Handbook of geometry and all that...
(A work in progress...)

The Dream Team

Contents

Preface

This handbook is intended to be a self-contained reference for the most fundamental methods of differential and hyperbolic geometry, together with some applications to physics.

It must not be intended as an exhaustive treatise but as a (hopefully) clear exposition of these topics. In particular, we have tried to reduce to the minimum one of the major problems reported by students when learning for the first time differential geometry: notation! Formulae in differential geometry can easily become notationally unbearable if a bad choice of notation is performed. This implies that some reasonable shortcut must be implicitly assumed to avoid this problem and keep equations as readable and meaningful as possible.

Coherently with our main concern, the major sources of inspiration for our handbook (among others, that are duly quoted) are listed below.

- The extremely clear videos about differential geometry by Francesco Bottacin, professor at the university of Padova, Italy. They are available online (in Italian) at the following url: `https://www.math.unipd.it/~bottacin/geomdiff.htm`. A great deal of this handbook can be thought as a free translation of his notes and videos. Professor Bottacin is warmly acknowledged.

- J. Lee's treatise: 'Introduction to smooth manifolds' [10], one of the clearest, most complete, introductory books about differential geometry.

- C. Isham's splendid big little book [8], for once, a book about mathematical concepts written for physicists that does not treat them as 'dummies'.

- J.G. Ratcliffe's book: 'Foundations of hyperbolic manifolds' [15], to our knowledge, the treatise on hyperbolic geometry hat fits best with the spirit of this handbook.

Of course, every mistake in this document must be referred to the authors of each chapter and not to the books and material quoted above.

The authors.

Part I:

Introduction to differential geometry

In many cases, proofs based on coordinate free local representations in charts are clearer than proofs which are repleate with the claws of a rather unpleasant prying insect such as Γ^i_{jkl}.
S. Lang, 'Differential and Riemannian Manifolds', 1995

Chapter 1

Differential manifolds: definitions and basic properties (Edoardo Provenzi)

> Determinations of measure require magnitude to be independent of location, a state of things which can occur in more than one way.
>
> B. Riemann, 1854

In this first chapter we introduce the basic definitions and properties of differential manifolds. The reader not used to Einstein's convention for sum over repeated indices and differential calculus in $\mathbb{R}^n$ is referred to the appendices.

1.1 Differential manifolds

The first mathematician to conceive the idea of what we call today a differential manifold was Bernhard **Riemann** (1826 – 1866) who, in his groundbreaking 1854 habilitation defense [17], introduced the concept of an abstract manifold not necessarily embedded in a Euclidean space, as, instead, it was thought by his PhD advisor, the prince of mathematicians C.F. **Gauss** (1777 – 1855).

Riemann's ideas have been further refined until the modern definition of differential manifold that we report in this document, first introduced in the literature by Charles **Ehresmann** (1905 – 1979) [4] in 1943. In this definition a (finite dimensional) differential manifold is seen as a topological space (with some suitable requests to make calculus easier) with the fundamental requirement to be locally identifiable with a model space, which is a topological vector space.

The reason for considering topological vector spaces as local models lies in the fact that one of the fundamental elements of calculus, the derivative, represents a *local linearization* of a function, which explains the need of a linear structure on the model space that makes it a vector space. Moreover, the computation of derivatives requires the concept of limit, which implies that a topology coherent with the linear structure should be present. Finally, the fact that derivatives are defined in a local neighborhoods of points will allow us **transporting the differential structure of topological vector spaces to more general topological spaces that 'resemble' to them only locally**.

This local resemblance is provided by means of **homeomorphisms**, i.e. **bicontinuous** maps between topological spaces (continuous bijective functions with a continuous inverse).

Depending on the particular choice of topological vector space that is considered as local model, different differential manifolds can be defined. Classically, the local model is chosen to be $\mathbb{R}^n$, $n < +\infty$, but of course it can be $\mathbb{C}^n$ or an infinite-dimensional Frechet, Banach or Hilbert space and so on. Here, the local model will always be $\mathbb{R}^n$.

Before going through the details of differential manifolds, let us spend just a few words on topological manifolds.

Def. 1.1.1 (Topological manifold) *The couple given by a connected topological space M and a set of couples $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ (where A is an index set, U_α are open subsets of M), satisfying:*

- $M = \bigcup_{\alpha\in A} U_\alpha$, *i.e. the union of the sets U_α covers M*
- $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ *are homeomorphisms*[1],

is said to be a topological manifold of dimension n.

The definition of the dimension is well posed, in fact either there is a single homeomorphism that covers M, and so n is univocally defined, or at least the domain of two homeomorphisms has a non empty intersection. Suppose that these homeomorphisms are $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ and $\varphi_\beta : U_\beta \to \mathbb{R}^m$, with $U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \emptyset$. Then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \subseteq \mathbb{R}^n \to \varphi_\beta(U_{\alpha\beta}) \subset \mathbb{R}^m$ is a homeomorphism (as composition of homeomorphisms), this implies that $n = m$ because it cannot exist a homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$ if $n \neq m$, see e.g. [10]. Thus, n is an invariant in the definition of a topological manifold.

The fact that M is locally homeomorphic to an open set of $\mathbb{R}^n$ guarantees that, **locally, a topological manifold M defined as before has all the properties of** $\mathbb{R}^n$, e.g. M **is locally connected** (and locally connected by paths) and M **is locally compact**, i.e. every point $p \in M$ has a compact neighborhood, i.e., there exists an open set $U \subset M$ and a compact set $K \subset M$, such that $x \in U \subseteq K$. Other properties, e.g. the Hausdorff and second countable property, must be separately required.

Let us now move a step forward towards the concept of differential manifold.

Def. 1.1.2 *A topological space M is a* ***locally Euclidean space of dimension n*** $\in \mathbb{N}$, *$n < +\infty$, if:*

1. *it is a* ***Hausdorff space***[2]*: for every couple of elements $p, q \in M$, there exist two open neighborhoods U_p and U_q such that $U_p \cap U_q = \emptyset$;*
2. *it is* ***second countable***[3]*: there exists a countable collection $\mathcal{U} = \{U_i\}_{i=1}^\infty$ of open subsets of M such that any open subset of M can be written as a union of elements of some subfamily of $\mathcal{U}$;*

[1] i.e. bicontinuous functions: continuous invertible functions with continuous inverse, thus $\varphi_\alpha(U_\alpha)$ **is open in** $\mathbb{R}^n$ because it is the anti-image of the open U_α via the continuous map φ_α^{-1}.

[2] The Hausdorff property serves to assure that convergent sequences in M have a **unique limit**.

[3] The second countability is needed to assure the existence of a **partition of unity**, an essential tool to extend local objects to global ones.

3. *it is* ***locally homeomorphic to*** $\mathbb{R}^n$*: for every point* $p \in M$ *it exists an open neighborhood* $U \subseteq M$ *containing* p *and a homeomorphism:*

$$\begin{array}{rccl} \varphi : & U \subseteq M & \xrightarrow{\sim} & \varphi(U) \equiv V \subseteq \mathbb{R}^n \\ & p & \longmapsto & \varphi(p) = x = (x^1, \dots, x^n). \end{array}$$

The couple (U, φ) *is called a* ***local chart in p****, it is said to be* ***centered in p*** *if* $\varphi(p) = 0 \in \mathbb{R}^n$*.* U *is called* ***chart domain*** *and* φ ***chart map****.*

1.1.1 Local coordinates of a point

We are going to show that it is always possible to represent the position of any point p in a manifold M of dimension n with the coordinates of the local model $\mathbb{R}^n$ as long as we remain inside a chart domain U of a local chart (U, φ) in p.

The first step consists of course in applying the chart function φ to p to obtain the vector $x = \varphi(p)$ which lives in an open subset of $\mathbb{R}^n$ and the second step consists simply in extracting its components by using the functionals ε^j of the dual canonical basis of $\mathbb{R}^n$. The composition of these two steps gives rise to the following real-valued functions:

$$\begin{array}{rccl} x^j : & U \subseteq M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & x^j(p) = (\varepsilon^j \circ \varphi)(p). \end{array}$$

The x^j's are nothing but *the components functions of* φ *interpreted as a vector-valued function*[4], thus we can write:

$$\varphi \equiv (x^1, \dots, x^n), \quad \text{or} \quad \varphi \equiv (x^j)_{j=1}^n.$$

Def. 1.1.3 (Local coordinates) *The locally-defined real-valued functions*

$$x^j \equiv \varepsilon^j \circ \varphi : U \to \mathbb{R}$$

are called ***local coordinate functions*** *and the couple* $(U, (x^j))$ *is said to be a* ***local coordinate system*** *in* p*,* $j = 1, \dots, n$*.*

Notice the typical **abuse of notation** to write with x^j both the components of the image of $p \in M$ via the local chart φ w.r.t. the canonical basis of $\mathbb{R}^n$, which are *real numbers, and* the *real-valued functions* $\varepsilon^j \circ \varphi : U \to \mathbb{R}$.

On one side, this abuse of notation implies the weird formula $x^j(p) = x^j$, however, on the other side, in general it is clear when x^j refers to a function or a to real number and this notational simplification improves enormously the readability of expressions involving coordinates.

Following the idea of transporting the differential structure of $\mathbb{R}^n$ to a locally Euclidean space M, we must assure two things: the first is that **all the points of M are covered by a local chart**, the second is that **two intersecting charts** are compatible in the sense that the differential structure that they induce on M is **not in conflict**. The formalization of these ideas is given in the following definition.

Def. 1.1.4 (Atlas) *Given a locally Euclidean space* M *of dimension* n*, an* ***atlas*** *for* M *is a collection of charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$*, satisfying:*

[4] in fact some author denote them more correctly as φ^j instead of x^j.

1. ***Covering:*** *$\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ covers M, i.e.*
$$M = \bigcup_{\alpha \in A} U_\alpha$$

2. ***Compatibility:*** *whenever $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$, the function:*
$$\begin{array}{rccl} \eta_{\beta\alpha} := \varphi_\beta \circ \varphi_\alpha^{-1} : & \varphi_\alpha(U_{\alpha\beta}) \subseteq \mathbb{R}^n & \longrightarrow & \varphi_\beta(U_{\alpha\beta}) \subseteq \mathbb{R}^n \\ & x & \longmapsto & \tilde{x} := \eta_{\beta\alpha}(x) = \varphi_\beta(\varphi_\alpha^{-1}(x)), \end{array}$$
*is **smooth**, i.e. it belongs to $\mathscr{C}^\infty(\varphi_\alpha(U_{\alpha\beta}))$.*

The function $\eta_{\beta\alpha}$ is called **transition function** from the local representation $(U_\alpha, \varphi_\alpha)$ to (U_β, φ_β). It is invertible, being a composition of invertible functions, and its inverse is
$$\eta_{\beta\alpha}^{-1} = \eta_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}.$$

In general, showing that the charts domains of an atlas cover M and the smoothness of the chart maps is not a difficult task. **What requires much work is to verify** the compatibility, i.e. **that the transition functions are smooth**.

If the transition function $\eta_{\beta\alpha}$ is of class $\mathscr{C}^r$, then the compatibility will be called of class $\mathscr{C}^r$, but here we will always consider the smooth compatibility, unless otherwise stated.

By composing the transition functions with the elements of the canonical dual basis of $\mathbb{R}^n$ we obtain the functions that allow us transforming the local coordinates x^j of a point $p \in M$ w.r.t. the chart $(U_\alpha, \varphi_\alpha)$ to the local coordinates $\tilde{x}^j$ w.r.t. the chart (U_β, φ_β):
$$\begin{array}{rccl} \varepsilon^j \circ \eta_{\beta\alpha} : & \varphi_\alpha(U_{\alpha\beta}) \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x = (x^1, \ldots, x^n) & \longmapsto & \tilde{x}^j = (\varepsilon^j \circ \eta_{\beta\alpha})(x). \end{array}$$
Notice that $\varepsilon^j \circ \eta_{\beta\alpha}$ are nothing but the component functions of $\eta_{\beta\alpha}$ interpreted as vector-valued functions. Instead of denoting them as $\eta^j_{\beta\alpha}$, it is usual (in particular in Physics books) to write them simply with the symbol $\tilde{x}^j$:
$$\begin{array}{rccl} \tilde{x}^j : & \varphi_\alpha(U_{\alpha\beta}) \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x^i) & \longmapsto & \tilde{x}^j(x^i) = (\varepsilon^j \circ \eta_{\beta\alpha})(x^i), \end{array}$$
they are called the **local coordinate transformation functions**. The diagram below gives a graphical visualization of the objects just defined.

$$\begin{tikzcd}
& U_{\alpha\beta} \arrow[dr, "\varphi_\beta"] & \\
\mathbb{R}^n \supseteq \varphi_\alpha(U_{\alpha\beta}) \ni x \arrow[ur, "{\varphi_\alpha}^{-1}"] \arrow[rr, dashed, "\eta_{\alpha\beta}"] \arrow[drr, dashed, "\tilde{x}^j(x^i)"'] & & \tilde{x} \in \varphi_\beta(U_{\alpha\beta}) \subseteq \mathbb{R}^n \arrow[d, "\varepsilon^j"] \\
& & \tilde{x}^j \in \mathbb{R}
\end{tikzcd}$$

It should be clear from the context when $\tilde{x}^j$ represents a real number or a real-valued function, in any case, the weird notation $\tilde{x}^j = \tilde{x}^j(x^i)$ must be interpreted as follows:
$$\underset{\text{(real number)}}{\tilde{x}^j} = \underset{\text{(function } \mathbb{R}^n \to \mathbb{R})}{\tilde{x}^j} \underset{(\mathbb{R}^n \text{ vector})}{(x^i)},$$
and similarly for the inverse local coordinate transformation $x^i = x^i(\tilde{x}^j)$.

In general, a point in manifold M has always:

- a **local representation**, which lives in the local model $\mathbb{R}^n$, obtained by applying a local chart map;
- a **local coordinate representation**, which lives in $\mathbb{R}$ and it is obtained by further composing the local representation with the functionals of the canonical dual basis of the local model $\mathbb{R}^n$.

We will see that this considerations can be extended also to other objects defined on M, e.g. functions.

The compatibility between local charts can be equivalently stated in coordinates. To understand why, let us first recall the classical inverse function theorem of ordinary calculus in $\mathbb{R}^n$.

Theorem 1.1.1 (Inverse mapping theorem in $\mathbb{R}^n$) *Let:*

- $\Omega \subset \mathbb{R}^n$ *be an open set;*
- $f : \Omega \to \mathbb{R}^n$, $f \in \mathscr{C}^k(\Omega)$, $k \geqslant 1$;
- $x_0 \in \Omega$ *such that*[5]*:*
$$\det(Jf(x_0)) \neq 0.$$

Then there exist two neighborhoods $U \subseteq \Omega$ of x_0 and $V \subseteq \mathbb{R}^n$ of $f(x_0)$ such that $f|_U : U \to V$ is a $\mathscr{C}^k$-diffeomorphism.

If we organize the partial derivatives of the local coordinate transformation functions $\tilde{x}^j : \mathbb{R}^n \to \mathbb{R}$ in **the matrix of functions** $\frac{\partial \tilde{x}^j}{\partial x^i} : \mathbb{R}^n \to \mathbb{R}$ defined by $J^j_i := \left(\frac{\partial \tilde{x}^j}{\partial x^i}\right)_{i,j=1,\ldots n}$, explicitly:

$$J := \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n} \end{pmatrix},$$

then, if the determinant of the Jacobian matrix $J(x)$ is not null for every $x \in \varphi_\alpha(U_{\alpha\beta})$, the charts are compatible, i.e.

Compatibility condition between local charts in coordinates:

$$\det J(x) \neq 0 \qquad \forall x \in \varphi_\alpha(U_{\alpha\beta}),$$

where $J(x) \in M(n, \mathbb{R})$, $J(x) = \mathrm{ev}_x \circ J = \left(\frac{\partial \tilde{x}^j}{\partial x^i}(x)\right)_{i,j=1,\ldots,n}$, ev_x being the evaluation map of the functions $\frac{\partial \tilde{x}^j}{\partial x^i}$ in x.

[5]The geometrical interpretation of this condition is the following: the fact that the Jacobian matrix of f in p_0 is non-singular guarantees that the total derivative $Df(x_0) \in \mathrm{End}(\mathbb{R}^n)$ is invertible. Since the differential map is the linear approximation of f in a neighborhood of x_0, the result of the theorem says that this is enough to guarantee that, if we consider a sufficiently small neighborhood of p_0, f itself is invertible and its inverse map has the same regularity as f.

Def. 1.1.5 (Equivalent atlases) *Two atlases of a locally Euclidean space are equivalent if all the local charts of the first atlas are compatible with all those of the second atlas.*

Many authors define **two atlases** of a locally Euclidean space **equivalent if their union is again an atlas** for the same locally Euclidean space. Of course the two definitions are equivalent because, if all the local charts of the first are compatible with those of the second, then the covering and compatibility properties are satisfied and so we get an atlas; vice-versa, if the union is an atlas, then, by definition the compatibility of charts must be satisfied.

The adjective *equivalent* is not used by chance, in fact it can be verified that being equivalent is an actual equivalence relation in the set of atlases of locally Euclidean spaces.

This fact gives us the possibility to define the concept of differential manifold without ambiguity.

Def. 1.1.6 (Differential (smooth) manifold) *A differential (smooth) manifold of dimension n is a couple $(M, \mathcal{A})$, where M is a locally Euclidean space of dimension n and $\mathcal{A}$ is an equivalence class of smooth atlases of M. A (smooth) maximal atlas, i.e. an atlas that is not contained in any other atlases, is said to provide a (smooth)* ***differential structure*** *for M.*

If the compatibility among local charts is only of class $\mathscr{C}^r$, then we will talk about a $\mathscr{C}^r$ differential manifold. If the compatibility is analytic, in symbols $\mathscr{C}^\omega$, the manifold is called **real analytic**.

Convention: in this document we will only consider smooth manifolds, so we will omit to specify the adjective 'smooth' from now on, unless otherwise explicitly stated.

This choice is not so reductive after all, in fact, a celebrated theorem due to the great geometer Hassler Whitney [20] states that **every differential manifold of class $\mathscr{C}^1$ can always be endowed with a real-analytic maximal atlas and with $\mathscr{C}^r$ maximal atlases**, for all $r \geqslant 1$, which make it either a real-analytic or a $\mathscr{C}^r$ manifold (hence also a smooth manifold). Moreover, all the $\mathscr{C}^r$ differential structures are equivalent. Thus, for a manifold the really important gap to pass is that from a $\mathscr{C}^0$-compatibility between local charts to a $\mathscr{C}^1$-compatibility, the more regular compatibility being assured to exist thanks to Whitney's theorem.

If the local model is $\mathbb{C}^n$ and not $\mathbb{R}^n$, then we will talk about a **complex manifold** of dimension n, in this case the transition functions are required to be holomorphic.

1.2 Examples of manifolds

Let us discuss some example of manifold:

1. **The trivial manifold**. $\mathbb{R}^n$ is a manifold with the **canonical single chart atlas** given by $(\mathbb{R}^n, id_{\mathbb{R}^n})$.

   To give an example of non-equivalent atlases, let us consider $\mathbb{R}$ and the atlas $(\mathbb{R}, \varphi)$, where $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = \begin{cases} x & x \leqslant 0 \\ 2x & x > 0 \end{cases}$. This atlas is not compatible with the canonical atlas, in fact the transition function $\eta = \varphi \circ id_{\mathbb{R}}^{-1} = \varphi \circ id_{\mathbb{R}} = \varphi$ is continuous but not derivable in $x = 0$.

2. **Open submanifold**. Any open subset $U \subseteq \mathbb{R}^n$ is a manifold with single chart atlas given by (U, id_U).

3. **Product manifold**. If M and N are manifolds of dimension m and n, respectively, with atlases:
$$\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}, \quad \mathscr{B} = \{(V_\beta, \psi_\beta)\}_{\beta \in B},$$
respectively, then
$$\mathscr{A} \times \mathscr{B} := \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{(\alpha,\beta) \in A \times B},$$
where $\varphi_\alpha \times \psi_\beta$ is the Cartesian product maps[6]
$$\begin{array}{rccl} \varphi_\alpha \times \psi_\beta : & U_\alpha \times V_\beta & \longrightarrow & \mathbb{R}^m \times \mathbb{R}^n \\ & (x,y) & \longmapsto & (\varphi_\alpha \times \psi_\beta)(x,y) = (\varphi_\alpha(x), \psi_\beta(y)), \end{array}$$
is an atlas that makes the Cartesian product $M \times N$ a manifold, called the **product manifold** of M and N. Since $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{m+n}$, **the dimension of the product manifold is the sum of the factor manifolds**: $\dim(M \times N) = m + n$.

4. **Vector spaces of finite dimension as manifolds**. Let V be a real vector space of finite dimension n. Any norm on V determines a topology, which is known to be independent of the choice of the norm. With this topology, V is a topological manifold of dimension n. A natural differential structure on V can be defined thanks to the isomorphism between V and its prototype $\mathbb{R}^n$. More precisely, if $E = (e_1, \ldots, e_n)$ is any basis of V, then $I : V \to \mathbb{R}^n$, $v = v^i e_i \mapsto (v^i)_{i=1}^n$, is a linear isomorphism and also a homeomorphism in the topology induced by the norm. It follows that (V, I) is a *global chart* for V that can be used as a single-chart atlas.

   Any other basis $\tilde{E} = (\tilde{e}_1, \ldots, \tilde{e}_n)$ will induce a new global chart for V given by $(V, \tilde{I})$, where $\tilde{I} : V \to \mathbb{R}^n$, $v = \tilde{v}^i \tilde{e}_i \mapsto (\tilde{v}^i)_{i=1}^n$. To find the transition functions between these two charts, let us first recall that the change-of-basis matrix $A = (a_i^j)$, defined by $e_i = a_i^j \tilde{e}_j$, is invertible. From the equation
$$v = \tilde{v}^j \tilde{e}_j = v^i e_i = v^i a_i^j \tilde{e}_j, \qquad \forall v \in V,$$
we deduce that $\tilde{v}^j = a_i^j v^i$, i.e. the coordinates of any $v \in V$ w.r.t. the two charts, are related by an invertible linear transformation, which is obviously a diffeomorphism in $\mathbb{R}^n$. As a consequence, V is a smooth manifold.

   The differential structure defined in this way is called the *standard differential structure of the real vector space V*.

5. **The manifold of matrices**. The group of $m \times n$ matrices with real entries $M(m \times n, \mathbb{R})$ is known to be isomorphic with $\mathbb{R}^{mn}$ via the lexicographic order of the matrix elements (ordered by either rows or columns), thus it is a manifold of dimension mn. $M(m \times n, \mathbb{C})$ is a $2mn$ dimensional real manifold.

[6] We have used the Cartesian product map , defined as follows: given $f : D_f \to R_f$ and $g : D_g \to R_g$, D and R are used for domain and range, the Cartesian product function between f and g is:
$$\begin{array}{rccl} f \times g : & D_f \times D_g & \longrightarrow & R_f \times R_g \\ & (x,y) & \longmapsto & (f \times g)(x,y) := (f(x), g(x)). \end{array}$$

6. **The manifold of invertible matrices**. $\mathrm{GL}(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}),\ \det(A) \neq 0\}$ is not only a subset of $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, but it is also *open* w.r.t. the topology of $\mathbb{R}^{n^2}$. In fact, $\mathrm{GL}(n,\mathbb{R}) = (\det^{-1}\{0\})^c$, i.e. it is the complementary set of the inverse image of 0 via the determinant function, being $\{0\}$ a closed set, $\det^{-1}\{0\}$ is closed because det is a continuous function, thus $\mathrm{GL}(n,\mathbb{R})$ is the complementary of a closed set, so it is an open set. As open subset of $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, **$\mathrm{GL}(n,\mathbb{R})$ is manifold of dimension n^2**. $\mathrm{GL}(n,\mathbb{C})$ is a $2n^2$ dimensional real manifold.

7. **The sphere as a manifold**. Proving that a spherical surface in $\mathbb{R}^{n+1}$, briefly a sphere, is a manifold is a classical and beautiful computation in differential geometry. Before considering the most general case, we start with the easiest one, i.e. that of the 1-dimensional sphere of radius 1, which has the advantage of showing us in a very clear geometrical way how to build an atlas. We will then extend this same construction to the n-dimensional case and to a generic radius $R > 0$.

   Let $S^1 := \{x \in \mathbb{R}^2\ :\ \|x\| = 1\}$, where $\|\ \|$ is the Euclidean norm, be the 1-dimensional unit sphere in $\mathbb{R}^2$, i.e. with radius equal to 1. We start by considering the following identification:
$$\pi := \{x \in \mathbb{R}^2,\ x = (x^1, 0)\} \cong \mathbb{R},$$
   then we define the north pole N, south pole S and a generic point p of the 1-dimensional sphere S^1 as follows:
$$\begin{cases} N = (0,1) = e_2 \ \text{(the second element of the canonical basis of } \mathbb{R}^2) \\ S = (0,-1) = -N \\ p = (p^1, p^2). \end{cases}$$
   Let us now consider $\mathcal{A} := \{(U_1, \varphi_N), (U_2, \varphi_S)\}$, where $U_1 := S^1 \backslash \{N\}$, $U_2 := S^1 \backslash \{S\}$, and
$$\begin{array}{rccl} \varphi_N : & U_1 & \longrightarrow & \pi \\ & (p^1,p^2) & \longmapsto & \varphi_N(p^1,p^2) := \frac{1}{1-p^2}\, p^1, \end{array} \qquad \begin{array}{rccl} \varphi_S : & U_2 & \longrightarrow & \pi \\ & (p^1,p^2) & \longmapsto & \varphi_S(p^1,p^2) := \frac{1}{1+p^2}\, p^1. \end{array}$$
   The functions φ_1 and φ_2 are called **stereographic projections** from the north and the south pole, respectively. Their geometrical meaning is represented in figure 1.1.

   The (unique) intersection between $\pi \cong \mathbb{R}$ and the straight line that connects $N = (0,1)$ with $p = (p^1, p^2)$ can be determined as follows: the Cartesian equation of this straight line is of course $y(p) = 1 + \frac{1-p^2}{0-p^1}(p-0)$, i.e. $y(p) = 1 - \frac{1-p^2}{p^1}p$, so the only value of $p^* \in \pi$ such that $y(p^*) = 0$ is $p^* = \frac{1}{1-p^2}p^1 = \varphi_N(p^1,p^2)$, thus the stereographic projection from the north pole is simply the point p^*. Analogous considerations can be done for the stereographic projection from the south pole, obtaining $\varphi_S(p^1,p^2) = \overline{p}$.

   We observe that the stereographic projection from N excludes from its domain N itself and maps the south pole to the origin of $\pi \cong \mathbb{R}$, in fact: $\varphi_N(S) = \varphi_1(0,-1) = \frac{1}{2}\,0 = 0$. The same considerations hold exchanging N with S and φ_N with φ_S. Of course $U_1 \cup U_2 = S^1$, so the covering property is verified by $\mathcal{A}$, we must check the compatibility. φ_N and φ_S are of course smooth and invertible on their respective domains, let us make the transition functions between them explicit in order to check if they are smooth.

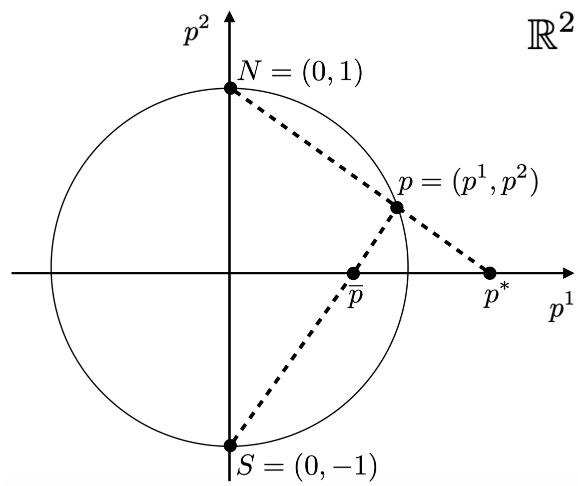


Figure 1.1: The stereographic projection from the north pole in 2D.

We start with φ_N: its inverse function is $\varphi_N^{-1} : \pi \to S^1 \backslash \{N\}$, $x \mapsto \varphi_N^{-1}(x) = p$, with p such that $\varphi_N(p) = x$, i.e. $\frac{p^1}{1-p^2} = x$. If we manage to write $p^2(x)$, i.e. p^2 as a function of x, then, considering that

$$p^1 = (1-p^2)x, \tag{1.1}$$

we manage to express also p^1 as a function of x, thus making φ_N^{-1} explicit. In order to do so, it is convenient to use the constraint that defines S^1, i.e. $\|p = (p^1, p^2)\| = 1 \iff \|(p^1, p^2)\|^2 = 1$, or:

$$(p^1)^2 + (p^2)^2 = 1 \iff (p^1)^2 = 1 - (p^2)^2 = (1-p^2)(1+p^2),$$

which, introduced in the square of eq. (1.1) gives:

$$\cancel{(1-p^2)}(1+p^2) = (1-p^2)^{\cancel{2}} x^2 \iff 1 + p^2 = x^2 - x^2 p^2 \iff p^2(x) = \frac{x^2-1}{x^2+1},$$

which, introduced in eq. (1.1) gives:

$$p^1(x) = (1 - p^2(x))x = \left(1 - \frac{x^2-1}{x^2+1}\right) x = \frac{2}{x^2+1} x.$$

Hence, the explicit expression of φ_N^{-1} is:

$$\begin{aligned} \varphi_N^{-1} : \quad \pi \cong \mathbb{R} &\longrightarrow S^1 \backslash \{N\} \\ x &\longmapsto \varphi_N^{-1}(x) = \big(p^1(x), p^2(x)\big) = \left(\tfrac{2}{x^2+1} x, \tfrac{x^2-1}{x^2+1}\right), \end{aligned}$$

analogously, we obtain:

$$\begin{aligned} \varphi_S^{-1} : \quad \pi \cong \mathbb{R} &\longrightarrow S^1 \backslash \{S\} \\ x &\longmapsto \varphi_S^{-1}(x) = \left(\tfrac{2}{x^2+1} x, \tfrac{1-x^2}{x^2+1}\right). \end{aligned}$$

We can now compute the transition functions explicitly to test if they are smooth: first of all we notice that, since $\varphi_N(S) = \varphi_S(N) = 0$, on the intersection $U_{1,2} := U_1 \cap U_2 =$

$S^1 \backslash \{N, S\}$ we have that $\varphi_N(U_{1,2}) = \varphi_S(U_{1,2}) = \mathbb{R}\backslash\{0\}$, so $\eta_{SN} := \varphi_S \circ \varphi_N^{-1} : \mathbb{R}\backslash\{0\} \to U_{1,2}$ and similarly for η_{NS}. By direct computation we have, for all $y \in \mathbb{R}\backslash\{0\}$,

$$\eta_{SN}(y) = \varphi_S(\varphi_N^{-1}(y)) = \varphi_S\left(\frac{2}{y^2+1}y, \frac{y^2-1}{y^2+1}\right) = \frac{\frac{2}{y^2+1}y}{1+\frac{y^2-1}{y^2+1}} = \frac{2y}{2y^2} = \frac{1}{y},$$

which is a smooth function on $\mathbb{R}\backslash\{0\}$, similarly:

$$\eta_{NS}(y) = \varphi_N(\varphi_S^{-1}(y)) = \varphi_N\left(\frac{2}{y^2+1}y, \frac{1-y^2}{y^2+1}\right) = \frac{\frac{2}{y^2+1}y}{1-\frac{1-y^2}{y^2+1}} = \frac{1}{y},$$

again, a smooth function on $\mathbb{R}\backslash\{0\}$. Thus, the transition functions between the charts defined by the stereographic projections are smooth, so $\mathcal{A}$ is an atlas for S^1, which acquires the status of **smooth manifold of dimension 1 with local model** $\mathbb{R}$.

Let us consider the general case. We call **sphere of radius** $R > 0$ the subset of $\mathbb{R}^{n+1}$ given by

$$\boxed{S_R^n = \{x \in \mathbb{R}^{n+1}, \ \|x\| = R\}}\,, \tag{1.2}$$

where $\| \ \|$ is the Euclidean norm. If $R = 1$ we simply write S^n. The sphere S_R^n is a n-dimensional manifold for every $R > 0$. To prove it, let us build an atlas with two charts and show that the transition functions are smooth. As before, we use the stereographic projections of the generic point $p \in S_R^n$ from the north N and the south S pole:

$$\begin{cases} N = (0, \ldots, 0, R) = Re_{n+1} \\ S = (0, \ldots, 0, -R) = -N \\ p = (p^1, \ldots, p^{n+1}) \end{cases}$$

onto the hyperplane

$$\pi := \{x \in \mathbb{R}^{n+1}, \ x = (x^1, \ldots, x^n, 0)\} \cong \mathbb{R}^n.$$

The first chart is: $(S_R^n \backslash \{N\}, \varphi_N)$, with

$$\begin{array}{rccl} \varphi_N: & S_R^n\backslash\{N\} & \longrightarrow & \pi \\ & p = (p^1, \ldots, p^{n+1}) & \longmapsto & \varphi_N(p) = \frac{R}{R-p^{n+1}}(p^1, \ldots, p^n). \end{array} \tag{1.3}$$

This time, to understand why the stereographic projection of p from the north pole N has this analytic form, instead of the Cartesian equation of the straight line connecting N to p, we consider (just to offer another possible view) its parametric equation, i.e. $x : \mathbb{R} \to \mathbb{R}^{n+1}$, $t \mapsto x(t) = N + t(p - N)$, notice that $x(0) = N$, $x(1) = p$. Since the coordinates of N are all zero unless the last one which is equal to R, the coordinates of $x(t)$ are

$$\begin{cases} x^1(t) = tp^1 \\ x^2(t) = tp^2 \\ \vdots \\ x^n(t) = tp^n \\ x^{n+1}(t) = R + t(p^{n+1} - R). \end{cases}$$

The point $\varphi_N(p) \in \pi$ is obtained by applying on the previous coordinates the constraint that defines π, i.e. by imposing $x^{n+1}(t) = 0$, or $x^{n+1} = R + t(p^{n+1} - R) = 0 \iff t = \frac{R}{R-p^{n+1}}$, so

$$\varphi_N(p) = (x^1(t), \ldots, x^n(t))\big|_{t=R/(R-p^{n+1})},$$

i.e. eq. (1.3).

Notice that $\varphi_N(N)$ is not defined[7] and that, if we take $p = S = (0, \ldots, 0, -R)$, then $p^i = 0$ for all $i = 1, \ldots, n$ and $p^{n+1} = -R$, so $\varphi_N(S) = (0, \ldots, 0)$, i.e. the stereographic projection from the north pole of the south pole is the origin of $\mathbb{R}^n$.

By the unicity of the intersection between the hyperplane π and the straight line passing through N and p, we have that φ_N is bijective.

The inverse of φ_N is defined as:

$$\begin{array}{rccc} \varphi_N^{-1}: & \pi & \longrightarrow & S^n_R \setminus \{N\} \\ & x = (x^1, \ldots, x^n) & \longmapsto & \varphi_N^{-1}(x) = p, \end{array}$$

where $x = \varphi_N(p)$, i.e. $(x^1, \ldots, x^n) = \frac{R}{R-p^{n+1}}(p^1, \ldots, p^n)$, thus

$$(p^1, \ldots, p^n) = \frac{R - p^{n+1}}{R}(x^1, \ldots, x^n), \tag{1.4}$$

which shows also for this general case that we just need to compute p^{n+1} as a function of x, i.e. $p^{n+1}(x^1, \ldots, x^n)$, to express also $p^1, \ldots, p^n$ as functions of $(x^1, \ldots, x^n)$ and thus finding the explicit expression of φ_N^{-1}.

As in the 1-dimensional case, we take advantage of the constraint that defines S^n_R, i.e. $p \in S^n_R$ if and only if $(p^1)^2 + \cdots + (p^n)^2 + (p^{n+1})^2 = R^2$, thus

$$(p^1)^2 + \cdots + (p^n)^2 = R^2 - (p^{n+1})^2 = (R - p^{n+1})(R + p^{n+1}). \tag{1.5}$$

If we compute the sum of the square components of both sides of eq. (1.4) we get:

$$(p^1)^2 + \cdots + (p^n)^2 = \frac{(R - p^{n+1})^2}{R^2}(x^1)^2 + \cdots + (x^n)^2 \underset{(x^{n+1}=0!)}{=} \frac{(R - p^{n+1})^2}{R^2}\|x\|^2,$$

but thanks to eq. (1.5),

$$\cancel{(R - p^{n+1})}(R + p^{n+1}) = \frac{(R - p^{n+1})^{\cancel{2}}}{R^2}\|x\|^2 \iff \frac{R + p^{n+1}}{R - p^{n+1}} = \frac{\|x\|^2}{R^2},$$

which, solved w.r.t. p^{n+1}, gives

$$p^{n+1}(x) = R\frac{\|x\|^2 - R^2}{\|x\|^2 + R^2}.$$

By inserting this expression for $p^{n+1}(x)$ in eq. (1.4) we get:

$$p^j(x) = 1 - \frac{\|x\|^2 - R^2}{\|x\|^2 + R^2}x^j = \frac{2R^2}{\|x\|^2 + R^2}x^j, \quad j = 1, \ldots, n,$$

[7] If φ_N would be defined on the whole sphere, it would create a homeomorphism between a compact set and a non compact one, $\mathbb{R}^n$, which is impossible. This observation leads to the conclusion that **it is not possible to have a single chart atlas for the sphere**, or any other compact manifold in $\mathbb{R}^n$.

thus

$$\varphi_N^{-1}(x^1,\dots,x^n) = \left(\frac{2R^2}{\|x\|^2+R^2}x^1,\dots,\frac{2R^2}{\|x\|^2+R^2}x^n, R\frac{\|x\|^2-R^2}{\|x\|^2+R^2}\right). \tag{1.6}$$

The stereographic projection from the south pole is built in the same way, we simply have to replace N with S, obtaining

$$\begin{array}{rccl} \varphi_S: & S^n_R\backslash\{S\} & \longrightarrow & \pi \\ & p=(p^1,\dots,p^{n+1}) & \longmapsto & \varphi_S(p)=\frac{R}{R+p^{n+1}}(p^1,\dots,p^n). \end{array}$$

with $\varphi_S(N)=(0,\dots,0)$ and

$$\begin{array}{rccl} \varphi_S^{-1}: & \pi & \longrightarrow & S^n_R\backslash\{S\} \\ & x=(x^1,\dots,x^n) & \longmapsto & \varphi_S^{-1}(x)=\frac{1}{\|x\|^2+R^2}(2R^2x^1,\dots,2R^2x^n,R(R^2-\|x\|^2)). \end{array}$$

Having at disposal the explicit expressions of φ_S, φ_N and their inverses, we can check the compatibility between them, i.e. that the transition functions are smooth on the intersection $S^n_R\backslash\{N,S\}$. Since $\varphi_N(S)=\varphi_S(N)=0$, we have

$$\varphi_N(S^n_R\backslash\{N,S\}) = \mathbb{R}^n\backslash\{0\} = \varphi_S(S^n_R\backslash\{N,S\})$$

so $\eta_{SN} := \varphi_S\circ\varphi_N^{-1} : \mathbb{R}^n\backslash\{0\}\to\mathbb{R}^n\backslash\{0\}$, $y\mapsto\eta_{SN}(y)$. We have:

$$\eta_{SN}(y) = \frac{R}{R+R\frac{\|y\|^2-R^2}{\|y\|^2+R^2}}\frac{1}{\|y\|^2+R^2}(2R^2y^1,\dots,2R^2y^n) \iff \eta_{SN}(y)=\frac{R^2}{\|y\|^2}y,$$

which is smooth because $y\neq 0$ in the domain of η_{SN}. Moreover, since $\eta_{NS}=\eta_{SN}^{-1}$, we have $\eta_{NS}(y)=\frac{\|y\|^2}{R^2}y = \frac{R^2}{\|y\|^2}y$, smooth as well. This shows that $((U_N,\varphi_N),(U_S,\varphi_S))$ is an atlas for S^n_R, called **stereographic atlas** and that S^n_R **is a smooth manifold of dimension** n **with local model** $\mathbb{R}^n$.

8. **An alternative (but compatible) atlas on the sphere**. There are other atlases, compatible with the stereographic atlas that can be built on the sphere. For the sake of clarity, let us consider S^1 to show an alternative (very redundant) atlas that can be proven by direct computation to be compatible with the stereographic atlas. This is the atlas: $\mathcal{B}=\{(U_i,\varphi_i),\ i=1,\dots,4\}$, where:

$$\begin{cases} U_1=\{(p^1,p^2)\in S^1\ :\ p^1>0\}, & \varphi_1(p^1,p^2):=p^2 \\ U_2=\{(p^1,p^2)\in S^1\ :\ p^2>0\}, & \varphi_{12}(p^1,p^2):=p^1 \\ U_3=\{(p^1,p^2)\in S^1\ :\ p^1<0\}, & \varphi_{13}(p^1,p^2):=p^2 \\ U_4=\{(p^1,p^2)\in S^1\ :\ p^2<0\}, & \varphi_{14}(p^1,p^2):=p^1. \end{cases}$$

9. **The n-torus**. Thanks to example 3. we can build the product manifold:

$$\mathbb{T}^n = S^1\times\cdots\times S^1$$

which is a compact manifold.

10. **The real projective manifold**. It is defined as follows:

$$\boxed{\mathbb{RP}^n := \mathbb{R}^{n+1}\backslash\{0\}/\sim},$$

where

$$\forall x, y \in \mathbb{R}^{n+1}\backslash\{0\},\ x \sim y \iff \exists \lambda \in \mathbb{R}\backslash\{0\}\ :\ y = \lambda x,$$

i.e. with $\sim$ we **identify any two non-zero vectors in $\mathbb{R}^{n+1}$ which are multiples of each other by a non zero real coefficient**: $(x^0, \dots, x^n) = (\lambda x^0, \dots, \lambda x^n)$.

So, the elements of the projective manifold will be equivalence classes of vectors in $\mathbb{R}^{n+1}\backslash\{0\}$ that lie on the same straight line passing through the origin[8].

Endowed with the quotient topology, $\mathbb{RP}^n$ is a topological manifold, we will prove that $\mathbb{RP}^n$ is also a differential manifold of dimension n and this will provide a first example of manifold that is not made up by a subset of points in $\mathbb{R}^d$, $d \geqslant 1$, as the elements of $\mathbb{RP}^n$ can be identified with straight lines in $\mathbb{R}^{n+1}$ and not points of $\mathbb{R}^{n+1}$!

A typical notation used when dealing with the projective manifold is the following:

$$(x^0 : \dots : x^n) := (\lambda x^0 : \dots : \lambda x^n) \qquad \forall \lambda \neq 0,$$

$(x^0 : \dots : x^n)$ are called **homogeneous coordinates** of an element in $\mathbb{RP}^n$.

Let us construct an atlas with compatible charts for $\mathbb{RP}^n$ by considering the following open domains:

$$U_i := \{(x^0 : \dots : x^n) \in \mathbb{RP}^n\ :\ x^i \neq 0\},$$

i.e. the i-th homogeneous coordinate of the elements belonging to U_i is non null (the others can be null or not, but the i-th surely not). There are $n+1$ such domains and they trivially cover $\mathbb{RP}^n$, i.e. $\mathbb{RP}^n = \bigcup_{i=0}^{n} U_i$, in fact, having removed 0 from $\mathbb{R}^{n+1}$, at least one homogeneous coordinate of an arbitrary element of $\mathbb{RP}^n$ must be different from 0, but then it belongs to a suitable U_i.

The chart maps on U_i are defined as follows:

$$\begin{array}{rccl} \varphi_i : & U_i & \xrightarrow{\sim} & \mathbb{R}^n \\ & (x^0 : \dots : x^n) & \longmapsto & \varphi_i(x^0 : \dots : x^n) := \left(\frac{x^0}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^n}{x^i}\right), \end{array}$$

analytically well defined because in U_i, $x_i \neq 0$. Notice that we only have n components in the image of φ_i because the i-th component gives $\frac{x^i}{x^i} = 1$, which is a fixed value that we remove from the image. φ_i does not depend on the particular representative in the equivalence class where $(x^0 : \dots : x^n)$ belongs, in fact:

$$U_i \ni (\lambda x^0 : \dots : \lambda x^i : \dots : \lambda x^n) \overset{\varphi_i}{\mapsto} \left(\frac{\lambda x^0}{\lambda x^i}, \dots, \frac{\lambda x^{i-1}}{\lambda x^i}, \frac{\lambda x^{i+1}}{\lambda x^i}, \dots, \frac{\lambda x^n}{\lambda x^i}\right) = \varphi_i(x^0 : \dots : x^n),$$

so that $\varphi_i(\lambda x^0 : \dots : \lambda x^n) = \varphi_i(x^0 : \dots : x^n)\ \forall \lambda \neq 0$.

[8]Actually, since we have eliminated 0 from $\mathbb{R}^{n+1}$, the vectors belong to two opposite half lines with origin in 0, but, of course, these half lines identify in a unique way a straight lines passing through the origin of $\mathbb{R}^{n+1}$. This identification will be implicitly assumed in the main text.

φ_i is invertible, its inverse being the map that restores the value 1 after the i-th position starting from the value 1 of the index:

$$\begin{array}{rccl}\varphi_i^{-1}: & \mathbb{R}^n & \xrightarrow{\sim} & U_i \\ & (y^1,\dots,y^n) & \longmapsto & \varphi_i(y^1,\dots,y^n)=(y^1:\cdots:y^i:1:y^{i+1}:\cdots:y^n).\end{array}$$

In fact,

$$\varphi_i^{-1}(\varphi_i(x^0:\cdots:x^n))=\varphi_i^{-1}\left(\frac{x^0}{x^i},\dots,\frac{x^{i-1}}{x^i},\frac{x^{i+1}}{x^i},\dots,\frac{x^n}{x^i}\right)=\left(\frac{x^0}{x^i},\dots,\frac{x^{i-1}}{x^i},1,\frac{x^{i+1}}{x^i},\dots,\frac{x^n}{x^i}\right),$$

where, since in the last expression we start from the index 0, the value 1 must be restored after the $(i-1)$-th position. By definition of homogeneous coordinates we have $\left(\frac{x^0}{x^i},\dots,\frac{x^{i-1}}{x^i},1,\frac{x^{i+1}}{x^i},\dots,\frac{x^n}{x^i}\right)=(x^0:\cdots:x^{i-1}:x^i:x^{i+1}:\dots x^n)$, so $\varphi_i^{-1}\circ\varphi_i=id_{U_i}$ and, by an analogous computation, we have $\varphi_i\circ\varphi_i^{-1}=id_{\mathbb{R}^n}$.

$\{(U_i,\varphi_i),\ i=0,\dots,n\}$ is a $(n+1)$-charts atlas for the projective manifold if we can show that these charts are compatible on the intersections of their domains. For that, notice that, when $i\neq j$, the condition $U_i\cap U_j\neq\varnothing$ implies, by definition of the sets U_i and U_j, that the i-th and the j-th homogeneous coordinates of the elements of $\mathbb{RP}^n$ belonging to $U_i\cap U_j$ are both $\neq 0$. If $i<j$, the transition functions can be written as follows:

$$\begin{aligned}\eta_{ij}&=\varphi_i\circ\varphi_j^{-1}(y^1,\dots,y^n)=\varphi_i(y^1:\cdots:y^j:1:y^{j+1}:\cdots:y^n)\\&=\left(\frac{y^1}{y^i},\dots,\frac{y^{i-1}}{y^i},\frac{y^{i+1}}{y^i},\dots,\frac{y^j}{y^i},\frac{1}{y^i},\frac{y^{j+1}}{y^i},\dots,\frac{y^n}{y^i}\right),\end{aligned}$$

if $j<i$, we simply exchange i with j in the previous expression. Notice the gap between the $(i-1)$-th and the $(i+1)$-th coordinate, which guarantees the correct number of components. η_{ij} is evidently smooth because y^i and y^j are non null. Since $\eta_{ji}=\varphi_j^{-1}\circ\varphi_i$, we get exactly the same functional expression with inverted indices, thus also η_{ji} is smooth. So, $\mathbb{RP}^n$ **is a differential manifold of dimension** n.

11. **Grassmannian manifolds**. We have seen that $\mathbb{RP}^n$ can be identified with the set of vector subspaces of order 1 (the straight lines passing through the origin) of $\mathbb{R}^{n+1}$. More generally, if V is a real n-dimensional vector space, we define:

$$\boxed{\mathrm{Gr}_k(V):=\{W\ :\ W \text{ is a vector subspace of dimension } k \text{ of } V\}}\,.$$

It can be proven that $\mathrm{Gr}_k(V)$ is a differential manifold of dimension $k(n-k)$, called the Grassmannian manifold of order k of V. It is clear that:

$$\boxed{\mathbb{RP}^n=\mathrm{Gr}_1(\mathbb{R}^{n+1})}\,.$$

12. $\mathbb{RP}^n$ **as a suitable quotient of the sphere** S^n. Consider a vector $x\in\mathbb{R}^{n+1}\backslash\{0\}$, then x and $\frac{x}{\|x\|}$, $\|\ \|$ being the Euclidean norm on $\mathbb{R}^{n+1}$, define the same element of $\mathbb{RP}^n$. However, $\frac{x}{\|x\|}$ belongs to the sphere $S^n=\{x\in\mathbb{R}^{n+1},\ \|x\|=1\}$, this very simple observation shows that we can always see $\mathbb{RP}^n$ as a subset of S^n and that the map $\pi:S^n\to\mathbb{RP}^n$, $x=(x^0,\dots,x^n)\mapsto(x^0:\cdots:x^n)$ is surjective.

Notice however that π is not injective, because $\pi(x) = \pi(-x)$ for all $x \in S^n$, in fact $-\frac{x}{\|x\|}$ belongs to the same equivalence class as x and $\frac{x}{\|x\|}$ in the projective manifold! x and $-x$ are called **antipodal points**. To remove the lack of injectivity, it is sufficient to identify the antipodal points on the sphere S^n, i.e. to operate the quotient $S^n/\sim$, $x \sim -x$ for all $x \in S^n$. It is not difficult to prove that $S^n/\sim$ **endowed with the quotient topology, is isomorphic, as a differential manifold, to** $\mathbb{RP}^n$.

This example shows **how much manifold can be modified by a quotient**: in this case, we pass from a spherical surface, to a set of straight lines passing through the origin!

1.2.1 Manifolds from the level-set theorem in $\mathbb{R}^{n+m}$

Noticeable examples of manifolds embedded in a Euclidean space of suitable dimension can be built thanks to the so-called level-set theorem, which is a consequence of the inverse mapping theorem.

Let us consider $f : \Omega \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ open, $f \in \mathscr{C}^1(\Omega)$.

