$
- $\kappa_1 \leq \kappa_2 \leq \ldots$

The isoperimetric method: terminology

Let $\Gamma = (V, E)$ be a graph and $X \subset V$:

- ✓ the **boundary**: $\partial_\Gamma(X) = \Gamma(X) \setminus X$,
- ✓ the **exterior**: $\nabla_\Gamma(X) = V \setminus (X \cup \Gamma(X))$.

If $\min\{|X|, |\nabla(X)|\} \geq k$, for some $X \subset V$: Γ is **k-separable**.

$$\kappa_k(\Gamma) = \min\{|\partial X| : X \subset V, \min\{|X|, |\nabla(X)|\} \geq k\}$$

$F \subset V$ is a **k-fragment** (of minimal cardinality is called **k-atom)**.

- $\min\{|F|, |\nabla(F)|\} \geq k$
- $|\partial(X)| = \kappa_k(\Gamma)$

The isoperimetric method: the intersection property

Theorem (Hamidoune, 1996)

Let $\Gamma = (V, E)$ be a finite k-separable graph. Let A be a k-atom and let F be a k-fragment.

(i) If $|A \cap F| \geq k$ then $A \subset F$. In particular,

(ii) two distinct k-atoms intersect in at most $k - 1$ elements.

The isoperimetric method: vertex-transitive graphs

Theorem (Hamidoune, 2000)

Let $\Gamma = (V, E)$ be a finite 2-separable **vertex-transitive** graph of degree d and let A be a 2-atom, with $|A| \geq 3$. Then one of the following holds:

(i) Any vertex of V is contained in **at most two** distinct 2-atoms.

(ii) $|A| \leq \kappa_2(\Gamma) - d + 2$.

The isoperimetric method: vertex-transitive graphs

Theorem (Hamidoune, 2000)

Let $\Gamma = (V, E)$ be a finite 2-separable **vertex-transitive** graph of degree d and let A be a 2-atom, with $|A| \geq 3$. Then one of the following holds:

(i) Any vertex of V is contained in **at most two** distinct 2-atoms.

(ii) $|A| \leq \kappa_2(\Gamma) - d + 2$.

In particular,

if $\kappa_2(\Gamma) \leq d$, then, either $|A| = 2$, or

any vertex of V is contained in at most two distinct 2-atoms.

The isoperimetric method: Vosperian and Superconnected

Lemma (folklore) A vosperian graph.

Let $\Gamma = (V, E)$ be a finite, d-regular graph, such that $|V| \neq d + 3$. The following conditions are equivalent:

(i) Γ is **not** 2-separable or $\kappa_2(\Gamma) \geq d + 1$.

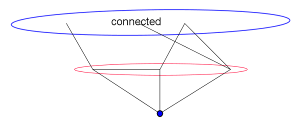
(ii) A 1-fragment of Γ has the form $\{x\}$ or $V \setminus (\{x\} \cup \Gamma(x))$.

(iii) A 1-fragment of Γ has size $= 1$ or $= |V| - d - 1$.

(iv) **Every** minimum cutset **isolates** a vertex and,

$$V \setminus (\{x\} \cup \Gamma(x))$$

is **connected** for every $x \in V$.

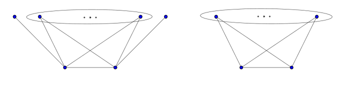


Vosperian vertex-transitive graphs

Theorem

Let $\mathbf{\Gamma} = (\mathbf{V}, \mathbf{E})$ be a **vt** graph of degree $\mathbf{d}$, $\mathbf{d} \leq |\mathbf{V}| - \mathbf{4}$. Let $\mathcal{A}$ be a subgroup of the automorphism group, which is transitive on $\mathbf{V}$ and let $\mathbf{v} \in \mathbf{V}$. Then $\mathbf{\Gamma}$ is **non-vosperian** if and only if one of the following conditions holds:

(i) There exists a **friend** pair. $\rightsquigarrow$



(ii) There exists a **twin** pair.

(iii) There are an $\alpha \in \mathcal{A}$ and a set $\mathbf{M}$, with $|\mathbf{M}| \geq \mathbf{2}$, $|\mathbf{M} \cup \partial(\mathbf{M})| \leq |\mathbf{V}| - \mathbf{2}$ and $|\partial(\mathbf{M})| \leq \mathbf{d}$, such that $\mathcal{A}_{\mathbf{v},\mathbf{M}} = \mathcal{A}_{\mathbf{M},\mathbf{M}} \cup \mathcal{A}_{\mathbf{M},\mathbf{M}}\alpha$.