Def. 1.2.1 *$x \in \Omega \subset \mathbb{R}^n$ is a **critical point** of f if the total derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ is not onto, i.e. if rank$(Df(x)) < m$. A **critical value** of f is the image via f of a critical point x of f, so $f(x) \in \mathbb{R}^m$. We denote with $\mathrm{Crit}(f) \subset \Omega$ the set of critical points of f. A **regular value** of f is an element in $f(\Omega)$ that is not critical for f.*

It is easy to see that $\mathrm{Crit}(f)$ is a closed subset of Ω. The following result gives a (not necessary) sufficient condition for a set to be a manifold.

Theorem 1.2.1 (Level set theorem in $\mathbb{R}^{n+m}$) *Let:*

- $\Omega \subseteq \mathbb{R}^{n+m}$ *open set*
- $f : \Omega \to \mathbb{R}^m$, $f \in \mathscr{C}^\infty(\Omega)$
- $a \in f(\Omega)$.

Then, the set

$$M_a = f^{-1}(a) \backslash \mathrm{Crit}(f),$$

i.e. the a-level set of f without the critical points, is a smooth manifold of dimension n (the difference between the dimension of the domain and the codomain of f), w.r.t. the differential structure inherited by $\mathbb{R}^{n+m}$.

Of course, if f does not have critical points, then $M_a = f^{-1}(a)$.

Thanks to this theorem we can prove quite easily that the most important matrix groups are differential manifolds.

- **SL**$(n, \mathbb{R})$ **as a manifold of dimension** $n^2 - 1$. The function to be considered here is the determinant of a $n \times n$ matrix with real entries:

$$\begin{array}{rccl} \det : & M(n, \mathbb{R}) \cong \mathbb{R}^{n^2} & \longrightarrow & \mathbb{R} \\ & A & \longmapsto & \det(A). \end{array}$$

If $A = (a_i^j) \in M(n, \mathbb{R})$, then, by Laplace's formula, $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_j^i \det(A_j^i)$, where $A_j^i \in M(n-1, \mathbb{R})$ is the submatrix of A obtained by eliminating the i-th row and the j-th column. Being a polynomial function, det is smooth. Moreover,

$$\frac{\partial \det}{\partial a_j^i}(A) = (-1)^{i+j} \det(A_j^i),$$

which shows that the critical points of det are given by the matrices $A \in M(n, \mathbb{R})$ whose sub-matrices $A_j^i \in M(n-1, \mathbb{R})$ have 0 determinant. In fact, in that case, the total derivative would not be onto: the partial derivatives are the entries of the Jacobian matrix and, if they are null, this matrix lacks to be full rank. This situation can happen only if A has rank strictly inferior to $n-1$, so:

$$\mathrm{Crit}(\det) = \{A \in M(n, \mathbb{R}) \ : \ \mathrm{rank}(A) \leqslant n-2\}.$$

Any $A \in \mathrm{Crit}(\det)$ has null determinant, thus the only critical value for det is 0. Since $\mathrm{SL}(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \ : \ \det(A) = 1\} = \det^{-1}\{1\}$, and 1 is a regular value for det, it follows that **$\mathbf{SL}(n, \mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$.**

As a consequence of this result, **$\mathbf{SL}(n, \mathbb{C})$ is a (real) manifold of dimension $2n^2 - 2$.**

An alternative proof consists in observing that, thanks to equation (**??**), the matrices of $SL(n, \mathbb{R})$ are not critical point for the determinant.

- With similar, but more sophisticated, techniques based on the rank theorem, it can be proven that:

  - **$\mathbf{O}(n)$ and $\mathbf{SO}(n)$ are manifolds of dimension $\frac{n(n-1)}{2}$;**
  - **$\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are (real) manifolds of dimension n^2.**

  We will show how to prove that $\mathrm{O}(n)$ is a manifold through the rank theorem in section 2.9.2 after discussing the concept of differential of functions between manifolds.

- We now show how easy it is to prove that the sphere $S_R^n = \{x \in \mathbb{R}^{n+1} \ : \ \|x\|^2 = R^2\}$ is a manifold of dimension n thanks to the level set theorem in comparison to the construction of the stereographic atlas. In fact, it is enough to consider the function that associates to each vector of $\mathbb{R}^{n+1}$ its squared Euclidean norm:

$$\begin{array}{rccl} f: & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) = \|x\|^2 = (x^1)^2 + \cdots + (x^{n+1})^2, \end{array}$$

  which is smooth and whose only critical value is 0, because $\frac{\partial f}{\partial x^i}(x) = 2x^i$, $i = 1, \ldots, n+1$. Thus, for all $R > 0$, the level set $f^{-1}(R^2) = S_R^n$ is a smooth manifold of dimension $(n+1) - 1 = n$.

1.3 Morphisms and diffeomorphims between manifolds

Manifolds are the arena of differential geometry, let us now analyze their morphisms, i.e. the transformations between manifolds that respect their properties regarding the differential structure. Smooth functions between manifolds are the morphisms of the category of smooth manifolds, while diffeomorphisms are its isomorphisms.

As usual, smoothness is defined through the use of local charts and compatibility among intersecting charts must be required.

Def. 1.3.1 *Given two manifolds M and N of dimensions m and n, respectively, and a function*

$$\begin{array}{rccl} f: & M & \longrightarrow & N \\ & p & \longmapsto & f(p) = q, \end{array}$$

*two local charts $(U_\alpha, \varphi_\alpha)$ in M and (V_β, ψ_β) in N are said to be **f-related** if $f(U_\alpha) \subseteq V_\beta$. Two atlases $\mathcal{A}$ and $\mathcal{B}$ of M and N, respectively, are f-related if every chart of one atlas is f-related with at least one chart of the other atlas.*

The following result shows that the continuity of f is sufficient to guarantee the existence of related atlases.

Theorem 1.3.1 *Given two manifolds M and N and a **continuous** function $f : M \to N$, it exists a couple of f-related atlases of M and N.*

Proof. The proof is constructive. Given any two atlases $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}_{\beta\in J}$ of M and N, respectively, a direct way to build an atlas $\tilde{\mathcal{A}}$ equivalent to $\mathcal{A}$ and f-related to $\mathcal{B}$ is to define $\tilde{\mathcal{A}} := \{(\tilde{U}_{\alpha\beta}, \tilde{\varphi}_{\alpha\beta})\}_{\alpha\in I,\, \beta\in J}$, with:

$$\begin{cases} \tilde{U}_{\alpha\beta} := U_\alpha \cap f^{-1}(V_\beta) \\ \tilde{\varphi}_{\alpha\beta} := \varphi_\alpha|_{\tilde{U}_{\alpha\beta}} . \end{cases}$$

In fact, thanks to the continuity of f, $f^{-1}(V_\beta)$ is an open subset of M and so $U_\alpha \cap f^{-1}(V_\beta)$ is an open subset included in (or coincident with) U_α. The charts $\tilde{\varphi}_{\alpha\beta}$ are compatible with the charts φ_α because the operation of restriction preserves the smoothness of the transition functions, thus the atlases $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are equivalent.

Moreover, $f(U_\alpha \cap f^{-1}(V_\beta)) \subseteq f(U_\alpha) \cap V_\beta \subseteq V_\beta$ thanks to well-known relationships between functions and sets, which guarantees that the atlases $\tilde{\mathcal{A}}$ and $\mathcal{B}$ are f-related. □

We can now define the important concept of local representation (or expression) of a function between manifolds.

Def. 1.3.2 (Local representation of a function between manifolds) *The local representation of $f : M \to N$ w.r.t. the f-related local charts $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) is the function:*

$$\boxed{f_{\beta\alpha} := \psi_\beta \circ f|_{U_\alpha} \circ \varphi_\alpha^{-1}} \, ,$$

$$\begin{array}{rccl} f_{\beta\alpha}: & \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m & \longrightarrow & \psi_\beta(f(U_\alpha)) \subseteq \mathbb{R}^n \\ & x \equiv (x^i)_{i=1}^m & \longmapsto & f_{\beta\alpha}(x) = y \equiv (y^j)_{j=1}^n . \end{array}$$

The following commutative diagram visualizes the local representation of a function.

$$\begin{tikzcd}
M \supseteq U_\alpha \arrow[r, "f|_{U_\alpha}"] & f(U_\alpha) \subseteq N \arrow[d, "\psi_\beta"] \\
\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m \arrow[u, "{\varphi_\alpha}^{-1}"] \arrow[r, dashed, "f_{\beta\alpha}"'] & \psi_\beta(f(U_\alpha)) \subseteq \mathbb{R}^n
\end{tikzcd}$$

$f_{\beta\alpha}$ is a function between open subsets of finite-dimensional real Euclidean spaces, thus we perfectly know what it means for such a function to be smooth. Its smoothness is used to define that of the function f itself.

Def. 1.3.3 (Smooth function between manifolds) *$f : M \to N$ is smooth if it exists a couple of f-related charts, $(U_\alpha, \varphi_\alpha)$ of M and (V_β, ψ_β) of N, such that $f_{\beta\alpha}$, the local representation of f w.r.t. these charts, is smooth.*

Notation: the symbol $\mathscr{C}^\infty(M, N)$ denotes the set of all smooth functions between M and N. If $N \equiv \mathbb{R}$ we simply write $\mathscr{C}^\infty(M)$.

As in standard differential calculus, smoothness implies continuity.

Theorem 1.3.2 *If $f : M \to N$ is smooth, then f is also continuous.*

Proof. Almost immediate: if $f : M \to N$ is smooth in any point $p \in M$ then, by definition of smoothness, it exists a couple of charts $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) such that $p \in U_\alpha$, $f(U_\alpha) \subseteq V_\beta$ and $f_{\beta\alpha} = \psi_\beta \circ f|_{U_\alpha} \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth, and thus continuous, because it is a map between real Euclidean spaces, where we know that smoothness implies continuity. But then, since φ_α and ψ_β are homeomorphisms, $\psi_\beta^{-1} \circ f_{\beta\alpha} \circ \varphi_\alpha$ is continuous too, as composition of continuous maps, but:

$$\psi_\beta^{-1} \circ \psi_\beta \circ f|_{U_\alpha} \circ \varphi_\alpha^{-1} \circ \varphi_\alpha = f|_{U_\alpha},$$

i.e. $f|_{U_\alpha}$ is continuous in an open neighborhood of any point $p \in M$, hence it is continuous on the whole manifold M that, we recall, is a topological manifold, so it intrinsically carries the notion of continuity w.r.t. its topology. □

The *definition of smoothness* just given is intrinsic, i.e. it does not depend on the f-related local charts considered: *once it is true for one couple of f-related local charts, it holds for all f-related local charts.*

To check this, fix any local chart (V_β, ψ_β) of N and consider two f-related overlapping local charts of M, $(U_\alpha, \varphi_\alpha)$ and $(U_{\alpha'}, \varphi_{\alpha'})$, i.e. $U_\alpha \cap U_{\alpha'} = U_{\alpha\alpha'} \neq \varnothing$ and $f(U_{\alpha\alpha'}) \subseteq V_\beta$. The chart maps are related by smooth transition functions $\eta_{\alpha'\alpha} = \varphi_{\alpha'} \circ \varphi_\alpha^{-1}$, thus $\varphi_\alpha^{-1} = \varphi_{\alpha'}^{-1} \circ \eta_{\alpha'\alpha}$. Hence, the local representations $f_{\beta\alpha} = \psi_\beta \circ f|_{U_{\alpha\alpha'}} \circ \varphi_\alpha^{-1}$ and $f_{\beta\alpha'} = \psi_\beta \circ f|_{U_{\alpha\alpha'}} \circ \varphi_{\alpha'}^{-1}$ satisfy:

$$f_{\beta\alpha} = \psi_\beta \circ f|_{U_{\alpha\alpha'}} \circ \varphi_{\alpha'}^{-1} \circ \eta_{\alpha'\alpha} = f_{\beta\alpha'} \circ \eta_{\alpha'\alpha},$$

which implies, thanks to the smoothness of $\eta_{\alpha'\alpha}$, that $f_{\beta\alpha}$ is smooth if and only if $f_{\beta\alpha'}$ is. The f-related couples of local charts considered, $((U_\alpha, \varphi_\alpha), (V_\beta, \psi_\beta))$ and $((U_{\alpha'}, \varphi_{\alpha'}), (V_\beta, \psi_\beta))$, are arbitrary, thus it is enough to check the smoothness of the local representation of f w.r.t. one couple of local maps to guarantee the validity of this property w.r.t. every other couple.

By composing $f_{\beta\alpha}$ with the functionals ε^j of the dual basis of $\mathbb{R}^n$, we get the real-valued functions:

$$\begin{array}{rrcl} f^j_{\beta\alpha} \equiv \varepsilon^j \circ f_{\beta\alpha} : & \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m & \longrightarrow & \mathbb{R} \\ & x = (x^i) & \longmapsto & f^j_{\alpha\beta}(x) = y^j, \end{array}$$

which, as always, are nothing but the scalar components of the $\mathbb{R}^n$-valued function $f_{\beta\alpha}$.

The functions $f^j_{\beta\alpha}$, $j = 1, \ldots, n$, represent **the local coordinate transformation functions between the local coordinates** (x^i) **of a point** $p \in M$ **and the local coordinates** $(y^j) = (f^j_{\beta\alpha}(x^i))$ **of the point** $q = f(p) \in N$.

With the usual abuse of notation, we write $f^j_{\beta\alpha} \equiv y^j$, so that:

$$y^j = y^j(x^i), \qquad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

The following diagram shows the action of the local coordinate transformation functions.

$$\begin{tikzcd} U_\alpha \arrow[rr, "f|_{U_\alpha}"] & & f(U_\alpha) \arrow[d, "\psi_\beta"] \\ x \in \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m \arrow[u, "{\varphi_\alpha}^{-1}"] \arrow[rr, dashed, "f_{\beta\alpha}"'] \arrow[drr, dashed, "y^j(x^i)"'] & & \psi_\beta(f(U_\alpha)) \subseteq \mathbb{R}^n \arrow[d, "\varepsilon^j"] \\ & & y^j \end{tikzcd}$$

Since the functionals ε^j are smooth, it follows that a function $f : M \to N$ **is smooth if and only if we can pass smoothly from a local coordinate description of a point** $x \in M$ **to a local coordinate description of the transformed point** $y = f(x) \in N$.

A special case is provided by functions for which $N = \mathbb{R}^n$, or an open subset of $\mathbb{R}^n$ (thus, in particular, for **scalar functions on** M when $n = 1$). In this case, the differential structure is provided by the canonical global atlas $(\mathbb{R}^n, id_{\mathbb{R}^n})$, so the composition with ψ_β is not necessary anymore and the **local representation of** $f : M \to \mathbb{R}^n$, is just $f_\alpha = f|_{U_\alpha} \circ \varphi_\alpha^{-1}$, that will be denote simply as

$$\boxed{f_\alpha = f \circ {\varphi_\alpha}^{-1}}\ . \tag{1.7}$$

The following commutative diagrams resume our considerations.

$$\begin{tikzcd} U_\alpha \arrow[r, "f|_{U_\alpha}"] & f(U_\alpha) \subseteq \mathbb{R}^n \arrow[d, "id_{f(U_\alpha)}"] & \underset{\text{(simplified as)}}{\Longleftrightarrow} & U_\alpha \arrow[r, "f"] & \mathbb{R}^n \\ \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^m \arrow[u, "{\varphi_\alpha}^{-1}"] \arrow[r, dashed, "f_\alpha"'] & f(U_\alpha) \subseteq \mathbb{R}^n & & \mathbb{R}^m \arrow[u, "{\varphi_\alpha}^{-1}"] \arrow[ur, dashed, "f_\alpha"'] & \end{tikzcd}$$

In the intersection of two charts $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) it holds that:

$$f_\alpha = f_\beta \circ \eta_{\beta\alpha}, \quad f_\beta = f_\alpha \circ \eta_{\alpha\beta},$$

as shown by the following diagram for the first formula, the second being analogous.

$$\begin{tikzcd}
 & U_\alpha \cap U_\beta \arrow[d, shift right, "\varphi_\beta"'] \arrow[dr, bend left, "f"] & \\
\mathbb{R}^m \arrow[ur, bend left, "{\varphi_\alpha}^{-1}"] \arrow[r, dashed, "\eta_{\beta\alpha}"] \arrow[rr, dashed, bend right, "f_\alpha"'] & \mathbb{R}^m \arrow[u, shift right, "{\varphi_\beta}^{-1}"'] \arrow[r, dashed, "f_\beta"] & \mathbb{R}^n
\end{tikzcd}$$

Another special case is provided by functions for which $M = \mathbb{R}^m$ or an open subset of $\mathbb{R}^m$, thus, in particular, for **curves in** N when $m = 1$, as recalled in the following definition.

Def. 1.3.4 (Path, or curve, in a manifold passing through a point) *The smooth function*[9] $\gamma : (-\varepsilon, \varepsilon) \subseteq \mathbb{R} \to M$, $\varepsilon > 0$, *is said to be a path, or curve, in* M *passing through the point* $p \in M$ *if* $\gamma(0) = p$.

In this case, the differential structure is provided by the canonical global atlas $(\mathbb{R}^m, id_{\mathbb{R}^m})$, so the composition with φ_α^{-1} is not necessary anymore and the **local representation of** $f : U \subseteq \mathbb{R}^m \to N$, such that $f(U) \subseteq V_\beta$ is just $f_\beta = \psi_\beta \circ f|_U$, that will be denote simply as

$$\boxed{f_\beta = \psi_\beta \circ f} \;. \tag{1.8}$$

$$\begin{tikzcd}
U \subseteq \mathbb{R}^m \arrow[r, "f|_U"] & f(U) \subseteq V_\beta \subseteq N \arrow[d, "\psi_\beta"] \\
U \subseteq \mathbb{R}^m \arrow[u, "id_U"] \arrow[r, dashed, "f_\beta"'] & \psi_\beta(f(U)) \subseteq \mathbb{R}^n
\end{tikzcd}
\quad \underset{\text{(simplified as)}}{\Longleftrightarrow} \quad
\begin{tikzcd}
U \arrow[r, "f"] \arrow[dr, dashed, "f_\beta"'] & N \arrow[d, "\psi_\beta"] \\
 & \mathbb{R}^n
\end{tikzcd}$$

To resume, the local representations of the previous special cases of functions between manifolds are:

$$\begin{cases} f_\alpha = f \circ \varphi_\alpha^{-1} & \forall f : M \to \mathbb{R}^n \\ f_\beta = \psi_\beta \circ f & \forall f : \mathbb{R}^m \to N. \end{cases}$$

We are now ready to define the concept of diffeomorphism.

Def. 1.3.5 (Global and local diffeomorphism) $f : M \to N$ *is a diffeomorphism if it is a smooth bijective function with smooth inverse* $f^{-1} : N \to M$, *in this case* M *and* N *are said to be* ***diffeomorphic manifolds****.*

$f : M \to N$ *is a local diffeomorphism if there exists an open subset* $U \subset M$ *such that* $f(U)$ *is open in* N *and* $f|_U : U \to f(U)$ *is a diffeomorphism.*

[9] $(-\varepsilon, \varepsilon)$ is to be considered as an open submanifold of $\mathbb{R}$.

The most **basic example of local diffeomorphism is easily provided by any chart map** $\varphi_\alpha : U_\alpha \subseteq M \to \varphi(U_\alpha) \subseteq \mathbb{R}^n$ of a manifold M of dimension n. By definition, φ_α is bijective, thus, the only property that we must check to verify that φ_α is a local diffeomorphism is its smoothness and that of its inverse $\varphi_\alpha^{-1} : \varphi(U_\alpha) \subseteq \mathbb{R}^n \to U_\alpha \subseteq M$.

It is clear that, in both cases, we can use formulae (1.7) and (1.8) to compute the local representations of φ_α and φ_α^{-1}, respectively. As the diagram below shows, **the local representation of a chart map and its inverse is provided by the identity function** $id_{\varphi_\alpha(U_\alpha)}$, which is of course smooth.

$$\begin{tikzcd} U_\alpha \arrow[r, "\varphi_\alpha"] & \varphi_\alpha(U_\alpha) \\ \varphi_\alpha(U_\alpha) \arrow[u, "\varphi_\alpha^{-1}"] \arrow[ur, dashed, "id_{\varphi_\alpha(U_\alpha)}"'] & \end{tikzcd} \qquad \begin{tikzcd} \varphi_\alpha(U_\alpha) \arrow[r, "\varphi_\alpha^{-1}"] \arrow[dr, dashed, "id_{\varphi_\alpha(U_\alpha)}"'] & U_\alpha \arrow[d, "\varphi_\alpha"] \\ & \varphi_\alpha(U_\alpha) \end{tikzcd}$$

Thus, each local chart map allows us to **diffeomorphically identify any open chart domain of M with an open subset of $\mathbb{R}^n$**. Moreover, **the transition functions $\eta_{\beta\alpha}$ are local diffeomorphisms**, being composition of chart maps and their inverses.

We end this section by underlying the difference between identical and diffeomorphic manifolds.

Def. 1.3.6 (Identical manifolds) *Let M be a topological manifold and $(M, \mathcal{A}_1)$, $(M, \mathcal{A}_2)$ two manifolds over M with their corresponding maximal atlases. Then, $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ are said to be identical, as manifolds, if $id_M : (M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$ is a diffeomorphism w.r.t. the differential structures associated to $\mathcal{A}_1$ and $\mathcal{A}_2$.*

From the point of view of manifold classification, diffeomorphic manifolds are considered as equivalent. However, as the following example shows, in the same diffeomorphic class of manifolds, we can find manifolds that are not identical.

Example of diffeomorphic non-identical manifolds. We consider:

$$\begin{cases} M_1 = (\mathbb{R}, \varphi = id_\mathbb{R}) \\ M_2 = (\mathbb{R}, \psi), \ \psi(x) = x^3 \ \forall x \in \mathbb{R}. \end{cases}$$

To check if $id_\mathbb{R}$ is a diffeomorphism w.r.t. these two monochart atlases, we have to consider, as always, the local representation:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f = id_\mathbb{R}"] \arrow[d, "\varphi = id_\mathbb{R}"'] & \mathbb{R} \arrow[d, "\psi"] \\ \mathbb{R} \arrow[r, dashed, "\tilde{f}"'] & \mathbb{R} \end{tikzcd}$$

While $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}(x) = (\psi \circ id_\mathbb{R} \circ \varphi^{-1})(x) = \psi(x) = x^3$ is smooth, its inverse $\tilde{f}^{-1} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}^{-1}(y) = (\varphi \circ id_\mathbb{R} \circ \psi^{-1})(y) = \sqrt[3]{y}$ is not, because $(\tilde{f}^{-1})'(y) = 1/(3\sqrt[3]{y^2})$, which is not differentiable in $y = 0$.

Thus, M_1 and M_2 are not identical manifolds. However, they are diffeomorphic to each other, a simple diffeomorphism being $F : M_1 \to M_2$, $x \mapsto F(x) = \sqrt[3]{x}$. To check it, let us analyze again the local representation:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "F"] \arrow[d, "\varphi = id_{\mathbb{R}}"'] & \mathbb{R} \arrow[d, "\psi"] \\
\mathbb{R} \arrow[r, dashed, "\tilde{F}"'] & \mathbb{R}
\end{tikzcd}$$

Of course, $\tilde{F}(x) = (\sqrt[3]{x})^3 = x$ and $(\tilde{F})^{-1}(y) = (\sqrt[3]{y})^3 = y$, both smooth.

More generally,

- $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ is not a diffeomorphism for all $n \geqslant 1$, so polynomial functions on $\mathbb{R}$ are not diffeomorphisms because their inverse functions lack of smoothness.

- $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{1/n}$ is a diffeomorphism for all $n \in \mathbb{N}$ odd.

We list next some general interesting facts about differential structures:

- Any connected manifold M of dimension 1 is diffeomorphic to either S^1 or to $\mathbb{R}$. In particular, if M is compact (as a topological manifold), then it is diffeomorphic to S^1, otherwise it is diffeomorphic to $\mathbb{R}$.

- Every topological manifold of dimension $\leqslant 3$ admits a *unique differential structure up to diffeomorphisms.*

- For every topological manifold of dimension > 3 there exist compact topological manifolds that does not admit differentiable atlases.

- $\mathbb{R}^n$ admits a *unique differential structure up to diffeomorphisms for all* $n \neq 4$.

- Donaldson-Freedman's 1984 result: $\mathbb{R}^4$ admits *infinite non-countable non-diffeomorphic smooth structures.*

- S^7 has exactly 28 *non-diffeomorphic smooth structures* that can be explicitly written.

1.3.1 Introduction to Lie groups

We now have all the information that we need to introduce the hugely important concept of Lie group, that will be extensively treated later in this document.

We have seen that $M = \mathrm{GL}(n, \mathbb{R})$ is a smooth manifold of dimension n^2, as open subset of $M(n, \mathbb{R}^2) \cong \mathbb{R}^{n^2}$. We also know that $M \times M$ is a product manifold of dimension $2n^2$.

The matrix product function is:

$$\begin{array}{rccl} f: & M \times M & \longrightarrow & M \\ & (A, B) & \longmapsto & f(A, B) = C := A \cdot B, \end{array}$$

where $C = (c^i_j)_{i,j=1,\dots,n}$, with $c^i_j = a^i_h b^h_j$. The components of f are polynomial functions, hence they are smooth and so is f.

The inverse matrix function is:

$$\begin{array}{rccl} g: & M & \longrightarrow & M \\ & A & \longmapsto & g(A) = A^{-1} = \frac{A^*}{\det(A)}, \end{array}$$

where A^* is the *adjugate matrix* of A, i.e. the transpose of its cofactor matrix, defined by $C(A) = \Big((-1)^{i+j}\det(A^i_j)\Big)_{j=1,\dots,n}$, where A^i_j is, as we have already seen, the submatrix of A obtained by eliminating the i-th row and the j-th column. All the operations contained in A^* are smooth, plus the division by the determinant of A is smooth, so g is a smooth function.

Thus, the fundamental group operations, product and inversion, of M are smooth. Every group which has these properties is called a Lie group, as defined below.

Def. 1.3.7 (Lie group) *A topological group*[10] *G endowed with a differential structure that makes it a manifold and such that the product $G \times G \to G$, $(a,b) \mapsto a \cdot b$ and the inversion $G \to G$, $g \mapsto g^{-1}$ are smooth is called a Lie group. The dimension of a Lie group is its dimension as manifold.*

$\mathbb{R}^d$, considered as a group w.r.t. the operation of sum is a Lie group for all $d \geqslant 1$ and, thus, so is $M(n,\mathbb{R})$. Other examples of Lie groups are given by the so-called classical matrix Lie groups, which are listed below.

Classical real matrix groups

- $\mathrm{GL}(n,\mathbb{R}) = \{g \in M(n,\mathbb{R}) \ : \ \det(g) \neq 0\}$ (**general linear group**)
- $\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \ : \ \det(g) = 1\}$ (**special linear group**)
- $\mathrm{O}(n) = \{g \in \mathrm{GL}(n,\mathbb{R}) \ : \ \forall x,y \in \mathbb{R}^n, \ \langle gx, gy\rangle = \langle x,y\rangle\} = \{g \in \mathrm{GL}(n,\mathbb{R}) \ : \ g^t = g^{-1}\}$ (**orthogonal group**[11], it is the group of all the **isometries of** $\mathbb{R}^n$)
- $\mathrm{SO}(n) = \{g \in \mathrm{O}(n) \ : \ \det(g) = 1\}$ (**special orthogonal group**)

Classical complex matrix groups

- $\mathrm{GL}(n,\mathbb{C}) = \{g \in M(n,\mathbb{C}) \ : \ \det(g) \neq 0\}$ (**general linear complex group**)
- $\mathrm{SL}(n,\mathbb{C}) = \{g \in \mathrm{GL}(n,\mathbb{C}) \ : \ \det(g) = 1\}$ (**special linear complex group**)
- $\mathrm{U}(n) = \{g \in \mathrm{GL}(n,\mathbb{C}) \ : \ \forall x,y \in \mathbb{C}^n, \ \langle gx, gy\rangle = \langle x,y\rangle\} = \{g \in \mathrm{GL}(n,\mathbb{C}) \ : \ g^\dagger = g^{-1}\}$ (**unitary group**[12] it is the group of all the **isometries of** $\mathbb{C}^n$)
- $\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \ : \ \det(g) = 1\}$ (**special unitary group**)

[10] i.e. a group G that is also a topological space such that the product and the inversion maps are continuous.
[11] In this definition $\langle \ , \ \rangle$ is the Euclidean product of $\mathbb{R}^n$.
[12] In this definition $\langle \ , \ \rangle$ is the Euclidean product of $\mathbb{C}^n$ and $g^\dagger = \overline{g}^t$ is the adjoint matrix of g.

1.3.2 S^1, SO(2) and U(1) as isomorphic mono-dimensional Lie groups

We can easily show that the unit sphere S^1, the groups SO(2) and $U(1)$ are isomorphic Lie groups by using the isomorphism between $\mathbb{R}^2$ and $\mathbb{C}$:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\sim} & \mathbb{C} \\ (a,b) & \longmapsto & z = a+ib. \end{array} \tag{1.9}$$

In fact,

$$S^1 = \{(a,b) \in \mathbb{R}^2 \ : \ a^2+b^2 = 1\} \subset \mathbb{R}^2,$$

is the unit circle in $\mathbb{R}^2$, and $\mathrm{U}(1) = \{z \in \mathbb{C} \ : \ \forall x, y \in \mathbb{C}^n, \ \langle zx, zy\rangle = \langle x, y\rangle\}$, but thanks to the sequilinearity of the complex scalar product, $\langle zx, zy\rangle = |z|^2\langle x, y\rangle = \langle x, y\rangle$ if and only if $|z|^2 = 1$, i.e. $|z| = 1$, thus:

$$\mathrm{U}(1) = \{z \in \mathbb{C} \ : \ |z| = 1\} \subset \mathbb{C},$$

can be identified with the multiplicative group of complex numbers with unit modulus: if $|z_1| = |z_2| = 1$, then $|z_1 z_2| = 1$ and $|z^{-1}| = 1$ whenever $|z| = 1$, thus the multiplicative group structure of $U(1)$ is evident. Since $|z| = 1 \iff |z|^2 = a^2+b^2 = 1$, it is clear that if we restrict the isomorphism (1.9) to S^1, we obtain the following isomorphism:

$$\begin{array}{ccc} \mathbb{R}^2 \supset S^1 & \xrightarrow{\sim} & \mathrm{U}(1) \subset \mathbb{C} \\ (a,b) & \longmapsto & z = a+ib. \end{array}$$

Thanks to this identification, S^1 inherits the group structure from U(1) and, vice-versa, U(1) inherits a manifold structure from S^1. It can be proven that the manifold and group structures are compatible, in the sense of definition 1.3.7, so S^1 and U(1) are Lie groups. Since the dimension of S^1 is 1, **S^1 and U(1) are mono-dimensional compact Lie groups**.

We can push the isomorphism even further by considering the group SO(2). We recall that the matrices of this group can be characterized very easily. In fact, given any 2×2 real matrix with unit determinant A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we have that $A^t = A^{-1} \iff a = d$ and $c = -b$, i.e. we car rewrite SO(2) as follows:

$$\mathrm{SO}(2) = \left\{ A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ \det(A) = a^2+b^2 = 1 \right\},$$

but then the correspondence

$$\begin{array}{ccc} \mathbb{R}^2 \supset S^1 & \xrightarrow{\sim} & \mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R}) \\ (a,b) & \longmapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \end{array}$$

is an isomorphism. Moreover, for all $\vartheta \in [0, 2\pi)$, if we set $a = \cos\vartheta$ and $b = \sin\vartheta$ or $b = -\sin\vartheta$, then $a^2+b^2 = 1$, so we can explicitly characterize the matrices of SO(2) as follows:

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \ : \ \vartheta \in [0, 2\pi) \right\} = \left\{ \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \ : \ \vartheta \in [0, 2\pi) \right\}.$$

As a consequence, we have **three isomorphic mono-dimensional Lie groups**, S^1, $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ that can be explicitly characterized by one free parameter $\vartheta \in [0, 2\pi)$ as follows:

$$S^1 = \{(a,b) \in \mathbb{R}^2 \ : \ a = \cos\vartheta,\ b = \sin\vartheta,\ \vartheta \in [0,2\pi)\} \subset \mathbb{R}^2$$

$$\mathrm{U}(1) = \{z \in \mathbb{C} \ : \ z = \cos\vartheta + i\sin\vartheta,\ \vartheta \in [0,2\pi)\} \subset \mathbb{C}$$

$$\mathrm{SO}(2) = \left\{ A \in \mathrm{SL}(2,\mathbb{R}) \ : \ A = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix},\ \vartheta \in [0,2\pi) \right\} \subset \mathrm{SL}(2,\mathbb{R}).$$

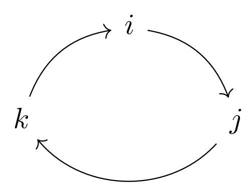
1.3.3 S^3, $\mathbb{H}_1$ and SU(2) as isomorphic Lie groups of dimension 3

We pass from S^1 to S^3 without considering S^2, in fact it can be proven that S^2 **is not a Lie group**.

The isometries that we have discussed in the previous section follow from the natural identification between $\mathbb{R}^2$ and $\mathbb{C}$, those that we analyze here follow from the natural identification between $\mathbb{R}^4$ and the non-Abelian division algebra (thus also a group) of **quaternions** $\mathbb{H}$:

$$\begin{array}{ccc} \mathbb{R}^4 & \xrightarrow{\sim} & \mathbb{H} \\ (a,b,c,d) & \longmapsto & z = a + ib + jc + kd, \end{array} \tag{1.10}$$

where $i^2 = j^2 = k^2 = -1$ and the multiplication of the quaternionic units i, j, k follows this diagram:



if we multiply the quaternionic units in the sense of the arrows, we get as result the next quaternionic unit multiplied by $+1$, if we multiply the quaternionic units following the opposite sense w.r.t. the arrows, we obtain the next quaternionic unit multiplied by -1. For example, $ij = k$, $ji = -k$, $ik = -j$, $jk = i$, and so on.

The conjugate quaternion of $z = a + ib + jc + kd$ is $\overline{z} := a - ib - jc - kd$ and its modulus is the non negative real number $|z|$ such that: $|z|^2 := z\overline{z} = a^2 + b^2 + c^2 + d^2$.

The set of **quaternions with unit modulus** is denoted by

$$\mathbb{H}_1 := \{z = a + ib + jc + kd \in \mathbb{H} \ : \ |z| = 1 \iff a^2 + b^2 + c^2 + d^2 = 1\} \subset \mathbb{H}.$$

By recalling that the sphere S^3 is defined as:

$$S^3 = \{(a,b,c,d) \in \mathbb{R}^4 \ : \ a^2 + b^2 + c^2 + d^2 = 1\} \subset \mathbb{R}^4,$$

it is clear that if we restrict the identification defined by (1.10) to $S^3 \subset \mathbb{R}^4$ we get a natural identification between S^3 and $\mathbb{H}_1$:

$$\begin{array}{ccc} \mathbb{R}^4 \supset S^3 & \xrightarrow{\sim} & \mathbb{H}_1 \subset \mathbb{H} \\ (a,b,c,d) & \longmapsto & z = a + ib + jc + kd. \end{array}$$

Thanks to this isomorphism, S^3 inherits the group structure from $\mathbb{H}_1$ and, vice-versa, $\mathbb{H}_1$ inherits a manifold structure from S^3. As for the case of S^1 and U(1), it can be proven that the manifold and group structures are compatible, thus making S^3 and $\mathbb{H}_1$ Lie groups. Since the dimension of S^3 is 3, S^3 and $\mathbb{H}_1$ are Lie groups of dimension 3.

As before, we can find a further isomorphism with a matrix group: SU(2). In order to formalize this, we need to introduce the **Pauli matrices**:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.11}$$

For $\ell = 1, 2, 3$, the matrices σ_ℓ are complex, Hermitian ($\overline{\sigma_\ell}^t = \sigma_\ell$) and unitary ($\overline{\sigma_\ell}^t = \sigma_\ell^{-1}$), so it also holds that $\sigma_\ell = \sigma_\ell^{-1}$. Actually, the set $(I_2, \sigma_1, \sigma_2, \sigma_3)$ is a **basis** for $U(2)$, the real vector space of 2×2 Hermitian matrices.

By direct computation, we get that:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2, \quad \sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_1 = -i\sigma_3, \ldots$$

These properties are reminiscent of those of the quaternionic units, they perfectly agree with them if we multiply the Pauli matrices by i, for in that case we get:

$$\tilde{\sigma}_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}_2 = i\sigma_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{\sigma}_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$\tilde{\sigma}_1^2 = \tilde{\sigma}_2^2 = \tilde{\sigma}_3^2 = -I_2, \quad \tilde{\sigma}_1\tilde{\sigma}_2 = -\tilde{\sigma}_3, \quad \tilde{\sigma}_2\tilde{\sigma}_1 = \tilde{\sigma}_3, \quad \tilde{\sigma}_2\tilde{\sigma}_3 = -\tilde{\sigma}_1, \ldots$$

By comparison with the quaternions, we can establish these correspondences:

$$\begin{cases} 1 \leftrightarrow I_2 \\ i \leftrightarrow \tilde{\sigma}_3 \\ j \leftrightarrow \tilde{\sigma}_2 \\ k \leftrightarrow \tilde{\sigma}_1. \end{cases}$$

This allow us to represent the quaternions via matrices, in fact:

$$z = 1 \cdot a + i \cdot b + j \cdot c + k \cdot d \iff z = I_2 a + \tilde{\sigma}_3 b + \tilde{\sigma}_2 c + \tilde{\sigma}_1 d = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} =: A_z.$$

Moreover, by direct computation, we have:

$$\det(A_z) = a^2 + b^2 + c^2 + d^2 = |z|^2. \tag{1.12}$$

We notice that A_z is a matrix of the type:

$$M = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

with $\det(M) = |\alpha|^2 + |\beta|^2$ and $\overline{M}^t M = \det(M) I_2$, so:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \ |\alpha|^2 + |\beta|^2 = 1 \right\} = \left\{ \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}, \ a^2 + b^2 + c^2 + d^2 = 1 \right\}.$$

From eq. (1.12) we get the (group) isomorphism

$$\mathbb{H}_1 \cong \mathrm{SU}(2).$$

The matrices $(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$ are **anti-Hermitian**, i.e. $\overline{\tilde{\sigma}_\ell}^t = -\tilde{\sigma}_\ell$, $\ell = 1, 2, 3$. We will show that they constitute a basis for the Lie algebra of SU(2): $\mathfrak{su}(2) \cong T_e\,\mathrm{SU}(2)$, the tangent space to SU(2) at e, the unit element of the group. Since SU(2) has dimension 3 as a manifold, its tangent space has dimension 3 as well and so does its Lie algebra.

1.4 Covering and universal covering

The concept of covering (or cover) is very important in differential geometry, in particular in Lie group theory.

The definition of covering can be puzzling at first sight, thus we prefer to discuss a very simple example that will serve as a motivation for the definition.

Consider $S^1_R \subset \mathbb{R}^2$, $R > 0$, and $\mathbb{R}$, then the map

$$\begin{array}{rccl} \pi : & \mathbb{R} & \longrightarrow & S^1_R \\ & t & \longmapsto & \pi(t) = (R\cos t, R\sin t) \end{array}$$

is smooth and surjective, i.e. via π we can *cover* smoothly the whole manifold S^1_R. However, π is not injective, thus, if we consider any open subset $U \subset S^1_R$, the counter-image $\pi^{-1}(U)$ will be composed by infinitely many open subsets of $\mathbb{R}$. For example, to fix the ideas, consider the open arc A of the circle of radius R which goes from $(R, 0)$ to $(0, R)$, then $\pi^{-1}(A)$ is the following union of disjoint open intervals in $\mathbb{R}$:

$$\pi^{-1}(A) = \bigcup_{k\in\mathbb{Z}} (2k\pi, \pi/2 + 2k\pi).$$

For a fixed value $k \in \mathbb{Z}$, the interval $I_k = (2k\pi, \pi/2 + 2k\pi)$ is a connected set in $\mathbb{R}$ and the restriction of π on I_k is a diffeomorphism between I_k and A.

These considerations motivate the definition of covering.

Def. 1.4.1 (Covering) *Given the manifold M, a covering of M is the couple $(\tilde{M}, \pi)$, where $\tilde{M}$ is a manifold and $\pi : \tilde{M} \to M$ verifies the following properties:*

1. *π is smooth and surjective*

2. *for all $p \in M$ it exists an open connected neighborhood $U \subset M$ of p such that the restriction of π to all the connected components $\tilde{U} \subset \tilde{M}$ of $\pi^{-1}(U)$ is a diffeomorphism between $\tilde{U}$ and U.*

If $\tilde{M}$ is simply connected[13]*, then we say that $(\tilde{M}, \pi)$ is the* ***universal covering***[14] *of M.*

The components of $\pi^{-1}(U)$ are called the ***sheets of the covering****.*

1.4.1 $\mathbb{R}$ and $\mathbb{R}^n$ as the universal covering of S^1_R and the torus $\mathbb{T}^n$

$\mathbb{R}$ is simply connected and we have seen that it is a covering of S^1_R, it follows that $\mathbb{R}$ **is the universal covering of S^1_R**. This is the 1-dimensional case of a more general covering involving $\mathbb{R}^n$ and the torus $\mathbb{T}^n$.

Fixed any lattice $\Lambda \subset \mathbb{R}^n$, we can define an equivalence relation $\sim_\Lambda$ in $\mathbb{R}^n$ by identifying the elements of $\mathbb{R}^n$ that belong to the opposite edges, as depicted in Figure 1.2.

$(\mathbb{R}^n, \pi)$, where $\pi := \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n/\sim_\Lambda$, $x \mapsto \pi(x) = [x]$, is the universal covering of the n-dimensional torus $\mathbb{T}^n$.

[13] i.e. M, as topological space, is such that any continuous loop contained in M is homotopic to a point.

[14] If it exists, the universal covering, can be proven to be unique up to homeomorphisms.

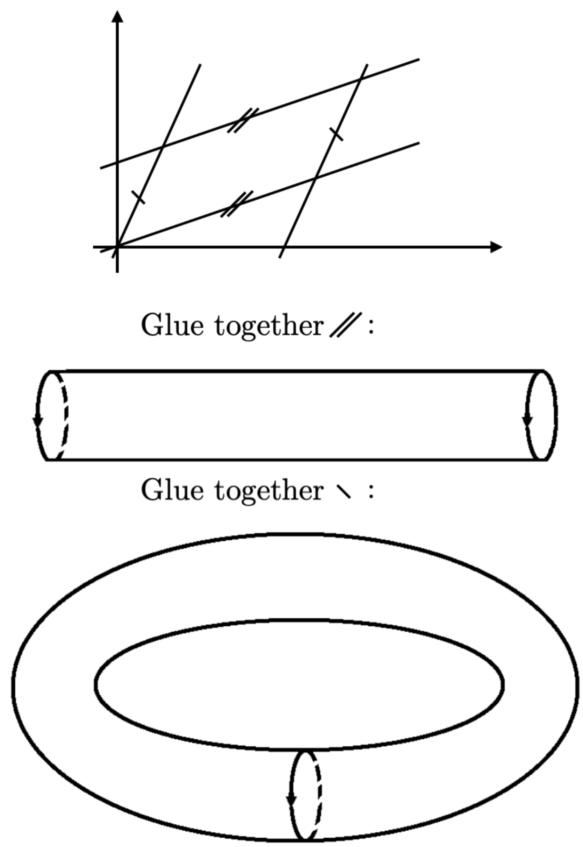


Figure 1.2: The construction of the torus $\mathbb{T}^2$.

1.4.2 SU(2) as the two-sheets universal covering of SO(3)

The Lie group SU(2) is diffeomorphic to S^3, which is simply connected, thus it is simply connected itself. We prove that it is the universal covering of SO(3), the Lie group of proper rotations in $\mathbb{R}^3$.

To this aim, the quaternions will help again. In fact, the first thing that we need to do is to identify $\mathbb{R}^3$ with the quaternions with null real part $\mathbb{H}_0 := \{ib + jc + kd,\ b, c, d \in \mathbb{R}\} \subset \mathbb{H}$. The map:

$$\begin{array}{rccl} q: & \mathbb{R}^3 & \longrightarrow & \mathbb{H}_0 \\ & x = (x, y, z) & \longmapsto & q(x) := ix + jy + kz, \end{array}$$

is a natural isomorphism.

Next, fixed any quaternion with unit modulus $z \in \mathbb{H}_1$, $|z| = 1$, it can be verified with straightforward computations that the map

$$\begin{array}{rccl} R_z: & \mathbb{H}_0 & \longrightarrow & \mathbb{H}_0 \\ & q(x) & \longmapsto & R_z(q(x)) := zq(x)z^{-1} \end{array}$$

is well-posed because $zq(x)z^{-1}$ has null real part. Moreover:

$$|R_z(q(x))| = |zq(x)z^{-1}| = |z|\ |q(x)|\ |z^{-1}| = |q(x)|,$$

i.e. R_z is an isometry of $\mathbb{H}_0 \cong \mathbb{R}^3$. If we interpret R_z as a linear map, then $R_z \in \mathrm{O}(3)$. By direct computation, it can be verified that $\det(R_z) = 1$, so $R_z \in \mathrm{SO}(3)$, i.e. R_z is a proper rotation.

The matrix associated to R_z, $z = a + ib + jc + kd$, $z \in \mathbb{H}_1$, i.e. $a^2 + b^2 + c^2 + d^2 = 1$, w.r.t. the canonical basis of $\mathbb{R}^3$ is:

$$\mathbf{R}_z = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix},$$

$\det(\mathbf{R}_z) = (a^2 + b^2 + c^2 + d^2)^3 = 1$.

Actually, it can be proven that all matrix of SO(3) can be written as the matrix above, so the map $\mathbb{H}_1 \ni z \mapsto \mathbf{R}_z \in \mathrm{SO}(3)$ is onto.

Finally, from the fact that each entry of the matrix $\mathbf{R}_z$ is a polynomial of order two of the coefficients of z, it follows with simple calculations that $\mathbf{R}_z = \mathbf{R}_{z'} \iff z = z'$ or $z = -z'$. Thus, the correspondence $\mathbb{H}_1 \ni z \mapsto \mathbf{R}_z \in \mathrm{SO}(3)$ is 2:1.

To resume, we have proven that SU(2) is the universal covering[15] of SO(3). We can say more: the onto map $z \mapsto \mathbf{R}_z$ is also a homomorphism of groups:

$$\begin{array}{rccc} \pi: & \mathbb{H}_1 \cong \mathrm{SU}(2) & \twoheadrightarrow & \mathrm{SO}(3) \\ & z \cong A_z & \longmapsto & \pi(z) := \mathbf{R}_z, \end{array}$$

with

$$\ker(\pi) = \{I_2, -I_2\}, \qquad I_2 : \text{identity of } M(2, \mathbb{C}),$$

so that, by the homomorphism theorem, we have the isomorphism:

$$\mathrm{SU}(2)/\{I_2, -I_2\} \cong \mathrm{SO}(3).$$

Thinking about SU(2) and SO(3) as Lie groups, $\pi : \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ defines a two-sheets covering (since the counter-image of $\mathbf{R}_z$ by π is $\pi^{-1}(\mathbf{R}_z) = \{z, -z\}$).

Finally, if we identify SU(2) with S^3, the quotient $\mathrm{SU}(2)/\{I_2, -I_2\}$ becomes the quotient of S^3 w.r.t. the equivalence relation $\sim_{\leftrightarrow}$ given by antipodal points identification on the 3-sphere. However, as we have seen in section 1.2, this quotient procedure gives rise to the real 3-dimensional projective space $\mathbb{RP}^3$, thus:

$$\mathbb{RP}^3 \;\cong\; S^3/\sim_{\leftrightarrow} \;\cong\; \mathrm{SU}(2)/\{I_2, -I_2\} \;\cong\; \mathrm{SO}(3)$$

and, thanks to these identifications, even the **3-dimensional real projective space** $\mathbb{RP}^3$ **acquires a Lie group structure**!

[15] In Physics, and in the formalism of Clifford algebras, the universal covering of SO(3) is called **Spin**(3).

1.5 Partition of the unity

Partitions of the unity are very important in differential geometry, because they allows us to extend the definition of objects from a local neighborhood of a point to the whole manifold. This is used, just to give an idea, for connections and Riemannian metrics.

Let us start with the following useful function displayed in Figure 1.3:

$$h : \mathbb{R} \to \mathbb{R}, \quad h(t) = \begin{cases} 0 & \text{if } t \leqslant 0 \\ e^{\frac{1}{t}} & \text{if } t > 0 \end{cases}$$

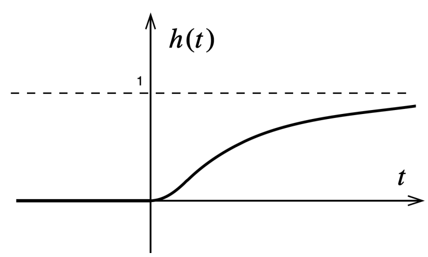


Figure 1.3: The h function.

The properties of h are listed below:

- $h(t) \in [0, 1)\ \forall t \in \mathbb{R}$ and $h(t) \underset{t \to +\infty}{\to} 0$;
- h is increasing;
- $h \in \mathscr{C}^\infty(\mathbb{R})$.

With this smooth function h, we can cook up other one, depicted on the left hand side of Figure 1.4:

$$\eta : \mathbb{R} \to \mathbb{R}, \quad \eta(t) = \frac{h(1 - |t|^2)}{h(1 - |t|^2) + h(|t|^2 - \frac{1}{4})}$$

with the following characteristics:

- $\eta(t) \geqslant 0\ \forall t \in \mathbb{R}$;
- $h \in \mathscr{C}^\infty(\mathbb{R})$;
- $\eta(t) = 1$ (exactly 1) in $[-\frac{1}{2}, \frac{1}{2}]$, in fact, for $t \in [-\frac{1}{2}, \frac{1}{2}]$, $|t|^2 \leqslant \frac{1}{4}$, so $h(|t|^2 - \frac{1}{4}) = 0$, because of the definition of h;
- $\eta(t) = 0$ for $t \geqslant 1$ or $t \leqslant -1$, in fact, in this case $1 - |t|^2 \geqslant 0$, so that $h(1 - |t|^2) = 0$ by definition in these intervals.

The extension to $\mathbb{R}^n$ is the following (for $n = 2$ the graph is depicted on the right hand side of Figure 1.4):

$$\eta : \mathbb{R}^n \to \mathbb{R}, \quad \eta(x) = \frac{h(1 - \|x\|^2)}{h(1 - \|x\|^2) + h(\|x\|^2 - \frac{1}{4})}, \quad \eta \in \mathscr{C}^\infty(\mathbb{R}^n),$$

- $\eta(t) \geqslant 0 \ \forall t \in \mathbb{R}$;
- $\eta(x) = 1$ (exactly 1) in $\overline{B(0, \frac{1}{2})}$;
- $\eta(x) = 0 \ \forall x \in \mathbb{R}^n \backslash B(0, 1)$.

η is called the **bump function**.

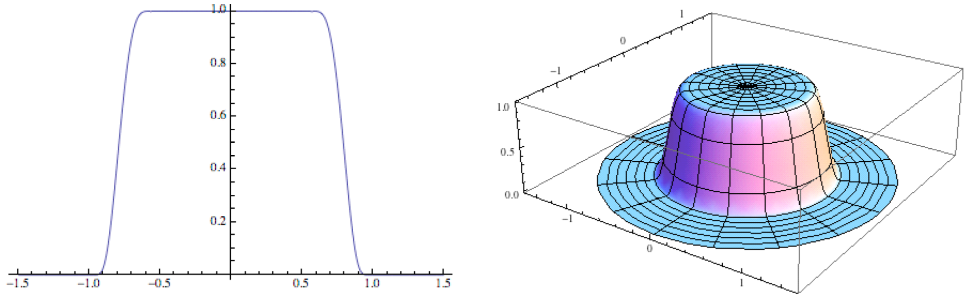


Figure 1.4: From left to right, the bump function η function for $n = 1$ and $n = 2$.

We recall that, given a topological space X, the support of a function $f : X \to \mathbb{R}$ is the closed subset of X defined by $\mathrm{supp}(f) = \overline{\{x \in X \ : \ f(x) \neq 0\}}$.

The following result is central in the theory of partitions of unity.

Theorem 1.5.1 *Let M be a smooth manifold and:*

- *$K \subset M$ a compact subset of M;*
- *$V \subset M$ an open subset of M containing K: $K \subset V$.*

Then, there exists a smooth function $g : M \to \mathbb{R}$ such that

$$\begin{cases} g|_K \equiv 1 \\ \mathrm{supp}(g) \subset V \implies g|_{M \backslash V} \equiv 0. \end{cases}$$

Thus, g is a generalization of the bump function to M: g is identically 1 on K, identically 0 on $M \backslash V$ and it takes intermediate (unknown) values on $V \backslash K$.

The proof is constructive.

Corollary 1.5.1 *For every point $p \in M$ and every open neighborhood $V \subseteq M$ of p, it exist $f, g \in \mathscr{C}^\infty(M)$ such that*

$$\begin{cases} f(p) = 0 \\ f|_{M \backslash V} \equiv 1 \end{cases}, \quad \begin{cases} g(p) = 1 \\ g|_{M \backslash V} \equiv 0 \end{cases}.$$

Proof. It is enough to choose $K = \{p\}$, obviously compact, in the previous theorem: we obtain a function $g \in \mathscr{C}^\infty(M)$ such that $g(p) = 1$ and $g|_{M \backslash V} \equiv 0$. Then, by setting $f(x) = 1 - g(x)$ for all $x \in M$, we obtain the thesis. □

Let us now introduce a handy symbol that will give a sort of generalization of *smooth functions for maps not necessarily defined on open sets.*

Def. 1.5.1 *Let $S \subset M$ be **any subset** of M. Then we denote with $\mathscr{C}^\infty(S)$ the set of continuous real-valued functions $f : S \to \mathbb{R}$ that can be obtained by restriction of a smooth function $\tilde{f} : V \to \mathbb{R}$, V open and $S \subset V$, i.e. $f = \tilde{f}\Big|_S$.*

We use immediately this concept to show that **any $\mathscr{C}^\infty$ function defined on a compact subset of a manifold M can be extended to a smooth function on the whole** M... with *a sort of smooth padding with zeros*!

Theorem 1.5.2 (Extension theorem for smooth functions) *Let $K \subset M$ be **a compact** subset of the smooth manifold M and let $f \in \mathscr{C}^\infty(M)$. Let also $K \subset W$, W open in M. Then, it exists $\hat{f} \in \mathscr{C}^\infty(M)$ such that:*

- $\hat{f}\Big|_K = f$;
- $\operatorname{supp}(\hat{f}) \subset W$, *so that* $\hat{f}\Big|_{M \setminus W} \equiv 0$.

Proof. By definition, f extends to $\tilde{f} \in \mathscr{C}^\infty(U)$, for some U open in M, $K \subset U$.

We set $V = U \cap W$ and we consider $g \in \mathscr{C}^\infty(M)$ such that $g|_K \equiv 1$ and $\operatorname{supp}(g) \subset V$, which exists thanks to the previous result.

We define

$$\begin{aligned} \hat{f} : \quad M &\longrightarrow \mathbb{R} \\ q &\longmapsto \hat{f}(q) = \begin{cases} g(q)\tilde{f}(q) & q \in V \\ 0 & q \in M \setminus V. \end{cases} \end{aligned}$$

$\hat{f}$ is smooth and $\hat{f}\Big|_K \equiv f$ because $g(q) = 1$ for all $q \in K$. Moreover, $\tilde{f}(q) = f(q)$ for all $q \in K$ and, finally, $\hat{f}\Big|_{M \setminus W} = 0$, because either $\hat{f}$ is evaluated outside V, or, in any case, g is 0. □

The last concept that we need is that of cover.

Def. 1.5.2 (Cover) *Let X be a topological space. A cover of X is a family of subsets $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X such that $X = \bigcup_{\alpha \in A} U_\alpha$. The cover is said to be:*

- ***open**, if all the sets U_α are open;*
- ***locally finite**, if every $p \in X$ has a neighborhood $U \subset X$ such that $U \cap U_\alpha \neq \varnothing$ only for a finite number of indices α.*

*Another covering $\mathcal{V} = \{V_\beta\}_{\beta \in J}$ is a **refining** of $\mathcal{U}$ if $\forall \beta \in J \; \exists \, \alpha \in I$ such that $V_\beta \subset U_\alpha$, i.e. if the subsets of $\mathcal{V}$ are smaller than those of $\mathcal{U}$.*

Def. 1.5.3 (Partition of unity) *Let M be a smooth manifold. A partition of unity on M is a family of functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in I}$, I finite or infinite set, such that:*

1. *$\rho_\alpha \in \mathscr{C}^\infty(M)$;*
2. *$\rho_\alpha(p) \in [0, 1]$ $\forall p \in M$, $\forall \alpha \in I$;*
3. *$\{supp(\rho_\alpha)\}_{\alpha \in I}$ is a locally finite covering of M;*

4. $\sum\limits_{\alpha \in I} \rho_\alpha(p) = 1 \ \forall p \in M$.

The partition of unity is subordinated to the open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ *of* M *if* $supp(\rho_\alpha) \subset U_\alpha$ $\forall \alpha \in I$.

The last property explains the name. The third property implies that $\sum\limits_{\alpha \in I} \rho_\alpha(p)$ is always a finite sum of real numbers, and not a series.

The fundamental result about partitions of unity is the following. The proof relies on the fact that the topological space underlying a smooth manifold is required to be second countable.

Theorem 1.5.3 *Every open covering of a smooth manifold admits a partition of unity subordinated to it.*

For the proof see [10].

Chapter 2

Tangent vector and tangent space to a manifold at a point (Edoardo Provenzi)

Inspirational epithap wanted...

...

Disclaimer: the reader is invited to get acquainted with the notations and concepts discussed in Appendix B about ordinary differential calculus in $\mathbb{R}^n$ before reading this chapter.

A firm understanding of the concept of tangent vector and tangent space to a point of a manifold is the most important step towards the comprehension of more advanced concepts of differential geometry.

There are at least five different, but (of course) equivalent[1] ways to define a tangent vector to a point of a manifold. Each one has advantages and disadvantages, but all of them must be known. A thorough analysis of the equivalence between these definitions is available in [9].

1. **Geometrical definition**: **tangent vectors as equivalent class of curves**. It is an intuitive definition, but not the easiest one to use in proofs or for its notation;

2. **Algebraic definition $\sharp$ 1**: **tangent vectors as derivations of smooth scalar functions**. It is probably the most widely used in the literature, thanks to its notational and conceptual simplicity. It is the one that we will use more commonly throughout this document.

3. **Algebraic definition $\sharp$ 2**: **tangent vectors as derivations of germs of smooth functions**. It is similar to the previous one, it has the advantage to make the local nature of tangent vectors even clearer and of being extendable to real-analytic and complex manifolds, but it has the disadvantages of being even more abstract and with a less simple notation.

4. **Physicists' definition**: **tangent vectors as equivalence classes of n-tuples**. It is mainly used by physicists and engineers, it uses the fact that tangent vectors verify a peculiar way of transforming under coordinate transformations.

[1] A perfect equivalence holds only for finite-dimensional manifolds. If the manifold dimension is infinite, the situation is trickier.

5. **Jets definition**: it is a quite abstract definition, that we will not discuss here, but it as a great importance in modern versions of calculus of variations, covariant geometric field theory and general relativity.

2.1 Geometric definition of tangent vectors

We start introducing tangent vectors with the most geometrical way. Later, we will discuss the algebraic and the physicists' way and prove their equivalence.

Following [8], let us be guided by the very easy example of the unit spheres S^1 and S^2 depicted in Fig. 2.1 (courtesy of Eric Shapiro) to understand how to define tangent vectors.

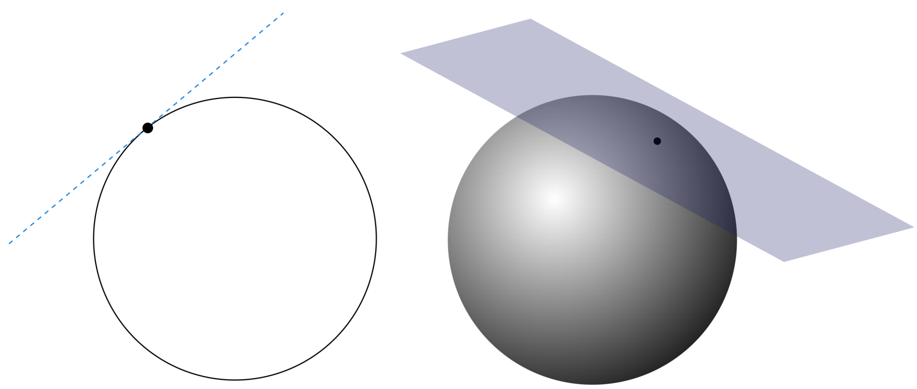


Figure 2.1: Intuitive depiction of tangent line to a circle (left) and tangent plane to a sphere (right).

We see that, while S^1 and S^2 are manifolds of dimension 1 and 2, respectively, the tangent line to a point of S^1 and the tangent plane to a point of S^2 live in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. While this may not be a problem for manifolds naturally embedded in $\mathbb{R}^{n+1}$ as the sphere S^n, for a generic abstract manifold[2] M of dimension n it is desirable to have an *intrinsic* definition of tangent vector and space, that does not make use of a larger structure.

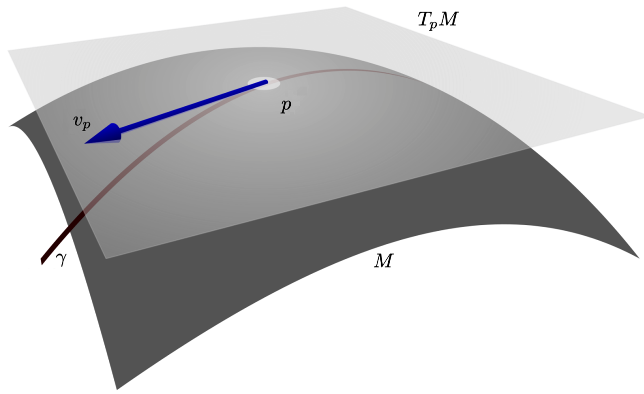
It turns out that manifold-valued paths are exactly what we need to provide such an intrinsic definition.

Given a path γ passing through $p \in M$, the tangent vector to γ in p, i.e. the velocity at which γ passes through p, will be also tangent to M at p, since the image of γ lies in M, as shown in the picture below.

To make this intuition precise, we must first define what the tangent vector to a path in M is. As always, since we know how to compute the tangent vector of a path in the local model $\mathbb{R}^n$, we can consider any local chart (U, φ) in p and build a path in $\mathbb{R}^n$ simply by composing γ with $\varphi|_{\gamma(-\varepsilon,\varepsilon)\cap U}$, that we will still denote with φ for simplicity:

$$\begin{array}{rccc} \varphi \circ \gamma : & (-\varepsilon, \varepsilon) & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & (\varphi \circ \gamma)(t), \end{array}$$

[2]By **Whitney's embedding theorem** [10], every n-dimensional manifold M can be embedded in $\mathbb{R}^{2n+1}$, however, the fact that this embedding exists, does not mean that it is convenient to think about M as an embedded submanifold of $\mathbb{R}^{2n+1}$. For example, in general relativity, spacetime is a 4-dimensional manifold and it meaningless to embed it into $\mathbb{R}^9 \ldots$



since $(\varphi \circ \gamma)(0) = \varphi(p) = x \in \mathbb{R}^n$, $\varphi \circ \gamma$ is a path in $\mathbb{R}^n$ passing through $x = \varphi(p)$.

Using the standard definition of calculus, the tangent vector to the curve $\varphi \circ \gamma$ at x is:

$$(\varphi \circ \gamma)^{\cdot}(0) := \frac{(\varphi \circ \gamma)(t) - (\varphi \circ \gamma)(0)}{t} \equiv \left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma)(t).$$

Of course, in general, there may be other curves passing through p with the property that their local representations via φ have the same tangent vector as γ.

The following basic lemma shows that, remarkably, if the local representations any two curves passing through p have the same tangent vector in $\mathbb{R}^n$ w.r.t. a given local chart in p, then this holds for any other local chart in p.

Lemma 2.1.1 *Let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ be two overlapping charts in p and γ, σ two paths passing through p. Define:*

$$\gamma_\alpha := \varphi_\alpha \circ \gamma,\ \gamma_\beta := \varphi_\beta \circ \gamma \ \text{ and } \ \sigma_\alpha := \varphi_\alpha \circ \sigma,\ \sigma_\beta := \varphi_\beta \circ \sigma.$$

Then:

$$\dot{\gamma_\alpha}(0) = \dot{\sigma_\alpha}(0) \iff \dot{\gamma_\beta}(0) = \dot{\sigma_\beta}(0).$$

Proof. With the notations of the Lemma we have:

$$\dot{\gamma_\beta}(0) = (\varphi_\beta \circ \gamma)^{\cdot}(0) = (\varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \gamma)^{\cdot}(0) = (\eta_{\beta\alpha} \circ \gamma_\alpha)^{\cdot}(0),$$

where $\eta_{\beta\alpha}$ is the (smooth) transition function between charts. Thanks to eq. (B.7) we have:

$$\dot{\gamma_\beta}(0) = D(\eta_{\beta\alpha} \circ \gamma_\alpha)(0)1 \underset{\text{(chain rule)}}{=} D\eta_{\beta\alpha}(\gamma_\alpha(0))D\gamma_\alpha(0)1 = D\eta_{\beta\alpha}(x)\dot{\gamma_\alpha}(0). \tag{2.1}$$

Of course, the same holds for the path η, i.e. $\dot{\sigma_\beta}(0) = D\eta_{\beta\alpha}(x)\dot{\sigma_\alpha}(0)$. By the linearity of the operator $D\eta_{\beta\alpha}(x)$, it follows that:

$$\dot{\gamma_\beta}(0) - \dot{\sigma_\beta}(0) = D\eta_{\beta\alpha}(x)(\dot{\gamma_\alpha}(0) - \dot{\sigma_\alpha}(0)).$$

Now, $\eta_{\beta\alpha}$ is a local diffeomorphism, thus $D\eta_{\beta\alpha}(x)$ is a linear isomorphism (the Jacobian matrix of $\eta_{\beta\alpha}$ in x has non null determinant), thus $\dot{\gamma_\alpha}(0) - \dot{\sigma_\alpha}(0) = 0$ if and only if $\dot{\gamma_\beta}(0) - \dot{\sigma_\beta}(0) = 0$, which proves the theorem. □

This lemma implies that the equality of the tangent vector in $\mathbb{R}^n$ for the local representation of two curves in M passing through the same point is an *intrinsic property of the manifold* M, meaning that it does not depend on the local chart chosen. This property allows us to define an equivalence relationship in the set of curves and also the first, geometric, definition of tangent vector to a manifold at a certain point.

Def. 2.1.1 (Tangentially equivalent, or tangent, curves) *Let M be an n-dimensional manifold and $p \in M$ fixed. Two paths γ, σ in M passing through p are tangent, or tangentially equivalent, if they identify the same tangent vector in $\mathbb{R}^n$ when composed with any local chart φ in p, i.e.*

$$(\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0) = (\varphi \circ \sigma)^{\boldsymbol{\cdot}}(0).$$

Being defined via an equality, the fact of being tangentially equivalent is easily seen to be indeed an equivalence relationship in the set of curves in M passing through p.

Def. 2.1.2 (Geometric tangent vectors and tangent space to M at p) *A (geometric) tangent vector to M at p is a tangentially equivalence class of curves passing through p, denoted with $[\gamma]$. The (geometric) tangent space to M at p, denoted with $T_p^{\text{geom}}M$ is the set of all tangent vectors to M at p.*

Remark: a slightly different definition of tangent vector can be obtained in a similar manner, replacing the local charts with smooth scalar functions, in this case we define two paths γ and η to be equivalent if, for all $f \in \mathscr{C}^\infty(M)$, $(f \circ \gamma)^{\boldsymbol{\cdot}}(0) = (f \circ \sigma)^{\boldsymbol{\cdot}}(0)$, where both $f \circ \gamma$ and $f \circ \sigma$ are scalar functions of a real variable. In this case we say that γ and η have a **contact of first order** in p (a **contact of order zero** being simply the fact that the pass through the same point, i.e. $\gamma(0) = \eta(0) = p$).

The set of curves in M passing through p quotiented w.r.t. the tangential equivalence turns out to be a copy of $\mathbb{R}^n$, as stated in the following result.

Theorem 2.1.1 *Fixed a local chart (U, φ) in $p \in M$, the map*

$$\begin{array}{rccl} I_{p,\varphi}: & T_p^{\text{geom}}M & \xrightarrow{\sim} & \mathbb{R}^n \\ & [\gamma] & \longmapsto & I_{p,\varphi}([\gamma]) = (\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0), \end{array}$$

which associates to a tangentially equivalence class of paths passing through p their common tangent vector $(\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0)$ in $\mathbb{R}^n$ w.r.t. the local chart φ, is a bijection.

Proof. Injectivity is obvious: different tangential classes of curves are associated to different tangent vectors in $\mathbb{R}^n$.

To prove surjectivity, fixed any $v \in \mathbb{R}^n$, we must prove that there exists $[\gamma] \in T_p^{\text{geom}}M$ such that $I_{p,\varphi}([\gamma]) = (\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0) = v$. This can be done very simply by transporting to M via φ^{-1} the segment of straight line passing through $x = \varphi(p)$ and directed as v, i.e. $r_{x,v}|_{(-\varepsilon,\varepsilon)} : \mathbb{R} \to \mathbb{R}^n$, $r_{x,v}(t) := x + tv$, where $\varepsilon > 0$ is small enough so that $r_{x,v}(-\varepsilon, \varepsilon)$ is contained in $\varphi(U)$:

$$\gamma = \varphi^{-1} \circ r_{x,v}|_{(-\varepsilon,\varepsilon)} : (-\varepsilon, \varepsilon) \to M, \quad \boxed{\gamma(t) = \varphi^{-1}(x + tv)}, \qquad t \in (-\varepsilon, \varepsilon). \tag{2.2}$$

γ is such that $\gamma(0) = \varphi^{-1} \circ r_{x,v}|_{(-\varepsilon,\varepsilon)}(0) = \varphi^{-1}(x) = \varphi^{-1}(\varphi(p)) = p$, hence, to prove surjectivity it remains only to check that the tangent vector in $\mathbb{R}^n$ of the local representation of γ associated to φ, i.e. $(\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0)$, coincides with v:

$$I_{p,\varphi}([\gamma]) = (\varphi \circ \gamma)^{\boldsymbol{\cdot}}(0) = (\varphi \circ \varphi^{-1}(x + tv))^{\boldsymbol{\cdot}}(0) = (x + tv)^{\boldsymbol{\cdot}}(0) = v.$$

□

$T_p^{\text{geom}}M$ cannot be canonically identified with $\mathbb{R}^n$ because $I_{p,\varphi}$ depends both on the point p and the local chart φ: changing the point p on M and/or the local chart φ changes the identification with $\mathbb{R}^n$.

Since the elements of $T_p^{\text{geom}}M$ are called tangent vectors, we expect $T_p^{\text{geom}}M$ to be a vector space, this is actually the case. The linear structure of $T_p^{\text{geom}}M$ is borrowed from that of $\mathbb{R}^n$ thanks to the bijection provided by $I_{p,\varphi}$.

Linear structure of $T_p^{\text{geom}}M$:

$$[\gamma] + [\sigma] := I_{p,\varphi}^{-1}(I_{p,\varphi}([\gamma]) + I_{p,\varphi}([\sigma])), \quad [\gamma], [\sigma] \in T_p^{\text{geom}}M$$
$$k[\gamma] := I_{p,\varphi}^{-1}(k I_{p,\varphi}([\gamma])), \quad k \in \mathbb{R}.$$

This definition of linear structure seems to depend on φ, however it does not, it is intrinsic. We prove this for the sum, an analogous proof holds for the product by a real coefficient.

Using the hypotheses and notations of Lemma 2.1.1, we have:

$$I_{p,\varphi_\beta}([\gamma]) = \dot{\gamma_\beta}(0) \underset{(2.1)}{=} D\eta_{\beta\alpha}(x)\dot{\gamma_\alpha}(0) = (D\eta_{\beta\alpha}(x) \circ I_{\varphi_\alpha,p})([\gamma]),$$

since this holds for all $[\gamma] \in T_p^{\text{geom}}M$, we have:

$$I_{p,\varphi_\beta} = D\eta_{\beta\alpha}(x) \circ I_{\varphi_\alpha,p} \iff I_{p,\varphi_\beta}^{-1} = I_{\varphi_\alpha,p}^{-1} \circ (D\eta_{\beta\alpha}(x))^{-1}, \qquad x = \varphi_\alpha(p). \tag{2.3}$$

If we denote temporarily with $+_\alpha$ and $+_\beta$ the sum brought to $T_p^{\text{geom}}M$ by the local charts φ_α and φ_β, respectively, then:

$$\begin{aligned}
[\gamma] +_\beta [\sigma] &= I_{p,\varphi_\beta}^{-1}(I_{p,\varphi_\beta}([\gamma]) + I_{p,\varphi_\beta}([\sigma])) \\
&\underset{(2.3)}{=} I_{\varphi_\alpha,p}^{-1} \circ (D\eta_{\beta\alpha}(x))^{-1}((D\eta_{\beta\alpha}(x) \circ I_{\varphi_\alpha,p})([\gamma]) + (D(\eta_{\beta\alpha}(x)) \circ I_{\varphi_\alpha,p})([\sigma])) \\
&\underset{(\text{lin. of } D\eta_{\beta\alpha}(x)}{=} (I_{\varphi_\alpha,p}^{-1} \circ (D\eta_{\beta\alpha}(x))^{-1} \circ D\eta_{\beta\alpha}(x))(I_{\varphi_\alpha,p}([\gamma]) + I_{\varphi_\alpha,p}([\sigma])) \\
&= I_{\varphi_\alpha,p}^{-1}(I_{\varphi_\alpha,p}([\gamma]) + I_{\varphi_\alpha,p}([\sigma])) \\
&= [\gamma] +_\alpha [\sigma].
\end{aligned}$$

The bijection $I_{p,\varphi}$ becomes a linear isomorphism between $T_p^{\text{geom}}M$ and $\mathbb{R}^n$ and thus it can be used to transport a basis of $\mathbb{R}^n$ to a basis of $T_p^{\text{geom}}M$. The easiest one is of course the canonical basis of $\mathbb{R}^n$, thanks to the proof of the surjectivity of $I_{p,\varphi}$, we have that the **basis of geometric tangent vectors of $T_p^{\text{geom}}M$ associated to the canonical basis of $\mathbb{R}^n$** is:

$$\left([\varphi^{-1} \circ r_{x,e_1}|_{(-\varepsilon,\varepsilon)}], \ldots, [\varphi^{-1} \circ r_{x,e_n}|_{(-\varepsilon,\varepsilon)}]\right) \equiv \left([t \mapsto \varphi^{-1}(x + te_1)], \ldots, [t \mapsto \varphi^{-1}(x + te_n)]\right), \tag{2.4}$$

where φ is any local chart in p and $r_{x,e_i}|_{(-\varepsilon,\varepsilon)} : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is the straight line segment passing through $x = \varphi(p)$, contained in $\varphi(U)$ and parallel to the i-th coordinate axis, $i = 1, \ldots, n$.

Finally, $I_{p,\varphi}$ can be used also to transport any norm of $\mathbb{R}^n$ to $T_p^{\text{geom}}M$. In this case, the norm on $T_p^{\text{geom}}M$ depends on φ, however, topologically speaking, this creates no problem at all because it is well-known that all norms on finite-dimensional vector spaces are equivalent, in particular, they are equivalent to the Euclidean norm.

To resume, $T_p^{\textbf{geom}}M$ **is a normed vector space isomorphic to a copy of** $\mathbb{R}^n$ **but not canonically**.

Before passing to the algebraic definition of tangent vectors and tangent space, we introduce the concept of differential of push forward of a geometric tangent vector.

Def. 2.1.3 (Differential (or push-forward, or tangent map) of a smooth function) *Given the smooth function $f : M \to N$, the differential (or push forward, or tangent map) of f at p is the map that transforms a tangentially equivalence class of paths passing through p to a tangentially equivalence class of paths passing through $f(p)$ simply by composition, i.e. :*

$$\begin{array}{rccl} df_p \equiv f_* : & T_p^{\text{geom}}M & \longrightarrow & T_{f(p)}^{\text{geom}}N \\ & [\gamma] & \longmapsto & df_p([\gamma]) \equiv f_*([\gamma]) = [f \circ \gamma]. \end{array} \tag{2.5}$$

With a quite technical computation that uses the definition of the linear structure of $T_p^{\text{geom}}M$, it can be proven that df_p is a linear operator. The non manifestly linear nature of $T_p^{\text{geom}}M$ and of the differential of geometric tangent vectors is one of the main reasons why mathematicians searched for an alternative definition, the algebraic one, which makes linearity manifest, as we are going to discuss in the next section.

2.2 Algebraic definition of tangent vectors

The link between the geometric and algebraic definition of tangent vectors on a manifold passes through the following considerations. An element of $\mathbb{R}^n$ can be interpreted either as a point, say $x \in \mathbb{R}^n$, and as a vector $v \in \mathbb{R}^n$, once these ones are fixed, there is just one way to define the directional derivative $D_v f(x)$ of a scalar valued function $f : \mathbb{R}^n \to \mathbb{R}$ in x along the direction defined by v, as discussed in Appendix B.

Directional derivatives are linear, they satisfy Leibniz's rule when applied to the product of two functions and they are null when a function is constant along the direction of derivation.

It turns out that these properties are necessary and sufficient to identify a tangent vector on a manifold in an algebraic way. This alternative vision, as it will be proven, is fully equivalent to the geometric one previously discussed.

This algebraic abstraction of a tangent vector is typical in modern mathematics and it can be considered as the analogous of the algebraic abstraction that leads to the definition of scalar product in an arbitrary vector space: in that case, the properties of bilinearity, symmetry and positive-definiteness are necessary and sufficient to identify a form on a real vector space as a scalar product. The advantages of this abstractions are known, e.g. the possibility to define a scalar product for vector spaces of any dimension and whose elements are not necessarily vectors in the Euclidean sense, but also polynomials, functions and so on.

The algebraic abstraction of the concept of tangent vector starts with the definition of a derivation on the set of smooth real scalar functions from M to $\mathbb{R}$, denoted with $\mathscr{C}^\infty(M)$, in a point $p \in M$. $\mathscr{C}^\infty(M)$ is a real algebra w.r.t. the point-wise linear operations and multiplication.

Def. 2.2.1 (Derivation on $\mathscr{C}^\infty(M)$ in a point) *Let $f, g \in \mathscr{C}^\infty(M)$. Fixed $p \in M$, a derivation on $\mathscr{C}^\infty(M)$ in $p \in M$ is a linear functional $v : \mathscr{C}^\infty(M) \to \mathbb{R}$ that satisfies the following* ***Leibniz rule****:*

$$\boxed{v(fg) = f(p)v(g) + g(p)v(f)}\,, \qquad \forall f, g \in \mathscr{C}^\infty(M).$$

The set of derivations on $\mathscr{C}^\infty(M)$ in $p \in M$ is easily proven to be a real vector space, w.r.t. the point-wise linear operations, that we denote with $\mathrm{Der}_p(M)$.

The Leibniz rule implies two basic properties of derivations.

Lemma 2.2.1 *Let $v \in Der_p(M)$ and $f, g \in \mathscr{C}^\infty(M)$. Then:*

1. *v sets to 0 constant functions: if $k_c \equiv c$, i.e. $k_c(q) = c \in \mathbb{R}$ for all $q \in M$, then $v(k_c) = 0$;*
2. *If f, g take null values in the application point p, i.e. $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Proof.

1.: let $k_1 \equiv 1$, then:

$$\begin{aligned} v(k_c) &= v(k_c k_1) = v(c k_1) \underset{v \text{ lin.}}{=} c v(k_1) = c v(k_1 \cdot k_1) \underset{\text{Leibniz}}{=} c(k_1(p)v(k_1) + k_1(p)v(k_1)) \\ &= 2cv(k_1) = 2v(ck_1) = 2v(k_c), \end{aligned}$$

i.e. $v(k_c) = 0$.

2.: $v(fg) = f(p)v(g) + g(p)v(f) = 0v(g) + 0v(f) = 0.$ □

The following property is of fundamental importance: it says that **derivations act locally**, in the sense that only the values taken by a smooth function on an arbitrarily small open neighborhood of the application point matter to define the action of the derivation.

Theorem 2.2.1 *Let $v \in Der_p(M)$ and $f, g \in \mathscr{C}^\infty(M)$. If there exists any open neighborhood $U \subseteq M$ of p such that $f|_U = g|_U$, then $v(f) = v(g)$.*

Proof. By hypothesis, $f - g$ is a smooth function on M that vanishes in U. Thanks to proposition 1.5.1, we know that it exists a smooth function $h \in \mathscr{C}^\infty(M)$ such that $h(p) = 1$ and $h|_{M \setminus U} \equiv 0$, then the product function $(f - g)h$ is zero, thus, thanks to the Leibniz property:

$$0 = v((f-g)h) = v(f-g)\cancelto{1}{h(p)} + (f-g)(p)v(h) = v(f) - v(g) + \cancelto{0}{(f(p) - g(p))}\, v(h),$$

i.e. $v(f) = v(g)$. □

We are now ready to define the concept of algebraic tangent vector.

Def. 2.2.2 (Algebraic tangent vector and space) *The vector space $T_p^{\text{alg}} M$, called algebraic tangent space to the manifold M at the point $p \in M$, is the vector space $\mathrm{Der}_p(M)$ of derivations on $\mathscr{C}^\infty(M)$ in $p \in M$:*

$$\boxed{T_p^{\text{alg}} M := \mathrm{Der}_p(M)}\,.$$

An element of $T_p^{\text{alg}} M$, i.e. a derivation on $\mathscr{C}^\infty(M)$ in $p \in M$, will be called an algebraic tangent vector to M in p.

v is a linear functional, i.e. $v \in \mathscr{C}^\infty(M)^*$, the dual space of $\mathscr{C}^\infty(M)$ (interpreted as a real vector space). $\mathscr{C}^\infty(M)^*$ is an infinite-dimensional vector space, however, as we will show, the Leibniz property is such a strong constraint to be satisfied that the linear functionals that satisfy it, i.e. those composing the subspace $\mathrm{Der}_p(M) \subset \mathscr{C}^\infty(M)^*$, form a n-dimensional vector space, n being the dimension of M.

When we have discussed the case of geometric tangent vectors, we have proven an analogous dimensional reduction, in that case it was operated by the quotient w.r.t. the tangential equivalence between paths on the set of paths in M passing through a point. This is a first indication of the fact that geometric and algebraic tangent vectors are equivalent concepts.

Proving that the algebraic tangent space to a manifold at a point is a n-dimensional vector space is more difficult than for its geometric counterpart. Multiple proofs are available in the literature, the line of reasoning that we have chosen to follow in this document is not the shortest, but it has the advantage that the intermediate steps are fairly easy to prove:

1. first of all, we prove the result in the trivial case of $M = \mathbb{R}^n$;
2. then, we define the algebraic version of the differential (or push-forward) of a smooth function and analyze its remarkable properties;
3. by fusing the previous steps, the proof that $T_p^{\mathrm{alg}}M$ is (not canonically) isomorphic to $\mathbb{R}^n$ will be almost immediate.

To prove that the algebraic tangent space to $\mathbb{R}^n$, or an open subset of $\mathbb{R}^n$, at a point x_0 is isomorphic to a copy of $\mathbb{R}^n$, we need the following intermediate result, which says that every smooth function f on $\mathbb{R}^n$ is associated to a n-tuple of smooth functions that coincide with the partial derivatives of f in x_0 and, moreover, this n-tuple of smooth functions allows for a sort of first order expansion of f in a sufficiently small open neighborhood of x_0.

Lemma 2.2.2 *Let $x_0 = (x_0^1, \dots, x_0^n) \in \mathbb{R}^n$ and $f \in \mathscr{C}^\infty(M)$, then there exist n smooth functions $g_1, \dots, g_n \in \mathscr{C}^\infty(V)$, where V is an open neighborhood V of x_0, such that:*

$$g_j(x_0) = \frac{\partial f}{\partial x^j}(x_0)$$

and

$$f(x) = f(x_0) + \sum_{j=1}^n (x^j - x_0^j) g_j(x),$$

for all $x \in V$.

Proof. V can be considered as *star-shaped*, i.e. the straight line segment between x_0 and $x \in V$ defined by $x_0 + t(x - x_0)$ for all $t \in [0,1]$ is entirely included in V; if it is not, then we can restrict it to a star-shaped open subset of $\mathbb{R}^n$ and work on this new neighborhood of x_0. Thanks to this remark, the expression $f(x_0 + t(x - x_0))$ is well-defined for all $x \in V$ and we can re-write the difference $f(x) - f(x_0)$ as follows:

$$f(x) - f(x_0) = [f(x_0 + t(x - x_0))]_{t=0}^{t=1} = \int_0^1 d(f(x_0 + t(x - x_0))),$$

thanks to the fundamental theorem of integral calculus. On the other side we have

$$\int_0^1 d(f(x_0 + t(x - x_0))) = \int_0^1 \frac{\partial}{\partial t} f(x_0 + t(x - x_0))dt,$$

we can expand the derivative under the integral by using the chain rule:

$$\frac{\partial}{\partial t} f(x_0+t(x-x_0)) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(x_0+t(x-x_0))\frac{\partial(x_0 + t(x^j - x_0))}{\partial t} = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(x_0+t(x-x_0))(x^j-x_0)$$

so that

$$f(x) - f(x_0) = \int_0^1 \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(x_0 + t(x - x_0))(x^j - x_0)dt = \sum_{j=1}^{n}(x^j - x_0) \int_0^1 \frac{\partial f}{\partial x^j}(x_0 + t(x - x_0))dt.$$

Since f is smooth, the integral exists and it is finite, and (since integration increases of one degree the regularity of the integrand) the functions g_j defined as follows:

$$g_j(x) = \int_0^1 \frac{\partial f}{\partial x^j}(x_0 + t(x - x_0))dt, \qquad \forall x \in V,$$

are smooth on V. Each g_j verifies both $f(x) = f(x_0) + \sum_{j=1}^{n}(x^j - x_0)g_j(x)$ and

$$g(x_0) = \int_0^1 \frac{\partial f}{\partial x^j}(x_0 + t(x_0 - x_0))dt = \int_0^1 \frac{\partial f}{\partial x^j}(x_0)dt = \frac{\partial f}{\partial x^j}(x_0) \int_0^1 dt = \frac{\partial f}{\partial x^j}(x_0),$$

thus proving the lemma. □

Theorem 2.2.2 *Let $V \subseteq \mathbb{R}^n$ be an open set and $x_0 \in V$, then the following map is an isomorphism of vector spaces:*

$$\begin{array}{rccl} \iota : & \mathbb{R}^n & \xrightarrow{\sim} & T^{\text{alg}}_{x_0}V \\ & v = (v^j) & \longmapsto & \iota(v) := \sum_{j=1}^{n} v^j \left.\frac{\partial}{\partial x^j}\right|_{x_0} \equiv D_v|_{x_0}, \end{array}$$

where the derivation $D_v|_{x_0} : \mathscr{C}^\infty(V) \to \mathbb{R}$ is nothing but the linear functional on $\mathscr{C}^\infty(V)$ that, when applied to a smooth scalar function f on V, provides its directional derivative along v in x_0:

$$D_v|_{x_0}(f) = D_v f(x_0) = \sum_{j=1}^{n} v^j \frac{\partial f}{\partial x^j}(x_0).$$

In particular, $T^{\text{alg}}_{x_0}V$ is a n-dimensional vector space.

Proof. The fact that ι is linear can be checked directly and it follows from the linearity of the operations involved in its definition.

<u>ι is one-to-one</u>: since ι is linear, to prove that it is injective we simply have to check that $\ker(\iota) = \{0_{\mathbb{R}^n}\}$. For that, it is sufficient to show that, if $v = (v^j) \neq 0_{\mathbb{R}^n}$, i.e. at least one

component is non null, say, $v^k \neq 0$, then the corresponding derivation $\iota(v)$ is not the null derivation, i.e. the derivation that sets all smooth scalar functions on V to 0.

To verify that, it is enough to consider the k-th canonical element of the dual basis of $\mathbb{R}^n$, i.e. $\varepsilon^k : V \to \mathbb{R}$, $\varepsilon^k(x) := x^k$. Of course $\varepsilon^k \in \mathscr{C}^\infty(V)$ because the projection on the k-th axis is smooth, so we can apply $\iota(v)$ to it, obtaining:

$$\iota(v)(\varepsilon^k) := \sum_{j=1}^n v^j \left.\frac{\partial}{\partial x^j}\right|_{x_0} \varepsilon^k = \sum_{j=1}^n v^j \frac{\partial \varepsilon^k(x_0)}{\partial x^j} = \sum_{j=1}^n v^j \frac{\partial x_0^k}{\partial x^j} = \sum_{j=1}^n v^j \delta_j^k = v^k \neq 0,$$

and so ι is one-to-one (note that x_0 is not a constant, but a variable, for this reason $\frac{\partial x_0^k}{\partial x^j} = \delta_k^j$).

<u>ι is onto</u>: we must show that, for every $D \in \mathrm{Der}_p(V)$, it exists a vector $v = (v^j) \in \mathbb{R}^n$ such that $D = \iota(v) = \sum\limits_{j=1}^n v^j \left.\frac{\partial}{\partial x^j}\right|_{x_0}$. To this aim, we use the previous lemma, expanding an arbitrary $f \in \mathscr{C}^\infty(V)$ as follows:

$$f(x) = f(x_0) + \sum_{j=1}^n (x^j - x_0^j) g_j(x)$$

in a neighborhood of x_0 inside V. This expression can be re-written as a functional equation, namely:

$$f = k_{f(x_0)} + \sum_{j=1}^n (\varepsilon^j - k_{x_0^j}) g_j,$$

where $k_{f(x_0)}(x) \equiv f(x_0)$ and $k_{x_0^j}(x) \equiv x_0^j$ are constant functions, and $\varepsilon^j(x) = x^j$. Applying D on f we get, by linearity, $D(f) = \cancelto{0}{D(k_{f(x_0)})} + \sum\limits_{j=1}^n D\left((\varepsilon^j - k_{x_0^j})g_j\right)$, having used the fact that a derivation sets to 0 constant functions. Now, by using Leibniz's rule:

$$D(f) = \sum_{j=1}^n \left[D(\varepsilon^j - k_{x_0^j}) g_j(x_0) + \cancelto{0}{(\varepsilon^j - k_{x_0^j})(x_0)}\, D g_j \right],$$

where the second term between square brackets vanishes because $(\varepsilon^j - k_{x_0^j})(x_0) = \varepsilon^j(x_0) - k_{x_0^j}(x_0) = x_0^j - x_0^j = 0$. So, using again the linearity of D, the nullification of constant functions and the fact that $g_j(x_0) = \frac{\partial f}{\partial x^j}(x_0)$ (thanks to the previous lemma), we have:

$$D(f) = \sum_{j=1}^n (D(\varepsilon^j) - \cancelto{0}{D(k_{x_0^j})}) g_j(x_0) = \sum_{j=1}^n D(\varepsilon^j) g_j(x_0) = \sum_{j=1}^n D(\varepsilon^j) \frac{\partial f}{\partial x^j}(x_0),$$

since f is arbitrary, we can write $D = \sum\limits_{j=1}^n D(\varepsilon^j) \left.\frac{\partial}{\partial x^j}\right|_{x_0}$, thus, to obtain $D = \iota(v)$ we simply have to consider the vector $v = (v^j) \in \mathbb{R}^n$ whose components satisfy:

$$\boxed{v^j := D(\varepsilon^j)}\,, \qquad j = 1, \dots, n. \tag{2.6}$$

□

Corollary 2.2.1 *For any fixed $x_0 \in \mathbb{R}^n$, the n derivations on $\mathscr{C}^\infty(\mathbb{R}^n)$ given by*

$$\left(\frac{\partial}{\partial x^1}\Big|_{x_0}, \ldots, \frac{\partial}{\partial x^n}\Big|_{x_0} \right) \equiv \left(D_{e_1}|_{x_0}, \ldots, D_{e_n}|_{x_0} \right)$$

form a basis of $T^{\mathrm{alg}}_{x_0}\mathbb{R}^n$.

Proof. Almost immediate: since the linear isomorphism ι of the previous theorem maps basis to basis, if we apply it to $(e_1, \ldots, e_n)$, the canonical basis of $\mathbb{R}^n$, we obtain a basis of $T^{\mathrm{alg}}_{x_0}\mathbb{R}^n$. Since the components of the canonical basis elements are all 0 except for only one, the images of $(e_1, \ldots, e_n)$ are exactly the evaluation in x_0 of the directional derivatives along each Cartesian axis, i.e. $ev_{x_0} \circ \frac{\partial}{\partial x^j} \equiv \frac{\partial}{\partial x^j}\big|_{x_0}$, $j = 1, \ldots, n$. □

2.2.1 The (algebraic) differential of a smooth function between manifolds

As we have already seen in the case of geometric tangent vectors, every smooth map f between manifolds M and N can be 'lifted' to a linear map between the tangent spaces of M and N called either differential, tangent map or (point-wise) push forward.

Here we provide the definition of differential when the tangent spaces are defined algebraically. Its properties will prove to be of fundamental importance.

Def. 2.2.3 (Differential of a smooth function – algebraic case) *Let $f : M \to N$ be a smooth function and $p \in M$, the differential of f in p is the linear function defined in this way:*

$$\begin{array}{rccc} df_p : & T^{alg}_p M & \longrightarrow & T^{alg}_{f(p)} N \\ & v & \longmapsto & df_p(v), \end{array}$$

where $df_p(v)$ is the derivation at $f(p)$ defined as follows:

$$\begin{array}{rccc} df_p(v) : & \mathscr{C}^\infty(N) & \longrightarrow & \mathbb{R} \\ & \phi & \longmapsto & \boxed{df_p(v)(\phi) = v(\phi \circ f)} \end{array}. \tag{2.7}$$

The composition between a scalar function with a map between manifolds appears often in differential geometry, for this reason it bears a special name and symbol.

Def. 2.2.4 (Pull-back of scalar functions) *Let $f : M \to N$ be a smooth function and $\phi : N \to \mathbb{R}$ a scalar function on N. Then, we can define a scalar function on M simply by composition with f:*

$$\begin{array}{rccc} f^* : & \mathscr{C}^\infty(N) & \longrightarrow & \mathscr{C}^\infty(M) \\ & \phi & \longmapsto & \boxed{f^*(\phi) = \phi \circ f} \end{array}.$$

f^* is called the pull-back via f because it pulls-back a scalar function on N, the codomain of f, to a scalar function on M, the domain of f. Of course, $(id_M)^*(\phi) = \phi$ for all $\phi \in \mathscr{C}^\infty(M)$, so:

$$(id_M)^*(\phi) = id_{\mathscr{C}^\infty(M)}. \tag{2.8}$$

Note that, if $f \in \mathscr{C}^\infty(M,N)$, $g \in \mathscr{C}^\infty(N,P)$ and $\phi \in \mathscr{C}^\infty(P)$, then $(g \circ f)^*(\phi) := \phi \circ (g \circ f)$, but $(f^* \circ g^*)(\phi) = f^*(g^*(\phi)) = f^*(\phi \circ g) = \phi \circ (g \circ f)$, thus

$$\boxed{(g \circ f)^* = f^* \circ g^*}\ . \tag{2.9}$$

When using the pull-back, the differential of a smooth map becomes:

$$df_p(v)(\phi) = (v \circ f^*)(\phi) \qquad \Longleftrightarrow \qquad df_p(v)(\phi) = v(f^*(\phi)), \qquad \forall \phi \in \mathscr{C}^\infty(N),$$

or, since the previous equation holds for every $\phi \in \mathscr{C}^\infty(N)$,

$$df_p(v) = v \circ f^* \qquad \Longleftrightarrow \qquad df_p(v) = v(f^*). \tag{2.10}$$

If we use the push forward notation (in which the point p is omitted) to push a tangent vector to M at p towards a tangent vector to N at $f(p)$, then we get the evocative expression below:

$$\begin{array}{rccl} f_*: & T_p^{\text{alg}}M & \longrightarrow & T_{f(p)}^{\text{alg}}N \\ & v & \longmapsto & \boxed{f_*(v) = v(f^*)}\ . \end{array}$$

The principal properties of the differential are listed in the following proposition.

Theorem 2.2.3 (Properties of the differential) *For all $p \in M$ the following properties hold.*

1. $d(id_M)_p = id_{T_p^{alg}M}$;
2. ***Chain rule for differential:*** *if $f \in \mathscr{C}^\infty(M,N)$ and $g \in \mathscr{C}^\infty(N,P)$, then the differential of the composed function $g \circ f : M \to P$ is the linear map $d(g \circ f)_p : T_p^{alg}M \to T_{g(f(p))}^{alg}P$ such that:*
$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$
3. *If $U \subseteq M$ is an open set containing p and $\iota : U \hookrightarrow M$ is the canonical inclusion in M, then $d\iota_p : T_p^{alg}U \to T_p^{alg}M$ is a canonical linear isomorphism.*
4. *If f is a <u>local</u> diffeomorphism defined on an open subset $U \subseteq M$ with values in $f(U) \subseteq N$, then $df_p : T_p^{alg}M \to T_{f(p)}^{alg}N$ is a (globally defined) linear isomorphism and*
$$(df_p)^{-1} = d(f^{-1})_{f(p)}. \tag{2.11}$$

Proof.

1. By (2.10) we get $d(id_M)_p(v) = v \circ (id_M)^*$ for all $v \in T_p^{\text{alg}}M$, moreover, thanks to (2.8) we have $(id_M)^* = id_{\mathscr{C}^\infty(M)}$, thus $d(id_M)_p(v) = v$, i.e. $d(id_M)_p = id_{T_p^{\text{alg}}M}$.

2. Let $v \in T_p^{\text{alg}}M$, arbitrary, then:

$$\begin{aligned} d(g \circ f)_p(v) &\underset{(2.10)}{=} v \circ (g \circ f)^* \underset{(2.9)}{=} v \circ (f^* \circ g^*) = (v \circ f^*) \circ g^* \\ &\underset{(2.10)}{=} df_p(v) \circ g^* \underset{(2.10)}{=} dg_{f(p)}(df_p(v)) \\ &= (dg_{f(p)} \circ df_p)(v). \end{aligned}$$

3. We will prove injectivity and surjectivity of $d\iota_p$.

Injectivity: we must show that the kernel of the linear map $d\iota_p$ is reduced to the zero derivation. For that, let us consider an arbitrary $v \in T_p^{\text{alg}}U$ and suppose that $d\iota_p(v) = 0$, we must show that this implies $v = 0$. To this aim, let B be an open neighborhood of p such that $\overline{B} \subseteq U$, then the extension theorem for smooth function (th. 1.5.2) assures us that any $f \in \mathscr{C}^\infty(U)$ can be extended to $\tilde{f} \in \mathscr{C}^\infty(M)$ in such a way that $\tilde{f}\Big|_{\overline{B}} \equiv f|_{\overline{B}}$. This implies that f and $\tilde{f}\Big|_U$ are smooth functions on U that agree on B, which is an open neighborhood of p, but then theorem 2.2.1 implies $v(f) = v(\tilde{f}\Big|_U)$. Now, $\tilde{f}\Big|_U$ is nothing but $\tilde{f} \circ \iota$, so

$$v(f) = v(\tilde{f} \circ \iota) \underset{(2.7)}{=} d\iota_p(v)(\tilde{f}) = 0,$$

because, by hypothesis, $d\iota_p(v) = 0$, the null derivation. Since $f \in \mathscr{C}^\infty(U)$ is arbitrary, $v = 0$ and so $d\iota_p$ is injective.

Surjectivity: consider an arbitrary $w \in T_p^{\text{alg}}M$, we must prove that it exists $v \in T_p^{\text{alg}}U$ such that $w = d\iota_p(v)$. We define such derivation as follows: $v(f) := w(\tilde{f})$ where $\tilde{f}$ in any smooth function on M such that $\tilde{f}\Big|_{\overline{B}} = f|_{\overline{B}}$.

Thanks to theorem 2.2.1, this definition of v does not depend on the choice of $\tilde{f}$ and it is of course a derivation of $\mathscr{C}^\infty(U)$ at p, thanks to the fact that w is linear and verifies the Leibniz rule. Finally, fixed any arbitrary function $g \in \mathscr{C}^\infty(M)$, we have that g, $g \circ \iota$ and $\widetilde{g \circ \iota}$ agree on B, thus:

$$d\iota_p(v)(g) \underset{(2.7)}{=} v(g \circ \iota) := w(\widetilde{g \circ \iota}) = w(g),$$

since g is arbitrary, we have that $w = d\iota_p(v)$ and so $d\iota_p$ is also surjective.