(where, $\mathcal{A}_{\mathbf{X},\mathbf{Y}} = \{\mathbf{f} \in \mathcal{A} : \mathbf{f}(\mathbf{X}) \subset \mathbf{Y}\}$)

Twin reduction

Theorem

Let Γ be a **vt** with $\kappa(\Gamma) = d$. Then,

$\overline{\Gamma}$ is **superconnected** if and only if Γ is **superconnected**,

where $\overline{\Gamma} = \Gamma / \sim$ and $x \sim y$ if and only if $\Gamma(x) = \Gamma(y)$.

Superconnected vertex-transitive graphs

Theorem

Let Γ be a **vt** graph of degree d **without** a **twin** pair. If **every** small cutset (cardinality $\leq d$) consists of some vertex neighbors, then for every $x \in V$, $V \setminus (\{x\} \cup \Gamma(x))$ is **connected**. In particular,

Γ is **vosperian** if and only if it is **superconnected**.

Superconnected vertex-transitive graphs

Theorem

Let Γ be a **vt** graph of degree d **without** a **twin** pair. If **every** small cutset (cardinality $\leq d$) consists of some vertex neighbors, then for every $x \in V$, $V \setminus (\{x\} \cup \Gamma(x))$ is **connected**. In particular,

Γ is **vosperian** if and only if it is **superconnected**.

Corollary

Let Γ be a **vt** graph of degree d, such that $\gcd(|V|, d) = 1$. Then,

Γ is **vosperian** if and only if it is **superconnected**.

Superconnected vertex-transitive graphs

Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

Superconnected vertex-transitive graphs

Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

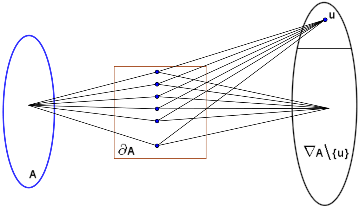
Assume $A| \geq 3$. In particular, A is connected. Let $u \in V$ such that $\partial(A) = \Gamma(u)$. Thus $u \in \nabla(A)$.

Superconnected vertex-transitive graphs

Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

Assume $\mathbf{A}| \geq 3$. In particular, $\mathbf{A}$ is connected. Let $\mathbf{u} \in \mathbf{V}$ such that $\partial(\mathbf{A}) = \Gamma(\mathbf{u})$. Thus $\mathbf{u} \in \nabla(\mathbf{A})$.



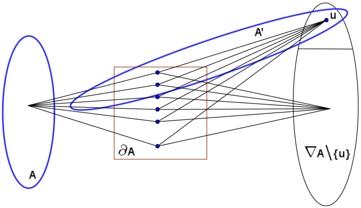
Superconnected vertex-transitive graphs

Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

Assume $A| \geq 3$. In particular, A is connected. Let $u \in V$ such that $\partial(A) = \Gamma(u)$. Thus $u \in \nabla(A)$.

Let A' be a 2-atom, $u \in A'$.



Superconnected vertex-transitive graphs

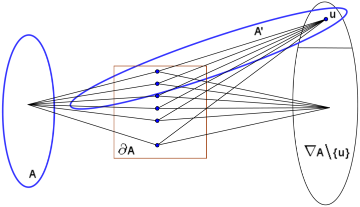
Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

Assume $A| \geq 3$. In particular, A is connected. Let $u \in V$ such that $\partial(A) = \Gamma(u)$. Thus $u \in \nabla(A)$.

Let A' be a 2-atom, $u \in A'$.
There exists u' such that

$$\partial(A') = \Gamma(u').$$



Superconnected vertex-transitive graphs

Lemma

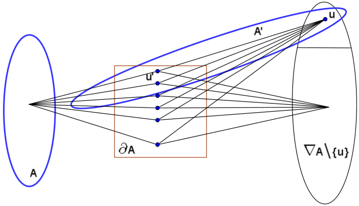
Let $\boldsymbol{\Gamma}$ be a **superconnected vt** graph. If $\boldsymbol{\Gamma}$ is **non-vosperian** then the cardinality of a $\mathbf{2}$-atom is $\mathbf{2}$.