4. It is an easy consequence of the previous points. In fact, if f is a local diffeomorphism between U and $f(U)$, then it exists $f^{-1} : f(U) \to U$, such that $f^{-1} \circ f = id_U$, thus:

$$d(id_U)_p = d(f^{-1} \circ f)_p \underset{(2.)}{=} d(f^{-1})_{f(p)} \circ df_p.$$

On the other side, thanks to 1., $d(id_U)_p = id_{T_p^{\text{alg}}U}$ and, thanks to 3., $T_p^{\text{alg}}U \cong T_p^{\text{alg}}M$, thus $d(id_U)_p = id_{T_p^{\text{alg}}M}$. So, equating the two expressions for $d(id_U)_p$ that we have determined, we find $d(f^{-1})_{f(p)} \circ df_p = id_{T_p^{\text{alg}}M}$. Exchanging f with f^{-1} we get, with analogous considerations, $df_p \circ d(f^{-1})_{f(p)} = id_{T_p^{\text{alg}}N}$, thus proving 4. □

Property 3. allows us to identify in a canonical way the tangent space at a point to an open neighborhood of p with the tangent space at the same point to the whole manifold: the derivation $d\iota_p(v)$ is the same derivation as v in p acting on smooth scalar functions defined on the whole manifold M instead of those defined on U.

This is not surprising at all, since, as proven in proposition 2.2.1, the action of a derivation in a given point on a scalar function depends only on the values of the function in an arbitrarily small neighborhood of that point. From now on, we will implicitly accept the following natural identification:

$$\boxed{T_p^{\text{alg}}U \cong T_p^{\text{alg}}M} \qquad U \subseteq M,\ U \text{ open}.$$

As previously stated, thanks to $T_x^{\text{alg}}\mathbb{R}^n \cong \mathbb{R}^n$ for all $x \in \mathbb{R}^n$ and to the properties of the differential, we can very easily prove the isomorphism between $T_p^{\text{alg}}M$ and $\mathbb{R}^n$ just by considering the differential of an arbitrary chart map.

Theorem 2.2.4 *If M is a manifold of dimension n, then, fixed any $p \in M$, it exists a non-canonical linear isomorphism of vector spaces such that:*

$$\boxed{T_p^{\text{alg}}M \cong \mathbb{R}^n}$$

so, in particular, $\dim(T_p^{\text{alg}}M) = n$.

Proof. If (U, φ) is an arbitrary chart in p such that $\varphi(p) = x$, then $\varphi : U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ is a local diffeomorphism. By property 4. of the differential, $d\varphi_p : T_p^{\text{alg}}M \to T_x^{\text{alg}}\mathbb{R}^n$ is a linear isomorphism of vector spaces. Since this isomorphism depends on the chart φ, it is not canonical. □

2.2.2 A basis for $T_p^{\text{alg}}M$

Since $T_p^{\text{alg}}M$ is a n-dimensional vector space, it is natural to search for an explicit basis of tangent vectors.

In proposition 2.2.1 we have seen that, in the identification between $\mathbb{R}^n$ and $T_x^{\text{alg}}\mathbb{R}^n$, the canonical basis of $\mathbb{R}^n$ is identified with the basis of evaluations in x of the partial derivatives:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\sim} & T_x^{\text{alg}}\mathbb{R}^n \\ (e_1, \dots, e_n) & \longleftrightarrow & \left(\frac{\partial}{\partial x^1}\Big|_x, \dots, \frac{\partial}{\partial x^n}\Big|_x \right). \end{array}$$

Now, once selected a point $p \in M$ and a local chart (U, φ) in p such that $\varphi(p) = x \in \mathbb{R}^n$, we have just seen that the differential of φ in p is a linear isomorphism between $T_p^{\text{alg}}M$ and $T_p^{\text{alg}}\mathbb{R}^n \cong \mathbb{R}^n$, thus its inverse $(d\varphi_p)^{-1} : T_p^{\text{alg}}\mathbb{R}^n \cong \mathbb{R}^n \to T_p^{\text{alg}}M$ is a linear isomorphism too and, as such, it maps bases to bases.

As a consequence, **we can use $(d\varphi_p)^{-1}$ to transport the canonical basis of $\mathbb{R}^n$ (or, equivalently, the basis of $T_x^{\text{alg}}\mathbb{R}^n$ given by the evaluations in x of the partial derivatives) to a basis of $T_p^{\text{alg}}M$.**

So, for all $j = 1, \dots, n$:

$$\begin{array}{rccc} (d\varphi_p)^{-1} : & \mathbb{R}^n \cong T_x^{\text{alg}}\mathbb{R}^n & \xrightarrow{\sim} & T_p^{\text{alg}}M \\ & (e_j) \cong \left(\frac{\partial}{\partial x^j}\Big|_x \right) & \longleftrightarrow & (d\varphi_p)^{-1} \left(\frac{\partial}{\partial x^j}\Big|_x \right), \end{array}$$

the explicit action of the derivation $(d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^j}\Big|_x\right)$ on an arbitrary smooth scalar function $f \in \mathscr{C}^\infty(M)$ can be computed thanks to eq. (2.11), that in this case gives $(d\varphi_p)^{-1} = d(\varphi^{-1})_{\varphi(p)} = d(\varphi^{-1})_x$, so that:

$$(d\varphi_p)^{-1} \left(\frac{\partial}{\partial x^j}\bigg|_x \right)(f) = d(\varphi^{-1})_x \left(\frac{\partial}{\partial x^j}\bigg|_x \right)(f) := \frac{\partial}{\partial x^j}\bigg|_x (f \circ \varphi^{-1}) \equiv \frac{\partial (f \circ \varphi^{-1})}{\partial x^j}(x),$$

but $f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is nothing but **the local expression of f w.r.t. the chart** (U, φ) and the real numbers $\frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(x)$, $j = 1, \ldots, n$, represent the value of the directional derivatives of $f \circ \varphi^{-1}$ in the point $x \in \mathbb{R}^n$ along the unit canonical basis vectors e_j of $\mathbb{R}^n$.

The derivations $(d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^j}\big|_x\right)_{j=1}^n$ constitute a basis of $T_p^{\text{alg}}M$ and, to simplify the heavy notation, they are usually written as follows:

$$\partial_j\big|_p \equiv \left.\frac{\partial}{\partial x^j}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^j}\right|_x\right) = d(\varphi^{-1})_x\left(\left.\frac{\partial}{\partial x^j}\right|_x\right), \quad x = \varphi(p). \tag{2.12}$$

We resume what just discussed in the following theorem.

Theorem 2.2.5 (Coordinate tangent vectors to M at p) *Fixed $p \in M$ and a local chart (U, φ) in it such that $\varphi(p) = x$, the derivations of $T_p^{\text{alg}}M$ given by $(\partial_j\big|_p)_{j=1}^n$, or $\left(\frac{\partial}{\partial x^j}\big|_p\right)_{j=1}^n$, defined by:*

$$\begin{aligned} \partial_j\big|_p : \ \mathscr{C}^\infty(M) &\longrightarrow \mathbb{R} \\ f \quad &\longmapsto \boxed{\partial_j\big|_p(f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(x)}, \end{aligned} \tag{2.13}$$

or,

$$\begin{aligned} \left.\tfrac{\partial}{\partial x^j}\right|_p : \ \mathscr{C}^\infty(M) &\longrightarrow \mathbb{R} \\ f \quad &\longmapsto \boxed{\left.\frac{\partial}{\partial x^j}\right|_p(f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(x)}. \end{aligned} \tag{2.14}$$

form a basis of $T_p^{\text{alg}}M$. They are called coordinate tangent vectors to M at p.

Both notations are further simplified by writing:

$$\partial_j\big|_p(f) \equiv \partial_j f\big|_p = \frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(x) \quad \text{and} \quad \left.\frac{\partial}{\partial x^j}\right|_p(f) \equiv \left.\frac{\partial f}{\partial x^j}\right|_p = \frac{\partial(f\circ\varphi^{-1})}{\partial x^j}(x).$$

We remark again that **the real value obtained by applying th j-th coordinate tangent vector to M at p on a smooth function f on M is just the value of the partial derivative of the local expression of f** (and not of f itself!) **along the j-th axis in** $x = \varphi(p)$.

This is the reason why the expression $\left.\frac{\partial f}{\partial x^j}\right|_p$ **must not not be intepreted as the partial derivative of f in p in the usual sense**, because f is defined on M, not on $\mathbb{R}^n$! The notation $\partial_j f\big|_p$ may be used to avoid this misinterpretation, however, the notation $\left.\frac{\partial f}{\partial x^j}\right|_p$ has the advantage to make the chain rule 'visually easier' to handle, as we will see later.

The basis of coordinate tangent vectors will be **the key to understand the link between the algebraic definition of tangent vectors and the physicist's one**.

Remark: the derivations $(\partial_1\big|_p, \ldots, \partial_n\big|_p)$ are defined by applying the linear isomorphism $(d\varphi_p)^{-1}$ to the canonical basis of $\mathbb{R}^n$, so they cannot be considered as a canonical basis for $T_p^{\text{alg}}M$ (which does not exist), because **they depend on the choice of the local chart φ** *in* p! Different charts will produce, in general, different basis for $T_p^{\text{alg}}M$.

Moreover, **as p varies in M, the tangent spaces** T_pM, in spite of being isomorphic to $\mathbb{R}^n$, are not canonically isomorphic to each other and they **must be considered as different copies of $\mathbb{R}^n$ attached to each point p of the manifold M**.

2.2.3 Coordinate formula for the differential

Fixed a local chart $(U, \varphi \equiv (x^j))$ in $p \in M$ such that $\varphi(p) = x$, a tangent vector $v \in T_p^{\text{alg}}M$ can be written *uniquely* as a linear combination of the basis of coordinate tangent vectors as follows:

$$v = v^j \, \partial_j\big|_p \equiv v^j \, \frac{\partial}{\partial x^j}\bigg|_p ,$$

the real numbers v^j, $j = 1, \ldots, n$ are called the **components** of v on the basis of coordinate tangent vectors of $T_p^{\text{alg}}M$.

The following result establishes that the components of v characterize not only v as a derivation belonging to $T_p^{\text{alg}}M$, but also its expression in coordinates, furthermore, it gives a simple rule to explicitly compute the components v^j once the action of v on scalar functions is known.

Theorem 2.2.6 *With the notations of this section, it hold that:*

$$T_p^{\text{alg}}M \ni v = v^j \, \frac{\partial}{\partial x^j}\bigg|_p \quad \Longleftrightarrow \quad d\varphi_p(v) = v^j \, \frac{\partial}{\partial x^j}\bigg|_x \in T_x^{\text{alg}}\mathbb{R}^n, \tag{2.15}$$

moreover, the components of v are obtained by applying the derivation v to the local coordinate functions $x^j = \varepsilon^j \circ \varphi : U \to \mathbb{R}$, i.e.

$$\boxed{v^j = v(x^j)} \,. \tag{2.16}$$

Proof.

$\boxed{\Longrightarrow}$: let $v = v^j \, \frac{\partial}{\partial x^j}\big|_p \in T_p^{\text{alg}}M$, then the isomorphism $d\varphi_p$ allows us to obtain the tangent vector $d\varphi_p(v) \in T_x^{\text{alg}}\mathbb{R}^n$, whose action on smooth scalar functions $\phi \in \mathscr{C}^\infty(\mathbb{R}^n)$ is $d\varphi_p(v)(\phi) = v(\phi \circ \varphi)$, but, by definition of differential and by linearity, we have:

$$d\varphi_p\left(v^j \, \frac{\partial}{\partial x^j}\bigg|_p\right)(\phi) = v^j \, \frac{\partial}{\partial x^j}\bigg|_p (\phi \circ \varphi) \underset{(2.14)}{=} v^j \, \frac{\partial(\phi \circ \varphi \circ \varphi^{-1})}{\partial x^j}\bigg|_x = v^j \, \frac{\partial \phi}{\partial x^j}(x) = v^j \, \frac{\partial}{\partial x^j}\bigg|_x (\phi),$$

since this holds for all $\phi \in \mathscr{C}^\infty(\mathbb{R}^n)$, we have proven that $v = v^j \, \frac{\partial}{\partial x^j}\big|_p \in T_p^{\text{alg}}M$ implies $d\varphi_p(v) = v^j \, \frac{\partial}{\partial x^j}\big|_x \in T_x^{\text{alg}}\mathbb{R}^n$.

$\boxed{\Longleftarrow}$: suppose that $d\varphi_p(v) = v^j \, \frac{\partial}{\partial x^j}\big|_x \in T_x^{\text{alg}}\mathbb{R}^n$, then, applying the inverse linear isomorphism $(d\varphi_p)^{-1} = d(\varphi^{-1})_x : T_x^{\text{alg}}\mathbb{R}^n \to T_p^{\text{alg}}M$ we get:

$$v = (d\varphi_p)^{-1}(d\varphi_p(v)) = v^j \, d(\varphi^{-1})_x \left(\frac{\partial}{\partial x^j}\bigg|_x\right),$$

i.e. for all $\phi \in \mathscr{C}^\infty(U)$,

$$v(\phi) = v^j \, d(\varphi^{-1})_x \left(\frac{\partial}{\partial x^j}\bigg|_x\right)(\phi) = \frac{\partial}{\partial x^j}\bigg|_x (\phi \circ \varphi^{-1}) = v^j \frac{\partial(\phi \circ \varphi^{-1})}{\partial x^j}(x) \underset{(2.14)}{=} v^j \, \frac{\partial}{\partial x^j}\bigg|_p (\phi),$$

since this holds for all $\phi \in \mathscr{C}^\infty(U)$, we have proven that $d\varphi_p(v) = v^j \, \frac{\partial}{\partial x^j}\big|_x \in T_x^{\text{alg}}\mathbb{R}^n$ implies $v = v^j \, \frac{\partial}{\partial x^j}\big|_p \in T_p^{\text{alg}}M$.

Finally, since the coefficients v^j appear in two formulas, let us show (redundantly, by instructively) how to recover eq. (2.16) from both expressions. One strategy is to recall formula (2.6), which says that the real coefficients v^j are computed by applying the derivation $d\varphi_p(v)$ to the elements of the dual canonical basis of $\mathbb{R}^n$, i.e. $v^j = d\varphi_p(v)(\varepsilon^j) := v(\varepsilon^j \circ \varphi) = v(x^j)$.

Another strategy consists in the following brute force computation:

$$v(x^j) = v^k \left.\frac{\partial}{\partial x^k}\right|_p (x^j) = v^k \frac{\partial(\varepsilon^j \circ \varphi \circ \varphi^{-1})}{\partial x^k}(x) = v^k \frac{\partial x^j}{\partial x^k} = v^k \delta^j_k = v^j.$$

□

2.2.4 Differential of scalar functions and curves

Two special cases must be examined in relation with the differential: the first is when f is a scalar function, so that its *codomain* is a subset of $\mathbb{R}$, the other is when f is a path, so that its *domain* is a subset of $\mathbb{R}$.

Differential of a scalar function

Let us start with the case of a scalar function $\phi \in \mathscr{C}(M)$. Since ϕ is already a scalar function, we do not need to resort to other auxiliary scalar functions as in definition 2.2.3 and we can simply write:

$$\begin{array}{rccl} d\phi_p : & T^{\text{alg}}_p M & \longrightarrow & T^{\text{alg}}_{\phi(p)}\mathbb{R} \cong \mathbb{R} \\ & v & \longmapsto & \boxed{d\phi_p(v) := v(\phi)}\ . \end{array} \tag{2.17}$$

Since $T^{\text{alg}}_{\phi(p)}\mathbb{R}$ is a tangent space to $\mathbb{R}$ at a point and $v(\phi)$ is a real number, an explanation is needed to justify the previous definition. Note that $T^{\text{alg}}_{\phi(p)}\mathbb{R} = \text{span}\left(\frac{d}{dt}\big|_{\phi(p)}\right)$ and $\mathbb{R} = \text{span}(1)$, thus $T^{\text{alg}}_{\phi(p)}$ and $\mathbb{R}$ can be canonically identified via the following correspondence:

$$\begin{array}{ccc} T^{\text{alg}}_{\phi(p)}\mathbb{R} & \xrightarrow{\sim} & \mathbb{R} \\ \frac{d}{dt}\big|_{\phi(p)} & \longleftrightarrow & 1, \end{array}$$

so:

$$T^{\text{alg}}_{\phi(p)}\mathbb{R} \ni v(\phi) \left.\frac{d}{dt}\right|_{\phi(p)} \cong v(\phi)1 = v(\phi) \in \mathbb{R}.$$

It is custom to avoid specifying this canonical identification and to write the differential of a scalar function simply as in eq. (2.17).

Differential of a curve

Let us now consider $\gamma : (-\varepsilon, \varepsilon) \to M$ and $t_0 \in (-\varepsilon, \varepsilon)$. This time, via the identification $\mathbb{R} \simeq T_{t_0}\mathbb{R}$, we can identify t_0 with $\frac{d}{dt}\big|_{t_0}$, so that

$$\begin{array}{rccl} d\gamma_{t_0} : & T^{\text{alg}}_{t_0}\mathbb{R} \cong \mathbb{R} & \longrightarrow & T^{\text{alg}}_{\gamma(t_0)}M \\ & t_0 \equiv \frac{d}{dt}\big|_{t_0} & \longmapsto & d\gamma_{t_0}(\frac{d}{dt}\big|_{t_0}), \end{array}$$

where the action of $d\gamma_{t_0}(\frac{d}{dt}\big|_{t_0})$ on smooth functions on M is the canonical one for the differential, i.e.

$$\begin{array}{rcll} d\gamma_{t_0}(\frac{d}{dt}\big|_{t_0}) : & \mathscr{C}^\infty(M) & \longrightarrow & \mathbb{R} \\ & \phi & \longmapsto & d\gamma_{t_0}\left(\frac{d}{dt}\big|_{t_0}\right)(\phi) = \frac{d}{dt}\big|_{t_0}(\phi\circ\gamma) \equiv (\phi\circ\gamma)^\cdot(t_0). \end{array}$$

It us common to simplify the quite heavy notation as follows:

$$\boxed{d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right) \equiv \dot\gamma(t_0)}\ , \tag{2.18}$$

so that, when it is applied to a scalar function $\phi \in \mathscr{C}^\infty(M)$ it verifies:

$$\boxed{\dot\gamma(t_0)(\phi) := (\phi\circ\gamma)^\cdot(t_0)}\ .$$

Def. 2.2.5 (Velocity of a curve at a point) *The tangent vector $\dot\gamma(t_0) \in T^{\mathrm{alg}}_{\gamma(t_0)}M$ is called velocity of γ at t_0.*

As usual, if we want to find out the coordinate expression for $\dot\gamma(t_0) \in T^{\mathrm{alg}}_{\gamma(t_0)}M$, we must fix a local chart $(U, \varphi \equiv (x^i))$ in $p = \gamma(t_0) \in M$, such that $\gamma(-\varepsilon, \varepsilon) \subseteq U$.

Then, since $\dot\gamma(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\big|_{t_0}\right)$, the translation of the first equation of (2.15) and of eq. (2.16) into the present context gives:

$$\dot\gamma(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)(x^j)\ \frac{\partial}{\partial x^j}\bigg|_{\gamma(t_0)},$$

but $x^j = \varepsilon^j \circ \varphi$, so, by definition of differential:

$$d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)(x^j) = \frac{d}{dt}\bigg|_{t_0}(\varepsilon^j\circ\varphi\circ\gamma),$$

we notice that $\varphi\circ\gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and $\varepsilon^j\circ\varphi\circ\gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}$ are nothing but its n component functions which are usually indicated with γ^j, thus, the coordinate expression for the tangent vector of the curve γ in t_0 is:

$$\boxed{\dot\gamma(t_0) = \frac{d\gamma^j}{dt}(t_0)\ \frac{\partial}{\partial x^j}\bigg|_{\gamma(t_0)}}\ , \quad \gamma^j \equiv \varepsilon^j\circ\varphi\circ\gamma. \tag{2.19}$$

We finish this section by proving a result which shows that velocity vectors behave as expected under composition with smooth maps.

Theorem 2.2.7 (Velocity vector of a composite curve) *Let $f : M \to N$ be a smooth map, $\gamma : (-\varepsilon,\varepsilon) \to M$ be a smooth curve in M and $f\circ\gamma : (-\varepsilon,\varepsilon) \to N$ the composite curve in N. The velocity vector of $f\circ\gamma$ at any $t_0 \in (-\varepsilon,\varepsilon)$ satisfies:*

$$(f\circ\gamma)^\cdot(t_0) = df_{\gamma(t_0)}(\dot\gamma(t_0)).$$

Proof. By definition of velocity vector and thanks to the chain rule for the differential we have:

$$(f \circ \gamma)^{\bullet}(t_0) = d(f \circ \gamma)_{t_0} \left(\left.\frac{d}{dt}\right|_{t_0} \right) = df_{\gamma(t_0)} \circ d\gamma_{t_0} \left(\left.\frac{d}{dt}\right|_{t_0} \right) = df_{\gamma(t_0)}(\dot{\gamma}(t_0)).$$

□

This seemingly innocent result has a very useful consequence: from left to right, it tells us how to compute the velocity vector of a composite curve via the differential. But, if read the other way round, it allows us **to compute the differential of a function in terms of the velocity vector of a curve**! Let us see under which condition this is true: given a smooth function $f : M \to N$ and a point $p \in M$, to compute $df_p(v)$, $v \in T_p^{\mathrm{alg}}M$ with this technique we need a curve γ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

If such γ exists then, by using the previous result, the computation of the differential of f in p can be performed in terms of velocity vector of the composite curve $f \circ \gamma$ as follows:

$$df_p(v) = (f \circ \gamma)^{\bullet}(0), \quad v = \dot{\gamma}(0). \tag{2.20}$$

Actually, we are going to prove that the condition that we have pointed out is always verified. This result has a major importance also because it provides the bridge between the geometric and the algebraic definition of tangent vectors in differential geometry.

2.2.5 Equivalence between geometric and algebraic tangent vectors

We can finally prove that the definition of geometric and algebraic tangent vectors to a manifold at a point are completely equivalent.

Theorem 2.2.8 *Let $p \in M$ and let γ be a curve in M passing through p, i.e. $\gamma(0) = p$. Then, the map*

$$\begin{array}{rccl} I: & T_p^{\mathrm{geom}}M & \xrightarrow{\sim} & T_p^{\mathrm{alg}}M \\ & [\gamma] & \longmapsto & I[\gamma] := \dot{\gamma}(0), \end{array}$$

where $\dot{\gamma}(0) \equiv d\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0} \right)$ is the velocity vector of any $\gamma \in [\gamma]$, is an isomorphism of vector spaces. Thus, ***all tangent vectors to a manifold at a point are the velocity vector of a curve passing through that point****.*

Proof. First of all, let us prove that I is <u>well-defined</u>. Consider $\gamma_1, \gamma_2 \in [\gamma]$, we must verify that $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$. To this aim, consider a local chart $(U, \varphi \equiv (x^j))$ in $p = \gamma(0)$ and the local coordinate expressions of γ_1 and γ_2 given by $\gamma_1^j := \varepsilon^j \circ \varphi \circ \gamma_1$ and $\gamma_2^j := \varepsilon^j \circ \varphi \circ \gamma_2$. Then, by using the coordinate expression of the velocity vector, eq. (2.19), we get:

$$\dot{\gamma}_1(0) = \frac{d\gamma_1^j}{dt}(0) \left.\frac{\partial}{\partial x^j}\right|_p \qquad \text{and} \qquad \dot{\gamma}_2(0) = \frac{d\gamma_2^j}{dt}(0) \left.\frac{\partial}{\partial x^j}\right|_p .$$

γ_1 and γ_2 belong to the same tangentially equivalence class of curves (cfr. section 2.1), thus, by definition, $(\varphi \circ \gamma_1)^{\bullet}(0) = (\varphi \circ \gamma_2)^{\bullet}(0)$, i.e. $\frac{d\gamma_1^j}{dt}(0) = \frac{d\gamma_2^j}{dt}(0)$, for all $j = 1, \ldots, n$, since these values are nothing but the components of $(\varphi \circ \gamma_1)^{\bullet}(0)$ and $(\varphi \circ \gamma_2)^{\bullet}(0)$, respectively. It follows that $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ have the same decomposition on the coordinate tangent vector basis, hence, by the uniqueness of this decomposition, $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$.

This argument also shows that I is injective: if $[\gamma] \neq [\sigma]$, $\sigma(0) = p$, then, by definition, $\dot{\gamma}(0) \neq \dot{\sigma}(0)$. The linearity of I can be verified by direct computation and follows easily from the linearity of $\dot{\gamma}(0)$.

The only property that remains to be checked in the surjectivity of I, i.e. that for every $v \in T_p^{\text{alg}}M$, $v = v^j \frac{\partial}{\partial x^j}\big|_p$, it exists $[\gamma] \in T_p^{\text{geom}}M$ such that $I([\gamma]) = v$. We have already proven that I is well-defined, thus we can concentrate just on searching a representative curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, i.e. $\frac{d\gamma^j}{dt}(0) = v^j$ for all $j = 1, \ldots, n$.

To solve this problem we take inspiration from eq. (2.2) and we define the curve

$$\begin{array}{rccl} \gamma : & (-\varepsilon, \varepsilon) & \longrightarrow & U \\ & t & \longmapsto & \gamma(t) = \varphi^{-1}(x + t(v^1, \ldots, v^n)), \quad x = \varphi(p), \end{array}$$

which satisfies $\gamma(0) = p$ and, $\forall t \in (-\varepsilon, \varepsilon)$, $\forall j = 1, \ldots, n$:

$$\gamma^j(t) = (\varepsilon^j \circ \varphi \circ \gamma)(t) = \varepsilon^j(\varphi(\varphi^{-1}(x + t(v^1, \ldots, v^n))) = \varepsilon^j(x + t(v^1, \ldots, v^n)) = x^j + tv^j.$$

Finally, thanks to eq. (2.19), we get:

$$\dot{\gamma}(0) = \frac{d\gamma^j}{dt}(0) \left.\frac{\partial}{\partial x^j}\right|_p = \frac{d(x^j + tv^j)}{dt}(0) \left.\frac{\partial}{\partial x^j}\right|_p = v^j \left.\frac{\partial}{\partial x^j}\right|_p = v.$$

□

Starting from now, we will drop the specification 'geom' and 'alg' from the notation of tangent space and we will **write simply T_pM for the tangent space to M at p.**

It will be clear from the context which kind of vector we are considering and, in any case, we know how to pass from one to the other and vice-versa. In particular, we have made the observations that led to eq. (2.20) rigorous and we can resume them in the following proposition.

Theorem 2.2.9 *Let $f \in \mathscr{C}^\infty(M, N)$ and $p \in M$. Let also $(U, \varphi \equiv (x^j))$ be a local chart in p such that $\varphi(p) = x \in \mathbb{R}^n$. If $v \in T_pM$ has the following local coordinate expression $v = v^j \frac{\partial}{\partial x^j}\big|_p$ w.r.t. this local chart, then it holds that:*

$$\boxed{df_p(v) = (f \circ \gamma)^{\cdot}(0)}\,, \tag{2.21}$$

with $\gamma(t) = \varphi^{-1}(x + t(v^1, \ldots, v^n))$, for all $t \in \mathbb{R}$ such that $\gamma(t) \in U$.

In particular, the class of tangentially equivalent paths that are in one-to-one correspondence with the coordinate tangent vectors $\frac{\partial}{\partial x^j}\big|_p$ is:

$$\left.\frac{\partial}{\partial x^j}\right|_p \cong [t \mapsto \varphi^{-1}(x + te^j)], \qquad x = \varphi(p),$$

where e^j is the j-th element of the canonical basis of $\mathbb{R}^n$. This result confirms what we have already established in eq. (2.4) and underline once more that the tangent vectors $\frac{\partial}{\partial x^j}\big|_p$ are locally defined, they depend on the choice of the coordinate system defined by the chart (U, φ) and they are associated to the vectors of the canonical basis of $\mathbb{R}^n$.

In the particular case $M = \mathbb{R}^n$ or of a real vector space V we have global charts and we can state the previous result as follows.

Corollary 2.2.2 *Let V, W be two finite dimensional real vector spaces, $f \in \mathscr{C}^\infty(V, W)$ and $x \in V$. Then it holds that:*

$$\boxed{df_x(v) = \left.\frac{d}{dt}\right|_{t=0} f(x + tv)}\,, \qquad \forall v \in T_xV. \tag{2.22}$$

2.2.6 Relationship between the differential and the total derivative on vector spaces

We are now going to show that the differential of a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, Ω open, coincides with its total derivative in the sense defined in Appendix B.

Suppose $x \in \Omega$, then, by proposition 2.2.1, we can write any $v \in T_x\Omega \cong T_x\mathbb{R}^n$ as $v = v^j \left.\frac{\partial}{\partial x^j}\right|_x$. By definition of differential of a scalar function, i.e. (2.17), we get:

$$df_x(v) = v(f) = v^j \left.\frac{\partial}{\partial x^j}\right|_x (f) = v^j \frac{\partial f}{\partial x^j}(x) = D_v f(x),$$

where $D_v f(x)$ is the directional derivative of f in the direction defined by v, identified with a vector of $\mathbb{R}^n$ thanks to the canonical isomorphism $T_x\mathbb{R}^n \cong \mathbb{R}^n$.

However, in Appendix B it is proven that the $D_v f(x)$ is obtained by applying the total derivative of f in x to the vector v: $Df(x)(v) = D_v f(x)$ and this holds for all $v \in \mathbb{R}^n$.

As a consequence, **we can canonically identify the differential of a scalar function defined on an open $\Omega \subseteq \mathbb{R}^n$ at any point with its total derivative in the same point**:

$$\boxed{df_x = Df(x)}\,, \qquad \forall f \in \mathscr{C}^\infty(\Omega).$$

The same identification holds for functions as $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, Ω open: as always, one considers the component functions of $f = (f^1, \ldots, f^m)$, that are scalar functions to which one can apply the result just proven.

We will use this result to compute some remarkable differentials in section 2.9.

2.3 Matrix expression of the differential in coordinates

$df_p : T_pM \to T_{f(p)}N$ is a linear operator between finite dimensional vector spaces, thus we can represent it as a matrix. To understand how to do it, we first examine the trivial case of $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$.

Given $f : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$, U open and f smooth, once we fix any $x \in U$, we have just seen that the differential operator $df_x : \mathbb{R}^n \to \mathbb{R}^m$ coincides with the total derivative of f in x, which is represented in matrix form by the Jacobian matrix of f in x. It is an instructive exercise to explicitly verify that this is actually the case.

If we denote with $(x^i)_{i=1}^m$ and $(y^j)_{j=1}^n$ the coordinates in U and V respectively, then the coordinate tangent vectors $\left(\left.\frac{\partial}{\partial x^i}\right|_x\right)_{i=1}^m$ and $\left(\left.\frac{\partial}{\partial y^j}\right|_{f(x)}\right)_{j=1}^n$ form a basis of $T_x\mathbb{R}^m$ and $T_{f(x)}\mathbb{R}^n$, respectively.

To find the matrix expression of df_x w.r.t. these bases we know that we must apply df_x to the vectors of the first basis and the express the results as a linear combination of the vectors of the second basis. The coefficients of this combination are the columns of the matrix that represents df_x w.r.t. the chosen bases.

Note that $df_x\left(\left.\frac{\partial}{\partial x^i}\right|_x\right) \in T_{f(x)}V$, thus it is a derivation on $\mathscr{C}^\infty(V)$, so, to make its action explicit, we have to fix an arbitrary smooth scalar function $g \in \mathscr{C}^\infty(V)$ and write:

$$df_x\left(\left.\frac{\partial}{\partial x^i}\right|_x\right)(g) := \left.\frac{\partial}{\partial x^i}\right|_x (g \circ f) \underset{\text{(chain rule)}}{=} \frac{\partial g}{\partial y^j}(f(x))\frac{\partial f^j}{\partial x^i}(x) = \frac{\partial f^j}{\partial x^i}(x)\frac{\partial g}{\partial y^j}(f(x)),$$

re-writing conveniently $\frac{\partial g}{\partial y^j}(f(x)) = \left.\frac{\partial}{\partial y^j}\right|_{f(x)}(g)$ to make the coordinate tangent vector basis of $T_{f(x)}\mathbb{R}^n$ appear explicitly, we get:

$$df_x\left(\left.\frac{\partial}{\partial x^i}\right|_x\right)(g) = \left(\frac{\partial f^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{f(x)}\right)(g),$$

since g is arbitrary, we have:

$$\boxed{df_x\left(\left.\frac{\partial}{\partial x^i}\right|_x\right) = \frac{\partial f^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{f(x)}}\,. \tag{2.23}$$

We have obtained what we wanted: the explicit expression of the coordinate tangent vector basis of $T_x\mathbb{R}^m$ transformed by df_x and expressed as a linear combination of the coordinate tangent vector basis of $T_{f(x)}\mathbb{R}^n$.

The coefficients of the linear combination are the partial derivatives of the component functions f^j of f in x, it follow that **the matrix expression of df_x is exactly the Jacobian matrix of f in x**:

$$Jf(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \dots & \frac{\partial f^1}{\partial x^m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial x^1}(x) & \dots & \frac{\partial f^n}{\partial x^m}(x) \end{pmatrix} = \begin{pmatrix} \nabla f^1(x) \\ \vdots \\ \nabla f^n(x) \end{pmatrix}.$$

Let us now consider the more general situation of a smooth function $f : M \to N$ between manifolds of dimension m and n, respectively.

As always, the idea is to select a couple of f-related charts (U, φ) in M containing p and (V, ψ) in N containing $f(p)$ and to consider the local representation of f, i.e. $\tilde{f} = \psi \circ f \circ \varphi^{-1}$, as in the following diagram:

$$\begin{tikzcd} M \supseteq U \arrow[r, "f"] \arrow[d, bend right, "\varphi"'] & V \subseteq N \arrow[d, bend right, "\psi"'] \\ \mathbb{R}^m \supseteq \varphi(U) \arrow[r, dashed, "\tilde{f}"] \arrow[u, bend right, "\varphi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n. \arrow[u, bend right, "\psi^{-1}"'] \end{tikzcd}$$

We write $\varphi(p) = x \equiv (x^i) \in \varphi(U)$ and $\tilde{f}(x) = \psi(f(p)) = \psi(f(\varphi^{-1}(x))) = y \equiv (y^j) \in \psi(V)$.

Eq. (2.23) implies:

$$d\tilde{f}_x\left(\left.\frac{\partial}{\partial x^i}\right|_x\right) = \frac{\partial \tilde{f}^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{\tilde{f}(x)}. \tag{2.24}$$

Moreover, by definition of $\tilde{f}$ we get: $f \circ \varphi^{-1} = \psi^{-1} \circ \tilde{f}$, thus $d(f \circ \varphi^{-1}) = d(\psi^{-1} \circ \tilde{f})_x$, so, by the chain rule:

$$df_p \circ d(\varphi^{-1})_x = d(\psi^{-1})_{\tilde{f}(x)} \circ d\tilde{f}_x \tag{2.25}$$

and, thanks to property 4. of the differential (cfr. theorem 2.2.3), $d(\psi^{-1})_{\tilde{f}(x)} = d(\psi_{\psi^{-1}(\tilde{f}(x))})^{-1} = d(\psi_{f(\varphi^{-1}(x))})^{-1} = d(\psi_{f(p)})^{-1}$, i.e.

$$d(\psi^{-1})_{\tilde{f}(x)} = d(\psi_{f(p)})^{-1} \tag{2.26}$$

so:

$$\begin{aligned}
df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &\underset{(2.12)}{=} df_p\left((d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^j}\right|_x\right)\right) \underset{((d\varphi_p)^{-1}=d(\varphi^{-1})_x)}{=} df_p\left(d(\varphi^{-1})_x\left(\left.\frac{\partial}{\partial x^j}\right|_x\right)\right) \\
&= df_p \circ d(\varphi^{-1})_x\left(\left.\frac{\partial}{\partial x^j}\right|_x\right) \underset{(2.25)}{=} d(\psi^{-1})_{\tilde{f}(x)} \circ d\tilde{f}_x\left(\left.\frac{\partial}{\partial x^j}\right|_x\right) \\
&= d(\psi^{-1})_{\tilde{f}(x)}\left(d\tilde{f}_x\left(\left.\frac{\partial}{\partial x^j}\right|_x\right)\right) \underset{(2.24)}{=} d(\psi^{-1})_{\tilde{f}(x)}\left(\frac{\partial \tilde{f}^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{\tilde{f}(x)}\right) \\
&\underset{(\text{linearity})}{=} \frac{\partial \tilde{f}^j}{\partial x^i}(x)\, d(\psi^{-1})_{\tilde{f}(x)}\left(\left.\frac{\partial}{\partial y^j}\right|_{\tilde{f}(x)}\right) \underset{(2.26)}{=} \frac{\partial \tilde{f}^j}{\partial x^i}(x)\, d(\psi_{f(p)})^{-1}\left(\left.\frac{\partial}{\partial y^j}\right|_{\tilde{f}(x)}\right) \\
&\underset{(2.12)}{=} \frac{\partial \tilde{f}^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{f(p)},
\end{aligned}$$

so:

$$\boxed{df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \tilde{f}^j}{\partial x^i}(x)\left.\frac{\partial}{\partial y^j}\right|_{f(p)}}\,. \tag{2.27}$$

If we compare eqs. (2.23) and (2.27), we see that the only difference is that the real coefficients in the latter are given by the partial derivatives of the local expression f w.r.t. the charts chosen. Thus, also in the general case of a smooth function between manifolds, **the matrix expression of the differential of f in p (relative to the coordinate tangent vectors) is given by a Jacobian matrix, but, in this case, of the local expression of f computed in $x = \varphi(p)$**:

$$J\tilde{f}(x) = \begin{pmatrix} \frac{\partial \tilde{f}^1}{\partial x^1}(x) & \dots & \frac{\partial \tilde{f}^1}{\partial x^n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}^n}{\partial x^1}(x) & \dots & \frac{\partial \tilde{f}^n}{\partial x^n}(x) \end{pmatrix} = \begin{pmatrix} \nabla \tilde{f}^1(x) \\ \vdots \\ \nabla \tilde{f}^n(x) \end{pmatrix}.$$

2.4 The inverse mapping and implicit function theorems for manifolds

The result just obtained has a powerful consequence: **all the properties of standard differential calculus on $\mathbb{R}^n$ that are based on hypotheses made on the Jacobian matrix of a smooth function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$**, Ω open, **are also valid, locally, for smooth functions between manifolds**.

In this section we concentrate on two of the most important results of standard differential calculus on $\mathbb{R}^n$: the inverse mapping and the implicit function theorems.

We have already quoted the first, its extension can be stated as follows.

Theorem 2.4.1 (Inverse mapping theorem for manifolds) *Let $f : M \to N$ be a smooth function and $p \in M$ a point such that $df_p : T_pM \to T_{f(p)}N$ is an isomorphism. Then, there exist two open neighborhoods $U \subseteq M$ and $V \subseteq N$ of p and $f(p)$, respectively, such that $f|_U$ is a diffeomorphism.*

Proof. First of all notice that df_p, as a linear map, can be an isomorphism between vector spaces if and only if they have the same dimension, which implies that $\dim(M) = \dim(N)$.

Select two charts (U, φ) and (V, ψ) and consider the local representation $\tilde{f}$ of f. Since the Jacobian matrix of $\tilde{f}$ is the local representation of df_p and df_p is an isomorphism, $J\tilde{f}_{\varphi(p)}$ is invertible. Thanks to this, the standard inverse function theorem can be applied to $\tilde{f}$ and so $f|_U$ is a diffeomorphism. □

Let us now pass to the implicit function theorem by first recalling its classical statement, which tells us, in a very involved way, when we can locally solve an equation as $\phi(x, y) = z_0 \in \mathbb{R}$ and express y as a function of x.

Theorem 2.4.2 (Implicit function theorem in $\mathbb{R}^n$) *Hypotheses:*

- $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$*: open set;*
- $(x^1, \dots, x^n, y^1, \dots, y^m)$*: coordinates in U;*
- $\phi : U \to \mathbb{R}^m$*: differentiable function;*
- $(x_0, y_0) \in U$ *such that the matrix* $\left(\frac{\partial \phi^i}{\partial y^j}(x_0, y_0)\right)_{i,j}$ *is invertible.*

Thesis:

- $\exists$ *two open neighborhoods $V_0 \subseteq \mathbb{R}^n$ of x_0 and $W_0 \subseteq \mathbb{R}^m$ of y_0;*
- $\exists$ *a differentiable function $F : V_0 \to W_0$,*

such that, if $\phi(x_0, y_0) = z_0 \in \mathbb{R}$, the level set $\phi^{-1}(z_0) \cap (V_0 \times W_0)$ coincides with the graph of F, i.e.

$$\forall (x, y) \in V_0 \times W_0 \ : \ \phi(x, y) = z_0 \iff y = F(x).$$

Theorem 2.4.3 (Implicit function theorem for manifolds) *Hypotheses:*

- M, N*: smooth manifolds;*
- $\phi : M \times N \to N$*: smooth function;*
- $\forall p \in M$*, let*

$$\begin{array}{rccc} \phi_p : & N & \longrightarrow & N \\ & q & \longmapsto & \phi_p(q) = \phi(p, q); \end{array}$$

- *$d(\phi_{p_0})_{q_0} : T_{q_0} \to T_{r_0}Y$, where $r_0 = \phi(p_0, q_0)$, is an invertible linear map.*

Thesis: it exists two open neighborhoods $V_0 \subseteq M$ of p_0 and $W_0 \subseteq N$ of q_0 and a smooth function $F : V_0 \to W_0$ such that $\phi^{-1}(z_0) \cap (V_0 \cap W_0)$ coincides with the graph of F, i.e.

$$\forall (p,q) \in V_0 \times W_0 \ : \ \phi(p,q) = r_0 \iff q = F(p).$$

Proof. As for the inverse function theorem, by using two charts we can transport the problem to $\mathbb{R}^n$, where the standard hypotheses of the implicit function theorem hold. □

2.5 Alternative definitions of tangent vectors

In this section we complement the definition of geometric and algebraic tangent vector to a manifold at a point with other two definitions: the first is used mainly by pure mathematicians, the second mainly by physicists and engineers.

2.5.1 Tangent vectors as derivations on the algebra of germs of smooth functions

The name 'germ' is derived from 'cereal germ', which is the reproductive part of the cereal inside the seed. It is clearly used to indicate the 'heart' of a structure. It is a general concept related to topological spaces, where locality can be defined. In this section we will consider only the elements of the theory of germs that are strictly needed to give an alternative definition of tangent vectors, but the theory of germs is much more profound and not just related to differential geometry.

Def. 2.5.1 (Function element) *A smooth function element on a manifold M is an ordered pair (f, U), where U is an open subset of M and $f : U \to \mathbb{R}$ is a smooth scalar function.*

Fixed any point $p \in M$, it is possible to define an equivalence relation on the set of all smooth function elements whose domains contain p as follows: given $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$, $(f, U) \sim (g, V)$ if it exists an open neighborhood W of p such that:

$$W \subseteq U \cap V \quad \text{ and } \quad f|_W = g|_W \,,$$

i.e. if f and g coincide on some open neighborhood of p, however small, contained in the intersection of their domains.

Def. 2.5.2 (Germ of f at p) *The germ of f at p is the equivalence class of function elements (f, U) w.r.t. the equivalence relation defined above. The set of all germs of smooth functions at p is denoted by $\mathscr{C}^\infty_p(M)$.*

The germ of a function element (f, U) at p is denoted simply by $[f]_p$: in fact, there is no need to include the domain U in the notation because, by definition, the same germ is represented by the restriction of f to any open neighborhood of p.

$\mathscr{C}^\infty_p$ is a real vector space and an associative algebra under the point-wise defined linear operations and multiplication (of course, the sum and the multiplication are defined on the function element that has the intersection of the two functions as second entry in the couple).

We can now define the key concept of derivation on the algebra of germs.

Def. 2.5.3 (Derivation on the algebra of germs of smooth functions) *A derivation v on $\mathscr{C}^\infty_p(M)$ is a linear functional $v : \mathscr{C}^\infty_p(M) \to \mathbb{R}$ satisfying the following Leibniz rule:*

$$v([fg]_p) = (p)v([g]_p) + g(p)v([f]_p).$$

Derivations on $\mathscr{C}^\infty_p(M)$ form naturally a vector space that is denoted by $\mathscr{D}_p(M)$. Some author define the tangent space to M at p as the vector space $\mathscr{D}_p(M)$. The equivalence with the definition of tangent space in terms of derivations on $\mathscr{C}^\infty(M)$ is quite easy to prove thanks to the locality of derivations expressed by theorem 2.2.1.

Theorem 2.5.1 *The map*

$$\begin{array}{rccl} I: & \mathscr{D}_p(M) & \xrightarrow{\sim} & T_p^{\text{alg}}M \\ & v & \longmapsto & I(v), \end{array} \qquad \begin{array}{rccl} I(v): & \mathscr{C}^\infty(M) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & I(v)(f) := v([f]_p), \end{array}$$

is a natural linear isomorphism of vector spaces that allows us to identify algebraic tangent vectors to M at p with derivations on $\mathscr{C}^\infty_p(M)$.

Proof. Linearity clearly follows from the linearity of the derivation v. The injectivity of I is a consequence of the fact that, if $I(v) = 0$ (the identically null derivation on $\mathscr{C}^\infty(M)$), then, by definition, $v([f]_p) = 0$ for all $f \in \mathscr{C}^\infty(M)$, but this means that v is the null derivation on $\mathscr{C}^\infty_p(M)$, thus $\ker(I)$ is trivial.

Finally, to prove that I is surjective, we must verify that for any $w \in T_p^{alg}M$ there exists $v \in \mathscr{D}_p(M)$ such that $w = I(v)$. Thanks to theorem 2.2.1, such a $v \in \mathscr{D}_p(M)$ can simply be defined as follows:

$$v([f]_p) := w(f),$$

in fact, by definition of germ of smooth functions, if $f, g \in [f]_p$, then f and g are smooth scalar functions that coincide when restricted on an arbitrary small open neighborhood of p, so theorem 2.2.1 assures us that $w(f) = w(g)$, which guarantees that the definition of v is well-posed. Since w is a derivation on $\mathscr{C}^\infty(M)$, v also acts as a derivation on $\mathscr{C}^\infty_p(M)$, i.e. $v \in \mathscr{D}_p(M)$. □

Recall that, in order to obtain theorem 2.2.1, we had to make use of the theory of partitions of the unity and bump functions, thus, one immediate advantage of the use of germs to define tangent vectors is that we can avoid resorting to that theory and prove the same propositions with a less number of intermediate steps. We preferred to postpone until now the definition of tangent vectors via the theory of germs to avoid working with equivalence classes and to keep the notation as simple as possible.

2.5.2 Physicists' definition of tangent vectors

We introduce here **the oldest definition of tangent vector**, which is still the most widely used even today by the majority of physicists and engineers.

The construction is based on the decomposition of a tangent vector $v \in T_pM$ on the coordinates tangent vectors basis, which, as we have seen, is determined once we fix a local chart. Suppose, however, that p belongs to the intersection of two local charts (U, φ) and $(\tilde{U}, \tilde{\varphi})$, then we can decompose v w.r.t. the basis

- $\left(\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right)$, where $\frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\Big|_x$

  or w.r.t. the basis

- $\left(\frac{\partial}{\partial \tilde{x}^1}\Big|_p, \ldots, \frac{\partial}{\partial \tilde{x}^n}\Big|_p\right)$, where $\frac{\partial}{\partial \tilde{x}^j}\Big|_p (f) = \frac{\partial(f\circ\tilde{\varphi}^{-1})}{\partial \tilde{x}^j}\Big|_x$,

for all $f \in \mathscr{C}^\infty(U \cap \tilde{U})$.

Since the tangent vector v in p remains the same, we must have:

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p, \tag{2.28}$$

where, due to the uniqueness of the decomposition of a vector over a basis, the components v^i are uniquely associated to the coordinates on M defined by local chart (U, φ) and the components $\tilde{v}^j$ are uniquely associated to those defined by $(\tilde{U}, \tilde{\varphi})$. It is natural to ask oneself how the coefficients v^i and $\tilde{v}^j$ are related.

To answer this question, let us recall that the transition functions between these charts are, respectively:

$$\begin{array}{rccc} x^i = \varepsilon^i \circ \varphi \circ \tilde{\varphi}^{-1}: & \tilde{\varphi}(U \cap \tilde{U}) \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & \tilde{x} = \tilde{\varphi}(p) & \longmapsto & x^i(\tilde{x}) = \varepsilon^i(\varphi(p)), \end{array}$$

and

$$\begin{array}{rccc} \tilde{x}^j = \varepsilon^j \circ \tilde{\varphi} \circ \varphi^{-1}: & \varphi(U \cap \tilde{U}) \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x = \varphi(p) & \longmapsto & \tilde{x}^j(x) = \varepsilon^j(\tilde{\varphi}(p)), \end{array}$$

$i, j = 1, \ldots, n$.

The tool to obtain the explicit coordinate transformations $v^i \mapsto \tilde{v}^j$ and $\tilde{v}^j \mapsto v^i$ are the following formulae:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p, \tag{2.29}$$

and

$$\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p = \frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}) \left.\frac{\partial}{\partial x^i}\right|_p, \tag{2.30}$$

typically quoted to be the result of the application of the chain rule, without any further comment.

It is an instructive computation to verify these formulae. We will do that for the first one, the method to get the second one is identical. Consider the differential:

$$d(\tilde{\varphi}\circ\varphi^{-1})_x \left(\left.\frac{\partial}{\partial x^i}\right|_x \right) \underset{(2.23)}{=} \frac{\partial(\varepsilon^j \circ \tilde{\varphi} \circ \varphi^{-1})}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\tilde{x}} \equiv \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\tilde{x}}, \tag{2.31}$$

then, thanks to def. (2.12) of coordinate tangent vectors, we have:

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= d(\varphi^{-1})_x \left(\left.\frac{\partial}{\partial x^i}\right|_x \right) \quad (\varphi^{-1} = \tilde{\varphi}^{-1}\circ\tilde{\varphi}\circ\varphi^{-1} \text{ and the chain rule for differential imply}) \\ &= d(\tilde{\varphi}^{-1})_{\tilde{x}} \circ d(\tilde{\varphi}\circ\varphi^{-1})_x \left(\left.\frac{\partial}{\partial x^i}\right|_x \right) \underset{(2.31)}{=} d(\tilde{\varphi}^{-1})_{\tilde{x}} \left(\frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\tilde{x}} \right) \underset{(2.23)}{=} \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p, \end{aligned}$$

which confirms eq. (2.29).

If we insert the expressions (2.29) and (2.30) in (2.28), we get

$$v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \iff v^i \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p = \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}) \left.\frac{\partial}{\partial x^i}\right|_p .$$

By the uniqueness of the decomposition of a vector on a basis, we have that

$$v = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p = v^i \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p$$

implies:

$$\boxed{\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(x) v^i} \iff \tilde{v}^j = J_i^j(\tilde{x})\Big|_x v^i, \qquad i, j = 1, \ldots, n, \tag{2.32}$$

where $J_i^j(\tilde{x})$ is the $n \times n$ *matrix of functions* that contains the partial derivatives of the function $\tilde{x} = \tilde{\varphi} \circ \varphi^{-1} : \varphi(U \cap \tilde{U}) \subseteq \mathbb{R}^n \to \tilde{\varphi}(U \cap \tilde{U}) \subseteq \mathbb{R}^n$: each rows contains the gradient of the function $\tilde{x}^j = \varepsilon^j \circ \tilde{x}$:

$$J_i^j(\tilde{x}) = \begin{pmatrix} \nabla \tilde{x}^1 \\ \vdots \\ \nabla \tilde{x}^n \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{x}^n}{\partial x^1} & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n} \end{pmatrix},$$

once evaluated in x, this becomes a matrix of real entries that represents the Jacobian matrix $J_i^j(\tilde{x})\Big|_x$ of the function $\tilde{x}$ in x. Since transition functions are invertible, $J_i^j(\tilde{x})\Big|_x \in \mathrm{GL}(n, \mathbb{R})$.

On the other side, the equality

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}) \left.\frac{\partial}{\partial x^i}\right|_p$$

implies:

$$\boxed{v^i = \frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}) \tilde{v}^j} \iff v^i = {}^t\!\left(J_i^j(\tilde{x})\Big|_x \right)^{-1} \tilde{v}^j, \qquad i, j = 1, \ldots, n, \tag{2.33}$$

where ${}^t\!\left(J_i^j(\tilde{x})\Big|_x \right)^{-1}$ is **the inverse and transposed** (notice the position of the indices) of the Jacobian matrix $J_i^j(\tilde{x})\Big|_x$. The inversion is to be expected because the transition functions $\tilde{\varphi} \circ \varphi^{-1}$ and $\varphi \circ \tilde{\varphi}^{-1}$ are one the inverse of each other.

The rule (2.32) is called **gradient or contravariant transformation** and it is an **intrinsic property of tangent vectors** (no additional properties of structures have been used to obtain (2.32) other than those related to tangent vectors).

This motivates why tangent vectors can be alternatively defined as ordered n-tuples of real scalars that undergo the contravariant transformation (2.32) under local coordinate changes.

Example: consider the polar coordinates $(x^1, x^2) = (r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$ on the plane $\mathbb{R}^2$, the point $p = (2, \pi/2)$ and the tangent vector $v \in T_p\mathbb{R}^2$ expressed by:

$$v = 3 \left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p .$$

We want to find the expression of v w.r.t. Cartesian coordinates. The transition map between polar and Cartesian coordinates in an open neighborhood of $p \in \mathbb{R}^2$ is

$$\begin{cases} \tilde{x}^1 = x = r\cos\theta \\ \tilde{x}^2 = y = r\sin\theta. \end{cases}$$

The vector v can be expressed as follows:

$$v = 3\left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p = \tilde{v}^1 \left.\frac{\partial}{\partial x}\right|_p + \tilde{v}^2 \left.\frac{\partial}{\partial y}\right|_p,$$

with $v^1 = 3$ and $v^2 = -1$. By means of eq. (2.32) we get:

$$\begin{aligned} \tilde{v}^1 &= \frac{\partial \tilde{x}^1(r,\theta)}{\partial x^1}(2,\pi/2)v^1 + \frac{\partial \tilde{x}^1(r,\theta)}{\partial x^2}(2,\pi/2)v^2 = 3\frac{\partial(r\cos\theta)}{\partial r}(2,\pi/2) - \frac{\partial(r\cos\theta)}{\partial\theta}(2,\pi/2) \\ &= (3\cos\theta)|_{(2,\pi/2)} + (r\sin\theta)|_{(2,\pi/2)} = 3\cos(\pi/2) + 2\sin(\pi/2) = 2, \end{aligned}$$

and

$$\begin{aligned} \tilde{v}^2 &= \frac{\partial \tilde{x}^2(r,\theta)}{\partial x^1}(2,\pi/2)v^1 + \frac{\partial \tilde{x}^2(r,\theta)}{\partial x^2}(2,\pi/2)v^2 = 3\frac{\partial(r\sin\theta)}{\partial r}(2,\pi/2) - \frac{\partial(r\sin\theta)}{\partial\theta}(2,\pi/2) \\ &= (3\sin\theta)|_{(2,\pi/2)} - (r\cos\theta)|_{(2,\pi/2)} = 3, \end{aligned}$$

thus:

$$v = 2\left.\frac{\partial}{\partial x}\right|_p + 3\left.\frac{\partial}{\partial y}\right|_p.$$

Remark: the notation $\left.\frac{\partial}{\partial x^i}\right|_p$ must not lead to think that the coordinate tangent vector $\left.\frac{\partial}{\partial x^i}\right|_p$ depends only on x^i: in fact it depends on the entire coordinate system. The geometrical reason underlying this is the fact that $\left.\frac{\partial}{\partial x^i}\right|_p$ is the derivation whose action on a smooth scalar function is defined by taking the partial derivative of the local expression of this function w.r.t. x^i, i.e. by letting x^i vary and fixing all the other local coordinates x^j, $j \neq i$. So, if we change the coordinates x^j, they are not constant anymore and this, in general, affects $\left.\frac{\partial}{\partial x^i}\right|_p$.

We illustrate this fact with the following concrete example: consider $\mathbb{R}^2$ with the standard Cartesian coordinates (x, y) and let $p = (1, 0) \in \mathbb{R}^2$, expressed w.r.t. the standard coordinates. Now, perform the coordinate change defined by

$$\begin{cases} \tilde{x} = x \\ \tilde{y} = y + x^3. \end{cases}$$

Our aim is to show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p,$$

in spite of the fact that $x = \tilde{x}$.

First of all notice that the coordinates $(\tilde{x}, \tilde{y})$ are smooth and global on $\mathbb{R}^2$ since the inverse of the coordinate change $(x, y) \mapsto (\tilde{x} = x, \tilde{y} = y + x^3)$ is $(\tilde{x}, \tilde{y}) \mapsto (x = \tilde{x}, y = \tilde{y} - \tilde{x}^3)$. Thanks to eq. (2.29), we have:

$$\left.\frac{\partial}{\partial x}\right|_p = \frac{\partial x}{\partial x}(1,0)\left.\frac{\partial}{\partial \tilde{x}}\right|_p + \frac{\partial(y+x^3)}{\partial x}(1,0)\left.\frac{\partial}{\partial \tilde{y}}\right|_p = \left.\frac{\partial}{\partial \tilde{x}}\right|_p + (3x^2)|_{(1,0)}\left.\frac{\partial}{\partial \tilde{y}}\right|_p = \left.\frac{\partial}{\partial \tilde{x}}\right|_p + 3\left.\frac{\partial}{\partial \tilde{y}}\right|_p,$$

thus $\frac{\partial}{\partial x}\big|_p \neq \frac{\partial}{\partial \tilde{x}}\big|_p$.

From now on, **any definition of tangent vector to a manifold at a point** (geometric, algebraic, via germs of smooth functions or the physicists' one) **will be considered as equivalent**.

2.6 Canonical identification between vector spaces and their tangent spaces and differential of linear functions

We have proven that, for every $p \in M$, T_pM is isomorphic to $\mathbb{R}^n$, which can be considered the (non canonical) prototype of any tangent space to a manifold of dimension n at a given point.

On the other side, $\mathbb{R}^n$ is also the (non canonical) prototype of another object: a real vector space V of dimension n: once we fix a basis of V, the map that links a vector of V to the vector of $\mathbb{R}^n$ given by its components w.r.t. the chosen basis is a linear isomorphism (non canonical because it depends on the basis).

Thanks to the interplay between these two non canonical isomorphisms, we can obtain a third (canonical!) one: we are going to prove that any finite-dimensional vector space V over $\mathbb{R}$ is canonically isomorphic to its tangent space at any point.

In order to prove this, we must play with the dual nature of V: it can be considered as a vector space or as a smooth manifold w.r.t. its standard differential structure defined in section 1.2.

Once we fix any vector $u \in V$, we can consider a particularly natural derivation on $\mathcal{C}^\infty(V)$: the directional derivative of a smooth scalar function $\phi \in \mathcal{C}^\infty(V)$:

- in $x \in V$, where here u is considered as a point of *the manifold* V;
- w.r.t. to the direction defined by any $v \in V$, where v is considered as a vector of *the vector space* V.

By definition, we have:

$$\begin{array}{rccl} D_v|_x : & \mathcal{C}^\infty(V) & \longrightarrow & \mathbb{R} \\ & \phi & \longmapsto & D_v|_u(\phi) = \frac{d}{dt}\big|_{t=0}\phi(x+tv), \end{array}$$

where the operation $x + tv$ is guaranteed to be well-defined in V for all $t \in \mathbb{R}$ because of the vector space structure of V.

$D_v|_x$ plays a central role in the proof of the following result.

Theorem 2.6.1 *Let V, W be any two real finite-dimensional vector spaces with their standard smooth manifold structure. For each point x of V, the map*

$$\begin{array}{rccl} I_x : & V & \xrightarrow{\sim} & T_xV \\ & v & \longmapsto & I_x(v) := D_v|_x\,,\ D_v|_x(\phi) = \frac{d}{dt}\big|_{t=0}\phi(x+tv) \quad \forall \phi \in \mathscr{C}^\infty(V), \end{array}$$

is a canonical linear isomorphism such that, for any linear map $L : V \to W$, the following diagram commutes:

$$
\begin{tikzcd}
V \arrow[r, "I_x"] \arrow[d, "L"'] & T_x V \arrow[d, "dL_x"] \\
W \arrow[r, "I_{Lx}"'] & T_{Lx} W
\end{tikzcd}
\qquad \iff \qquad dL_x \circ I_x = I_{Lx} \circ L.
$$

So, the explicitly formula for the differential of a linear function between vector spaces is:

$$dL_x(D_v|_x) = D_{Lv}|_{Lx}$$

or, by identifying $D_v|_x$ with v,

$$dL_x(v) = D_{Lv}|_{Lx}\,.$$

Proof. The linearity of I_x is a direct consequence of the linearity of $D_v|_x$. Let us now prove that I_x is a bijection.

Injectivity: suppose $v_1, v_2 \in V$ are such that $D_{v_1}|_x = D_{v_2}|_x$, then, thanks to the fact that $D_v|_x$ is linear w.r.t. v, we have that $D_{(v_1-v_2)}\big|_x \equiv 0$, the 0 derivation. Then, for all $\phi \in \mathscr{C}^\infty(V)$:

$$D_{(v_1-v_2)}\big|_x(\phi) = \frac{d}{dt}\bigg|_{t=0} \phi(x + t(v_1 - v_2)) = 0.$$

Now we note that the linear functionals $\ell : V \to \mathbb{R}$ living in the dual V^* of V are of course smooth scalar functions on V, i.e. they belong to $\mathscr{C}^\infty(V)$, so we can consider the action of $D_{(v_1-v_2)}$ on $\ell \in V^*$:

$$0 = \frac{d}{dt}\bigg|_{t=0} \ell(x + t(v_1 - v_2)) = \frac{d}{dt}\bigg|_{t=0} (\ell(x) + t\ell(v_1 - v_2)) = \cancelto{0}{\frac{d}{dt}\bigg|_{t=0} \ell(x)} + \frac{d}{dt}\bigg|_{t=0} t\ell(v_1 - v_2),$$

i.e. $\ell(v_1 - v_2) = 0$ for all $\ell \in V^*$. However, thanks to the finite-dimensional Riesz representation theorem, we know that $V \cong V^*$ and that for all $\ell \in V^*$ it exists only one vector $w_\ell \in V$ such that $\ell(v) = \langle v, w_\ell \rangle$. Thus, the equation $\ell(v_1 - v_2) = 0$ for all $\ell \in V^*$ can be reformulated as $\langle v_1 - v_2, w_\ell \rangle = 0$ for all $w_\ell \in V$, but the only vector orthogonal to all other vectors is the null vector, so $v_1 - v_2 = 0$, or $v_1 = v_2$, thus implying the injectivity of I_x.

Surjectivity: we can conveniently use the equivalence between $T_x^{\text{geom}}V$ and $T_x^{\text{alg}}V$ and prove surjectivity by considering geometric tangent vectors. The proof then simply consists in observing that any tangentially equivalent class of curves passing through x with velocity vector v clearly contains the curve $t \mapsto x + tv$.

Finally, suppose $L : V \to W$ to be a linear map, then L is of course smooth because its components w.r.t. any choice of basis (which also play the role of charts for vector spaces, as seen in chapter 1) for V and W are linear functions of the coordinates. By definition of differential and thanks to the linearity of L we get, for all $\phi \in \mathscr{C}^\infty(V)$:

$$
\begin{aligned}
dL_x(D_v|_x)(\phi) := D_v|_x(\phi \circ L) &= \frac{d}{dt}\bigg|_{t=0} \phi(L(x + tv)) = \frac{d}{dt}\bigg|_{t=0} \phi(Lx + tLv) \\
&= D_{Lv}|_{Lu}(\phi),
\end{aligned}
$$

i.e. $dL_x(D_v|_x) = D_{Lv}|_{Lx}$. □

To fact that I_x is a canonical isomorphism (i.e. independent of any choice of basis) justifies why, in differential geometry, *the tangent vector t to a vector space at any point v, which is a vector, is identified with the vector v itself.*

An immediate, and very important, consequence of this fact is that, if U is an open submanifold of a real finite-dimensional vector space V, then $T_xU \cong T_xV \cong V$, so we obtain a canonical identification of each tangent space to U with V itself. As a noticeable example, since $GL(n,\mathbb{R})$ is an open submanifold of the vector space $M(n,\mathbb{R})$, the following result holds.

Theorem 2.6.2 *For all $X \in GL(n,\mathbb{R})$ it holds that:*

$$T_X\, GL(n,\mathbb{R}) \cong M(n,\mathbb{R}),$$

i.e. the tangent space to the vector space of real invertible matrices of dimension n is the vector space of all real square matrices of dimension n.

There is another natural, and very useful, identification for tangent spaces to a product manifold, as stated in the following proposition.

Theorem 2.6.3 *Let $M_1, \ldots, M_N$ be smooth manifolds, and let $\pi_j : M_1 \times \cdots \times M_N \to M_j$, be the projection onto the j-th factor, for each $j = 1, \ldots, N$. For any point $p = (p_1, \ldots, p_N) \in M_1 \times \cdots \times M_N$ the map*

$$\begin{array}{ccc} T_p(M_1 \times \cdots \times M_N) & \xrightarrow{\sim} & T_{p_1}M_1 \oplus \cdots \oplus T_{p_N}M_N \\ v & \longmapsto & (d(\pi_1)_p(v), \ldots, d(\pi_N)_p(v)), \end{array}$$

is a canonical isomorphism.

For example, $T_{(p,q)}(M \times N)$ can be identified with $T_pM \oplus T_qN$ and T_pM and T_qN can be treated as subspaces of $T_{(p,q)}(M \times N)$.

2.7 Immersion, submersion, embedding and the problem of compatibility between differential structures

The substructures of a manifold show some subtleties that is important to underline.

First of all, let us define the rank of a smooth map in an analogous way as we did for a smooth function between Euclidean spaces.

Def. 2.7.1 *Let $f : M \to N$ be a smooth map between manifolds. The **rank** of f in $p \in M$ is the rank of the linear function $df_p : T_pM \to T_{f(p)}N$.*

Equivalently, fixed any local chart (U, φ) in p, the rank of f is the rank of the Jacobian matrix of the local expression $\tilde{f}$ of f in $x = \varphi(p)$.

*If the rank of f remains constant for every point $p \in M$, then f is said to have **constant rank**.*

Def. 2.7.2 *The smooth map $f : M \to N$ is a/an:*

- ***Immersion**: if df_p is injective for all $p \in M$;*

- ***Submersion***: *if df_p is surjective for all $p \in M$;*
- ***Embedding***[3]: *if it is an immersion and $f : M \to f(M)$ is a homeomorphism.*

Examples:

1. The curve
$$\begin{array}{rccl} \alpha : & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & t & \longmapsto & \alpha(t) = (t^2, t^3), \end{array}$$
is injective, but $\frac{d\alpha}{dt} = (2t^2, 3t^2)$ is null for $t = 0$, so $d\alpha|_{t=0}$ is not injective;

2. The curve
$$\begin{array}{rccl} \beta : & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & t & \longmapsto & \alpha(t) = (t^3 - 4t, t^2 - 4), \end{array}$$
is not injective, e.g. $\beta(-2) = \beta(2) = (0, 0)$, but $\frac{d\beta}{dt} = (3t - 4, 2t)$ is never null in both coordinates, so β is an immersion, but not an embedding because it is not injective;

3. The curve
$$\begin{array}{rccl} \gamma : & (-\pi/2, 3\pi/2) & \longrightarrow & \mathbb{R}^2 \\ & t & \longmapsto & \alpha(t) = (\sin(2t), \cos(t)), \end{array}$$
γ is injective and $\frac{d\gamma}{dt} = (2\cos(2t), -\sin(t)) \neq (0, 0)$ $\forall t \in (-\pi/2, 3\pi/2)$, thus it is an immersion. However, the domain of γ is an open set in $\mathbb{R}$ and its codomain is a compact subset of $\mathbb{R}^2$, thus γ cannot be a homeomorphism between its domain and its codomain.

The curve γ, usually called **lemniscate**, or ‘the 8’ for its shape, shows that *even an injective immersion can fail to be an embedding.* However, the next theorem guarantees that *every immersion is*, at least, *local embedding.*

Theorem 2.7.1 *Let $f : M \to N$ be a smooth map between manifolds. If f is an immersion, then, for all $p \in M$, it exists an open neighborhood $U \subseteq M$ of p such that $f|_U : U \to f(U) \subseteq N$ is an embedding.*

The most important consequence of the previous result is that, *if $f : M \to N$ is an injective immersion, it is always possible to endow $f(M)$ with a differential structure induced by that of M.* In fact, let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be a smooth atlas for M such that $f|_{U_\alpha}$ is a homeomorphism with its image $f(U_\alpha) \subseteq f(M)$, then, since $\varphi : U_\alpha \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ are homeomorphisms, we get that $\{(f(U_\alpha), \varphi_\alpha \circ f^{-1}|_{f(U_\alpha)})\}_{\alpha\in A}$ is a smooth atlas for $f(M)$.

Thus, on $f(M)$ we have two differential structures, namely, the one naturally inherited as a subset of N and the one induced by M in the way described above. It turns out that these differential structures can lack of compatibility because the underlying topologies may fail to be equivalent. This is clearly exemplified by the curve γ: the counter-image of an open neighborhood of the central point of the 8, in $\mathbb{R}^2$, is the union of three open intervals in $\mathbb{R}$, while for the topology of $\mathbb{R}$ an open neighborhood is just an open interval.

In general, it can be difficult to establish if an injective immersion is an embedding, with the exception of the compact case, as stated below.

[3] An embedding is a sort of topologically coherent immersion. In French it is called *plongement*.

Theorem 2.7.2 *Let $f : M \to N$ be a smooth map between manifolds. If f is an injective immersion and M is compact (as topological manifold), then f is an embedding.*

The considerations above explain why we find two types of definitions for submanifolds in differential geometry.

Def. 2.7.3 (Embedded submanifold) *Let E, M be two smooth manifolds such that $E \subset M$. If the canonical inclusion $\iota : E \hookrightarrow M$ is an embedding, then E is said to be an embedded submanifold of M.*

Def. 2.7.4 (Immersed submanifold) *Let $f : M \to N$ be a smooth map between manifolds. If f is an injective immersion, then $f(M) \subset N$, endowed with the differential structure induced by M, is said to be a manifold immersed in N.*

Convention: without any further specification, a submanifold has to be intended as an embedded submanifold.

A classical example of an immersed submanifold of $\mathbb{R}^2$ that is not an embedding is the spire (coil) that envelops the torus with irrational step.

2.8 Characterization of the tangent space to a level set of a smooth function

It is possible to give a very useful characterization of the tangent space at a point to a level set of smooth functions thanks to the following result, whose proof can be found in [10], page 81 (th. 4.12).

Theorem 2.8.1 (The rank theorem) *Let M and N be smooth manifolds with dimension m and n, respectively. Let $f : M \to N$ be a smooth function with constant rank r. Then, for every $p \in M$ there exist local charts (U, φ) centered in p and (V, ψ) centered in $f(p)$, with $f(U) \subset V$, such that the local expression of f w.r.t. these charts is particularly simple, namely:*

$$\tilde{f}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0),$$

i.e. $\tilde{f}$ acts as the identity on the first r entries and it is identically 0 in the last $n - r$.

In particular, if f is a submersion, then $r = n$ and so

$$\tilde{f}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n),$$

while, if f is an immersion, then $r = m$ and so

$$\tilde{f}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

The rank theorem justifies the following definition.

Def. 2.8.1 *Let $S \subset M$ be a submanifold of dimension k of M. A local chart (U, φ) of M is said to be adapted to S if either $U \cap S = \varnothing$, or $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{\vec{0}\})$, where this notation means that the part of the submanifold S contained in U is mapped by φ to 0, i.e. $x^{k+1} = \ldots = x^n = 0$. An atlas of M is adapted to S is every chart of it is adapted to S.*

Theorem 2.8.2 *Embedded submanifolds always admit adapted charts.*

We can extend the definitions, given in chapter 1, of critical and regular point of a function defined between Euclidean spaces to functions between abstract manifolds.

Theorem 2.8.3 *Let M and N be smooth manifolds with dimension m and n, respectively. Let $f : M \to N$ be a smooth function.*

- *$p \in M$ is a **critical point** of f if $df_p : T_pM \to T_{f(p)}N$ is not surjective. The image, via f, of a critical point of f is a **critical value** for f.*
- *A regular value of f is an element of $f(M)$ that is not a critical value.*

We denote with $\mathrm{Crit}(f) \subset M$ the set of critical points of f.

We need a last definition before stating and proving the main result of this section.

Def. 2.8.2 (Level set of a smooth function) *A level set of $f : M \to N$ is a subset of M of the type $f^{-1}(q) := \{p \in M \ : \ f(p) = q\}$, where $q \in f(M)$.*

Theorem 2.8.4 (Level set theorem for manifolds) *Let M and N be smooth manifolds with dimension $n + k$ and n, respectively, $k \geqslant 0$. Let $f : M \to N$ be a smooth function.*

1. *For all $a \in f(M)$, the set*

   $$M_a = f^{-1}(a) \backslash \mathrm{Crit}(f) \qquad a-\text{level set via } f \text{ minus the critical points}$$

   is an embedded submanifold of dimension k of M. In particular, if a is a regular value for f, $f^{-1}(a)$ is a k-dimensional embedded submanifold of M.

2. *If $p \in M_a$, then the tangent space T_pM_a is the kernel of $df_p : T_pM \to T_{f(p)}N$:*

   $$\boxed{T_pM_a = \ker(df_a)} \ . \tag{2.34}$$

3. *If, in particular, $N = \mathbb{R}$, then $f \in \mathcal{C}^\infty(M)$ and T_pM_a is given by the derivations $D \in T_pM$ that nullify smooth scalar functions: $D(f) = 0$ for all $f \in \mathcal{C}^\infty(M)$.*

Proof.

1. By using local charts, we can reduce the problem to the local representation of f, which is a function defined on an open subset of $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$. For such a function we can apply the level set theorem 1.2.1 in Euclidean spaces discussed in the first chapter.

2. Let $\iota : M_a \to M$ be the canonical inclusion of M_a in M. By theorem 2.2.3 we know that $d\iota_p : T_pM_a \to T_pM$ is a canonical linear isomorphism, thus we can identify T_pM_a with T_pM and so 2. is equivalent to $d\iota_p(T_pM_a) = \ker(df_p)$.

Since p is a regular point, $\dim(M_a) = k$, so $\dim(T_pM_a) = k$, moreover df_p is surjective, hence $\dim(\mathrm{Im}(df_p)) = n$ and the rank+nullity theorem implies

$$\dim(T_pM) = \dim(\ker(df_p)) + \dim(\mathrm{Im}(df_p)),$$

but $\dim(T_pM) = \dim(M) = n + k$, so $\dim(\ker(df_p)) = n + k - n = k$.

Thanks to the fact that $\dim(\ker(df_p)) = \dim(T_pM_a)$, to prove that T_pM_a and $\ker(df_p)$ are isomorphic it is sufficient to show that one space is included in the other. We chose arbitrarily to show that $T_pM_a \cong d\iota(T_pM_a) \subseteq \ker(df_p)$.

To do that, let us consider a derivation $v \in T_pM_a$, then $d\iota_p : T_pM_a \to T_{\iota(p)}M = T_pM$, so $d\iota_p(v) \in T_pM$ and we can apply $df_p : T_pM \to T_{f(p)}N$ to $d\iota_p(v)$, obtaining an element of $T_{f(p)}N$, i.e. $df_p(d\iota_p) \in T_{f(p)}N$. In order to understand its action, we need to apply it to a smooth scalar function $\phi \in \mathscr{C}^\infty(N)$:

$$df_p(d\iota(v))(\phi) \underset{\text{(chain rule)}}{=} d(f \circ \iota)_p(v)(\phi) := v(\phi \circ f \circ \iota),$$

but $f \circ \iota : M_a \to N$ is nothing but $f|_{M_a}$, so

$$df_p(d\iota(v))(\phi) = v(\phi \circ f|_{M_a}) = 0,$$

because $f|_{M_a}$ is, by definition of M_a, a constant function identically equal to a, and $\phi \circ f|_{M_a}$ is the constant function identically equal to $\phi(a)$, so $v(\phi \circ f|_{M_a}) = 0$ because derivations set to 0 constant functions.

This is true for all $\phi \in \mathscr{C}^\infty(N)$, so $df_p(d\iota_p(v)) = 0$, i.e. $T_pM_a \subseteq \ker(df_p)$.

3. Immediate consequence of 2. □

2.9 Explicit calculations of tangent spaces

In this section we are going to compute some remarkable differential and apply the result to obtain the explicit characterization of tangent spaces. In order to do that, we will mix the level set theorem with the results that we have discussed about the differential.

2.9.1 The tangent space to the sphere at a point

We are going to verify that the tangent space to a sphere at a point x is the hyperplane orthogonal to the radius connecting the center to x, as intuitively expected from the depiction in fig. 2.1.

We recall that the n-sphere of radius $R > 0$ is $S^n_R = \{x \in \mathbb{R}^{n+1} \ : \ \|x\|^2 = R^2\}$, thus it is natural to consider the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto f(x) = \|x\|^2$ to obtain S^n_R as a level set: $S^n_R = f^{-1}(R^2)$.

We know that in this case the differential of f coincides with its total derivative, i.e. for all $x \in \mathbb{R}^{n+1}$, $df_x = Df(x)$, to compute it we simply observe that:

$$f(x+ty) = \|x+ty\|^2 = \langle x+ty, x+ty\rangle = \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 = f(x) + Df(x)ty + o(t),$$

so $df_x(y) = Df(x)y = 2\langle x,y\rangle$ for all $y \in \mathbb{R}^{n+1}$.

By the level set theorem we get:

$$T_xS^n_R = \ker(df_x) = \{y \in \mathbb{R}^{n+1} \ : \ \langle x,y\rangle = 0\},$$

which confirms that $T_xS^n_R$ is nothing but the hyperplane in $\mathbb{R}^{n+1}$ passing through x and orthogonal to the radius of the sphere connecting x to 0.

2.9.2 The Lie group $\mathbf{O}(n)$ as an embedded submanifold of $M(n, \mathbb{R})$ and its Lie algebra $\mathfrak{o}(n)$

Here we prove that $\mathrm{O}(n) = \{A \in M(n, \mathbb{R}) \, : \, A^t A = I_n\}$, the orthogonal group, is a manifold of dimension $\frac{n(n-1)}{2}$ and we make its tangent space at any point explicit. The constraint that defines orthogonal matrices leads us naturally to consider the following function:

$$\begin{array}{rccl} f: & M(n, \mathbb{R}) & \longrightarrow & \mathrm{Sym}(n, \mathbb{R}) \\ & X & \longmapsto & f(X) = X^t X, \end{array} \tag{2.35}$$

because we can easily identify $\mathrm{O}(n)$ as the f-level set of the the identity matrix I_n, in fact:

$$f^{-1}\{I_n\} = \{X \in M(n, \mathbb{R}) \, : \, f(X) = X^t X = I_n\} \equiv \mathrm{O}(n).$$

In order to apply the level set theorem, let us compute the differential of f. Both $M(n, \mathbb{R})$ and $\mathrm{Sym}(n, \mathbb{R})$ are vector spaces, thus we can canonically identify the tangent spaces to $M(n, \mathbb{R})$ and $\mathrm{Sym}(n, \mathbb{R})$ at any point (matrix) with the vector spaces themselves. With this identification in mind, for all $X \in M(n, \mathbb{R})$, $df_X : M(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ and, thanks to eq. (2.22), for all $A \in M(n, \mathbb{R})$ we have:

$$\begin{aligned} df_X(A) &= \left.\frac{d}{dt}\right|_{t=0} f(X + tA) = \left.\frac{d}{dt}\right|_{t=0} \big((X + tA)^t (X + tA)\big) \\ &= \left.\frac{d}{dt}\right|_{t=0} \big(X^t X + t(X^t A + A^t X) + t^2 A^t A\big) \\ &= \cancelto{0}{\left.\frac{d}{dt}\right|_{t=0} \big(X^t X\big)} + \big(X^t A + A^t X\big) + \cancelto{0}{\big(2t A^t A\big)\big|_{t=0}} \\ &= X^t A + A^t X, \end{aligned}$$

i.e. $df_X(A) = X^t A + A^t X$, which is, as it should be, a symmetric matrix.

Remark: this result could have been obtained also by identifying the differential with the total derivative and observing that:

$$f(X + tA) = X^t X + t(X^t A + A^t X) + t^2 A^t A = f(X) + Df(X)tA + o(t),$$

so that $d_X f(A) = Df(X)A = X^t A + A^t X$.

Now that the differential is explicit, let us analyze its surjectivity: for every $B \in \mathrm{Sym}(n, \mathbb{R})$ we must determine under what condition on X it exists at least one $A \in M(n, \mathbb{R})$ such that $B = df_X(A) = X^t A + A^t X$.

To obtain this result first notice that B is symmetric, so we can write:

$$B = \frac{1}{2}B + \frac{1}{2}B = \frac{1}{2}B + \frac{1}{2}B^t,$$

that must be compared to

$$B = X^t A + A^t X = X^t A + (X^t A^t)^t,$$

the two expressions are compatible if and only if $X^t A = \frac{1}{2}B$, if X is invertible, then we can solve this equation obtaining $A = \frac{1}{2}(X^t)^{-1}B$. Thus $df_X : T_X\mathrm{GL}(n,\mathbb{R}) \cong M(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R})$ is surjective for all $X \in \mathrm{GL}(n,\mathbb{R})$, since every symmetric $n \times n$ real matrix B can be written as $df_X(\frac{1}{2}(X^t)^{-1}B)$, where $X \in \mathrm{GL}(n,\mathbb{R})$.

The identity I_n is symmetric, an orthogonal matrix X is invertible and $I_n = f(X)$, thus I_n is a regular value for f and the level set theorem can be applied to guarantee that $\mathrm{O}(n) = f^{-1}(I_n)$ is an embedded submanifold of $M(n,\mathbb{R})$ of dimension $\dim(\mathrm{O}(n)) = \dim(M(n,\mathbb{R})) - \dim(\mathrm{Sym}(n,\mathbb{R}))$.

The dimension of $\mathrm{Sym}(n,\mathbb{R})$ can be recovered by observing that if we want to identify a symmetric matrix of order n we must specify $\frac{n(n+1)}{2}$ real values: $n^2 - n$ is the totality of matrix elements minus those lying on the diagonal, if we divide this number by 2 we obtain the matrix element above (or below) the diagonal, to these elements we must add back the diagonal entries, thus arriving to $\frac{n^2-n}{2} + n = \frac{n(n+1)}{2}$. Hence, $\mathrm{Sym}(n,\mathbb{R})$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$ and so it has dimension $\frac{n(n+1)}{2}$ as a manifold.

It follows that the dimension of $\mathrm{O}(n)$ as embedded submanifold of $M(n,\mathbb{R})$ is:

$$\dim(\mathrm{O}(n)) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Finally, thanks to (2.34), we can compute the tangent space to $\mathrm{O}(n)$ as follows:

$$T_X\mathrm{O}(n) = \ker(df_X) = \{A \in M(n,\mathbb{R}) \ : \ X^t A + A^t X = 0\}, \quad \forall X \in \mathrm{O}(n),$$

i.e. matrices $A \in M(n,\mathbb{R})$ such that $X^t A$ is skew-symmetric, thus, in particular, if $X = I_n$,

$$T_{I_n}\mathrm{O}(n) = \{A \in M(n,\mathbb{R}) \ : \ A + A^t = 0 \iff A^t = -A\},$$

i.e. the tangent space at the identity element of $\mathrm{O}(n)$ can be identified with the space of **skew-symmetric matrices**.

We will see in the chapter dedicated to Lie groups that $T_{I_n}\mathrm{O}(n)$ can be identified with the Lie algebra of the Lie group $\mathrm{O}(n)$, that will be denoted with the symbol $\mathfrak{o}(n)$:

$$\mathrm{O}(n) = \{A \in M(n,\mathbb{R}) \ : \ A^t = A^{-1}\}, \quad \mathfrak{o}(n) = \{A \in M(n,\mathbb{R}) \ : \ A^t = -A\}.$$

Remark: if A were a positive real number a, then we could compute the logarithm of A^{-1}, obtaining $\log A^{-1} = -\log A$, which suggests that the elements of $\mathrm{O}(n)$ could be considered as the exponential of the elements of $\mathfrak{o}(n)$. We will see that, indeed, it exists a fundamental function, called again *exponential*, that relates Lie algebras and Lie groups.

Chapter 3

Tangent, cotangent and vector bundles (Edoardo Provenzi)

Inspirational epithap wanted...

...

The simple act of taking the union of the tangent spaces to a manifold in all its points generates another manifold, the tangent bundle, with double the dimension of the original one, and with a surprisingly rich intrinsic structure that happens to be the prototype of the so-called vector bundles.

3.1 The tangent bundle over a manifold

We have seen that the tangent spaces T_pM and T_qM to a smooth manifold M of dimension n in two different points p and q are not canonically isomorphic and so they cannot be identified, in spite of the fact that they are both two copies of $\mathbb{R}^n$.

The union of the tangent spaces to M as we vary the point on M is then a *disjoint* one. The canonical symbol to denote this disjoint union is:

$$TM = \bigsqcup_{p\in M} T_pM = \{(p,v) \,:\, p \in M,\ v \in T_pM\}.$$

This space comes equipped with a natural projection:

$$\begin{array}{rccl} \pi: & TM & \longrightarrow & M \\ & (p,v) & \longmapsto & \pi(p,v) = p. \end{array}$$

Def. 3.1.1 *TM is called the* ***tangent bundle*** *of the smooth manifold M. The* ***fiber*** *over $p \in M$ is the set:*

$$\pi^{-1}(p) = \{(p,v) \,:\, v \in T_pM\} \cong T_pM.$$

The most important geometrical characteristic of the tangent bundle is its *local triviality*, i.e. the fact that, locally, it is diffeomorphic to the Cartesian product between a chart domain and $\mathbb{R}^n$, the local model of M.

Local triviality is easily understood if we consider a local chart (U, φ) in $p \in M$ and the restriction of TM to U, defined by

$$TM|_U = \bigsqcup_{p \in U} T_p M.$$

As we have seen in chapter 2, the act of fixing a local chart (U, φ) in $p \in M$ induces the non-canonical linear isomorphism $d\varphi_p : T_p M \xrightarrow{\sim} \mathbb{R}^n$ defined by $d\varphi_p(\partial_i|_p) = e_i$, where e_i is the i-th element of the canonical basis of $\mathbb{R}^n$, $i = 1, \ldots, n$, thus the extension on the whole tangent space to M at p is given by the correspondence: $T_p M \ni v = v^i \, \partial_i|_p \longleftrightarrow (v^i)_{i=1}^n \in \mathbb{R}^n$.

This holds for every point $p \in U$, so we can extend this non-canonical identification to all U as follows:

$$\begin{array}{rccc} id_U \times d\varphi_p : & TM|_U = \bigsqcup_{p \in U} T_p M & \xrightarrow{\sim} & U \times \mathbb{R}^n \\ & (p, (v^i \, \partial_i|_p)_{i=1}^n) & \longmapsto & (p, (v^i)_{i=1}^n). \end{array}$$

Finally, each chart map sends $U \subseteq M$ diffeomorphically to $\varphi(U) \subseteq \mathbb{R}^n$, thus we can further identify $U \times \mathbb{R}^n$ with an open subset $\varphi(U) \times \mathbb{R}^n$ of $\mathbb{R}^{2n}$ as follows:

$$\begin{array}{rccc} \varphi \times id_{\mathbb{R}^n} : & U \times \mathbb{R}^n & \xrightarrow{\sim} & \varphi(U) \times \mathbb{R}^n \\ & (p, (v^i)_{i=1}^n) & \longmapsto & (x, (v^i)_{i=1}^n), \quad x = \varphi(p). \end{array}$$

By composition we obtain:

$$\begin{array}{rccc} \Phi \equiv (\varphi \times id_{\mathbb{R}^n}) \circ (id_U \times d\varphi_p) : & TM|_U & \xrightarrow{\sim} & \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \\ & (p, (v^i \, \partial_i|_p)_{i=1}^n) & \longmapsto & (x, (v^i)_{i=1}^n), \quad x = \varphi(p), \end{array}$$

which shows that the couple $(TM|_U, \Phi)$ is *a local chart for* TM with local coordinates obtained by replacing φ by its component functions $x^i \equiv (\varepsilon^i \circ \varphi)_{i=1}^n$, i.e.

$$\begin{array}{rccc} ((x^1, \ldots, x^n) \times id_{\mathbb{R}^n}) \circ (id_U \times d\varphi_p) : & TM|_U & \xrightarrow{\sim} & \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n} \\ & (p, (v^i \, \partial_i|_p)_{i=1}^n) & \longmapsto & (x^i(p), v^i)_{i=1}^n). \end{array}$$

Def. 3.1.2 *Given a local coordinate system $(U, \varphi \equiv (x^i))$ in $p \in M$, the coordinates defined by $(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n)$, such that $v \in T_p M$ is written as $v = v^j \, \partial_j|_p$, are called the **natural local coordinates on the tangent bundle** TM.*

As we vary U in an atlas of M, we obtain a covering of TM and the charts can be proven to be compatible, so that they constitute an atlas for TM, see [10] proposition 3.18 page 66 for the technical proof. As a consequence, TM **is a** $2n$**-dimensional smooth manifold**.

As we will see later, the property of being diffeomorphic to the Cartesian product $U \times \mathbb{R}^n$ that the tangent bundle TM is so important to be one of the conditions included in the definition of a general vector bundle. The map $id_U \times d\varphi_p : TM|_U \xrightarrow{\sim} U \times \mathbb{R}^n$ is called a **local trivialization of the vector bundle** TM.

The next remark will have a great importance for the general theory of vector bundles: let us concentrate on the local trivialization $TM|_{U_{\alpha\beta}} \cong U_{\alpha\beta} \times \mathbb{R}^n$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is the intersection of two chart domains for M with chart maps φ_α and φ_β, respectively. We know that the compatibility between charts is equivalent to the request that the Jacobian matrix $J_{\eta_{\alpha\beta}}(x)$ of $\eta_{\alpha\beta}$ evaluated in any $x \in \varphi_\beta(p)$, for all $p \in U_{\alpha\beta}$, is non singular, i.e. it belongs to

$\mathrm{GL}(n,\mathbb{R})$. This means that each tangent bundle comes equipped with the following smooth functions

$$\begin{array}{rccl} \tau_{\alpha\beta}: & U_\alpha \cap U_\beta & \longrightarrow & \mathrm{GL}(n,\mathbb{R}) \\ & p & \longmapsto & \tau_{\alpha\beta}(p) = J_{\eta_{\alpha\beta}}(x), \quad x = \varphi_\beta(p), \end{array}$$

which can be easily seen to satisfy the following properties:

$$\begin{cases} \tau_{\alpha\alpha}(p) = I_n, \ \forall\, p \in U_\alpha \cap U_\beta \\ \tau_{\alpha\beta}(p) = \tau_{\beta\alpha}(p)^{-1}, \ \forall\, p \in U_\alpha \cap U_\beta \\ \tau_{\alpha\beta}(p) \circ \tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \ \forall\, p \in U_\alpha \cap U_\beta \cap U_\gamma \end{cases},$$

thanks to the corresponding features of the transition functions $\eta_{\alpha\beta}$ between charts.

The functions $\tau_{\alpha\beta}$ are called **transition functions between the local trivializations** of TM given by $TM|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$ and $TM|_{U_\beta} \cong U_\beta \times \mathbb{R}^n$.

The importance of the transition functions between the local trivializations is that they permit to construct the manifold structure of a collection of vector spaces attached to points of a manifold in a sense that will be specified more rigorously later in this chapter.

Remark: notice that, in spite of bearing the same name and of being related as described above, the transition functions $\eta_{\alpha\beta} : \varphi_\beta(U_{\alpha\beta}) \subseteq \mathbb{R}^n \to \varphi_\alpha(U_{\alpha\beta}) \subseteq \mathbb{R}^n$ between two charts of M and the transition functions $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$ between two local trivializations of TM are very different objects and must not be confused.

Def. 3.1.3 (Global differential) *If $f : M \to N$ is a smooth map between smooth manifolds M and N, then the map $df : TM \to TN$ such that $df|_{T_pM} = df_p$ is called the **global differential** or global tangent to f.*

Theorem 3.1.1 *If $f : M \to N$ is a smooth map, then its global differential $df : TM \to TN$ is a smooth map.*

Proof. It is sufficient to recall eq. (2.27), which gives the local expression of the differential of f in a point $p \in M$ in coordinates as:

$$df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial \tilde{f}^j}{\partial x^i}(x)\,\left.\frac{\partial}{\partial y^j}\right|_{f(p)},$$

where $\tilde{f}^j$ are the component functions of the local expressions of f. Thus, the coordinate representation of df in terms of the natural coordinates of TM and TN is:

$$df(x^1,\ldots,x^n,v^1,\ldots,v^n) = \left(\tilde{f}^1(x),\ldots,\tilde{f}^n(x),\frac{\partial \tilde{f}^1}{\partial x^j}(x)v^j,\ldots,\frac{\partial \tilde{f}^n}{\partial x^j}(x)v^j\right),$$

$x = (x^1,\ldots,x^n)$. The smoothness of f implies that of the coordinate representation. □

The properties of df listed below follow easily from those of the differential in a point.

Theorem 3.1.2 (Properties of the global differential) *Given smooth maps $f : M \to N$ and $g : N \to P$ the following properties hold.*

1. $d(id_M) = id_{TM}$.

2. ***Chain rule for the global differential:***

$$d(g \circ f) = dg \circ df.$$

3. *If f is a diffeomorphism, then $df : TM \to TN$ is a diffeomorphism and $(df)^{-1} = d(f^{-1})$.*

Thanks to 3. it is not ambiguous to write simply df^{-1} for the inverse of the global differential of a smooth function.

3.1.1 The tangent bundle as the configuration space of a classical mechanical system

A state of a classical mechanical system is given by specifying a configuration, i.e. the position and the speed of the system particles at a given time. These data are necessary and sufficient to give the initial conditions to write the system of differential equations given by Newton's second law of motion (or its equivalent Lagrangian or Hamiltonian formulations).

If the configuration space is assumed to be a smooth manifold Q, then the state space is the tangent bundle TQ. Thanks to local triviality, if $\dim(Q) = n$, a state at the time t_0 can be locally described via these coordinates:

$$\left(q^1(t_0), \ldots, q^n(t_0), \dot{q}^1(t_0), \ldots, \dot{q}^n(t_0)\right),$$

where $q^i \equiv x^i$ in physical notation, and $q^i(t_0) \equiv \frac{dq^i(t)}{dt}(t_0)$.

3.2 Vector bundles

The tangent bundle is the prototype of a category of smooth bundles called vector bundles, to which this section is dedicated.

Before introducing the formal definition, we stress that the main idea underlying a vector bundle is to construct **a family of vector spaces E_p parameterized by points p of a manifold** M (or, as it is often said, *attached* to these points) **in such a way that these vector spaces fit together to form another manifold**, which is called a vector bundle over M. We can study with the techniques of differential geometry this new manifold, which turns out to carry a richer and more interesting structure than the original one.

The next definition contains all the information needed to 'glue together' the copies of the vector spaces attached to each point of M to form a vector bundle.

Def. 3.2.1 (Vector bundle) *A (real) vector bundle of **rank r** over a smooth manifold M of dimension $n \geqslant r$, called **base space**, is described by the triple (E, M, π), where E is a smooth manifold, called the **total space** of the bundle, and $\pi : E \to M$ is a smooth **surjective** map, such that:*

1. *for all $p \in M$, the **fiber** $E_p := \pi^{-1}(p)$ is a real vector space of dimension r;*

2. every $p \in M$ admits an open neighborhood $U \subseteq M$ and a diffeomorphism

$$\chi : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r,$$

*called **local trivialization**, such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, "pr_1"] \\ & U & \end{tikzcd} \qquad \Longleftrightarrow \qquad pr_1 \circ \chi = \pi.$$

3. for all $p \in U$, the function $\chi|_p : E_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$ is a linear isomorphism.

*Vector bundles of rank 1 are called **line bundles**.*

In literature, to denote (or even to define) vector bundles it is common to use either the notation (E, M, π) or $\pi : E \to M$ or simply E, depending on what has to be emphasized. We will follow this tradition.

The simplest example of vector bundle is obtained when the family of vector spaces is constant, i.e., when there is a canonical, fixed, vector space E such that $E_p = E$ for all $p \in M$: in this case there is just one copy of E for each $p \in M$ and these copies fit together to form the vector bundle $M \times E$ over M. Due to the extreme simplicity of this construction, such a vector bundle is called **trivial**.

The **tangent bundle of a manifold M of dimension n is a vector bundle of rank n.** This fundamental example shows that, in general, vector bundles are only *locally* trivial.

Any non globally trivial bundle requires more than one local trivialization, thus it is natural to ask oneself what happens in the overlap of any two local trivializations. The following result shows that, thanks to the requests 2. and 3. in the definition of vector bundle, the composition of two local trivializations on the overlap domain has a particularly simple expression.

Theorem 3.2.1 *Let $\pi : E \to M$ be a vector bundle of rank r over M and suppose that $\chi_1 : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ and $\chi_2 : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^r$ are two local trivializations of E with non empty intersection $U_\alpha \cap U_\beta \neq \varnothing$. Then, there exists a smooth map*

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$$

such that the composition $\chi_1 \circ \chi_2^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$ can be written as

$$\chi_1 \circ \chi_2^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v),$$

i.e. it acts as the identity on the first entry and linearly on the second entry, with the application of the non-singular matrix $\tau(p) \in \mathrm{GL}(r, \mathbb{R})$ on the vector $v \in \mathbb{R}^r$.

Proof. Thanks to property 2. in the definition of vector bundle, the following diagram commutes (we have not written the restriction of the local trivializations to $\pi^{-1}(U_\alpha \cap U_\beta)$ for notational simplicity).

$$
\begin{tikzcd}
(U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[r, bend left, "\chi_2^{-1}"] \arrow[d, dashed, "\sigma"'] \arrow[dr, "pr_1"] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\chi_2"'] \arrow[r, "\chi_1"] \arrow[d, "\pi"] & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[dl, "pr_1"'] \\
\mathbb{R}^r & U_\alpha \cap U_\beta &
\end{tikzcd}
$$

This implies that $pr_1 \circ (\chi_1 \circ \chi_2^{-1}) = pr_1$, i.e. $\chi_1 \circ \chi_2^{-1}$ acts as the identity of the first entry, so that the only significant action of the composition $\chi_1 \circ \chi_2^{-1}$ is on the second entry, which belongs to $\mathbb{R}^r$, we denote this action with the smooth map $\sigma : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to \mathbb{R}^r$ so that

$$\chi_1 \circ \chi_2^{-1}(p, v) = (p, \sigma(p, v)).$$

Property 3. in the definition of vector bundle implies that, for every fixed $p \in U_\alpha \cap U_\beta$, the map $\mathbb{R}^r \ni v \mapsto \sigma(p, v) \in \mathbb{R}^r$ is a linear isomorphism, thus its action can be associated to a non-singular matrix $\tau(p) \in \mathrm{GL}(r, \mathbb{R})$ such that $\sigma(p, v) = \tau(p)v$.

The smoothness of τ is a technical matter left as an exercise. $\square$

Def. 3.2.2 *The smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ of the previous theorem is called transition function between the local trivializations χ_1 and χ_2 of the vector bundle $\pi : E \to M$.*

As we have seen before, when $E = TM$, the transition functions map every p in $U_\alpha \cap U_\beta$ in the Jacobian matrix evaluated in $\varphi_\beta(p)$ of the transition function $\eta_{\alpha\beta}$ between two charts φ_α and φ_β of M. Moreover, as for the case of the tangent bundle, it is simple to verify that the transition functions $\tau_{\alpha\alpha}$ satisfy the so-called **cocycle relations** (identical to those of the tangent bundle, with the only difference of the dimension $r \leqslant n$ for the matrix):

$$\begin{cases} \eta_{\alpha\alpha}(p) = I_r \\ \eta_{\alpha\beta}(p) = \eta_{\beta\alpha}(p)^{-1} \\ \eta_{\alpha\beta}(p)\eta_{\beta\gamma}(p) = \eta_{\alpha\gamma}(p) \end{cases},$$

for all $p \in U_\alpha \cap U_\beta$ (the first two properties) and for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$ (the third one).

The importance of the transition functions can be fully understood by the following results, which shows how to provide a vector bundle structure to a collection of vector spaces with fixed dimension attached to the points of a manifold via the transition functions.

Theorem 3.2.2 *Suppose we are given a manifold M and a collection of real vector spaces E_p of fixed dimension r attached to each point $p \in M$. Let then:*

- $E := \bigsqcup_{p \in M} E_p$;
- $\pi : E \to M$, *such that $\pi|_{E_p}$ maps all elements of E_p to p.*

Suppose furthermore that we are given:

1. *an open cover $\{U_\alpha\}_{\alpha \in A}$ of M;*

2. *for each $\alpha \in A$, a bijective map $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ such that $\chi_\alpha|_{E_p}$ is a linear isomorphism between E_p and $\{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$;*

3. *for each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ such that $\chi_\alpha \circ \chi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$ for all $p \in U_\alpha \cap U_\beta$ and $v \in \mathbb{R}^r$.*

Then there exists a unique topology and smooth structure on E that make it a smooth manifold and a vector bundle of rank r over M, with projection π and smooth local trivializations $\{(U_\alpha, \chi_\alpha)\}$.