Assume $\mathbf{A}| \geq \mathbf{3}$. In particular, $\mathbf{A}$ is connected. Let $\mathbf{u} \in \mathbf{V}$ such that $\partial(\mathbf{A}) = \boldsymbol{\Gamma}(\mathbf{u})$. Thus $\mathbf{u} \in \nabla(\mathbf{A})$.

Let $\mathbf{A}'$ be a $\mathbf{2}$-atom, $\mathbf{u} \in \mathbf{A}'$.
There exists $\mathbf{u}'$ such that

$$\partial(\mathbf{A}') = \boldsymbol{\Gamma}(\mathbf{u}').$$

Thus, $\mathbf{u}' \in \mathbf{A}'$!!!



Superconnected vertex-transitive graphs

Lemma

Let Γ be a **superconnected vt** graph. If Γ is **non-vosperian** then the cardinality of a 2-atom is 2.

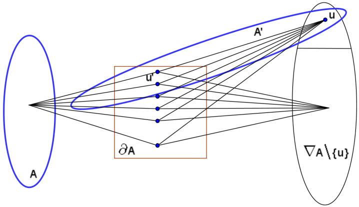
Assume $A| \geq 3$. In particular, A is connected. Let $u \in V$ such that $\partial(A) = \Gamma(u)$. Thus $u \in \nabla(A)$.

Let A' be a 2-atom, $u \in A'$.
There exists u' such that

$$\partial(A') = \Gamma(u').$$

Thus, $u' \in A'$!!!

$$(\ldots)$$



A **superconnected non-vosperian vt** graph contains a **twin pair**!

2-atoms in non-vosperian Cayley graphs

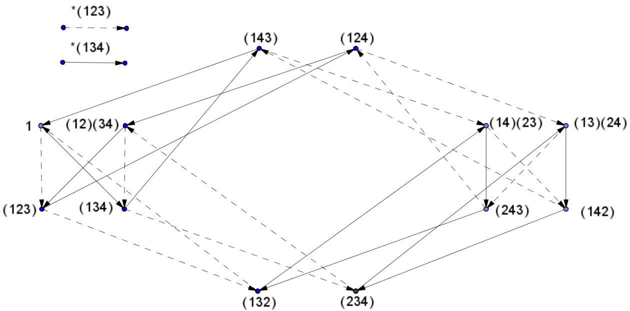
Proposition

Let $\mathbf{S}$ be a generating subset of a finite group $\mathbf{G}$ with $\mathbf{1} \notin \mathbf{S}$ and $\mathbf{S} = \mathbf{S}^{-1}$, such that $\mathbf{Cay(G, S)}$ is **non-vosperian**. Let $\mathbf{A}$ be a $\mathbf{2}$-atom of $\mathbf{Cay(G, S)}$ with $\mathbf{1} \in \mathbf{A}$. Then there are a subgroup $\mathbf{H}$ and an $\mathbf{a} \in \mathbf{G}$ such that $\mathbf{A} = \mathbf{H} \cup \mathbf{Ha}$.

2-atoms in non-vosperian Cayley graphs

Proposition

Let $\mathbf{S}$ be a generating subset of a finite group $\mathbf{G}$ with $\mathbf{1} \notin \mathbf{S}$ and $\mathbf{S} = \mathbf{S}^{-1}$, such that $\mathbf{Cay(G, S)}$ is **non-vosperian**. Let $\mathbf{A}$ be a $\mathbf{2}$-atom of $\mathbf{Cay(G, S)}$ with $\mathbf{1} \in \mathbf{A}$. Then there are a subgroup $\mathbf{H}$ and an $\mathbf{a} \in \mathbf{G}$ such that $\mathbf{A} = \mathbf{H} \cup \mathbf{Ha}$.



Vosperian Cayley graphs

Theorem

Let $\mathbf{S}$ be a generating subset of a finite group $\mathbf{G}$, with $\mathbf{1} \notin \mathbf{S}$, $\mathbf{S} = \mathbf{S}^{-1}$, $|\mathbf{S}| \leq |\mathbf{G}| - \mathbf{4}$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \{\mathbf{1}\}$. Then $\mathbf{\Gamma} = \mathrm{Cay}(\mathbf{G}, \mathbf{S})$ is **vosperian** if and only if for every $\mathbf{r} \in \mathbf{G} \setminus \mathbf{1}$,