The proof is quite technical and we omit it, the interested reader can find it in [10], Lemma 10.6 page 253.

Without this results, in order to give a vector bundle structure on a collection of vector spaces attached to points of a manifold, one should have to build a manifold topology and a smooth structure on their disjoint union, then construct the local trivializations and show that they satisfy all the properties of definition 3.2.1. This is, in general, a much longer and complicated procedure than the one described in the theorem above.

3.2.1 Operations on vector bundles

The operations that can be done on vector spaces can be extended to vector bundles. The key to do that is simply to perform these operations on the fibers, which are vector spaces.

Def. 3.2.3 (Whitney (direct) sum of vector bundles) *Given a smooth manifold M and two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ of rank r_1 and r_2, respectively, the **Whitney sum** of E_1 and E_2 is the vector bundle over M of **rank** $r_1 + r_2$ whose fiber at each point $p \in M$ is the direct sum $(E_1)_p \oplus (E_2)_p$.*

It can be proven that, with this definition, we get indeed a vector bundle with total space

$$E_1 \oplus E_2 = \bigsqcup_{p \in M} ((E_1)_p \oplus (E_2)_p).$$

The transition functions for this bundle are $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r_1 + r_2, \mathbb{R})$, where, for each $p \in M$, $\tau_{\alpha\beta}(p)$ is a block diagonal matrix of the form $\begin{pmatrix} (\tau_1)_{\alpha\beta}(p) & 0 \\ 0 & (\tau_2)_{\alpha\beta}(p) \end{pmatrix}$.

Def. 3.2.4 (Restriction of a vector bundle) *Given a smooth manifold M, a smooth vector bundle $\pi : E \to M$ of rank r and an immersed or embedded subset $S \subset M$, the restriction of E to S is the vector bundle with total space $E_S = \bigsqcup_{p \in S} E_p$ and projection $\pi_S = \pi|_{E_S}$.*

It can be proven that $\pi_S : E_S \to M$ is a smooth vector bundle. As a particular case, the restricted vector bundle $TM|_S$ is called the **ambient vector bundle** over M.

Def. 3.2.5 (Dual of a vector bundle) *Let E be a vector bundle of rank r over the manifold M. Then, its dual vector bundle is:*

$$E^* = \bigsqcup_{p \in M} {E_p}^*.$$

${E_p}^* = \mathrm{Hom}(E_p, \mathbb{R})$ is the dual vector space of E_p. The projection is again the map $\pi : E^* \to M$ such that its restriction to every E_p^* sends its elements to p. The rank of E^* is r. The transition functions are given by $\tau : U \to \mathrm{GL}(r, \mathbb{R})$, $\tau^*(p) = (\tau(p)^{-1})^t$ for all $p \in U$.

When we operate the dualization procedure to the tangent bundle of a manifold, we obtain a very important object, that we discuss in the next section.

3.3 The cotangent bundle over a manifold

Before formalizing the concept of cotangent bundle, let us extend to generic finite-dimensional real vector spaces what stated in Appendix B about the relationship between $\mathbb{R}^n$ equipped with its canonical basis and its dual space $(\mathbb{R}^n)^*$ equipped with the canonical dual basis.

If V is an n-dimensional real vector space, then, by convention, we call its elements $v \in V$ **vectors** and we write them in matrix form as a $n \times 1$ matrix, i.e. as **column vectors**.

Instead, the elements of its dual space V^*, i.e. linear functionals $\omega : V \to \mathbb{R}$, are called **covectors** and they are indicated in matrix form as a $1 \times n$ matrix, i.e. as **row vectors**.

We know that the dual basis $(\varepsilon^1, \dots, \varepsilon^n)$ of $(\mathbb{R}^n)^*$ is associated to the canonical basis $(e_1, \dots, e_n)$ via $\varepsilon^i(e_j) = \delta^i_j$, so that $\varepsilon^i(v^j e_j) = v^i$, the same holds for generic vector spaces and bases.

More precisely, if $(e_1, \dots, e_n)$ is *any basis* of V, the corresponding dual basis of V^*, denoted again $(\varepsilon^1, \dots, \varepsilon^n)$, is defined by:

$$\varepsilon^i(e_j) := \delta^i_j,$$

which implies that, if $v = v^j e_j$, then

$$\varepsilon^i(v^j e_j) = v^j \varepsilon^i(e_j) = v^j \delta^i_j = v^i.$$

So, also for generic vector spaces, the i-th element of the dual basis $(\varepsilon^1, \dots, \varepsilon^n)$ acts simply as the projection on the direction defined by the i-th vector e_i of a fixed basis of V.

A generic covector $\omega \in V^*$ will be written in terms of the basis $(\varepsilon^1, \dots, \varepsilon^n)$ as $\omega = \omega_i \varepsilon^i$, with the components $\omega_i \in \mathbb{R}$ satisfying

$$\omega(e_i) = \omega_j \varepsilon^j(e_i) = \omega_j \delta^j_i = \omega_i,$$

i.e. the components of ω are determined simply by applying it to all the elements of the basis $(e_1, \dots, e_n)$. As a consequence, the action of ω on a generic vector $v = v^j e_j$ is the following: $\omega(v) = (\omega_i \varepsilon^i)(v^j e_j) = \omega_i v^j \varepsilon^i(e_j) = \omega_i v^j \delta^i_j = \omega_i v^i$. The fact that

$$\omega(v) = \omega_i v^i \tag{3.1}$$

explains the convention of writing ω in matrix form as a row vector and v as a column vector.

Other useful facts that is worthwhile recalling are listed below:

- **Transpose** (or **dual**) **map**: if $A : V \to W$ is a linear operator between two finite dimensional real vector spaces V and W, the the linear map

$$\begin{array}{rccc} A^t : & W^* & \longrightarrow & V^* \\ & \omega & \longmapsto & A^t(\omega) \end{array}, \qquad \begin{array}{rccl} A^t(\omega) : & V & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & A^t(\omega)(v) := \omega(Av), \end{array}$$

  is called the transpose (or dual) map of A.

- The transpose map verifies $(A \circ B)^t = B^t \circ A^t$ and $(id_V)^t = id_{V^*}$, with obvious meaning of the symbols used.

- The **bidual**, or second dual space of V is $V^{**} := (V^*)^*$. For finite-dimensional vector spaces, V and its bidual V^{**} are naturally isomorphic via the map:

$$\begin{array}{cccc} \xi : & V & \overset{\sim}{\longrightarrow} & V^{**} \\ & v & \longmapsto & \xi(v) \end{array}, \qquad \begin{array}{cccc} \xi(v) : & V^* & \longrightarrow & \mathbb{R} \\ & \omega & \longmapsto & \xi(v)(\omega) := \omega(v), \end{array} \tag{3.2}$$

  the isomorphism being natural because only the intrinsic elements of the spaces involved have been used to define it and nothing else, in particular without the choice of a basis.

- Because of eqs. (3.1) and (3.2), the real number can be interpreted either as the application of the linear functional $\omega \in V^*$ to the vector $v \in V$, or as the application of the linear functional $\xi(v) \in V^{**}$ to the covector $\omega \in V^*$. Because of the natural identification between V and V^{**}, it is custom to omit ξ and to write simply v, which, with this omission, acquires the double role of vector of V and linear functional over V^*. Due to this double role, the real number $\omega(v) = \xi(v)(\omega) \equiv v(\omega)$ is often written in a more symmetric-looking way as follows:

$$\langle w, v\rangle := \omega(v), \quad \langle v, w\rangle := \xi(v)(\omega) \equiv v(\omega),$$

  called **pairing** between v and ω. The pairing $\langle \varepsilon^i, e_j\rangle = \delta^i_j$ is called **canonical pairing** between bases of $V \cong V^{**}$ and V^*.

The definition of cotangent bundle over a smooth manifold M is identical to that of tangent bundle, the only difference being that the tangent spaces are replaced by their duals.

Def. 3.3.1 *$T^*_pM = \mathrm{Hom}(T_pM, \mathbb{R})$ is the dual of T_pM, called the **cotangent space to** M **at** p. An element $\omega \in T^*_pM$ is called **cotangent vector** to M in p, or **covector**, or **differential form**.*

Def. 3.3.2 (Cotangent bundle) *The cotangent bundle over M, denoted with T^*M is given by the following disjoint union of cotangent spaces at different $p \in M$:*

$$T^*M = \bigsqcup_{p\in M} T^*_pM = \{(p, \omega)\ :\ p \in M,\ \omega \in T^*_pM\}, \quad \pi|_{T^*_pM}(\omega) := p.$$

Analogously to what we did for the tangent bundle, we can prove that **the cotangent bundle is manifold of dimension $2n$ and a vector bundle of rank n.**

In the case of tangent spaces, we have seen that the act of fixing a local coordinate system $(U, \varphi \equiv (x^1, \dots, x^n))$ in $p \in M$ induces the basis $(\partial_1|_p, \dots, \partial_n|_p)$ of T_pM. We are going to prove that the dual basis of T^*_pM can be built by taking the differential of the coordinate functions $x^i : U \subseteq M \to \mathbb{R}$, $x^i(p) = (\varepsilon^i \circ \varphi)(p) = \varepsilon^i(x^1, \dots, x^n) = x^i$. Being scalar functions, we must apply eq. (2.17) to get

$$dx^i\big|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \left.\frac{\partial}{\partial x^j}\right|_p (x^i) \underset{(2.14)}{=} \frac{\partial(x^i \circ \varphi^{-1})}{\partial x^j}(x) = \frac{\partial(\varepsilon^i \circ \varphi \circ \varphi^{-1})}{\partial x^j}(x) = \frac{\partial \varepsilon^i}{\partial x^j}(x) = \frac{\partial x^i}{\partial x^j} = \delta^i_j,$$

so, the linear functionals of T_pM given by

$$\begin{array}{rccl} dx^i\big|_p : & T_pM & \longrightarrow & T_{x^i}\mathbb{R} \cong \mathbb{R} \\ & \partial_j|_p & \longmapsto & dx^i\big|_p\,(\partial_j|_p) = \delta^i_j, \end{array}$$

verify the pairing

$$\langle dx^i\big|_p\,, \partial_j|_p\rangle = \delta^i_j$$

which means that they are the dual basis of the coordinate tangent vectors $\partial_j|_p$. This justifies the following definition.

Def. 3.3.3 (Coordinate cotangent vectors) *The vectors $(dx^1\big|_p, \ldots, dx^n|_p)$ are called coordinate cotangent vectors and they form the **standard basis of** T_p^*M dually associated to the basis of coordinate tangent vectors $(\partial_1|_p, \ldots, \partial_n|_p)$ of T_pM.*

Once established $(dx^i\big|_p)$ as the standard basis of T_p^*M, we infer, from what recalled above for a general vector space, that:

- every cotangent vector $\omega \in T_p^*M$ can be expressed as the following linear combination:

$$\omega = \omega_i\; dx^i\big|_p\,, \qquad \omega_i = \omega(\partial_i|_p) \in \mathbb{R},$$

- the action of $dx^i\big|_p$ on the generic tangent vector $v = v^j\; \partial_j|_p \in T_pM$ is simply the extraction of the i-th component w.r.t. the coordinate tangent vectors of T_pM:

$$dx^i\big|_p\,(v^j\; \partial_j|_p) = v^i.$$

Analogously as for the tangent bundle, we can define the local coordinates of the cotangent bundle as follows.

Def. 3.3.4 *Given a local coordinate system $(U, \varphi \equiv (x^i))$ in $p \in M$, the coordinates defined by $(x^1(p), \ldots, x^n(p), \omega_1, \ldots, \omega_n)$, such that $\omega \in T_p^*M$ is written as $\omega = \omega_i\; dx^i\big|_p$, are called the **natural local coordinates on the cotangent bundle** T^*M.*

We summarize below the results that we obtained so far about the local expressions of a tangent and cotangent vectors.

- Given a local chart (U, φ) of $p \in M$ with local coordinate functions x^i,

$$\begin{array}{rccl} x^i = \varepsilon^i \circ \varphi : & U \subseteq M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & \varepsilon^i(\varphi(p)). \end{array}$$

- The basis of T_pM induced by this chart is

$$\Big(\partial_1|_p\,, \ldots, \partial_n|_p\Big)\,.$$

- The dual basis of T_p^*M is

$$\Big(dx^1\big|_p\,, \ldots, dx^n|_p\Big)\,.$$

- They verify the following pairing: $\langle dx^i, \partial_j|_p\rangle = \delta^i_j$.
- A generic tangent vector $v \in T_pM$ will be written as:

$$v = v^j \, \partial_j|_p \,, \qquad v^j \in \mathbb{R},\ j = 1, \dots, n.$$

- A generic cotangent vector (or covector, or differential form) $\omega \in T_pM$ will be written as:

$$\omega = \omega_i \, dx^i\big|_p \,, \qquad \omega_i = \omega(\,\partial_i|_p) \in \mathbb{R},\ i = 1, \dots, n.$$

Many times, in the physical and engineering literature, *the specification of the basis is omitted and the position of the indices is used to qualify the object*:

- **tangent vector** $(v^1, \dots, v^n)$ **- components with indices above**
- **covector or differential form** $(\omega_1, \dots, \omega_n)$ **- components with indices below**.

In the trivial case of $M = \mathbb{R}^n$ we have at disposal the single chart atlas $(\mathbb{R}^n, \varphi \equiv id_{\mathbb{R}^n})$ which allows us to canonically identify $\left(\partial_1|_p \,, \dots, \, \partial_n|_p\right)$ with the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and $\left(dx^1\big|_p \,, \dots, \, dx^n|_p\right)$ with the dual canonical basis $(\varepsilon^1, \dots, \varepsilon^n)$.

3.3.1 A noticeable example of cotangent vector: the differential of a scalar function at a point

Let $\phi : M \; to \mathbb{R}$ be a smooth scalar function and $p \in M$. Since $d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R} \cong \mathbb{R}$ is linear, we clearly have that $d\phi_p \in T^*_pM$, i.e. $d\phi_p$ is a cotangent vector to M at p.

Fixed a local chart $(U, \varphi \equiv (x^i))$ in p such that $\varphi(p) = x$, we can of course express $d\phi_p$ as a linear combination of the coordinate cotangent vectors:

$$d\phi_p = \omega_i \, dx^i\big|_p \,,$$

and we know that:

$$\omega_i = d\phi_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) := \left.\frac{\partial}{\partial x^i}\right|_p (\phi) = \frac{\partial(\phi \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) = \frac{\partial \tilde{\phi}}{\partial x^i}(x),$$

where $\tilde{\phi} = \phi \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ is the local representation of ϕ.

Thus, the explicit expression of the cotangent vector $d\phi_p$ is:

$$\boxed{d\phi_p = \frac{\partial \tilde{\phi}}{\partial x^i}(x) \, dx^i\big|_p} \,. \tag{3.3}$$

3.3.2 Transformation rule for the local coordinates of cotangent vectors

Here we analyze how the components of a cotangent vector change when we change the local coordinates in a point. This is the analog for cotangent vectors of what we have already did in section 2.5.2 for tangent vectors and thus it can be thought as a sort of physicist definition of cotangent vectors.

Suppose that $p \in M$ belongs to the intersection of two local charts $(U, \varphi \equiv (x^i))$ and $(\tilde{U}, \tilde{\varphi} \equiv (\tilde{x}^j))$, then we can decompose $\omega \in T_p^* M$ w.r.t. the basis $(dx^1\big|_p, \ldots, dx^n|_p)$ or w.r.t. the basis $(d\tilde{x}^1\big|_p, \ldots, d\tilde{x}^n|_p)$ obtaining, respectively,

$$\omega = \omega_i \; dx^i\big|_p = \tilde{\omega}_j \; d\tilde{x}^j\big|_p .$$

As we have just seen, the coefficients of the cotangent vectors can be obtained by applying ω on the coordinate tangent vectors:

$$\omega_i = \omega \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) \quad \text{and} \quad \tilde{\omega}_j = \omega \left(\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \right).$$

Recall now from eq. (2.29) that

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p ,$$

we get

$$\omega_i = \omega \left(\frac{\partial \tilde{x}^j}{\partial x^i}(x) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(x) \, \omega \left(\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(x) \, \tilde{\omega}_j.$$

Similarly, by using eq. (2.30) and repeating the calculations above on $\tilde{\omega}_j$ we obtain:

$$\tilde{\omega}_j = \frac{\partial x^i}{\partial \tilde{x}^j}(x) \, \omega_i.$$

As we said in section 2.5.2, in the early days of differential geometry (and still nowadays in the physicist and engineering setting), a tangent vector was interpreted as the assignment of an n-tuple of real numbers associated to each coordinate system following precise transformation rules when we change from one coordinate system to another. It is thus important to compare the transformation rules of the components of a tangent and cotangent vector:

Tangent vectors:

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(x) v^i, \quad v^i = \frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}) \tilde{v}^j$$

Cotangent vectors:

$$\tilde{\omega}_j = \frac{\partial x^i}{\partial \tilde{x}^j}(x)\,\omega_i, \quad \omega_i = \frac{\partial \tilde{x}^j}{\partial x^i}(x)\,\tilde{\omega}_j.$$

Since eq. (2.29) for the transformation of the coordinated tangent vectors is a direct consequence of the chain rule, mathematicians considered it as a sort of 'standard' for the transformation under change of coordinate system and called covariant, from the Latin prefix *co-*, which means *with*, so that covariant means that an object 'vary with' the standard transformation rule.

It can be seen that cotangent vectors follow the standard transformation rule, eq. (2.29), while tangent vectors follow the opposite rule. For this reason, it is still customary to say that:

- **cotangent** vectors are **covariant** vectors;
- **tangent** vectors are **contravariant** vectors.

Despite the same nomenclature, this has nothing to do with covariant and contravariant functors of category theory.

3.4 Local and global sections of a vector bundle

In Physics, when we talk about a vector field we mean a vector attached to each point of a certain region in space. This concept can be made rigorous in differential geometry thanks to the definition of sections of vector bundles. Being quite simple, we will first introduce the abstract concept of section on a general vector bundle and then we will specialize it on the tangent and cotangent bundles.

Let us consider a vector bundle $\pi : E \to M$ over the smooth manifold M and a neighborhood U of a point $p \in M$. The most natural vectors associated to p are those belonging to the fiber over it, i.e. $\pi^{-1}(p) = E_p$, because each $v \in E_p$ projects on p via π. Thus, a function that associates points of U to vectors belonging the fibers over them is also a natural object. Of course, to be able to perform differential calculus over this object, we require it to be smooth, i.e. we demand that the vector assignment is smooth when we pass from one point to another.

The definition of local section gives a mathematical formalization to what just said.

Def. 3.4.1 (Local section or local vector field) *A local section (or a local vector field) of E on an open set $U \subseteq M$ is a smooth function $\sigma : U \to E$ such that $\pi \circ \sigma = \iota_U$, i.e. such that the following diagram commutes:*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ U \arrow[ur, "\sigma"] \arrow[r, hook, "\iota_U"'] & M \end{tikzcd}$$

i.e. $\pi \circ \sigma = \iota_U$, where ι is the canonical inclusion of U in M.

Notice that the definition contains exactly the information that we wanted to formalize, in fact, thanks to the local triviality of E, $\sigma(p) = (p, v) \in U \times E_p$, so

$$(\pi \circ \sigma)(p) = \pi(p, v) = p = \iota(p).$$

In this way, we do not attach to p any vector, but a vector v belonging to the fiber over p, which is called the *significant part* of the section σ, because it is the only information that allows us to distinguish it from another section on U.

The set of all sections of E on U is denoted with the following symbol:

$$\boxed{\Gamma(U, E) = \{\sigma : U \to E,\ \pi \circ \sigma = \iota_U\}}\ .$$

$\Gamma(U, E)$ is an Abelian group w.r.t. the sum of sections on U defined as follows: if $\sigma_1, \sigma_2 \in \Gamma(U, E)$, with $\sigma_1(p) = (p, v_1|_p)$ and $\sigma_2(p) = (p, v_2|_p)$, then $(\sigma_1 + \sigma_2)(p) = (p, v_1|_p + v_2|_p)$, which makes perfect sense because both $v_1|_p$ and $v_2|_p$ belong to the same vector space $\pi^{-1}(p)$, so we can add them together meaningfully.

If it is possible to define σ on the entire manifold M, then we get the global sections.

Def. 3.4.2 (Global section or global vector field) *A global section (or a global vector field) of E on M is a smooth function $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$, i.e. such that the following diagram commutes.*

$$\begin{tikzcd}
 & E \arrow[d, "\pi"] \\
M \arrow[ur, "\sigma"] \arrow[r, "id_M"'] & M
\end{tikzcd}$$

Convention: without further specification, a section on a vector bundle will be considered as global.

The set of all sections of E on M is denoted with the following symbol:

$$\boxed{\Gamma(E) = \{\sigma : M \to E,\ \pi \circ \sigma = id_M\}}\ .$$

Noticeable examples of sections, or vector fields, are obtained by considering $E = TM$ and $E = T^*M$, the tangent and cotangent bundle of M, respectively.

3.4.1 Tangent vector fields

In this and in the next subsection we will omit the adjective local or global, since the definitions and results hold for both situations, with evident adjustments.

Def. 3.4.3 *A (tangent) vector field is a smooth assignment $X : M \to TM$, to each point $p \in M$, of a tangent vector to M at p, i.e. $\pi \circ X = id_M$,*

$$\begin{array}{rccl} X: & M & \longrightarrow & TM \\ & p & \longmapsto & X(p) \equiv (p, X_p), \end{array}$$

with $X_p \in T_pM$, the significant part of the (tangent) vector field X.

Due to its importance, $\Gamma(TM)$, the set of all sections of TM, is denoted with a particular symbol:

$$\boxed{\mathfrak{X}(M) = \tau(M) = \{\sigma : M \to TM,\ \pi \circ \sigma = id_M\}}\ . \tag{3.4}$$

It is custom to omit 'tangent' and write only vector field when it is clear that the vector bundle that we are considering is TM. We will use this convention.

The basic properties of $\mathfrak{X}(M)$ are listed in the following result.

Theorem 3.4.1 *The following assertions hold.*

- $\mathfrak{X}(M)$ *is a real vector space under point-wise addition and scalar multiplication, i.e.*

$$(aX + bY)_p := aX_p + bY_p, \qquad X, Y \in \mathfrak{X}(M),\ a, b \in \mathbb{R}.$$

  The 0 element of $\mathfrak{X}(M)$ *is the null vector field, that attaches to any* $p \in M$ *the 0 tangent vector of* T_pM.

- *If* $f \in \mathscr{C}^\infty(M)$ *and* $X \in \mathfrak{X}(M)$, *then* $fX : M \to TM$ *defined as:*

$$(fX)_p := f(p)X_p \qquad \forall p \in M,$$

  is a vector field.

- $\mathfrak{X}(M)$ *is a module over the ring* $\mathscr{C}^\infty(M)$.

Using the natural local coordinates of TM for every coordinate chart $(U, (x^i))$ we can write, for every $p \in M$,

$$X_p = (p, X^i(p)\,\partial_i|_p),$$

where the coefficients $X^i(p) \in \mathbb{R}$, in general, vary with p. This implies the existence of n functions $X^i : U \subseteq M \to \mathbb{R}$, called **component functions** of the vector field $X \in \mathfrak{X}(M)$ in the chart $(U, (x^i))$ such that, for all $p \in M$:

$$X_p = X^i(p)\,\left.\frac{\partial}{\partial x^i}\right|_p,$$

which is an equation involving tangent vectors of T_pM. Using the fact that $\mathfrak{X}(M)$ is a module over the ring $\mathscr{C}^\infty(M)$, this relationship can be written also as an equation involving vector fields, i.e.

$$X = X^i \frac{\partial}{\partial x^i} \equiv X^i \partial_i,$$

where

$$\begin{array}{rccl} \frac{\partial}{\partial x^i} : & U & \longrightarrow & TM \\ & p & \longmapsto & \frac{\partial}{\partial x^i}(p) := (p,\, \left.\frac{\partial}{\partial x^i}\right|_p) \equiv (p,\, \partial_i|_p), \end{array}$$

is called the **i-th coordinate tangent vector field**.

It is clear that the restriction of a vector field to a chart domain U is smooth if and only if the component functions w.r.t. that domain are smooth.

3.4.2 1-forms or cotangent vector fields

Sections of the cotangent bundle are a fundamental object in differential geometry and its applications.

Def. 3.4.4 (1-form or cotangent vector field) *A 1-form or cotangent vector field, is a smooth assignment* $\omega : M \to T^*M$, *to each point* $p \in M$, *of a cotangent vector to* M *at* p, *i.e.* $\pi \circ \omega = id_M$,

$$\begin{array}{rccl} \omega : & M & \longrightarrow & T^*M \\ & p & \longmapsto & \omega(p) \equiv (p, \omega_p), \end{array}$$

with $\omega_p \in T_p^*M$, *the significant part of the 1-form* ω.

The easiest example of 1-form is the differential of a smooth scalar function $f \in \mathscr{C}^\infty(M)$:

$$\begin{array}{rccl} df: & M & \longrightarrow & T^*M \\ & p & \longmapsto & df(p) \equiv (p, df_p). \end{array}$$

The set of all sections of T^*M is denoted which either

$$\boxed{\mathfrak{X}^*(M) = \Lambda(M) = \Omega(M)}\,. \tag{3.5}$$

As $\mathfrak{X}(M)$, also $\Lambda(M)$ is a real vector space w.r.t. point-wise operations and a module over the ring $\mathscr{C}^\infty(M)$ w.r.t. the operation

$$(f\omega)_p := f(p)\omega_p, \qquad \forall f \in \mathscr{C}^\infty(M),\ p \in M.$$

Using the natural local coordinates of TM for every coordinate chart $(U, (x^i))$ we can write, for every $p \in M$,

$$\omega_p = (p, \omega_i(p)\ dx^i\big|_p),$$

where the coefficients $\omega_i(p) \in \mathbb{R}$, in general, vary with p. This implies the existence of n functions $\omega_i : U \subseteq M \to \mathbb{R}$, called **component functions** of the 1-form $\omega \in \Lambda(M)$ in the chart $(U, \varphi \equiv (x^i))$ such that, for all $p \in M$:

$$\omega_p = \omega_i(p)\ dx^i\big|_p\,,$$

which is an equation involving cotangent vectors of T_p^*M. Using the fact that $\Lambda(M)$ is a module over the ring $\mathscr{C}^\infty(M)$, this relationship can be written also as an equation involving 1-forms, i.e.

$$\omega = \omega_i\ dx^i\big|_p \equiv \omega_i dx^i,$$

where

$$\begin{array}{rccl} dx^i: & U & \longrightarrow & T^*M \\ & p & \longmapsto & dx^i(p) := (p,\ dx^i\big|_p), \end{array}$$

is called the ***i*-th coordinate 1-form**.

It is clear that the restriction of a 1-form to a chart domain U is smooth if and only if the component functions w.r.t. that domain are smooth.

In the particular case where $\omega = df$, $f \in \mathscr{C}^\infty(M)$, thanks to (3.3) it holds $df_p = \frac{\partial \tilde{f}}{\partial x^i}(x)\ dx^i\big|_p$, so we can write

$$df = \frac{\partial \tilde{f}}{\partial x^i} dx^i$$

i.e. the component functions of the differential of a scalar function are the partial derivatives of the local representation $\tilde{f} = f \circ \varphi^{-1}$ of the scalar function itself w.r.t. the chart. If $M = \mathbb{R}^n$, we can use the single chart atlas $(\mathbb{R}^n, id_{\mathbb{R}^n})$ and $\tilde{f} = f$, thus the previous formula reduces to the well-known formula for the total derivative of ordinary differential calculus in $\mathbb{R}^n$.

Thanks to what just discussed, we get a criterion to decide weather a smooth scalar function on a manifold is constant or not, for the proof see [10] Proposition 11.22, page 282.

Theorem 3.4.2 (Criterion for constant scalar functions on a manifold) *$f \in \mathscr{C}^\infty(M)$ is constant on each connected components of M if and only if $df = 0$.*

This result suggests that we can interpret df as a 'small' change of $f \in \mathscr{C}^\infty(M)$ generated by small changes of its variables as in ordinary calculus in $\mathbb{R}^n$. This is the case, in fact, since we are interested in small changes, we can fix any point $p \in M$ and a local chart $(U, \varphi \equiv (x^i))$ in p, with $x = \varphi(p)$, so that we can associate f to its local representation $\tilde{f} = f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$ and consider $\Delta f := \tilde{f}(x + v) - \tilde{f}(x)$, where the norm of $v \in \mathbb{R}^n$ is sufficiently small. By the smoothness of f we can apply a Taylor expansion in x for $\tilde{f}$ and write:

$$\Delta f \approx \frac{\partial \tilde{f}}{\partial x^i}(x) v^i$$

but we know that the coordinate cotangent vectors $dx^i\big|_p$ act as component extractors on vectors, so $v^i = dx^i\big|_p(v)$ and thus

$$\Delta f \approx \frac{\partial \tilde{f}}{\partial x^i}(x)\, dx^i\big|_p(v) = df_p(v).$$

From this computation, we infer that df encodes the first-order variation of $f \in \mathscr{C}^\infty(M)$ in an intrinsic, coordinate-free way, on every manifold M.

Chapter 4

Tensor calculus (Edoardo Provenzi)

> It is well known that around the turn of the century Riemann's theory of metrical continua, which had fallen so completely into oblivion, was revivified and deepened by Ricci and Levi-Civita; and that the work of these two decisively advanced the formulation of general relativity.
>
> ALBERT EINSTEIN, 1955

Tensor calculus, invented by G. Ricci-Curbastro and T. Levi-Civita in 1900 [16], is omnipresent in differential geometry and its applications. In this chapter we give a very basic introduction to this topic, first discussing the tensor product for vector spaces and then specializing these concepts on the fibers of a vector bundle.

4.1 Tensor products of vector spaces and vectors

Let V, W be two real vector spaces of finite dimension m and n, respectively, $V^* = \mathrm{Hom}(V, \mathbb{R})$, $W^* = \mathrm{Hom}(W, \mathbb{R})$ their dual spaces and let $\mathrm{Bil}(V \times W)$ the vector space of bilinear forms $g : V \times W \to \mathbb{R}$ on $V \times W$, i.e. linear in one variable when the other is kept fixed.

The most natural way to build a bilinear form $g : V \times W \to \mathbb{R}$ is by considering the product of two linear forms $\varphi \in V^*$ and $\psi \in W^*$, i.e. $g(v, w) = \varphi(v)\psi(w)$, in fact, by definition of bilinearity, if we fix one variable, say w, then $\psi(w)$ becomes simply a real coefficient and the linearity of φ in v guarantees the linear behavior of g w.r.t. v; of course the same holds if we exchange the role of v and w thus guaranteeing the bilinearity of g.

The bilinear form arising in this way is called **tensor product of φ and ψ** and denoted with $\varphi \otimes \psi$:

$$\begin{array}{rccl} \varphi \otimes \psi : & V \times W & \longrightarrow & \mathbb{R} \\ & (v, w) & \longmapsto & \boxed{\varphi \otimes \psi(v, w) := \varphi(v)\psi(w)} \, . \end{array} \tag{4.1}$$

For example, if $V = W = \mathbb{R}^2$, $\varphi = \varepsilon^1$ and $\psi = \varepsilon^2$, where ε^i is the i-th element of the canonical basis of $(\mathbb{R}^2)^*$, then, for any $v, w \in \mathbb{R}^2$ such that $v = (v^1, v^2)$ and $w = (w^1, w^2)$, we have $\varepsilon^1 \otimes \varepsilon^2(v, w) = v^1 w^2$.

The naturalness of the bilinearity of the tensor product of linear forms raises the following question: is it possible to express *all* bilinear forms on $V \times W$ as tensor product of linear forms on V and W? The answer is affirmative (for the proof see [10], proposition 12.10 page 311). Thus, if we define **the tensor product of V^* and W^*** as the vector space (w.r.t. the point-wise linear operations)

$$\boxed{V^* \otimes W^* := \{\varphi \otimes \psi \mid \varphi \in V^*,\ \psi \in W^*\}},$$

we have the canonical identification

$$\boxed{V^* \otimes W^* \cong \ \mathrm{Bil}(V \times W)}.$$

It is straightforward to verify the following formulae, valid for each $\varphi_1, \varphi_2 \in V^*$, $\psi_1, \psi_2 \in W^*$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$:

$$(a_1\varphi_1 + a_2\varphi_2) \otimes \psi = a_1\varphi_1 \otimes \psi + a_2\varphi_2 \otimes \psi, \qquad \varphi \otimes (b_1\psi_1 + b_2\psi_2) = \varphi \otimes b_1\psi_1 + b_2\psi_2.$$

More generally, if we consider the basis $(\varphi^1, \ldots, \varphi^m)$ of V^* and $(\psi^1, \ldots, \psi^n)$ of W^*, then any $\varphi \in V^*$ and any $\psi \in W^*$ can be written as $\varphi = a_i\varphi^i$ and $\psi = b_j\psi^j$, $a_i, b_j \in \mathbb{R}$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, so

$$\varphi \otimes \psi = a_i\varphi^i \otimes b_j\psi^j \underset{\text{linearity}}{=} a_ib_j\varphi^i \otimes \psi^j,$$

which implies that

$$(\varphi^i \otimes \psi^j)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \text{ is a basis for } V^* \otimes W^*$$

and so $\dim(V^* \otimes W^*) = mn$.

As a consequence, every $g \in V^* \otimes W^* \cong \mathrm{Bil}(V \times W)$ can be univocally written as

$$\boxed{g = g_{ij}\varphi^i \otimes \psi^j},$$

$g_{ij} \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

The results just discussed can be extended to a finite set of vector spaces, obtaining the following canonical identification:

$$\bigotimes_{i=1}^{p} V_i^* \cong \ \mathrm{Mul}(\bigtimes_{i=1}^{p} V_i),$$

where $\mathrm{Mul}(\bigtimes_{i=1}^{p} V_i)$ is the vector space of p-multilinear forms, i.e. linear in each one of the p variables separately, when all the other $p-1$ are kept fixed.

Up to now we have considered tensor products of linear forms and dual vector spaces, we can define tensor product of vectors and vector spaces by considering the natural isomorphism between a finite dimensional real vector space V and its bidual $V^{**} = \mathrm{Hom}(V^*, \mathbb{R})$:

$$\begin{array}{ccc} V & \xrightarrow{\sim} & V^{**} \\ v & \longmapsto & \alpha_v, \end{array} \quad , \qquad \begin{array}{cccc} \alpha_v : & V* & \longrightarrow & \mathbb{R} \\ & \varphi & \longmapsto & \alpha_v(\varphi) = \varphi(v) \equiv \langle v, \varphi \rangle. \end{array}$$

By exchanging the role of V, W and V^*, W^* and thanks to the identification just recalled, we can define the **tensor product of two vectors** $v \in V$ **and** $w \in W$ as follows:

$$\begin{array}{cccc} v \otimes w : & V^* \times W^* & \longrightarrow & \mathbb{R} \\ & (\varphi, \psi) & \longmapsto & v \otimes w(\varphi, \psi) = \langle v, \varphi \rangle \langle w, \psi \rangle, \end{array}$$

i.e. the tensor product of two vectors $v, w \in V$ is a bilinear form on $V^* \times W^*$.

Thus, **both the tensor product of two linear forms and two vectors lead to bilinear forms**, what changes is just their domain.

As before, if we define **the tensor product of V and W** as the vector space (w.r.t. the point-wise linear operations)

$$\boxed{V \otimes W := \{v \otimes w \mid v \in V,\ \psi \in W\}}\ ,$$

we have the canonical identification

$$\boxed{V \otimes W \cong\ \mathrm{Bil}(V^* \times W^*)}\ .$$

As before, if $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are basis of V and W, respectively, then

$$(v_i \otimes w_j)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \text{ is a basis for } V \otimes W,$$

so $\dim(V \otimes W) = mn$ and a generic element $g \in V \otimes W \cong \mathrm{Bil}(V^* \times W^*)$ can be written as

$$\boxed{g = g^{ij} v_i \otimes w_j} \qquad g^{ij} \in \mathbb{R},\ i = 1, \ldots, m,\ j = 1, \ldots, n.$$

The formulae:

$$\begin{cases} g = g^{ij} v_i \otimes w_j & v \in V,\ w \in W \\ g = g_{ij} \varphi^i \otimes \psi^j & \varphi \in V^*,\ \psi \in W^* \end{cases}$$

are vastly used and the position of the indices reveal if we are dealing with the tensor product of vectors or linear forms. The real coefficients g^{ij} and g_{ij} can be organized in a $m \times n$ matrix, for this reason the tensor product is often (erroneously) defined as a matrix.

Finally, as in the previous discussion, we can generalize these results to any finite number of finite-dimensional vector spaces by obtaining:

$$\bigotimes_{i=1}^{p} V_i \cong\ \mathrm{Mul}(\bigtimes_{i=1}^{p} V_i^*).$$

Useful canonical isomorphisms are listed below for finite-dimensional real vector spaces:

$$V \otimes W \cong W \otimes V, \qquad \text{symmetry of } \otimes$$

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3), \qquad \text{associativity of } \otimes$$

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \otimes (V_2 \otimes W), \qquad \text{distributivity of } \otimes \text{ w.r.t. } \oplus,$$

more generally,

$$\bigoplus_{i=1}^{r} V_i \otimes \bigoplus_{i=1}^{s} W_i \cong \bigoplus_{\substack{i=1,\ldots,r \\ j=1,\ldots,s}} V_i \otimes W_j.$$

Of course we can consider the tensor product also of the dual of a vector space with another vector space or vice-versa. The result in this case is particularly important and it is underlined in the following result.

Theorem 4.1.1 *For any couple of finite-dimensional real vector spaces V and W the following natural identification holds:*

$$\boxed{V^* \otimes W \cong \mathrm{Hom}(V, W)}\ ,$$

where the natural isomorphism between the two spaces is defined on the generic basis element[1] *$\varphi \otimes w$ of $V^* \otimes W$ by:*

$$\begin{array}{rrcl} F: & V^* \otimes W & \longrightarrow & \mathrm{Hom}(V, W) \\ & \varphi \otimes w & \longmapsto & F(\varphi \otimes w) \equiv F_{\varphi,w}, \end{array} \qquad \begin{array}{rrcl} F_{\varphi,w}: & V & \longrightarrow & W \\ & v & \longmapsto & F_{\varphi,w}(v) = \varphi(v) w. \end{array} \tag{4.2}$$

Proof. Let $(v_1, \ldots, v_n)$ be a basis of V, $(v^1, \ldots, v^n)$ the dual basis of V^*, such that $v^i v_j = \delta^i_j$, and let $(w_1, \ldots, w_m)$ be a basis of W. These bases induce the basis $(v^i \otimes w_j)_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ of $V^* \otimes W$. The theorem will be proven if we show that F sends this basis to a basis of $\mathrm{Hom}(V, W)$.

To this aim, let us make the action of F explicit: if we apply $F(v^i \otimes w_j) \equiv F_{v^i,w_j} \equiv F^i_j \in \mathrm{Hom}(V, W)$ to an element v_k of the basis of V fixed above, then, thanks to eq. (4.2) we get

$$F^i_j(v_k) = v^i(v_k) w_j = \delta^i_k w_j. \tag{4.3}$$

Thus, we have to prove that the linear maps $(F^i_j)_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ form a basis of $\mathrm{Hom}(V, W)$.

For this, it is sufficient to consider an arbitrary $L \in \mathrm{Hom}(V, W)$ and represent it as a matrix $A = (a^j_k)$ w.r.t. the bases of V and W that we have fixed: by definition of matrix associated to a linear map, the coefficients a^j_i verify $L(v_i) = a^j_i w_j$ for every vector v_k of the basis of V. The linear combination of the maps F_{v^i,w_j} with the coefficients a^j_i, i.e. $a^j_i F^i_j$, is an element of $\mathrm{Hom}(V, W)$, let us apply this map on the generic vector v_k of the basis of V and see what we get:

$$(a^j_i F^i_j)(v_k) \underset{\text{linearity}}{=} a^j_i F^i_j(v_k) \underset{\text{eq. (4.3)}}{=} a^j_i \delta^i_k w_j = a^j_k w_j \underset{\text{def. of } L}{=} L(v_k),$$

we see that the action of the arbitrary linear map $L \in \mathrm{Hom}(V, W)$ on the arbitrary vector v_k of the basis of V is obtained by linear combination of the action of the linear maps F^i_j, hence they form a basis for $\mathrm{Hom}(V, W)$. □

Also this result permits to understand why tensors are often defined as matrices: after fixing a basis of V (and so, by duality, also of V^*) and of W, a linear application belonging to $\mathrm{Hom}(V, W)$ is a matrix.

4.2 Covariant and contravariant tensors. Tensor algebra of a vector space

In the previous section, we have seen that, starting from a real vector space V of finite dimension n, we can build many other spaces via tensor product. These spaces are given by *multilinear functions defined on copies of V and V^*.*

Here we introduce a compact notation and terminology canonically used:

[1] and then, of course, extended by linearity to the whole space.

- $T^0_0(V) = T^0(V) = T_0(V) = \mathbb{R}$
- $T^1_0(V) = T^1(V) = V$
- $T^p_0(V) = T^p(V) = V\underbrace{\otimes\cdots\otimes}_{p \text{ times}}V \quad \Longrightarrow \quad \dim T^p_0(V) = n^p$
- $T^0_1(V) = T_1(V) = V^*$
- $T^0_q(V) = T_q(V) = V^*\underbrace{\otimes\cdots\otimes}_{q \text{ times}}V^* \quad \Longrightarrow \quad \dim T^0_q(V) = n^p$
- $T^p_q(V) = T^p(V) \otimes T_q(V) = V\underbrace{\otimes\cdots\otimes}_{p \text{ times}}V \otimes V^*\underbrace{\otimes\cdots\otimes}_{q \text{ times}}V^*$
- $T^\bullet(V) = \bigoplus_{p \geqslant 0} T^p(V)$
- $T_\bullet(V) = \bigoplus_{q \geqslant 0} T_q(V)$
- $T(V) = \bigoplus_{p,q \geqslant 0} T^p_q(V)$, is called **tensor algebra** of V. .

Let us fix our attention on $T^p_q(V)$.

Def. 4.2.1 *An element $t \in T^p_q(V)$ is called a **p-contravariant and q-covariant tensor on** V.*

t is nothing but **a multilinear form** of the type:

$$t: \quad V^*\underbrace{\times\cdots\times}_{p \text{ times}}V^* \times V\underbrace{\times\cdots\times}_{q \text{ times}}V \quad \longrightarrow \quad \mathbb{R}.$$

To understand its action, let us fix as usual a basis $(v_1, \ldots, v_n)$ of V and the dual basis $(v^1, \ldots, v^n)$ of V^*, then:

- $(v_{i_1} \otimes \cdots \otimes v_{i_p})$ is a basis of $V\underbrace{\otimes\cdots\otimes}_{p \text{ times}}V$, with independent indices $i_1, \ldots, i_p = 1, \ldots, n$;
- $(v^{j_1} \otimes\cdots\otimes v^{j_q})$ is a basis of $V^*\underbrace{\otimes\cdots\otimes}_{q \text{ times}}V^*$, with independent indices $j_1, \ldots, j_q = 1, \ldots, n$;
- $\left(v_{i_1} \otimes \ldots v_{i_p} \otimes v^{j_1} \otimes \ldots v^{j_q}\right)_{\substack{i_1,\ldots,i_p=1,\ldots,n \\ j_1,\ldots,j_q=1,\ldots,n}}$ is a basis of $T^p_q(V)$.

Since we have $p+q$ vectors in the basis of $T_q^p(V)$, each of which is parameterized by an index whose variability is between 1 and n, we have $\underbrace{n \cdots n}_{(p+q)\text{ times}} = n^{p+q}$, so

$$\boxed{\dim(T_q^p(V)) = n^{p+q}}\ .$$

The generic decomposition of the tensor of $T_q^p(V)$ on the basis previously obtained is:

$$\boxed{t = a^{i_1,\dots,i_p}{}_{j_1,\dots,j_q} v_{i_1} \otimes \dots v_{i_p} \otimes v^{j_1} \otimes \dots v^{j_q}}\ ,$$

with $a^{i_1,\dots,i_p}{}_{j_1,\dots,j_q} \in \mathbb{R}$, of course they depend on the particular choice of the basis. Physicists omit the bases and use to write simply

$$t = \left(a^{i_1,\dots,i_p}{}_{j_1,\dots,j_q}\right),$$

the presence of p contravariant and q covariant indices of this sort of multi-dimensional matrix is enough to specify what type of tensor t is.

There is an obvious product between tensors… the tensor product:

$$\begin{array}{ccc} T_{q_1}^{p_1}(V) \times T_{q_2}^{p_2}(V) & \longrightarrow & T_{q_1+q_2}^{p_1+p_2}(V) \\ (t_1, t_2) & \longmapsto & t_1 \otimes t_2, \end{array}$$

It is possible to verify that, with this operation, $T(V)$ becomes an algebra.

4.3 Operations on tensors

Here we define the operations on tensors that can be found in differential geometry for different purposes. Let us start by justifying why we call $T(V)$ the tensor algebra.

4.3.1 Contraction

of type $\binom{r}{s}$: it is a linear function $C_s^r : T_q^p(V) \to T_{q-1}^{p-1}(V)$ that reduces the covariance and contravariance degree of a tensor. Moreover, it **generalizes the concept of trace to tensors**. For simplicity of notation, we can define the contraction on the basis elements of $T_q^p(V)$ (the definition is extended by linearity on the whole $T_q^p(V)$):

$$\boxed{C_s^r(v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes v^{j_1} \otimes \cdots \otimes v^{j_q}) := v^{j_s}(v_{i_r}) v_{i_1} \otimes \cdots \otimes \cancel{v_{i_r}} \otimes \cdots \otimes v_{i_p} \otimes v^{j_1} \otimes \cdots \otimes \cancel{v^{j_s}} \otimes \cdots \otimes v^{j_q}}$$

explanation:

- we consider the i_r-th element of the V basis and the j_s-th element of the dual basis of V;
- we compute, in a linear way, the real number $v^{j_s}(v_{i_r})$;
- we multiply this number to the tensor product basis of $T_q^p(V)$ taking out v_{i_r} and v^{j_s}… because they already served another purpose.

In coordinates, the contraction can be written as follows: if $t \in T^p_q(V)$, $t = \left(a^{i_1,\dots,i_p}{}_{j_1,\dots,j_q}\right)$,

$$C^r_s(t) = \left(a^{i_1\dots i_{r-1}ki_{r+1}\dots i_p}{}_{j_1\dots j_{s-1}kj_{s+1}\dots j_q}\right),$$

the same index k replaces the index i_r and j_s, so that a sum over k is intended!

We are now going to prove that **the operator** $C^1_1 : T^1_1(V) = V \otimes V^* \to T^0_0(V) \equiv \mathbb{R}$ **is simply the trace**. If $(v_1, \dots, v_n)$ is a basis of V and $(v^1, \dots, v^n)$ is the dual basis, then $t \in V \otimes V^*$, $t = a_{ij} v_i \otimes v^j$. By using the identification $V \otimes V^* \cong \mathrm{Hom}(V, V) \equiv \mathrm{End}(V)$, we can identify t with the linear function associated to the matrix $A = (a^i_j)$ w.r.t. the basis $(v_1, \dots, v_n)$. By definition, the action of C^1_1 is as follows:

$$C^1_1(t) = C^1_1(a_{\substack{i\\j}} v_i \otimes v^j) \underset{\text{linearity}}{=} a^i_j C^1_1(v_i \otimes v^j) := a^i_j v^j(v_i) \cancel{v_i} \otimes \cancel{v^j} = a^i_j \delta^j_i = a^i_i = \mathrm{Tr}(A).$$

4.3.2 Symmetrization and antisymmetrization

W e know that some particular multilinear forms are associated with important geometric concepts. For example, *symmetric bilinear forms* define real-valued scalar product, which can be used to define the *angle* between vectors and the concept of *orthogonality*; *alternating forms* define determinants, which are involved in the measure of *areas* and *volumes*.

Since tensors are multilinear forms, it makes sense to analyze the extension of these properties to tensors, this will be essential to build important objects such as the p-forms.

We will develop our analysis on $T^p(V)$, the one on $T_q(V)$ can be reproduced analogously. $t \in T^p(V)$ is such that:

$$\begin{aligned} t:\ \ V^* \underbrace{\times \cdots \times}_{p \text{ times}} V^* \quad &\longrightarrow \quad \mathbb{R} \\ (\alpha^1, \dots, \alpha^p) \quad &\longmapsto \quad t(\alpha^1, \dots, \alpha^p) = a^{i_1\dots i_p} \alpha^1(v_{i_1}) \cdots \alpha^p(v_{i_p}). \end{aligned}$$

We want to single out those multilinear forms t which are *symmetric*, i.e.

$$t(\alpha^{\sigma(1)}, \dots, \alpha^{\sigma(p)}) = t(\alpha^1, \dots, \alpha^p)$$

for every permutation of the set of indices $\{1, \dots, p\}$, and those which are *alternating*, i.e.

$$t(\alpha^{\sigma(1)}, \dots, \alpha^{\sigma(p)}) = \mathrm{sign}(\sigma) t(\alpha^1, \dots, \alpha^p),$$

where $\mathrm{sign}(\sigma) = (-1)^{N(\sigma)} \in \{-1, 1\}$, where $N(\sigma)$ is the number of *inversions* performed by σ, where an inversion is a switch of ordinal position between two indices after the application of σ. This means that:

$$\mathrm{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ performs an even number of inversions} \\ -1 & \text{if } \sigma \text{ performs an odd number of inversions.} \end{cases}$$

Some examples for $T^2(V)$: if $v, w \in V$, then:

- $t_0 = v \otimes w$ is, in general, not symmetric, nor alternating;

- $t_1 = v \otimes w + w \otimes v$ is **symmetric**, in fact the change $v \leftrightarrow w$ leaves t_1 unaffected;

· $t_2 = v \otimes w - w \otimes v$ is **alternating**, in fact the change $v \leftrightarrow w$ transforms t_2 to $-t_2$.

Notation:

$$\boxed{S^p(V)} \; : \quad \text{subspace of } T^p(V) \text{ of symmetric tensors on } V$$

$$\boxed{\Lambda^p(V) = \mathbb{A}^p(V)} \; : \quad \text{subspace of } T^p(V) \text{ of alternating tensors on } V$$

It can be proven that, if $\dim(V) = n$, then

$$\dim(T^p(V)) = n^p, \quad \dim(S^p(V)) = \binom{n+p-1}{p}, \quad \dim(\Lambda^p(V)) = \begin{cases} \binom{n+p-1}{p} & 0 \leqslant p \leqslant n \\ 0 & p > n \end{cases}.$$

The case of bilinear forms, i.e. $p = 2$ is special, let us see why:

$$\dim(T^2(V)) = n^2, \quad \dim(S^2(V)) = \frac{(n+1)n}{2}, \quad \dim(\Lambda^2(V)) = \frac{n(n-1)}{2},$$

so that $\dim(T^2(V)) = \dim(S^2(V)) + \dim(\Lambda^2(V))$, this is a consequence of the fact that every tensor $t \in T^2(V)$ can be written as the sum of a symmetric and an alternating tensor in a unique way as follows:

$$v \otimes w = \frac{v \otimes w + w \otimes v}{2} + \frac{v \otimes w - w \otimes v}{2} \iff t_0 = t_1 + t_2,$$

thus:

$$\boxed{T^2(V) = S^2(V) \oplus \Lambda^2(V)}\,.$$

For $n > 2$ *this is no longer true* because of a dimensional argument: $n^p \neq \binom{n+p-1}{p} + \binom{n}{p}$.

The operations that transform a generic tensor to a symmetric and and alternating one are called:

· **Symmetrization**: defined on the basis of $T^p(V)$ as follows

$$\begin{array}{rccl} S: & T^p(V) & \longrightarrow & S^p(V) \\ & v_1 \otimes \cdots \otimes v_p & \mapsto & S(v_1 \otimes \cdots \otimes v_p) = \frac{1}{p!} \sum\limits_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}, \end{array}$$

and extended by linearity to the whole space;

· **Antisymmetrization**: defined on the basis of $T^p(V)$ as follows

$$\begin{array}{rccl} A: & T^p(V) & \longrightarrow & \Lambda^p(V) \\ & v_1 \otimes \cdots \otimes v_p & \mapsto & S(v_1 \otimes \cdots \otimes v_p) = \frac{1}{p!} \sum\limits_{\sigma} \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}, \end{array}$$

and extended by linearity to the whole space.

The normalization factor $1/p!$ comes from the fact that $p!$ is the number of distinct permutations of a set of p elements and it is introduced so that S and A reduce to the identity operator if they act, respectively, on symmetric and alternating tensors. For example: if $t = t_1 \in S^2(V)$, then

$$S(t_1) = S(v \otimes w + w \otimes v) \underset{\text{linearity}}{=} S(v \otimes w) + S(w \otimes v) := \frac{1}{2}(v \otimes w + w \otimes v) + \frac{1}{2}(w \otimes v + v \otimes w) = t_1,$$

where, for each term, we have applied the only two permutations on a set of two elements: the identity and the switch $v \leftrightarrow w$.

4.3.3 Symmetric product and external product

The symmetric product of tensors brings a couple of symmetric tensors to another symmetric tensor, and the external product of tensors brings a couple of alternating tensors to another alternating tensor. Let us start with the **symmetric product**:

$$\begin{array}{rccl} \odot: & S^p(V) \times S^q(V) & \longrightarrow & S^{p+q}(V) \\ & (t_1, t_2) & \mapsto & t_1 \odot t_2 := \frac{(p+q)!}{p!q!} S(t_1 \otimes t_2), \end{array}$$

by construction, it holds $t_1 \odot t_2 = t_2 \odot t_1$, i.e. $\odot$ is a commutative operation.

If we want $\odot$ to be an internal operation, we have to make p and q 'disappear', which can be done by taking the direct sum:

$$S(V) = \bigoplus_{p \geqslant 0} S^p(V),$$

$(S(V), \odot)$ is called **symmetrical algebra of** V.

Example: let $v, w \in S^1(V) = V$, then

$$v \otimes w \in T^2(V),\ S(v \otimes w) = \frac{1}{2!}(v \otimes w + w \otimes v)$$

$$v \odot w = \frac{2!}{1!1!} \frac{1}{2!}(v \otimes w + w \otimes v) = v \otimes w + w \otimes v,$$

which shows the usefulness of the normalization coefficients.

Analogously, if $v_1, \ldots, v_r \in V$, then

$$v_1 \odot \cdots \odot v_r = \sum_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

Let us now define the **external product**:

$$\begin{array}{rccl} \wedge: & \Lambda^p \times \Lambda^q(V) & \longrightarrow & \Lambda^{p+q}(V) \\ & (t_1, t_2) & \mapsto & t_1 \wedge t_2 := \frac{(p+q)!}{p!q!} A(t_1 \otimes t_2), \end{array}$$

and

$$S(V) = \bigoplus_{p=0}^{n} S^p(V),$$

we stop at $n = \dim(V)$ because, for $p > n$, $\Lambda^p(V) = \{0\}$.

$(\Lambda(V), \wedge)$ is the **external algebra of** V.

Example: let $v, w \in \Lambda^1(V) = V$, then

$$v \otimes w \in T^2(V),\ A(v \otimes w) = \frac{1}{2!}(v \otimes w - w \otimes v)$$

$$v \wedge w = \frac{2!}{1!1!} \frac{1}{2!}(v \otimes w - w \otimes v) = v \otimes w - w \otimes v.$$

Analogously, if $v_1, \ldots, v_r \in V$, then

$$v_1 \wedge \cdots \wedge v_r = \sum_\sigma \operatorname{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

4.4 Tensor bundles and tensor fields

All the previous constructions and operations on tensors have been defined for an arbitrary real vector space V of finite dimension. Thus, they can be applied in the case of vector bundles, where the fiber over each point of the base manifold is, by definition, a vector space.

If $\pi : E \to M$ is a vector bundle, then we have already seen that the dual bundle is constructed by taking the union of the dual spaces E_p^* of the fibers E_p, as p varies in M. This permits to build in a natural way the tensor bundle.

Def. 4.4.1 (Tensor bundle) *The tensor bundle $T_q^p(E)$ is the vector bundle whose fibers over $p \in M$ are given by*

$$T_q^p(E_p) = T^p(E_p) \otimes T_q(E_p) = E_p \underbrace{\otimes \cdots \otimes}_{p \text{ times}} E_p \otimes E_p^* \underbrace{\otimes \cdots \otimes}_{q \text{ times}} E_p^*.$$

Def. 4.4.2 (Tensor field) *A (local or global) p-contravariant and q-covariant tensor field is a (local or global) section of $T_q^p(E)$.*

Analogously, we can define the algebras $T(E), S(E), \Lambda(E)$.

The most important example is given by the tangent and cotangent bundle $E = TM$, $E = T^*M$ of a manifold M and particularly important is the external algebra of the cotangent bundle to a manifold M:

$$\Lambda(T^*M) = \bigoplus_{p=0}^{n} \Lambda^p(T^*M),$$

called **external algebra of** M (omitting TM).

Def. 4.4.3 (k-form) *A k-form on a manifold M is a section of $\Lambda^k(T^*M)$, i.e. a smooth assignment of an alternating tensor on T^*M. The set of all k-forms on M is a vector space w.r.t. the point-wise linear operations that is denoted either $\mathbb{A}^k(M)$ or $\Omega^k(M)$.*

As always, let us look at these objects in the local coordinates of a point $p \in M$ induced by a chart $(U, \varphi = (x^1, \dots, x^n))$: we know that $(\partial_1|_p, \dots, \partial_n|_p)$ is a basis for T_pM and this holds for every $p \in U$, thus it is possible to define the sections of TM given by

$$\begin{array}{rccl} \partial_i : & U & \longrightarrow & TM \\ & p & \longmapsto & \partial_i(p) := \partial_i|_p, \ \text{ with } \pi(\partial_i|_p) = p, \end{array}$$

Def. 4.4.4 (Local frame for TM) *The set $(\partial_1, \dots, \partial_n)$ is called a local frame of TM on U.*

Similarly, by considering the dual basis $(dx^1|_p, \dots, dx^n|_p)$ of T^*M, we can define the sections of T^*M given by

$$\begin{array}{rccl} dx^i : & U & \longrightarrow & T^*M \\ & p & \longmapsto & dx^i(p) = dx^i|_p, \ \text{ with } \pi(dx^i|_p) = p, \end{array}$$

every dx^i is a 1-form.

Def. 4.4.5 (Local frame for T^*M) *The set $(dx^1, \dots, dx^n)$ is called a local frame of T^*M on U.*

It is possible to verify that

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k}), \quad 1 \leqslant i_1 < \cdots < i_k \leqslant n,$$

is a **local frame of $\Omega^k(T^*M)$**.

Notice that the condition $1 \leqslant i_1 < \cdots < i_k \leqslant n$ is imposed to guarantee that *the indices $i_1, \dots, i_k$ are different*, otherwise the external product would be zero because of its anti-symmetry, which can be easily show by taking just two external factors $dx^h \wedge dx^h = -dx^h \wedge dx^h$ which implies $dx^h = 0$. Of course the name of the indices can always be permuted to fulfill the ordering written above.

Every k-form ω can be written, locally, as follows:

$$\boxed{\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}} \,,$$

where $a_{i_1 \cdots i_k} : U \to \mathbb{R}$ are scalar functions on the local chart domain U.

Def. 4.4.6 (Closed and exact forms, potentials) *A k-form ω is closed if $d\omega = 0$, it is exact if it exists a $(k-1)$-form η, called potential form, such that $\omega = d\eta$.*

Thus, an exact form is in the image of d, and a closed form belongs to the kernel of d.

For example, a **2-forms** can be written as $\omega = \omega_{ij} dx^i \wedge dx^j$ and the matrix ω_{ij} containing its coefficients its *anti-symmetric*.

Chapter 5

All about vector fields: flux, Lie derivative and bracket, distributions and foliations (Edoardo Provenzi)

The scope of this chapter is to discuss some fundamental objects of differential geometry that are related with vector fields. We will formalize the relationship between vector fields and differential equations via the flux theorem, which will allow us to introduce the Lie bracket and derivative. Then we will introduce the concept of distribution (totally unrelated to the distributions of the analytical domain...) and foliation.

5.1 Vector fields and derivations

In (3.5), we have defined $\mathfrak{X}(M)$, the space of tangent vector fields on a smooth manifold M as the set of sections on the tangent bundle of M, i.e. $\mathfrak{X}(M) = \{\sigma : M \to TM,\ \pi \circ \sigma = id_M\}$.

We are now going to see an algebraic characterization of this space that is useful in many situations, e.g. for the definition of the Lie bracket 5.3.

Recall that we have defined T_pM, the space of tangent vectors on p to M, as $\mathrm{Der}_p(M)$, the space of derivations on M in p, i.e. linear Leibniz-like $\mathbb{R}$-functionals defined on the vector space $\mathscr{C}^\infty(M)$ of smooth scalar functions on M, with the additional property that they set to 0 constant functions.

If we want to extend the connection between derivations in a point p and tangent vectors to p to vector fields, we must get rid of the dependence of the derivation to the point p and give a more general definition.

Def. 5.1.1 (Derivation of an algebra) *Given a commutative algebra $\mathcal{A}$ on a field $\mathbb{K}$, we call derivation on $\mathcal{A}$ any linear function*[1] $D : \mathcal{A} \to \mathcal{A}$ *that satisfies the Leibniz rule, i.e.*

$$D(ab) = D(a)b + aD(b) \qquad \forall a, b \in \mathcal{A},$$

the juxtaposition of symbols means that we are multiplying by using the product of $\mathcal{A}$.

The set of all derivations on $\mathcal{A}$ is written as $\mathrm{Der}(\mathcal{A})$ and it is a vector space w.r.t. linear operations defined point-wise.

[1] Notice that, in this definition, $D \in \mathrm{End}(\mathcal{A})$, so D is not a functional but an endomorphism of $\mathcal{A}$.

In differential geometry, we have at disposal a commutative algebra: $\mathscr{C}^\infty(M)$, thus the vector space $\mathrm{Der}(\mathscr{C}^\infty(M))$ is perfectly defined and: $D(fg) = D(f)g + fD(g)$, $\forall f, g \in \mathscr{C}^\infty(M)$.

Remark 5.1.1 It is important to stress the difference between $\mathrm{Der}(\mathscr{C}^\infty(M))$ and $\mathrm{Der}_p(M)$:

- the derivations belonging to $\mathrm{Der}(\mathscr{C}^\infty(M))$ are endomorphisms D of $\mathscr{C}^\infty(M)$ which act globally on smooth scalar functions on M: $D : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$

- those belonging to $\mathrm{Der}_p(M)$ are the tangent vectors to M at p, so they are linear functionals v_p acting locally, in an open neighborhood of p: $v_p : \mathscr{C}^\infty(U) \to \mathbb{R}$.

In spite of being different objects, there is a clear correspondence between them: we can define tangent vectors independently of a specified point to by considering a section of the tangent bundle TM, i.e. a vector field on M, as formalized in the following result.

Theorem 5.1.1 *The vector space of vector fields on M and of derivations on $\mathscr{C}^\infty(M)$ are canonically isomorphic:*

$$\boxed{\mathfrak{X}(M) \cong Der(\mathscr{C}^\infty(M))}\ .$$

In the proof of this theorem we use the concepts and results that we have developed previously. Here we simply show how to build the isomorphism: consider the vector field

$$\begin{array}{rccl} X: & M & \longrightarrow & TM \\ & p & \longmapsto & X(p) \equiv X_p, \end{array}$$

such that $X_p \in T_pM \equiv \mathrm{Der}_p(M)$, i.e. X_p is a derivation at p, and then define the function

$$\begin{array}{rccl} X(f): & M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & X(f)(p) := X_p(f), \end{array}$$

but then

$$\begin{array}{rccl} D_X: & \mathscr{C}^\infty(M) & \longrightarrow & \mathscr{C}^\infty(M) \\ & f & \longmapsto & D_X(f) := X(f), \end{array}$$

is clearly a derivation on $\mathscr{C}^\infty(M)$.

Vice-versa, starting from the derivation $D : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, we can univocally define the vector field $X : M \to TM$, $p \mapsto X_p$, $X_p \in T_pM$ whose local expression in local coordinate system $(x^1, \dots, x^n)$ of $p \in M$ is:

$$X_p = D(x^j)(p)\ \partial_j\big|_p\,,$$

perfectly well-defined because $x^j : M \to \mathbb{R}$ are smooth functions and so $D(x^j) \in \mathscr{C}^\infty(M)$, so $D(x^j)(p) \in \mathbb{R}$.

5.2 Integral curves and flux of a vector field

In this section we point out the relationship between vector fields and differential equations. In order to accomplish this task, we first need to recall a classical result of the theory of differential equations in $\mathbb{R}^n$.

Theorem 5.2.1 ($\exists$! of the solution of a system of ordinary differential equations in $\mathbb{R}^n$) *Let $U \subseteq R^n$ an open set and let $(x^1, \ldots, x^n) : U \to \mathbb{R}$ be smooth functions. Then:*

- $\boxed{\exists}$ *: for all $t_0 \in \mathbb{R}$ and $x_0 \in U$ there exists $\delta > 0$ and an open subset $U_0 \subset U$, with $x_0 \in U_0$, such that, for all $x \in U_0$, there exists a curve $\gamma_x : (t - \delta_0, t + \delta_0) \to U$ which solves the following Cauchy problem:*

$$\begin{cases} \frac{d\gamma^j(t)}{dt} = x^j(\gamma(t)), & j = 1, \ldots, n \\ \gamma(t_0) = x_0. \end{cases} \tag{5.1}$$

- $\boxed{\textit{Smooth dependence on initial data}}$ *: the function*

$$\begin{array}{rccc} \Theta : & (t_0 - \delta, t_0 + \delta) \times U_0 & \longrightarrow & U \\ & (t, x) & \longmapsto & \Theta(t, x) = \gamma_x(t), \end{array}$$

  is smooth, i.e. $\gamma_x(t)$ is smooth w.r.t. $t \in (t_0 - \delta, t_0 + \delta)$ and γ_x is smooth in $x \in U_0$.

- $\boxed{!}$ *: two solutions of the Cauchy problem always coincide in the intersection of their domains.*

Since this result holds locally, we can imagine that it is possible to extend it to manifolds. This is indeed the case and to prove it we must introduce a suitable terminology.

Def. 5.2.1 (Integral curve of a vector field) *Given a smooth manifold M, let us consider:*

- $X \in \mathfrak{X}(M)$
- $p \in M$
- $I \subset \mathbb{R}$ *open and such that* $0 \in I$
- $\gamma : I \to M$ *smooth.*

Then γ is the integral curve of the vector field X passing through p if:

$$\begin{cases} \gamma'(t) = X(\gamma(t)), & \forall t \in I \\ \gamma(0) = p. \end{cases}$$

Geometrically, the fact that γ is the integral curve of X means that the tangent vector $\gamma'(t)$ to M at each element of its support $\{\gamma(t),\ t \in I\} \subset M$ coincides with the tangent vector assigned by the vector field X in the point $\gamma(t)$.

We can transform locally the search for integral curves of a vector field in the situation considered in theorem 5.2.1, as formalized in the following result.

Theorem 5.2.2 ($\exists$! of the integral curves of a vector field on a manifold) *Let $X \in \mathfrak{X}(M)$, $p \in M$ and $(U, \varphi = (x^j)_{j=1,\dots,n})$ a local chart in p. Then the assertions of theorem 5.2.1 holds if we replace the Cauchy problem (5.1) with the following:*

$$\begin{cases} \frac{d\tilde{\gamma}^j(t)}{dt} = X^j(\tilde{\gamma}(t)), \qquad j = 1, \dots, n \\ \tilde{\gamma}(t_0) = \varphi(p) \in \mathbb{R}^n, \end{cases} \tag{5.2}$$

where $\tilde{\gamma} : \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ and $X = X^j \partial_j$ is the decomposition of X induced by the local coordinates $(x^1, \dots, x^n)$.

Proof. The proof consists in composing γ with φ to get a curve $\tilde{\gamma} : \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with ε small enough so that $\gamma(-\varepsilon, \varepsilon) \subset U$. If we write its components as $(\tilde{\gamma}^1, \dots, \tilde{\gamma}^n)$, then

$$\tilde{\gamma}'(t) = (\tilde{\gamma}^j)'(t)\ \partial_j|_{\tilde{\gamma}(t)},$$

where $\partial_j|_{\tilde{\gamma}(t)} \in T_{\tilde{\gamma}(t)}\mathbb{R}^n \cong \mathbb{R}^n$.

It is now clear that γ is an integral curve of X if and only if $\tilde{\gamma}$ is a solution of the Cauchy problem in $\mathbb{R}^n$ written in (5.2) and so the theorem 5.2.1 can be applied. $\square$

Let us now use this result to prove a very powerful theorem, fundamental for differential geometry of smooth manifolds.

Theorem 5.2.3 (The flux theorem) *Let M be a manifold. For every $X \in \mathfrak{X}(M)$ there exists a unique open neighborhood $\mathcal{U}$ of $\{0\} \times M$ in $\mathbb{R} \times M$ and a unique smooth function $\Theta : \mathcal{U} \to M$ such that the following assertions hold.*

1. *For all* <u>*fixed*</u> $p \in M$, *the set*
$$\mathcal{U}^p := \{t \in \mathbb{R} \ : \ (t,p) \in \mathcal{U}\} \subseteq \mathbb{R}$$
is an open interval containing 0.

2. *For all* <u>*fixed*</u> $p \in M$, *the curve*
$$\begin{array}{rccc} \vartheta^p : & \mathcal{U}^p & \longrightarrow & M \\ & t & \longmapsto & \vartheta^p(t) = \Theta(t,p) \end{array}$$
is the only maximal integral curve of X passing through p (i.e. it cannot be extended to a larger domain remaining an integral curve of X).

3. *For all* <u>*fixed*</u> $t \in \mathbb{R}$, *the set*
$$\mathcal{U}_t := \{p \in M \ : \ (t,p) \in \mathcal{U}\} \subseteq M$$
is an open subset of M.

4. *For all* <u>*fixed*</u> $t \in \mathbb{R}$, *the curve*
$$\begin{array}{rccc} \vartheta_t : & \mathcal{U}_t & \longrightarrow & M \\ & p & \longmapsto & \vartheta_t(t) = \Theta(t,p) \end{array}$$
is a diffeomorphism such that $\boxed{\vartheta_t^{-1} = \vartheta_{-t}}$ *and* $\boxed{\vartheta_0 = id_{\mathcal{U}_t}}$. *Moreover, if $p \in \mathcal{U}_t$, then $p \in \mathcal{U}_{t+s}$ if and only if $\Theta(t,p) \in \mathcal{U}_s$ and in this case it holds that*
$$\boxed{\vartheta_s(\vartheta_t(p)) = \vartheta_{s+t}(p)}\ .$$

5. *For all $f \in \mathscr{C}^\infty(M)$ and all $p \in M$:*
$$\boxed{\frac{d}{dt}(f \circ \vartheta^p)\bigg|_{t=0} = X(f)(p)}\ .$$

6. *For all $(t,p) \in \mathcal{U}$:*
$$\boxed{d(\vartheta_t)_p(X_p) = X_{\vartheta_t(p)}}\ .$$

Before going through the details of the proof, let us remark that, **when we write $\vartheta_s(\vartheta_t(p))$ we are considering two different integral curves of X:** from p, we first follow the integral curve of X passing through p for a time t and we stop when we arrive at the point $\vartheta_t(p) \in M$. From here, we continue by following the integral curve of X passing through $\vartheta_t(p)$ for a time s and we stop when we arrive at the point $\vartheta_s(\vartheta_t(p)) \in M$.

This is the reason why the equality $\vartheta_s(\vartheta_t(p)) = \vartheta_{s+t}(p)$ is so strong: it says that, with the procedure just described, we arrive exactly to the same point as if we followed just the integral curve of X passing from p for a time $s + t$.
Proof.

1. Theorem 5.2.2 implies that, for all $p \in M$, there exists always an integral curve of X passing through it and that two integral curves of X passing through p coincide in the intersection of their domains. This allows us to simply define $\mathcal{U}^p$ as **the union of all the open intervals** $I \subseteq \mathbb{R}$ **containing** 0 **on which an integral curve** $\gamma : I \to M$ **of** X **passing through** p **is defined**. Being the union of open sets, $\mathcal{U}^p$ is open.

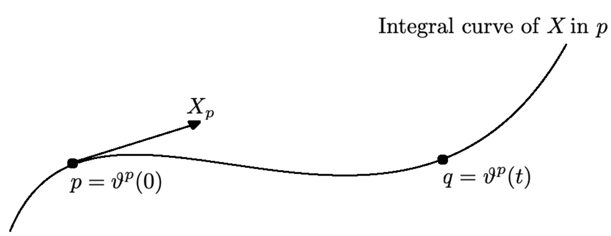
2. The previous argument implies that it exists $\vartheta^p : \mathcal{U}^p \to M$, integral curve of X passing through p and defined on the whole $\mathcal{U}^p$. This is the maximal integral curve and it is unique as a consequence of the unicity of the solution of the Cauchy problem (5.2). Moreover, this, together with the fact that we know how to construct $\mathcal{U}^p$, allow us to build $\mathcal{U}$ and Θ:

$$\mathcal{U} := \{(t, p) \in \mathbb{R} \times M \ : \ t \in \mathcal{U}^p\}, \quad \Theta : \mathcal{U} \to M, \ \Theta(t, p) := \vartheta^p(t).$$

Notice that this definition of $\mathcal{U}$ does not imply immediately that it is open, we will prove it later.

3. To prove that $\mathcal{U}$ is open and that Θ is smooth a quite technical use of theorem 5.2.2 must be performed, together with the result in 4. We skip these details and pass directly to the more interesting proof of 4.

4. The proof of 3. will be simpler if we first deal with the point 4. By definition $\mathcal{U}_0 = M$ and $\vartheta_0 = id_M$. Let $p \in M$ and $t \in \mathcal{U}^p$, we write $q = \vartheta^p(t)$, as represented in the figure below.



It is useful to perform a re-parameterization of ϑ^p as follows: let us define

$$\mathcal{U}^p - t := \{s \in \mathbb{R} \ : \ s + t \in \mathcal{U}^p\}$$

then, the curve

$$\begin{array}{rccl} \sigma: & \mathcal{U}^p - t & \longrightarrow & M \\ & s & \longmapsto & \sigma(s) = \vartheta^p(s+t) \end{array}$$

is still an integral curve of X because

$$\sigma'(s) = \frac{d\vartheta^p}{ds}(s + t) = X(\vartheta^p(s + t)) =: X(\sigma(s))$$

and σ passes through q, in fact $\sigma(0) = \vartheta^p(t) = q$.

We observe now that, by unicity of the integral curve, $\sigma(s) = \vartheta^q(s)$, so it must be:

$$\vartheta^{\vartheta^p(t)}(s) = \vartheta^q(s) = \sigma(s) =: \vartheta^p(s+t),$$

i.e. $\Theta(s, \Theta(t,p)) = \Theta(s+t, p)$ or $\vartheta_{s+t}(p) = \vartheta_s(\vartheta_t(p))$. Moreover, $\mathcal{U}^p - t \subseteq \mathcal{U}^q$.

Since $0 \in \mathcal{U}^p$, $0 - t = -t \in \mathcal{U}^q$, but $\mathcal{U}^q$ is the domain of ϑ^q, so the fact that $-t \in \mathcal{U}^q$ means that $\vartheta^q(-t) = p$. This formalizes the fact that, if we are placed in q, then we can turn back to p by 'reversing the time' of the quantity t.

If we interchange the couple $(-t, q)$ with (t, p) we get that $\mathcal{U}^q + t \subseteq \mathcal{U}^q$, thus it holds that $\mathcal{U}^q - t = \mathcal{U}^q = \mathcal{U}^{\Theta(t,p)}$. But then $q = \Theta(t,p) \in \mathcal{U}_s$ if and only if $p \in \mathcal{U}_{s+t}$, which concludes the proof of 4.

5. Since $\vartheta^p(0) = p$ and $(\vartheta^p)'(0) = X_p$, thanks to the definition of differential we have:

$$X(f)(p) = df_p(X_p) = \left.\frac{d}{dt}(f \circ \vartheta^p(t))\right|_{t=0}.$$

6. Let $(t_0, p_0) \in \mathcal{U}$ and $f \in \mathscr{C}^\infty(M)$, then:

$$\begin{aligned} d(\vartheta_{t_0})_{p_0}(X_{p_0})(f) &= X_{p_0}(f \circ \vartheta_{t_0}) \qquad \text{(def. of differential)} \\ &= \left.\frac{d}{dt}(f \circ \vartheta_{t_0} \circ \vartheta^{p_0}(t))\right|_{t=0} \qquad \text{(by using 5.)} \\ &= \left.\frac{d}{dt} f(\vartheta_{t_0+t})\right|_{t=0} = \left.\frac{d}{dt} f(\vartheta^{p_0}(t_0+t))\right|_{t=0} \\ &= X_{\vartheta^{p_0}(t_0)}(f) = X_{\vartheta_{t_0}(p_0)}(f), \end{aligned}$$

since the result hold for all $f \in \mathscr{C}^\infty(M)$, we have $d(\vartheta_t)_p(X_p) = X_{\vartheta_t(p)}$. □

The function Θ contains the information about all the integral curves of X passing through all the points of M. For this reason it deserves a special name and characterize certain special vector fields.

Def. 5.2.2 (Flux) *For every $X \in \mathfrak{X}(M)$, the function $\Theta : \mathcal{U} \subseteq \mathbb{R} \times M \to M$ is called the local flux of the vector field X.*

Def. 5.2.3 (Complete vector field) *$X \in \mathfrak{X}(M)$ is called complete if $\mathcal{U} = \mathbb{R} \times M \to M$, i.e. if all the integral curves of X are defined for all $t \in \mathbb{R}$.*

Def. 5.2.4 (X-invariant vector field) *Let $X, Y \in \mathfrak{X}(M)$, and Θ the local flux of X. Y is said to be X-invariant if, for all (t,p) belonging to the domain of the local flux of X, we have:*

$$d(\vartheta_t)_p(Y_p) = Y_{\vartheta_t(p)}, \qquad \vartheta_t(p) = \Theta(t,p).$$

Let us interpret this last definition: for all $p \in M$ we can evaluate Y in p, obtaining Y_p, a tangent vector to M. We then move along the integral curve of X for a time t, until arriving to the point $q = \vartheta_t(p)$. We can compare the tangent vector $Y_{\vartheta_t(p)}$ with the one that we obtain by applying the differential map to $\vartheta_t(p)$ calculated in Y_p, i.e. $d(\vartheta_t)_p(Y_p)$, which is a tangent vector to M at q, so it belongs to the same tangent space as $Y_{\vartheta_t(p)}$ and the comparison is meaningful. If it happens that these two tangent vectors are the same, then Y is X-invariant[2].

Thanks to the property 6. of the flux theorem, X **is** X**-invariant**.

[2]more synthetically: computing the tangent vector Y at $\vartheta_t(p)$ is the same as sending the tangent vector Y_p to $Y_{\vartheta_t(p)}$ via differential along the integral curve of X passing through p.

5.3 The Lie bracket

As previously said, the concept of Lie bracket shows the usefulness of interpreting vector fields as derivations on the ring of smooth scalar functions. In fact, if $X, Y \in \mathfrak{X}(M)$ are interpreted as derivations, i.e. $X, Y : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ are linear Leibniz-like operators, then they can be composed to get two new linear operators on $\mathfrak{X}(M)$, namely $X \circ Y$ and $Y \circ X$, this is a privilege that we do not have if we interpret X, Y as sections of TM. Linearity is obviously preserved by composition, however the Leibniz-like behavior is not, in fact, by using first the Leibniz behavior of Y and then of X we get:

$$(X \circ Y)(fg) = X(Y(fg)) = X(fY(g)+Y(f)g) = fX(Y(g))+X(f)Y(g)+X(g)Y(f)+X(Y(f))g,$$

this is different than $fX(Y(g)) + X(Y(F))g$, which is what we would expect from an hypothetical Leibniz-like behavior of $X \circ Y$. In fact, if we consider the geometrical meaning of the two intermediate terms of $X \circ Y$, we see that they act as a second-order differential operators, thus making, globally, $X \circ Y$ a second-order differential operator, instead of a first-order one, as it should be, a vector field is associated to the first order Cauchy problem (5.2).

Nonetheless, the intermediate terms of $X \circ Y$ are symmetrical w.r.t. the exchange of X with Y, thus, if we compute $Y \circ X$ and we subtract it from $X \circ Y$, we erase these spurious terms and we remain with a derivation. These considerations justify the following definition.

Def. 5.3.1 (Lie bracket) *Given $X, Y \in \mathfrak{X}(M)$, their Lie bracket is the vector field $[X, Y] \in \mathfrak{X}(M)$ defined by:*

$$\boxed{[X, Y] := X \circ Y - Y \circ X}\ .$$

*X, Y are said to **commute** if $[X, Y] = 0$, the null vector field.*

The properties of the Lie bracket of vector fields are listed below.

Theorem 5.3.1 *Let $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in \mathscr{C}^\infty(M)$ and $a, b \in \mathbb{R}$, then the following properties hold.*

1. $[Y, X] = -[X, Y]$.

2. $[aX + bY, z] = a[X, Z] + b[Y, Z]$ *and* $[Z, aX + bY] = a[Z, X] + b[Z, Y]$.

3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

4. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$, *in particular, of* $f \equiv 1$, $[X, gY] = g[X, Y] + X(g)Y$, *i.e.*

$$\begin{array}{rccc} [X, \cdot] : & \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & Y & \longmapsto & [X, Y], \end{array}$$

   is a derivation on $\mathfrak{X}(M)$.

5. *If $X = X^h\partial_h$ and $Y = Y^k\partial_k$ are the representations of X and Y in a local coordinate system, then the local coordinate expression for the Lie bracket $[X, Y]$ is the following:*

$$\boxed{[X, Y] = (X^h\partial_h Y^k - Y^h\partial_h X^k)\partial_k}.$$

   In particular, $[\partial_h, \partial_k] = 0$, as a consequence of Schwarz's theorem.

Proof.

1. and 2. Direct computation.

3. We have:

$$[X,[Y,Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX,$$
$$[Y,[Z,X]] = [Y, ZX - XZ] = YZX - YXZ - ZXY + XZY,$$
$$[Z,[X,Y]] = [Z, XY - YX] = ZXY - ZYX - XYZ + YXZ,$$

summing the left hand sides and the rightmost hand sides we get 0.

4. We have:

$$\begin{aligned}[fX, gY] &= fX(gY) - gY(fX) = fX(g)Y + fgXY - gY(f)X - gfYX \\ &= fg(XY - YX) + fX(g)Y - gY(f)X = fg[X,Y] + fX(g)Y - gY(f)X.\end{aligned}$$

5. We have:

$$\begin{aligned}[X,Y] &= [X^h\partial_h, Y^k\partial_k] \\ &\quad \text{(by linearity)} \\ &= X^h\partial_h(Y^k\partial_k) - Y^k\partial_k(X^h\partial_h) \\ &\quad \text{(applying the Leibniz rule for the action of the partial derivatives } \partial_h \text{ and } \partial_k\text{)} \\ &= X^h(\partial_h Y^k)\partial_k + X^hY^k\partial_h\partial_k - Y^k(\partial_k X^h)\partial_h - Y^kX^h\partial_k\partial_h \\ &\quad \text{(by Schwarz's theorem for second order partial derivatives)} \\ &= X^h(\partial_h Y^k)\partial_k + \cancel{X^hY^k\partial^2_{hk}} - \cancel{Y^kX^h\partial^2_{kh}} - Y^k(\partial_k X^h)\partial_h \\ &= X^h(\partial_h Y^k)\partial_k - Y^k(\partial_k X^h)\partial_h \\ &\quad \text{(by exchanging } h \text{ with } k \text{ in the second term)} \\ &= X^h(\partial_h Y^k)\partial_k - Y^h(\partial_h X^k)\partial_k \\ &= (X^h\partial_h Y^k - Y^h\partial_h X^k)\partial_k.\end{aligned}$$

□

Thanks to the properties just proven, the vector space of all vector fields on M endowed with the Lie bracket, i.e. $(\mathfrak{X}(M), [\ ,\ ])$ **is a Lie algebra**, as it is clear from the definition that follows.

Def. 5.3.2 (Lie algebra) *A vector space $\mathfrak{a}$ over a field $\mathbb{K}$ is a Lie algebra*[3] *if there exists a binary operation $[\ ,\ ] : \mathfrak{a} \to \mathfrak{a}$, called Lie bracket, that satisfies the following properties for all $a, b \in \mathbb{K}$ and all $x, y, z \in \mathfrak{a}$:*

1. *Anti-symmetry:* $[y, x] = -[x, y]$
2. *Bilinearity:* $[ax + by, z] = a[x, z] + b[y, z]$ *and* $[z, ax + by] = a[z, x] + b[z, y]$
3. *Jacoby identify:* $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$

By anti-symmetry it follows that $[x, x] = 0$ for all $x \in \mathfrak{a}$.

[3] A lower case fraktur letter is usually used to denote a Lie algebra.

5.4 The Lie derivative

The Lie derivative allows us to define the concept of **derivative w.r.t. a vector field** on a manifold. As we will see in chapter 7, this is *not exactly* the perfect generalization of the concept of directional derivative in $\mathbb{R}^n$ to abstract manifolds.

As always, let us first analyze the trivial case of $M = \mathbb{R}^n$. In this situation the tangent spaces to each point of M are canonically identified with $\mathbb{R}^n$, so a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ is simply a section of $T\mathbb{R}^n \cong \mathbb{R}^{2n}$, i.e. a smooth map $X : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, where U is an open neighborhood of $p \in \mathbb{R}^n$ and, as usual, we have made use of the identification $T_p\mathbb{R}^n \cong \mathbb{R}^n$. Thus the derivative of another vector field $Y \in \mathfrak{X}(\mathbb{R}^n)$ along an integral curve of X in p can be simply reduced to the directional derivative of Y in p along the direction given by the tangent vector $X_p \in \mathbb{R}^n$.

However, it is immediate to understand that these considerations do not work anymore in a non-trivial manifold M. Consider, in fact, the situation depicted in the figure 5.1.

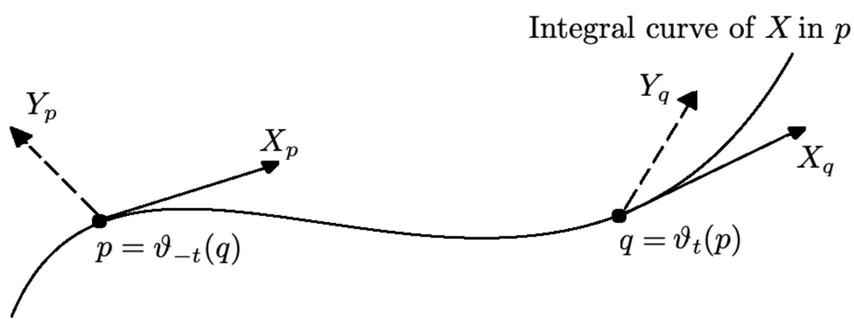


Figure 5.1: Comparing tangent vectors at different points of an integral curve.

If we want to estimate the rate of change of the vector field Y when we pass from $p = \vartheta_0(p)$ to $q = \vartheta_t(p)$, where $\vartheta_t(p) = \Theta(t,p)$, Θ being the local flux of X, then we should compute the quantity:

$$\lim_{t\to 0} \frac{Y_{q=\vartheta_t(p)} - Y_{p=\vartheta_0(p)}}{t},$$

but $Y_q \in T_qM$ and $Y_p \in T_pM$, thus the comparison $Y_q - Y_p$ is ill-posed because the two vectors live in different vector spaces!

The solution to this problem is to take back Y_q to the vector space T_pM along the integral curve of X. Notice that

$$\begin{array}{rccl} \vartheta_{-t}: & M & \longrightarrow & M \\ & q & \longmapsto & \vartheta_{-t}(q) = p, \end{array}$$

so, we clearly have to apply the differential to ϑ_{-t} to move the tangent vectors to the integral curve of X at q to bring them back to tangent vectors to the integral curve of X at p, i.e.

$$\begin{array}{rccl} d(\vartheta_{-t})_q: & T_qM & \longrightarrow & T_{\vartheta_{-t}(q)=p}M \\ & Y_q & \longmapsto & d(\vartheta_{-t})_q(Y_q), \end{array}$$

since $Y \circ \Theta$ is smooth, the function $t \mapsto d(\vartheta_{-t})_{\vartheta_t(p)}(Y_{\vartheta_t(p)})$ is a smooth curve in T_pM that depends smoothly on p. Notice that, in general, $d(\vartheta_{-t})_q(Y_q)$ will be different than Y_p, as depicted in figure 5.2, thus the difference between these two tangent vectors will be different than the null vector of T_pM. We can now formalize the concept of Lie derivative as follows.

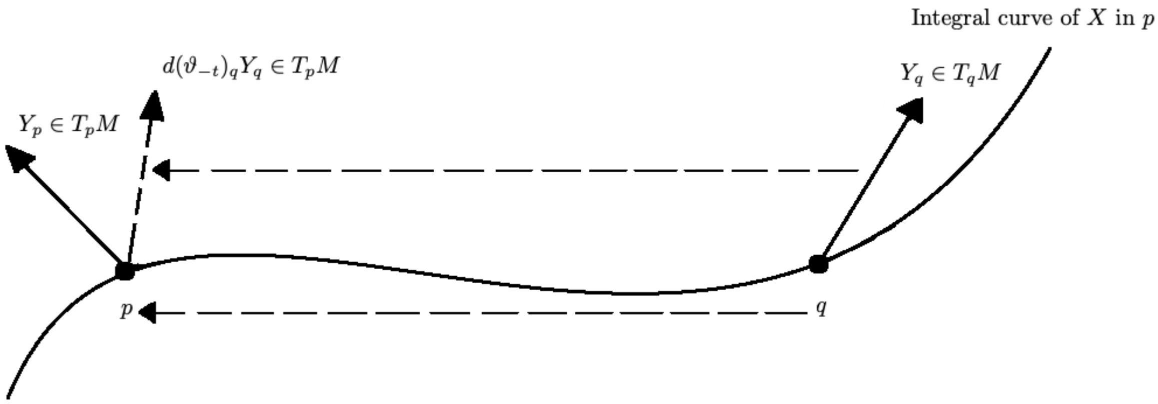


Figure 5.2: Construction of the Lie derivative of a vector field.

Def. 5.4.1 (Lie derivative of a vector field) *The Lie derivative of the vector field $Y \in \mathfrak{X}(M)$ along the vector field $X \in \mathfrak{X}(M)$ is the linear operator:*

$$\begin{array}{rccc} \pounds_X : & \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & Y & \longmapsto & \pounds_X Y, \end{array}$$

where

$$\pounds_X Y(p) := \lim_{t\to 0} \frac{d(\vartheta_{-t})_{\vartheta_t(p)}(Y_{\vartheta_t(p)}) - Y_p}{t} \equiv \frac{d}{dt}\left(d(\vartheta_{-t})_{\vartheta_t(p)}(Y_{\vartheta_t(p)})\right)\Big|_{t=0}. \tag{5.3}$$

which is called the Lie derivative of Y along X in the point $p \in M$.

It is clear that, **if Y is a X-invariant vector field, then** $\pounds_X Y = 0$, the null vector field.

Formula (5.3) is clearly not easy to handle, which is why mathematicians searched for a simpler expression, the result is surprising: thanks to the properties of the flux of vector fields, it can be proven that the Lie derivative is simply the Lie bracket!

Theorem 5.4.1 *For all $X, Y \in \mathfrak{X}(M)$, it holds that*

$$\boxed{\pounds_X Y = [X, Y]}\,.$$

The link between the Lie derivative and bracket shows that this latter hides a geometrical meaning that we investigate in the following subsection.

It is possible to generalize the concept of Lie derivative also to arbitrary tensor fields.

Def. 5.4.2 (Lie derivatives of scalar functions) *Given $X \in Der(\mathscr{C}^\infty) \cong \mathfrak{X}(M)$, the Lie derivative of a rank-0 tensors, i.e. a scalar fields $\phi \in \mathscr{C}^\infty(M) \equiv T^0_0(M)$, along X is the linear operator:*

$$\begin{array}{rccc} \pounds_X : & \mathscr{C}^\infty(M) & \longrightarrow & \mathscr{C}^\infty(M) \\ & \phi & \longmapsto & \boxed{\pounds_X \phi = X(\phi)}. \end{array}$$

Let us now pass to 1-forms $\omega \in \Omega^1(M) \equiv T_1^0(M)$: given a vector field $Y \in \mathfrak{X}(M) \equiv T_0^1(M)$, we can build a scalar field simply by applying ω to Y, in fact

$$\begin{array}{rccl} \omega(Y): & M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & \omega(Y)(p) := \omega_p(Y_p) \end{array}$$

is perfectly defined because $\omega_p \in T_p^*M$ and $Y_p \in T_pM$. In local coordinates, if $\omega = \omega_i dx^i$ and $Y = Y^j\partial_j$, with $\omega_i : M \to \mathbb{R}$ and $Y^j : M \to \mathbb{R}$ smooth coefficient functions, then

$$\omega(Y) = \omega_i Y^i,$$

in fact $\omega(Y) = \omega_i dx^i(Y^j\partial_j) = \omega_i Y^j dx^i(\partial_j) = \omega_i Y^j \delta^i_j = \omega_i Y^i$.

Let us impose that $\pounds_X(Y)$ verifies the Leibniz rule:

$$\pounds_X(\omega(Y)) = (\pounds_X\omega)Y + \omega(\pounds_X Y),$$

so $(\pounds_X\omega)Y = \pounds_X(\omega(Y)) - \omega(\pounds_X Y)$, but we already know how the Lie derivative is defined for scalar and vector fields, i.e. $\pounds_X(\omega(Y)) = X(\omega(Y))$ and $\pounds_X Y = [X, Y]$, respectively, thus we get:

$$(\pounds_X\omega)Y = X(\omega(Y)) - \omega([X, Y]).$$

This simple computation explains the definition of the Lie derivative of a 1-form as follows.

Def. 5.4.3 (Lie derivatives of a 1-form) *The Lie derivative of a 1-form $\omega \in \Omega^1(M) \equiv T_0^1(M)$ along X is the linear operator:*

$$\begin{array}{rccl} \pounds_X: & \Omega^1(M) & \longrightarrow & \Omega^1(M) \\ & \omega & \longmapsto & \pounds_X\omega, \end{array}$$

$$\begin{array}{rccl} \pounds_X\omega: & TM & \longrightarrow & \mathbb{R} \\ & Y & \longmapsto & \boxed{\pounds_X\omega(Y) := X(\omega(Y)) - \omega([X, Y])} \,. \end{array}$$

The general case is given by a tensor field $t \in T_q^p(M)$: if $Y_1, \ldots, Y_q \in T_0^1(M) = TM$ and $\omega_1, \ldots, \omega_p \in T_1^0(M) = \Omega^1(M)$, then $t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p) \in \mathscr{C}^\infty(M)$ and so:

$$\pounds_X(t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)) = X(t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)),$$

thus, to define the Lie derivative of t, we must impose, as before, the Leibniz behavior and solve for $\pounds_X t$:

$$\begin{aligned} \pounds_X(t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)) = {} & (\pounds_X t)(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p) + \\ & + t([X, Y_1], \ldots, Y_q, \omega_1, \ldots, \omega_p) + \ldots \\ & + t(Y_1, \ldots, Y_{q-1}, [X, Y_q], \omega_1, \ldots, \omega_p) \\ & + t(Y_1, \ldots, Y_q, \pounds_X\omega_1, \omega_2, \ldots, \omega_p) + \ldots \\ & + t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_{p-1}, \pounds_X\omega_p), \end{aligned}$$

i.e. the Lie derivative of the tensor field $t \in T_q^p(M)$ along $X \in \mathfrak{X}(M)$ is:

$$\begin{aligned} (\pounds_X t)(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p) := {} & X(t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)) \\ & - t([X, Y_1], \ldots, Y_q, \omega_1, \ldots, \omega_p) + \ldots \\ & - t(Y_1, \ldots, Y_{q-1}, [X, Y_q], \omega_1, \ldots, \omega_p) \\ & - t(Y_1, \ldots, Y_q, \pounds_X\omega_1, \omega_2, \ldots, \omega_p) + \ldots \\ & - t(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_{p-1}, \pounds_X\omega_p). \end{aligned}$$

5.4.1 Geometrical features of the Lie bracket

Given two vector fields $X, Y \in \mathfrak{X}(M)$, figure 5.3 depicts the following path:

- we start from $p \in M$ and we follow the integral curve <u>of X</u> passing through p for an amount of 'time' measured by the value h of the parameter t, arriving in q;

- we restart from q, but now we follow the integral curve <u>of Y</u> passing through q for the same amount of time h, arriving in r;

- we restart from r, following the integral curve <u>of X</u> passing through r for an amount of time $-h$, arriving in s;

- finally, from s, we follow the integral curve <u>of Y</u> passing through s for an amount of time $-h$, arriving in the point T that we indicate as $\gamma(h)$.

The curve $h \mapsto \gamma(h)$ is smooth and such that $\gamma(0) = p$.

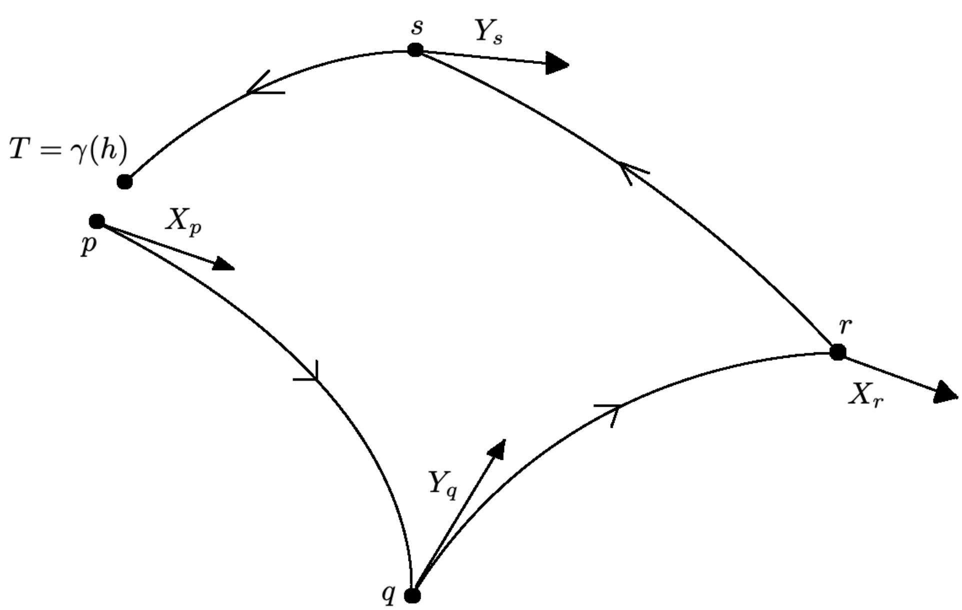


Figure 5.3: Geometrical interpretation of the Lie bracket.

In [18] we can find the proof of the following result.

Theorem 5.4.2 *With the notations above, it holds that:*

1. $\gamma'(0) = 0$, *i.e. at first order, the quadrilateral depicted in figure 5.3 is closed;*

2. $\gamma''(0) = 2[X, Y]$, *i.e. at second order, the obstruction to the closure of the quadrilateral depicted in figure 5.3 is measured by the Lie bracket between X and Y.*

5.4.2 Pushforward of a vector field by a diffeomorphism

The concept of Lie bracket (and so of Lie derivative) can be related to an operation that relates vector fields between manifolds: the pushforward.

To introduce this operation, consider a smooth map $f : M \to N$ and a vector field X on M, then, for each $p \in M$, $df_p : T_pM \to T_{f(p)}N$, but $X_p \in T_pM$, thus $df_p(X_p)$ is a tangent vector to N at $q = f(p)$.

However, depending on the properties of f, this technique may fail to defines a vector field Y on N. In fact, if f is not surjective, there is no rule to assign a tangent vector to N at the points $q \in N \backslash f(M)$. On the other hand, if f is not injective, then there are at least two distinct points $p_1, p_2 \in M$ such that $f(p_1) = f(p_2) = q \in N$, in this case $Y^1_{p_1} := df_{p_1}(X_{p_1})$ and $Y^2_{p_2} := df_{p_2}(X_{p_2})$ would be two possibly different tangent vectors to N at the same point q, thus creating an ambiguity in the assignment for the hypothetical vector field on N that we would like to create via f and X.

These considerations motivate why we can push a vector field forward to another manifold by means of a map if and only if the map is a diffeomorphism.

Def. 5.4.4 (Pushforward of a vector field) *Let M, N be to manifolds, $X \in \mathfrak{X}(M)$ and $f : M \to N$ a diffeomorphism. We call pushforward induced by f the linear map $f_* \equiv df$ defined as follows:*

$$\begin{array}{rccc} f_* \equiv df : & \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(N) \\ & X & \longmapsto & f_*(X) \equiv df, \end{array}$$

where, for all $q \in N$, $f_*(X)(q) \equiv f_*(X)_q$, or $df(X)(q) \equiv df(X)_q$, is defined as follows:

$$f_*(X)_q = df_{f^{-1}(q)}(X_{f^{-1}(q)}) \qquad or \qquad df(X)_q = df_{f^{-1}(q)}(X_{f^{-1}(q)}).$$

Again, we underline that the need of a diffeomorphism is clear from the definition: thanks to this, we can bring back any point $q \in N$ to a point $p = f^{-1}(q) \in M$ and use the vector field $X \in \mathfrak{X}(M)$.