(i) $\mathbf{G} \setminus \tilde{\mathbf{S}}$ **is not** an $\mathbf{r}$-**progression** and,

(ii) for every subgroup $\mathbf{H}$ and every $\mathbf{a} \in \mathbf{G}$ with $|\mathbf{H}| \geq \mathbf{2}$,

$$|(\mathbf{H} \cup \mathbf{Ha})\tilde{\mathbf{S}}| \geq \min(|\mathbf{G}| - \mathbf{1}, |\mathbf{H} \cup \mathbf{Ha}| + |\mathbf{S}| + \mathbf{1}).$$

Superconnected Cayley graphs

Corollary

Let S be an **aperiodic** generating subset of a finite group G, with $1 \notin S$, $S = S^{-1}$, $|S| \leq |G| - 4$ and $\tilde{S} = S \cup \{1\}$. Then $\mathrm{Cay}(G, S)$ is **superconnected** if and only if $G \setminus \tilde{S}$ **is not** an **r-progression**, and for every subgroup H and every $a \in G$ with $|H| \geq 2$,

$$|(H \cup Ha)\tilde{S}| \geq \min(|G| - 1, |H \cup Ha| + |S| + 1).$$

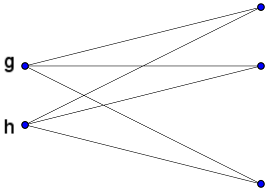
Superconnected Cayley graphs

Corollary

Let $\mathbf{S}$ be an **aperiodic** generating subset of a finite group $\mathbf{G}$, with $\mathbf{1} \notin \mathbf{S}$, $\mathbf{S} = \mathbf{S}^{-1}$, $|\mathbf{S}| \leq |\mathbf{G}| - \mathbf{4}$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \{\mathbf{1}\}$. Then $\mathrm{Cay}(\mathbf{G}, \mathbf{S})$ is **superconnected** if and only if $\mathbf{G} \setminus \tilde{\mathbf{S}}$ **is not** an **r-progression**, and for every subgroup $\mathbf{H}$ and every $\mathbf{a} \in \mathbf{G}$ with $|\mathbf{H}| \geq \mathbf{2}$,

$$|(\mathbf{H} \cup \mathbf{Ha})\tilde{\mathbf{S}}| \geq \min(|\mathbf{G}| - \mathbf{1}, |\mathbf{H} \cup \mathbf{Ha}| + |\mathbf{S}| + \mathbf{1}).$$

Proof If a twin pair exists



Superconnected Cayley graphs

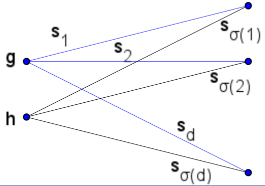
Corollary

Let $\mathbf{S}$ be an **aperiodic** generating subset of a finite group $\mathbf{G}$, with $\mathbf{1} \notin \mathbf{S}$, $\mathbf{S} = \mathbf{S}^{-1}$, $|\mathbf{S}| \leq |\mathbf{G}| - \mathbf{4}$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \{\mathbf{1}\}$. Then $\mathrm{Cay}(\mathbf{G}, \mathbf{S})$ is **superconnected** if and only if $\mathbf{G} \setminus \tilde{\mathbf{S}}$ **is not** an **r-progression**, and for every subgroup $\mathbf{H}$ and every $\mathbf{a} \in \mathbf{G}$ with $|\mathbf{H}| \geq \mathbf{2}$,

$$|(\mathbf{H} \cup \mathbf{Ha})\tilde{\mathbf{S}}| \geq \min(|\mathbf{G}| - \mathbf{1}, |\mathbf{H} \cup \mathbf{Ha}| + |\mathbf{S}| + \mathbf{1}).$$

Proof If a twin pair exists
$\Rightarrow \exists \sigma$ a permutation on $\mathbf{S}$,
defined by $\mathbf{gs_i} = \mathbf{hs}_{\sigma(\mathbf{i})} \forall \mathbf{i} = \mathbf{1}, \dots |\mathbf{S}|$.
Hence,

$$\mathbf{h}^{-1}\mathbf{gS} = \mathbf{S}$$



Superconnected and Vosperian digraphs

Let $\mathbf{\Gamma} = (\mathbf{V}, \mathbf{E})$ a strongly connected $\mathbf{d}$-regular digraph ($\mathbf{d} \leq |\mathbf{V}| - \mathbf{4}$).

$\Gamma = (V, E)$ is superconnected

Any $\mathbf{d}$-subset of $\mathbf{V}$ whose removal destroys the strong connectedness of $\mathbf{\Gamma}$ is either $\mathbf{\Gamma}(\mathbf{x})$ or $\mathbf{\Gamma}^{-}(\mathbf{x})$ for some $\mathbf{x}$.

$\Gamma = (V, E)$ is vosperian

Any $\mathbf{d}$-subset of $\mathbf{V}$ whose removal destroys the strong connectedness of $\mathbf{\Gamma}$ creates **exactly** two strongly connected components one of size $\mathbf{1}$.

Superconnected and Vosperian digraphs

Let $\Gamma = (V, E)$ a strongly connected d-regular digraph ($d \leq |V| - 4$).

$\Gamma = (V, E)$ is superconnected

Any d-subset of V whose removal destroys the strong connectedness of Γ is either $\Gamma(x)$ or $\Gamma^{-}(x)$ for some x.

$\Gamma = (V, E)$ is vosperian

Any d-subset of V whose removal destroys the strong connectedness of Γ creates **exactly** two strongly connected components one of size 1.

Vosperian $\Rightarrow$ Superconnected

Superconnected $\not\Rightarrow$ Vosperian

Vosperian Cayley digraphs

Theorem

Let S be a generating subset of a finite group G, with $1 \notin S$, $|S| \le |G| - 4$ and $\tilde{S} = S \cup \{1\}$. Then $\Gamma = \mathrm{Cay}(G, S)$ is **vosperian** if and only if for every $r \in G \setminus 1$,

(i) $G \setminus \tilde{S}$ **is not** an r-**progression** and,

(ii) for every subgroup H and $a \in G$ with $|H| \ge 2$,

$$\min(|(H \cup Ha)\tilde{S}|, |\tilde{S}(H \cup aH)|) \ge \min(|G| - 1, |H \cup Ha| + |S| + 1).$$

Superconnected Cayley digraphs

Theorem

Let $\mathbf{S}$ be an **aperiodic** generating subset of a finite group $\mathbf{G}$, with $\mathbf{1} \notin \mathbf{S}$, $|\mathbf{S}| \leq |\mathbf{G}| - \mathbf{4}$ and $\tilde{\mathbf{S}} = \mathbf{S} \cup \{\mathbf{1}\}$. Then $\mathbf{\Gamma} = \mathrm{Cay}(\mathbf{G}, \mathbf{S})$ is **superconnected** if and only if one of the following conditions holds:

(i) $\mathbf{G} \setminus \tilde{\mathbf{S}}$ **is not** a right $\mathbf{r}$-progression, for every $\mathbf{r} \in \mathbf{G} \setminus \mathbf{1}$, and moreover for every $\mathbf{a} \in \mathbf{G}$ and every subgroup $\mathbf{H}$ with $|\mathbf{H}| \geq \mathbf{2}$,

$$\min(|(\mathbf{H} \cup \mathbf{Ha})\tilde{\mathbf{S}}|, |\tilde{\mathbf{S}}(\mathbf{H} \cup \mathbf{aH})|) \geq \min(|\mathbf{G}| - \mathbf{1}, |\mathbf{H} \cup \mathbf{Ha}| + |\mathbf{S}| + \mathbf{1}.)$$

(ii) $\mathbf{G} \setminus \tilde{\mathbf{S}}$ **is** a right $\mathbf{r}$-progression with $\mathbf{r} \notin \mathbf{S}$ or $\mathbf{r}^{-1} \notin \mathbf{S}$.

Superconnected Cayley digraphs

Corollary

Let $\mathbf{S}$ be an **aperiodic** generating subset of a finite group $\mathbf{G}$, with $\mathbf{1} \notin \mathbf{S}$, and $|\mathbf{S}| \leq |\mathbf{G}|/\mathbf{2}$. Then $\mathbf{\Gamma} = \mathrm{Cay}(\mathbf{G}, \mathbf{S})$ is **superconnected** if and only if

(i) either $\mathbf{\Gamma}$ is **vosperian** or

(ii) there is a $\mathbf{r} \in \mathbf{G}$ such that $\mathbf{G} =< \mathbf{r} >$ and $\mathbf{S} = \{\mathbf{r}, \mathbf{r}^2, \ldots, \mathbf{r}^{|\mathbf{S}|}\}$.