If $f : M \to N$ is only a smooth map and not a diffeomorphism, then it is impossible to define the push-forward of vector fields, however, it is still possible to correlate them, in the sens formalized below.

Def. 5.4.5 (f-related vector fields) *Let $f : M \to N$ be a smooth function between manifolds, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. X and Y are f-related if, for all $p \in M$*

$$Y_{f(p)} = df_p(X_p),$$

i.e. if the tangent vectors determined by Y in the points of N belonging to the range of f coincide with the tangent vectors determined by the differential map of f applied to the tangent vectors determined by X in the points of M.

If f is a diffeomorphism, it is easy to see that $Y = f_*(X)$ is the only vector field on N f-related to X, see [10] Proposition 8.19 page 183.

The properties of f-related vector fields are listed in the following result, for the proof see [10] chapter 8.

Theorem 5.4.3 *Let $f : M \to N$ be a smooth function between manifolds, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$.*

1. *Y is f-related to X if and only if, for every $\phi \in \mathscr{C}^\infty(N)$, it holds that*

$$X(\phi \circ f) = Y(\phi) \circ f.$$

2. *If Y_1 is f-related to X_1 and Y_2 is f-correlated to X_2, then $[Y_1, Y_2]$ is f-correlated to $[X_1, X_2]$. In other words, the Lie bracket is compatible with the f-correlation.*

3. *If f is a diffeomorphism, then*

$$[f_*(X_1), f_*(X_2)] = f_*([X_1, X_2]),$$

i.e. the pushforward $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is compatible with the Lie bracket.

These properties are used to solve the following problem: suppose to have a local frame $X_1, \ldots, X_n$ for TM, $\dim(M) = n$, is there a set of conditions to guarantee that it exists a local chart (U, φ) of M such that $X_j = \partial_j$, $j = 1, \ldots, n$ on U?

A necessary condition can be found very easily: since $[\partial_i, \partial_j] = 0$ $\forall i, j = 1, \ldots, n$, it is necessary that $[X_i, X_j] = 0$ $\forall i, j = 1, \ldots, n$. In theorem 5.4.6 we will see that this condition is also sufficient, but to formulate it properly we have to define a new concept and to introduce intermediate results.

Def. 5.4.6 (Regular and singular points of a vector field) *Let $X \in \mathfrak{X}(M)$, a point $p \in M$ is said to be a regular point of the vector field X if $X_p \neq 0$, i.e. if the tangent vector to M assigned by X in p is not null, otherwise, if $X_p = 0$, p is called a singular point for X.*

Theorem 5.4.4 *Let $X \in \mathfrak{X}(M)$ and $p \in M$ a regular point for X. Then, it exists a local chart (U, φ) centered in p, i.e. $\varphi(p) = 0 \in \mathbb{R}^n$, such that:*

$$X|_U = \partial_1,$$

i.e. in an open neighborhood of p, the tangent vectors assigned by X are all parallel.

The following theorem says that if $[X, Y] = 0$, then the quadrilateral depicted in figure 5.3 is closed, not only at the second order, but at every order, i.e. the obstruction to its closure is totally contained in the Lie bracket.

Theorem 5.4.5 *Let $X, Y \in \mathfrak{X}(M)$ with flux $\Theta : \mathcal{U} :\to M$ and $\Psi : \mathcal{V} :\to M$, respectively. Then, the following assertions are equivalent:*

1. *$[X, Y] = 0$*

2. *Y is X-invariant*

3. *X is Y-invariant*

4. *$\psi_s \circ \vartheta_t = \vartheta_t \circ \psi_s$ as long as one of the two is defined, i.e. the fluxes of X and Y commute.*

We can now see that the condition of commuting is necessary and sufficient for linearly independent vector fields X_j in $\mathfrak{X}(M)$ to be written locally as ∂_j

Theorem 5.4.6 *Let $X_1, \ldots, X_k \in \mathfrak{X}(M)$ linearly independent vector fields in every point of M, thus $k \leqslant n = dim(M)$ (if $k = 1$ then $X_p \neq 0$ $\forall p \in M$). Then, the following properties are equivalent:*

1. *for all $p \in M$ it exists a local chart (U, φ) centered in p such that: $X_j|_U = \partial_j$, $\forall j = 1, \ldots, k$.*

2. *$[X_i, Y_j] = 0$ $\forall i, j = 1, \ldots, k$.*

5.5 Foliation of a manifold: distributions and the Frobenius theorem

TO BE WRITTEN...

Chapter 6

Riemannian and pseudo-Riemannian manifolds (Edoardo Provenzi)

We start with the introduction of the fundamental concept of Riemannian and pseudo-Riemannian metric.

6.1 Riemannian and pseudo-Riemannian metrics

A scalar product on a vector space allows us measuring the length of vectors and the angles between them. In differential geometry, the typical vector spaces that we have to deal with are the tangent spaces to each point p of a manifold M. If we assign a scalar product to each tangent space T_pM, i.e.

$$\begin{array}{rccc} g_p: & T_pM \times T_pM & \longrightarrow & \mathbb{R} \\ & (v,w) & \longmapsto & g_p(v,w) \end{array}$$

smoothly w.r.t. changes of $p \in M$, then we fix a so-called Riemannian metric on M.

Since the (real-valued) scalar product is bilinear, symmetric, i.e. $g_p(v,w) = g_p(w,v)$ for all $v, w \in T_pM$ and positive-definite, i.e. $g_p(v,v) \geqslant 0$ for all $v \in T_pM$, with $g_p(v,v) = 0$ if and only if $v = 0$, a Riemannian metric on M is nothing but a positive-definite symmetric tensor field on TM of type $(0,2)$, i.e. 2-covariant, as formalized by the definition below.

Def. 6.1.1 (Riemannian metric and manifold) *A Riemannian metric on a manifold M is a positive-definite tensor field $g \in S^0_2(M)$. A Riemannian manifold is a couple (M,g), where g is a Riemannian metric on M.*

The norm canonically induced by the scalar product g_p on T_pM will be denoted with $\| \, \|_p$:

$$\|v\|_p^2 = g_p(v,v) \qquad \forall v \in T_pM.$$

More generally, as it is required in relativistic theories, we can reduce the requests on g by dropping off the property of being positive, but keeping the fundamental property of **non-degeneracy**, i.e. $g_p(v,w) = 0 \ \forall w \in T_pM$ implies $v = 0$, i.e. the only vector g_p-orthogonal to all the other vectors of T_pM is the 0 vector of T_pM, in this case we get a pseudo-Riemannian metric.

Def. 6.1.2 (pseudo-Riemannian metric and manifold) *A pseudo-Riemannian metric on a manifold M is a non-degenerate tensor field $g \in S_2^0(M)$. A pseudo-Riemannian manifold is a couple (M, g), where g is a pseudo-Riemannian metric on M.*

An important concept related with pseudo-Riemannian metrics is their signature.

Def. 6.1.3 (Signature) *Given a pseudo-Riemannian metric g on a* ***connected*** *manifold M of dimension n, we say that g has signature (r, s), $r + s = n$, if the maximal dimension of a subspace of T_pM where g is:*

- *positive-definite is r;*
- *negative-definite is s.*

The definition is well-posed for connected manifolds because, by an argument of continuity, it can be proven that r and s do not depend on the point $p \in M$.

A particularly important case, that of relativistic theories, is that of signature $(1, n-1)$ or $(n-1, 1)$, in which case one says that g is a **Lorentz metric**, or that g has a **Lorentz signature**.

In a local chart $(U, \varphi = (x^1, \ldots, x^n))$, the metric, being a symmetric tensor field of type $(0, 2)$ can be written as:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu, \qquad g_{\mu\nu} \in \mathscr{C}^\infty(U),$$

where the matrix of functions $(g_{\mu\nu})$ is **symmetric** and positive-definite for a Riemannian metric, and symmetric and of signature (r, s) for a pseudo-Riemannian metric.

Since real symmetric matrices can be diagonalized, $g_{\mu\nu}$ can always put in the diagonal form $g_{\mu\nu} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i is the i-th eigenvalue of $g_{\mu\nu}$.

When $M = \mathbb{R}^n$, the tangent and cotangent bundle are canonically isomorphic and the canonical Euclidean metric induce by the Euclidean scalar product is such that $g_{\mu\nu} = g^{\mu\nu} = I_n$, i.e. the identity matrix of dimension n.

Remark about the notation: by symmetry, we could write $g = g_{\mu\nu} dx^\mu \odot dx^\nu$, where $\odot$ is the symmetric product, or, as it is typically done by physicists, $g = g_{\mu\nu} dx^\mu dx^\nu$, which is justified by the fact that the product is symmetric. Finally, many authors use the so-called Gauss' notation by writing ds^2 instead of g, so that we usually find the following notation for the metric:

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}.$$

By definition, the matrix $g_{\mu\nu}$ is invertible, since, in the Riemannian case, it is positive-definite and, in the pseudo-Riemannian case, it is non-degenerated.[1] The inverse is usually denote as $g^{\mu\nu}$, so that:

$$g^{\mu\gamma} g_{\gamma\nu} = \delta^\mu{}_\nu, \quad g_{\mu\gamma} g^{\gamma\nu} = \delta_\mu{}^\nu.$$

Apart from permitting the computation of the scalar product between tangent vectors, *a (pseudo)-Riemannian metric also allows us to canonically identify the tangent and the cotangent bundle* with the help of the following linear isomorphism:

$$\begin{array}{rccc} \flat_p : & T_pM & \xrightarrow{\sim} & T_p^*M \\ & v & \longmapsto & \flat_p(v), \end{array} \qquad \begin{array}{rccc} \flat_p(v) : & T_pM & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \flat_p(v)(w) := g_p(v, w). \end{array}$$

[1] To avoid specifying if we are discussing a Riemannian or pseudo-Riemannian metric, we will simply write (pseudo-)Riemannian metric by meaning that we can refer to both cases.

Being TM the disjoint union of T_pM when we vary the point $p \in M$, we can define a linear isomorphism $\flat$ of bundles $\flat : TM \xrightarrow{\sim} T^*M$ simply by requiring that $\flat|_{T_pM} = \flat_p$ for all $p \in M$.

Let us search for a local expression of $\flat$: let $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ and let $X = X^h \partial_h$ be a local section of TM, i.e. a local vector field, then the application of $\flat$ to X must give a local section of T^*M, i.e. a local covector field, or a 1-form on M that will be written as $\flat(X) = \alpha_j dx^j$ in local coordinates.

By definition of $\flat$, we have:

$$\flat(X)(\partial_k) = g(X, \partial_k) = (g_{\mu\nu} dx^\mu \otimes dx^\nu)(X^h \partial_h, \partial_k),$$

but $dx^\mu \otimes dx^\nu$ is a symmetric bilinear form, so we can move the coefficients X^h outside and write:

$$\flat(X)(\partial_k) = g_{\mu\nu} X^h (dx^\mu \otimes dx^\nu)(\partial_h, \partial_k).$$

Now, by definition of tensor product of two linear forms (cfr. (4.1)), we have that $(dx^\mu \otimes dx^\nu)(\partial_h, \partial_k) = dx^\mu(\partial_h) dx^\nu(\partial_k) = \delta^\mu{}_h \delta^\nu{}_k$, so:

$$\flat(X)(\partial_k) = g_{\mu\nu} X^h \delta^\mu{}_h \delta^\nu{}_k = g_{hk} X^h.$$

However, we also have:

$$\flat(X)(\partial_k) = \alpha_j dx^j(\partial_k) = \alpha_j \delta^j{}_k = \alpha_k,$$

thus $\alpha_k = g_{hk} X^h$, so, finally:

$$\boxed{\flat(X^h \partial_h) = g_{hk} X^h dx^k}\ .$$

Since the basis are the (fixed) standard basis of the tangent ant the cotangent bundle, it is custom to omit them and write simply the components, i.e.

$$\flat(X^h) = g_{hk} X^h = g_{kh} X^h,$$

by symmetry.

In conclusion, we can write:

$$\begin{array}{rccl} \flat: & TM & \xrightarrow{\sim} & T^*M \\ & (X^h) & \longmapsto & \flat(X^h) = (\alpha_k) = g_{kh} X^h. \end{array}$$

This formula explains why it is custom to say that $\flat$ is the isomorphism which transforms the components (X^h) of a local vector field to the components (α_k) of a local 1-form by '**lowering the indices with the metric tensor**'. The symbol $\flat$ ('flat' or 'bemolle') is chosen because in music it lowers in pitch by one semitone.

Analogously, the inverse isomorphism $\flat^{-1} \equiv \sharp : T^*M \xrightarrow{\sim} TM$ act like this:

$$\begin{array}{rccl} \sharp: & T^*M & \xrightarrow{\sim} & TM \\ & (\alpha_k) & \longmapsto & \sharp(\alpha_k) = (X^h) g^{hk} \alpha_k. \end{array}$$

It is custom to say that $\sharp$ is the isomorphism which transforms the components (α_k) of a local 1-form to the components (X^h) of a vector field by '**raising the indices by using the inverse metric tensor**', since $(X_h = g^{hk} \alpha_k)$. Again, the symbol $\sharp$ ('sharp' or 'diesis') is because in music it highers in pitch by one semitone.

Summarizing, in the presence of a (pseudo-)Riemannian metric we can transform vector fields to 1-forms and vice-versa, simply by applying the metric tensor and its inverse, respectively.

6.1.1 Noticeable example 1: the gradient of a scalar function

Let us apply the isomorphism $\sharp : T^*M \to TM$ to the differential of a smooth scalar function $\phi \in \mathscr{C}^\infty(M)$. We know that $d\phi$ is a section of T^*M, i.e. a 1-form, so, if we apply $\sharp$ to $d\phi$, we obtain a vector field, which turns out to be the generalization of the gradient to manifolds.

Def. 6.1.4 (Gradient of a scalar function) *Given a scalar function $\phi \in \mathscr{C}^\infty(M)$, its gradient* $\operatorname{grad}(\phi) \in \mathfrak{X}(M)$ *is the vector field defined by:*

$$\operatorname{grad}(\phi) := \sharp(d\phi).$$

In local coordinates, if $d\phi = (\partial_j \phi)\, dx^j$, then the action of $\sharp$ on the components is as follows: $\sharp(\partial_j \phi) = g^{ij}\,(\partial_j \phi)$, so that

$$\operatorname{grad}(\phi) = g^{ij}\,(\partial_j \phi)\,(\partial_i x^i),$$

coherently with the fact that $\operatorname{grad}(\phi)$ is a tangent vector, so it must be a linear combination of the ∂_i's. This shows that, for generic manifolds, **the presence of a (pseudo-)Riemannian metric is fundamental in order to define the gradient of a scalar function**.

This fact is hidden for the trivial case of $M = \mathbb{R}^n$ because, as already remarked, in that situation $g^{ij} = g_{ij} = I_n$ and so $\operatorname{grad}(\phi) = (\frac{\partial \phi}{\partial x^1}, \ldots, \frac{\partial \phi}{\partial x^n})$.

6.1.2 Noticeable example 2: symplectic manifolds, the Hamiltonian isomorphism and the Poisson bracket

Riemannian, or pseudo-Riemannian, metrics and manifolds are built via symmetric positive-definite, or non-degenerated, tensor fields of type $(0, 2)$. Another remarkable construction can be obtained by considering anti-symmetric non-degenerated tensor fields of type $(0, 2)$.

Def. 6.1.5 (Simplectic form and manifold) *A closed non-degenerated 2-form $\omega = \omega_{ij} dx^i dx^j$ is called a simplectic form on M and a couple (M, ω) is said to be a simplectic manifold.*

Also for simplectic manifold we can identify the tangent and the cotangent bundles with the analogous of the isomorphism $\sharp$ that, in this setting, is called the **Hamiltonian isomorphism**:

$$\begin{array}{rccl} H: & T^*M & \xrightarrow{\sim} & TM \\ & (\alpha_j) & \longmapsto & H(\alpha_j) = X^i,\ X^i = \omega^{ij}\alpha_j. \end{array}$$

We can repeat the same construction as before with $\sharp$ to obtain a vector field from the differential of a scalar function $\phi \in \mathscr{C}(M)$ but, this time, by using H instead of $\sharp$. What we obtain is $H(d\phi) \in \mathfrak{X}(M)$, which is called **Hamiltonian vector field** of the scalar function ϕ.

Since a 2-form ω takes as input two vector fields, it is interesting to see what happens if we consider the differential of two scalar functions $\phi, \psi \in \mathscr{C}^\infty(M)$, the Hamiltonian vector fields associated to them, i.e. $H(d\phi), H(d\psi)$, and then we apply ω. The result is the so-called **Poisson bracket**:

$$\boxed{\{\phi, \psi\} := \omega(H(d\phi), H(d\psi))}\ .$$

$\mathscr{C}^\infty(M)$ becomes a Lie algebra w.r.t. the Poisson bracket (just as the set of tangent vector fields on M becomes a Lie algebra w.r.t. the Lie bracket).

6.2 Existence of Riemannian metrics

We can now prove the existence of Riemannian metrics.

Theorem 6.2.1 *Every smooth manifold M admits a Riemannian metric.*

Proof. The idea is quite simple: we start with a local Riemannian metric and then we extend it to the whole manifold thanks to a partition of the unity. Let us discuss the technical details.

Consider an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of M and a partition of the unity $\{\rho_\alpha\}$ subordinated to the covering $\{U_\alpha\}$ (so that each ρ_α is identically 0 outside U_α).

On U_α it is very easy to induce a metric from the Euclidean metric of $\mathbb{R}^n$. To see how, consider a chart function $\varphi_\alpha : U_\alpha \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$, if $\varphi_\alpha \equiv (x^1, \ldots, x^n)$, then we know that the vector fields $(\partial_1, \ldots, \partial_n)$ provide a local frame for $TM|_{U_\alpha}$. Given $p \in U_\alpha$ and two tangent vectors $X_p, Y_p \in T_pM$, $X_p = X^i \partial_i\big|_p$, $Y_p = Y^j \partial_j\big|_p$, we define a scalar product between $\partial_i\big|_p$ and $\partial_j\big|_p$ by means of the Euclidean product $\langle\ ,\ \rangle$ of $\mathbb{R}^n$ as follows:

$$g_p^\alpha(\partial_i\big|_p, \partial_j\big|_p) := \langle \varphi_\alpha(\partial_i\big|_p), \varphi_\alpha(\partial_j\big|_p)\rangle = \langle e_i, e_j\rangle = \delta_{ij},$$

recalling that $\partial_i\big|_p = d\varphi_\alpha^{-1}\big|_p(e_i)$, e_i being the i-th element of the canonical basis of $\mathbb{R}^n$. The extension to any couple X_p, Y_p of tangent vectors in T_pM is performed by linearity:

$$g_p^\alpha(X|_p, Y_p) := X^i Y^j g_p^\alpha(\partial_i\big|_p, \partial_j\big|_p) = X^i Y^j \delta_{ij} = \sum_{i=1}^n X^i Y^i.$$

g_p^α is then a positive-definite bilinear form for all $p \in M$ and for all α. Now we glue together these scalar products to build a tensor field $g \in T_2^0(M)$ by defining:

$$g_p := \sum_\alpha \rho_\alpha(p) g_p^\alpha, \qquad \forall p \in M.$$

The definition is well-posed because the sum is actually finite since, for all $p \in M$, there is only a finite number of $\rho_\alpha(p)$ different from 0. Plus, $\rho_\alpha(p) \geqslant 0$ for all $p \in M$ and all α, thus the coefficients $\rho_\alpha(p)$ do not modify the positive-definiteness of the forms g_p^α and then g results in a positive-definite symmetric tensor field, i.e. a Riemannian metric on M. □

Remark: the proof just developed **works only to prove the existence of positive-definite (or negative-definite) Riemannian metrics on** M. It does not work if we want to build a pseudo-Riemannian metric on M with signature (r, s). In fact, even if the g^α have the same signature, the g resulting from the sum may not have the same signature and could even be degenerated.

6.3 Riemannian metrics and changes of coordinates

If $(U, \varphi \equiv (x^1, \ldots, x^n))$ and $(\tilde{U}, \tilde{\varphi} \equiv (\tilde{x}^1, \ldots, \tilde{x}^n))$ are two local charts, then in $U \cap \tilde{U}$ the transition functions allow us to express $\tilde{x}$ as a function of x and vice-versa. We already know that the differentials of $\tilde{x}$ and that of x are related by the Jacobian matrix of the function $\tilde{x} = \tilde{x}(x)$:

$$d\tilde{x}^h = \frac{\partial \tilde{x}^h}{\partial x^i} dx^i. \tag{6.1}$$

If we write the Riemannian metric g in terms of the $\tilde{x}$ and x coordinates, we have:

$$g = \tilde{g}_{hk} d\tilde{x}^h \otimes d\tilde{x}^k = g_{ij} dx^i \otimes dx^j,$$

where the matrices $(\tilde{g}_{hk})$ and (g_{ij}) represent the metric in the local coordinate systems $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n)$ and $x = (x^1, \ldots, x^n)$, respectively.

By replacing (6.1) in the expression of g we obtain:

$$g = \tilde{g}_{hk}\, d\tilde{x}^h \otimes d\tilde{x}^k = \tilde{g}_{hk} \left(\frac{\partial \tilde{x}^h}{\partial x^i} dx^i \right) \otimes \left(\frac{\partial \tilde{x}^k}{\partial x^j} dx^j \right) \underset{\text{bilinearity of } \otimes}{=} \left(\frac{\partial \tilde{x}^h}{\partial x^i}\, \tilde{g}_{hk}\, \frac{\partial \tilde{x}^k}{\partial x^j} \right) dx^i \otimes dx^j,$$

which, compared with $g = g_{ij} dx^i \otimes dx^j$ gives:

$$\boxed{g_{ij} = \frac{\partial \tilde{x}^h}{\partial x^i}\, \tilde{g}_{hk}\, \frac{\partial \tilde{x}^k}{\partial x^j}}\ .$$

If we use the matrix notation we can re-write this relationship as follows:

$$(g_{ij}) = \left(\frac{\partial \tilde{x}}{\partial x} \right)^t (\tilde{g}_{hk}) \left(\frac{\partial \tilde{x}}{\partial x} \right),$$

where $\left(\frac{\partial \tilde{x}}{\partial x} \right)$ is the Jacobian matrix of the transition function $\tilde{x}(x)$. This is coherent with the well-known linear algebra result which says that the matrices associated to symmetric bilinear forms transform, after a change of basis, by multiplication with the change of basis matrix on the right and its transposed (not its inverse) on the left.

This fact has an important consequence: the determinant of the matrices associated to the metric g in different coordinate systems are, in general, different, in fact:

$$\det(g_{ij}) = \det(\tilde{g}_{hk}) \left(\det \left(\frac{\partial \tilde{x}}{\partial x} \right) \right)^2,$$

i.e. they are related by the square of determinant of the Jacobian matrix of the transition function $\tilde{x}(x)$. The information that we can assure is that the sign of $\det(g_{ij})$ and $\det(\tilde{g}_{hk})$ is the same.

Chapter 7

Connections on vector bundles (Edoardo Provenzi)

Connections on manifolds are also called, in particular in the Physics literature, covariant derivatives. To motivate the exigence of introducing these objects let us start by showing a problem related with the Lie derivative that can be underlined already in the trivial case when the manifold M is an open set U in $\mathbb{R}^n$.

7.1 Motivation

Consider two vector fields $X, Y \in \mathfrak{X}(U)$, then, since all the tangent spaces in every point $p \in U$ can be canonically identified with $\mathbb{R}^n$, i.e. $T_pU \cong T_p\mathbb{R}^n \cong \mathbb{R}^n$, X and Y can be simply thought as vector-valued functions defined on U: $X, Y : U \subset \mathbb{R}^n \to \mathbb{R}^n$.

Thanks to this identification, the derivative of Y along X in every point $p \in U$ can be identified with the directional derivative of $Y : U \to \mathbb{R}^n$ in the direction defined by the vector $X(p) \equiv X_p$, we write:

$$\partial_X Y\big|_p := D_{X_p} Y(p) \equiv \lim_{\varepsilon \to 0} \frac{Y(p + \varepsilon\, X|_p) - Y_p}{\varepsilon},$$

having used definition (B.3).

Let us examine the properties of ∂_X: for all $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(U)$, $a, b \in \mathbb{R}$ and $f \in \mathscr{C}^\infty(U)$, we have

1. $\partial_X(aY_1 + bY_2) = a\partial_X Y_1 + b\partial_X Y_2$
2. $\partial_X(fY) = X(f)Y + f\partial_X Y$
3. $\partial_{aX_1+bX_2} Y = a\partial_{X_1} Y + b\partial_{X_2} Y$
4. $\partial_{fX} Y = f\partial_X Y$.

Property 1. follows simply from the linearity of the directional derivative. To understand property 2. notice that $fY : U \to \mathbb{R}^n$, $p \mapsto f(p)Y_p$ is the product of two functions, one real-valued the other vector-valued, defined on U, thus the Leibnitz rule must be applied and we get $\partial_X(fY)(p) = D_{X_p} f(p) Y_p + f(p) D_{X_p} Y(p)$, but $D_{X_p} Y(p) = \partial_X Y\big|_p$ and, regarding the first term, we must recall that $X \in \mathrm{Der}(\mathscr{C}^\infty(U))$, i.e. X can be interpreted also as a derivation whose action on the elements of $\mathscr{C}^\infty(U)$ is exactly the directional derivative, i.e. $X_p(f) = D_{X_p} f(p)$. So, $\partial_X(fY)(p) = X_p(f)Y_p + f(p)\, \partial_X Y|_p$ for all $p \in U$, i.e. property 2.

To resume: 1. & 2. $\implies$ $\partial_X(Y)$ **is $\mathbb{R}$-linear but not $\mathscr{C}^\infty(U)$-linear w.r.t.** Y.

Property 3. is an immediate consequences of the linearity of the directional derivative w.r.t. the directional vector, cfr. formula (B.10). To understand property 4. notice that $\partial_{fX}(Y)|_p = D_{(fX)(p)}(Y)(p) = D_{f(p)X_p}Y(p)$, but $f(p) \in \mathbb{R}$ and $X_p \in \mathbb{R}^n$, so the evaluation of fX in p simply gives a scalar multiple of the vector X_p and thus the property follows again from formula (B.10).

To resume: 3. & 4. $\implies$ $\partial_X(Y)$ **is both $\mathbb{R}$-linear and $\mathscr{C}^\infty(U)$-linear w.r.t.** X.

It is crucial to stress that the other operator that we have defined that implements the derivative of a vector field w.r.t. another one, i.e. the Lie derivative $\pounds_X Y$, does not possesses property 4., i.e. it is not $\mathscr{C}^\infty(U)$-linear, in fact, thanks to anti-symmetry and Leibniz rule:

$$\pounds_{fX} Y = [fX, Y] = -[Y, fX] = \pounds_Y fX = -Y(f)X - f\pounds_Y X = -Y(f)X + f\pounds_X Y,$$

thus $\pounds_{fX} Y = -Y(f)X + f\pounds_X Y$, i.e. $\pounds_{fX} Y \neq f\pounds_X Y$.

This shows that **the Lie derivative**, in spite of being a fundamental object that allows determining conditions to show the existence of integral submanifolds, **cannot be considered as the perfect analogue of the directional derivative** of a function defined on an open subset of $\mathbb{R}^n$.

Another limitation related to the Lie derivative is that it allows to compute the rate of variation of an object w.r.t. a vector field, only when this object is build from the tangent bundle to a manifold: in fact, vector and covector fields and tensors on a manifold are always built by starting from the tangent bundle. Thus, we cannot avoid the problem underlined above also when we take the Lie derivative of general tensors on a manifold.

The aim of connections (actually the linear ones) is solve both problems at once, i.e. to define a $\mathbb{R}$ and $\mathscr{C}^\infty(U)$- linear derivative along a vector field on M of the section of a general vector bundle E on M, not only of the tangent bundle TM.

7.2 Failed approach towards the generalization of the Lie derivative

It is highly instructive to discuss an approach that goes in the direction that we want, but that fails for one reason that will be underlined. The information learned from this failure will allow us understanding the correct path to follow.

Consider a generic vector bundle $\pi : E \to M$ of rank r on a manifold M, let $M \in \mathfrak{X}(M)$ and $s : U \to E$ be a section of E on an open set $U \subset M$. As stressed when we have defined the Lie derivative, given the integral curve γ of X passing through a given point $p \in U$, it does not make sense to compute the derivative of s along X in p as follows

$$\lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

simply because $s(\gamma(t)) \in E_{\gamma(t)}$ and $s(\gamma(0)) \in E_{\gamma(0)}$, which are two different vector spaces!

In the trivial case $M = \mathbb{R}^n$ all the fibers are canonically isomorphic to each other and the can perform the difference of vectors belonging to different fibers, but this is not possible if M is not trivial.

Vector bundles are not trivial, but they are always locally trivial, so a more refined idea could be to use local triviality to try to extend our definition of derivative. Let us see how long we can go by using this feature. We know that it exists an open cover $\{U_i\}$ with sets U_i small enough such that $E|_{U_i}$ is trivial, i.e. there are diffeomorphisms $\chi_i : E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{R}^r$. If we compose the section with the local trivializations χ_i as in the following commutative diagram

$$\begin{tikzcd}
E|_{U_i} \arrow[rr, "\chi_i"] & & U_i \times R^r \\
& U \cap U_i \arrow[ul, "s"] \arrow[ur, "\tilde{s}_i = \chi_i \circ s"'] &
\end{tikzcd}$$

then the advantage is that we obtain $\tilde{s}_i(p) = (p, \tilde{s}_i^1(p), \ldots, \tilde{s}_i^r(p))$ $\forall p \in U$, where each $\tilde{s}_i^k(p) : U \cap U_i \to \mathbb{R}$ is a simple smooth scalar function, for all $k = 1, \ldots, r$, and so we can apply any vector field $X \in \mathfrak{X}(M)$ (interpreted as a derivation of $\mathscr{C}^\infty(U \cap U_i)$) to these functions.

This seems to suggest that we could define the derivative $X(s)$ of the section s, on a small open neighborhood U_i of p, by deriving the functions $\tilde{s}_i^k$ as follows: $X(s)|_{U_i} \approx (X(\tilde{s}_i^1), \ldots, X(\tilde{s}_i^r))$, we use the symbol $\approx$ because we will see that this definition is not entirely correct. The problem with this definition is that, even if it is perfectly correct on U_i, we must assure its coherence when we consider another open cover $\{U_j\}$ and local trivializations χ_j on the intersections $U \cap (U_i \cap U_j)$ (that we will denote simply as $U_i \cap U_j$ to avoid a cumbersome notation). The following commutative diagram shows how the situation looks in this case.

$$\begin{tikzcd}
& E|_{U_i \cap U_j} \arrow[dl, "\chi_i"'] \arrow[dr, "\chi_j"] & \\
(U_i \times U_j) \times \mathbb{R}^r & & (U_i \cap U_j) \times \mathbb{R}^r \arrow[ll, "\eta_{ij}"'] \\
& U_i \cap U_j \arrow[ul, "\tilde{s}_i"] \arrow[ur, "\tilde{s}_j"'] \arrow[uu, dotted, bend right, "s" description] &
\end{tikzcd}$$

The two Cartesian products $(U_i \cap U_j) \times \mathbb{R}^r$ written on the left and on the right are characterized by different copies of $\mathbb{R}^r$ that host different coordinates of $\tilde{s}_i$ and $\tilde{s}_j$. They are related by the transition functions $\eta_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{R})$ and, for all $p \in U_i \cap U_j$, $\eta_{ij}(p)$ is an invertible matrix that represents the change of coordinates from the two copies of R^r, explicitly:

$$\tilde{s}_i(p) = \left(p, \begin{pmatrix} \tilde{s}_i^1(p) \\ \vdots \\ \tilde{s}_i^r(p) \end{pmatrix} \right) = \left(p, \eta_{ij}(p) \begin{pmatrix} \tilde{s}_j^1(p) \\ \vdots \\ \tilde{s}_j^r(p) \end{pmatrix} \right),$$

i.e. $\tilde{s}_i^k = \eta_{ij} \tilde{s}_j^k$, $k = 1, \ldots, r$.

The components $(X(\tilde{s}_i^1), \ldots, X(\tilde{s}_i^r))$ represent the local expressions of a section of $X(s)$ on E if, on $U_i \cap U_j$, they are related by the transition functions η_{ij} like this:

$$\begin{pmatrix} X(\tilde{s}_i^1) \\ \vdots \\ X(\tilde{s}_i^r) \end{pmatrix} = \eta_{ij} \begin{pmatrix} X(\tilde{s}_j^1) \\ \vdots \\ X(\tilde{s}_j^r) \end{pmatrix},$$

i.e. $X(\tilde{s}_i^k) = \eta_{ij} X(\tilde{s}_j^k)$, $k = 1, \dots, r$.

Let us see if this is really what happens by applying X (thought as a derivation of $\mathscr{C}^\infty(U_i \cap U_j)$) on both sides of the equation $\tilde{s}_i^k = \eta_{ij}\tilde{s}_j^k$. Thanks to the Leibniz-like behavior of X we find:

$$X(\tilde{s}_i^k) = \eta_{ij} X(\tilde{s}_j^k) + X(\eta_{ij})\tilde{s}_j^k, \qquad \forall k = 1, \dots, r,$$

this is different from the equation $X(\tilde{s}_i^k) = \eta_{ij} X(\tilde{s}_j^k)$ that we expected because of *the spurious term given by the derivatives of the transition functions* $X(\eta_{ij})\tilde{s}_j^k$.

This shows that the components $X(\tilde{s}_i^k)$ define a section representing the derivative of the section s along the vector field X if and only if $X(\eta_{ij}) = 0$ for all i, j. However, this, in general, is not true and so this approach defines an non-intrinsic object that depends on the local trivialization used. The only situation in which this construction works it when the transition functions η_{ij} are locally constant, so that their derivatives are null, in this case we talk about a **flat vector bundle**.

7.3 Connections on vector bundles

In the previous we have shown that:

1. the naive definition of the derivative of the section of a vector bundle w.r.t. a vector field as the limit of the incremental ratio makes no sense because we are comparing vectors belonging to different vector spaces;

2. a finer use of the local trivialization of the vector bundle leads us to an object that makes sense, but that cannot be considered as the derivative of a section because, in general, it depends on the trivialization itself.

The conclusion that we reach is that, unlike the Lie derivative, **there is no intrinsic way to define the derivative of the section of a vector bundle by using only the elements already present in the vector bundle structure**. We are forced to **introduce an external structure**, which is provided by the connection, as we define below (we recall that $\Gamma(E)$ is the set of all sections of the vector bundle $\pi : E \to M$).

Def. 7.3.1 (Connection) *A connection on a vector bundle $\pi : E \to M$ is a function*

$$\begin{array}{rccc} \nabla : & \mathfrak{X}(M) \times \Gamma(E) & \longrightarrow & \Gamma(E) \\ & (X, s) & \longmapsto & \nabla(X, s) \equiv \nabla_X s, \end{array}$$

that transforms the couple given by a vector field X on M and a section s of the bundle (E, M, π) in another section $\nabla_X s$ of the same bundle, in such a way that, for all $X, X_1, X_2 \in \mathfrak{X}(M)$, $f, f_1, f_2 \in \mathscr{C}^\infty(M)$, $s, s_1, s_2 \in \Gamma(E)$ and $k_1, k_2 \in \mathbb{R}$, the following properties are satisfied:

1. *$\mathscr{C}^\infty(M)$-linearity w.r.t. the vector field:* $\boxed{\nabla_{f_1X_1+f_2X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s}$

2. *$\mathbb{R}$-linearity w.r.t. the section:* $\boxed{\nabla_X(k_1s_1 + k_2s_2) = k_1\nabla_X s_1 + k_2\nabla_X s_2}$

3. *Leibniz property:* $\boxed{\nabla_X(fs) = f\nabla_X s + X(f)s}$.

These properties are obviously inspired by those of the directional derivative of a function defined on an open set of $\mathbb{R}^n$ that we have discussed in section 7.1 and are *imposed by hand* to make $\nabla_X s$ the correct generalization of the directional derivative in the trivial case.

Def. 7.3.2 (Covariant derivative) *The section $\nabla_X s$ is the covariant derivative of the section s along the vector field X.*

There is a special case that deserves a particular attention and a dedicated definition.

Def. 7.3.3 (Linear connection) *A connection on the tangent bundle TM to a manifold M is called a linear connection on M.*

Having defined a connection does not guarantee that such an object exists. In the second special case of a globally trivial vector bundle of rank r, i.e. $E = M \times \mathbb{R}^r$, a connection is easily seen to exist. In fact, a section $s \in \Gamma(M \times \mathbb{R}^r)$ can only have this form

$$\begin{array}{rccl} s: & M & \longrightarrow & M \times \mathbb{R}^r \\ & p & \longmapsto & s(p) = (p, (s^1(p), \dots, s^r(p))), \end{array}$$

where $s^i \in \mathscr{C}^\infty(M)$ for all $i = 1, \dots, r$. If $X \in \mathfrak{X}(M)$, then the *canonical section* define as follows

$$\begin{array}{rccl} \nabla_X s: & M & \longrightarrow & M \times \mathbb{R}^r \\ & p & \longmapsto & s(p) = (p, ((Xs^1)(p), \dots, (Xs^r)(p))), \end{array}$$

can be verified to be a covariant derivative of s along X (by direct verification of the defining properties), so $\nabla_X : \mathfrak{X}(M) \times \Gamma(M \times \mathbb{R}^r) \to \Gamma(M \times \mathbb{R}^r)$, $(X, s) \mapsto \nabla_X s$ is a connection on the trivial bundle $(M \times \mathbb{R}^r, M, \pi)$.

The following results shows, via a constructive proof that makes use of the partition of unity[1], that at least a connection (actually infinite, as we will see later) exist for all vector bundle.

Theorem 7.3.1 *Every vector bundle $\pi : E \to M$ admits a connection.*

Proof. We have just seen that, for a trivial bundle, a connection can be defined as above. We can always find an open cover (U_α) of M that corresponds to a local trivialization of the bundle, i.e. such that the functions χ_α, $E|_{U_\alpha} \xrightarrow{\chi_\alpha} U_\alpha \times \mathbb{R}^r$, are diffeomorphisms.

On $U_\alpha \times \mathbb{R}^r$ there is a canonical connection ∇^0_X as previously defined. Then, the function χ_α allows us to define a connection ∇^α, which depends on the local trivialization, on $E|_{U_\alpha}$.

To define ∇^α we must declare what is the covariant derivative of a section s of $E|_{U_\alpha}$: the first thing we need to do is to compose χ_α with s to obtain a section of the trivial bundle $U_\alpha \times \mathbb{R}^r$, then we can apply ∇^0_X to this section, obtaining another section of $U_\alpha \times \mathbb{R}^r$, by applying χ_α^{-1} we take this section back to $E|_{U_\alpha}$. Thus:

$$\nabla^\alpha_X s := \chi_\alpha^{-1}(\nabla^0_X(\chi_\alpha \circ s)),$$

[1] We notice that this proof cannot be used to guarantee the existence of connections on complex or algebraic manifolds because they do not possess a partition of unity. In fact, there is no alternative proof for those cases, i.e. the theorem is not valid, in general, for vector bundles of complex or algebraic manifolds.

the properties of a connection are easily proven to be satisfied by ∇_X^α thanks to the fact that ∇_X^0 is a connection.

The idea to extend the connection ∇_X^α from the restriction of E on U_α to all E consists in smoothly extend it to zero outside U_α and then to smoothly glue together all the ∇_X^α as U_α varies in the cover. This can be achieved thanks to a partition of the unity (ρ_α) subordinated to the cover (U_α). We recall that each ρ_α is a smooth function defined on M whose support is contained in U_α, i.e. $\rho_\alpha \equiv 0$ on $M \backslash U_\alpha$, and that the functions ρ_α sum up to 1.

Thanks to this, for all (global) section s of E on M, we can define its covariant derivative $\nabla_X s$ along X as follows:

$$\boxed{\nabla_X s := \sum_\alpha \rho_\alpha \, \nabla_X|_{U_\alpha} (s|_{U_\alpha}), \quad \text{with} \quad \rho_\alpha \, \nabla_X|_{U_\alpha} := \rho_\alpha \nabla_X^\alpha} \, .$$

It is customary to write the connection associated to the covariant derivative $\nabla_X s$ simply as $\nabla_X = \sum\limits_\alpha \rho_\alpha \nabla_X^\alpha$.

By direct computation, it can be proven that ∇_X just defined verifies all the properties of a connection. Here, we just verify the Leibniz property. For all $f \in \mathscr{C}^\infty(M)$ we have that

$$\begin{aligned}
\nabla_X(fs) &= \sum_\alpha \rho_\alpha \nabla_X^\alpha (f \, s|_{U_\alpha}) \\
&\quad (\nabla_X^\alpha \text{ is a connection}) \\
&= \sum_\alpha \rho_\alpha (f \, \nabla_X^\alpha (s|_{U_\alpha}) + X(f) \, s|_{U_\alpha}) \\
&\quad (f, X(f): \text{ independents of } \alpha) \\
&= f \sum_\alpha \rho_\alpha \, \nabla_X^\alpha (s|_{U_\alpha}) + X(f) \sum_\alpha \rho_\alpha \, s|_{U_\alpha} \\
&\quad (\rho_\alpha \, s|_{U_\alpha} = \rho_\alpha s) \\
&= f \nabla_X s + X(f) \sum_\alpha \rho_\alpha s = f \nabla_X s + X(f) s \sum_\alpha \rho_\alpha \\
&\quad (\sum_\alpha \rho_\alpha = 1) \\
&= f \nabla_X s + X(f) s.
\end{aligned}$$

Thus, the Leibniz property holds, the others are even simpler to check. □

As expected from the considerations at the beginning of this chapter, **the Lie derivative**

$$\begin{array}{rccc}
\pounds: & \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\
& (X, Y) & \longmapsto & \pounds_X Y = [X.Y],
\end{array}$$

is not a connection on TM. In fact, $\pounds_{fX} \neq f \pounds_X Y$ for $f \in \mathscr{C}^\infty(M)$. Thus, there is a sort of trade-off between Lie derivative and connection w.r.t. their properties: the Lie derivative is intrinsically defined on TM but it fails to be $\mathscr{C}^\infty$-linear, while a connection is not intrinsically defined on a vector bundle (neither on TM) but it has that property. So, there remains a degree of freedom in the choice of a connection. This ambiguity can be eliminated in special cases, e.g. when the vector bundle has a Riemannian structure, as we will see later with the concept of Levi-Civita connection.

It is useful to single out two noticeable properties of a connection ∇_X:

- for all section s of E on M and all point $p \in M$, $\nabla_X s(p)$ depends on the behavior of the section s in a neighborhood of p;

- instead, $\nabla_X s(p)$ depends only on the value of X in p, i.e. on $X_p \in T_pM$, the other tangent vectors in a neighborhood of p assigned by X are totally irrelevant.

Hence, **the behavior of $\nabla_X s(p)$ is local w.r.t. s and point-wise w.r.t. X.** These features of ∇_X are rigorously stated in the following proposition.

Theorem 7.3.2 (Structural properties of $\nabla_X s$) *Let $\pi : E \to M$ be a vector bundle and ∇ a connection on E.*

1. *If $X, \tilde{X} \in \mathfrak{X}(M)$ are such that $X_p = \tilde{X}_p$ and there exists an open neighborhood U of p such that $s, \tilde{s} \in \Gamma(M)$ are coincident on U, i.e. $s|_U = \tilde{s}|_U$, then*
$$\nabla_X s(p) = \nabla_{\tilde{X}} \tilde{s}(p).$$

2. *For all open set $U \subset M$ there exists only one connection on $E|_U$*
$$\begin{array}{rccc} \nabla^U : & \mathfrak{X}(M) \times \Gamma(U) & \longrightarrow & \Gamma(U) \\ & (X, s) & \longmapsto & \nabla^U(X, s), \end{array}$$
such that, for all $p \in U$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(M)$, we have:
$$\nabla^U_X\big|_U(p) = \nabla_X s(p).$$

3. *If, for all $X \in \mathfrak{X}(M)$ and $s, \tilde{s} \in \Gamma(M)$ it exists a path $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$, $\gamma'(0) = X_p$ and $s \circ \gamma = \tilde{s} \circ \gamma$, then $\nabla_X s(p) = \nabla_X \tilde{s}(p)$.*

The third property is a refinement of the first one: for two covariant derivatives to coincide in a point is it enough that they coincide on a 'small' arc of path passing through that point and having the vector X_p as tangent vector in p.

Proof. TO BE WRITTEN...Lezione 17, 1h02m. $\square$

7.3.1 Expression of a connection in local coordinates: the Christoffel symbols

Let (U, φ) be a local chart of M that trivializes E, i.e. such that $E|_U \xrightarrow[\chi]{\sim} U \times \mathbb{R}^r$.

By applying the inverse of χ to the couple given by a generic point $p \in U$ and an arbitrary vector of the canonical basis of $\mathbb{R}^r$, i.e. $(p, (0, \ldots, 0, 1, 0, \ldots, 0))$, where the value 1 is in the k-th position, $k = 1, \ldots, r$, we determine a local basis for $E|_U$, i.e. r local sections defined on U, that we denote with $(e_1, \ldots, e_r) \in \Gamma(U)$ for simplicity,
$$\begin{array}{rccl} e_k : & U & \longrightarrow & E|_U \\ & p & \longmapsto & \chi^{-1}(p, (0, \ldots, 0, 1, 0, \ldots, 0)), \end{array}$$

such that $(e_1(p), \dots, e_r(p))$ is a basis of the fiber E_p, for all $p \in U$.

Moreover, we know that the local chart (U, φ) determines a local basis of TM given by $(\partial_1, \dots, \partial_n)$, $n = \dim(M)$.

Let us then consider ∂_j, $j = 1, \dots, n$, as the vector field w.r.t. we want to define a connection and e_h, $h = 1, \dots, r$, as the section of E on which this connection acts. Then $\nabla_{\partial_j} e_h$ is again a section of E, by definition of connection. Hence, there must be suitable functions $\Gamma^k_{jh} \in \mathscr{C}^\infty(U)$ such that:

$$\boxed{\nabla_{\partial_j} e_h = \Gamma^k_{jh} e_k}\,, \qquad j = 1, \dots, n,\ h, k = 1, \dots, r, \tag{7.1}$$

notice that three indices are essential: k is the linear combination index and j, h take into account that the connection is defined w.r.t. the vector field ∂_j and it is applied on the basis section e_h.

Def. 7.3.4 (Connection coefficients - Christoffel symbols) *The function $\Gamma^k_{jh} \in \mathscr{C}^\infty(U)$ appearing in eq. (7.1) are called (local) connection coefficients. In the special case $E = TM$, $r = n$ and the connection coefficients are called Christoffel symbols.*

Def. 7.3.5 (Flat connections) *A connection is said to be flat if all its coefficients are identically 0.*

Let us verify that the connection coefficients determine completely the connection. For any $X \in \mathfrak{X}(M)$ and $s \in \Gamma(U)$ we have:

$$X = X^j \partial_j \quad \text{ and } \quad s = s^h e_h, \qquad X^j, s^h \in \mathscr{C}^\infty(U),$$

thus, by definition of connection and by using its properties,

$$\nabla_X s = \nabla_X(s^h e_h) = X(s^h) e_h + s^h \nabla_X e_h,$$

$\nabla_X e_h = \nabla_{X^j \partial_j} e_h = X^j \nabla_{\partial_j} e_h = X^j \Gamma^k_{jh} e_k$, thus, by renaming the summation index $X(s^h) e_h = X(s^k) e_k$, we get

$$\nabla_X s = X(s^k) e_k + s^h X^j \Gamma^k_{jh} e_k = (X(s^k) + \Gamma^k_{jh} X^j s^h) e_k.$$

In the literature sometimes we write simply $s = (s^k)$ avoiding the specification of the basis sections e_k. In this case we get the much easier formula to remember:

$$\boxed{\nabla_X(s^k) = X(s^k) + \Gamma^k_{jh} X^j s^h}\,, \tag{7.2}$$

in fact, it says that the covariant derivative is composed by two term:

- the first term is simply given by the action of X, interpreted as a derivation, applied on s^k (the equivalent of the directional derivative in $\mathbb{R}^n$);
- the additional term, i.e. the correction w.r.t. the classical directional derivative, is provided by a linear combination in which the connection coefficients appear. Thus, if the connection is trivial (i.e. its coefficients are all 0), then the covariant derivative and the directional derivative coincide.

7.3.2 Parallel sections

We now want to discuss the very important concept of parallel sections. In order to examine this, we need to discuss the properties of covariant derivatives in relation with curves. We begin with a definition.

Def. 7.3.6 (Section along a curve) *Let $\pi : E \to M$ be a vector bundle over M and let $\gamma : I \subseteq \mathbb{R} \to M$ be a path in M. A section of E along γ is a $\mathscr{C}^\infty$ function $s : I \to E$ such that $\forall t \in I$, $s(t) \in E_{\gamma(t)}$.*

Such a section is said to be extendable to a local section $s \in \Gamma(E, U)$ if there exist an open neighborhood U of the image of γ and a section $\tilde{s} \in \Gamma(E, U)$ such that $s(t) = \tilde{s}(\gamma(t))$ $\forall t \in I$.

Notation: the set of sections of E along γ forms a vector space, w.r.t. the point-wise linear operations, that is denoted by $\Gamma(E, \gamma)$.

Theorem 7.3.3 *Let $\gamma : I \to M$ be a path in M and ∇ a connection on E. Then, it exists a unique operator $D : \Gamma(E, \gamma) \to \Gamma(E, \gamma)$ such that:*

1. *D is $\mathbb{R}$-linear, i.e.*
$$D(a_1 s_1 + s_2 s_2) = a_1 D(s_1) + a_2 D(s_2), \qquad \forall a_1, a_2 \in \mathbb{R},\ s_1, s_2 \in \Gamma(E, \gamma).$$
2. *D satisfies the Leibniz rule:*
$$D(fs) = f's + fD(s), \qquad \forall f \in \mathscr{C}^\infty(I).$$
3. *If $s \in \Gamma(E, \gamma)$ is extendable and $\tilde{s}$ is an extension of s to an open neighborhood of the image of γ, then we have:*
$$Ds(t) = \nabla_{\gamma'(t)} \tilde{s}.$$

See [1] for the proof.

Def. 7.3.7 (Covariant derivative along a path) *The operator $D : \Gamma(E, \gamma) \to \Gamma(E, \gamma)$ is called the covariant derivative along the path γ. Ds is the covariant derivative of s along the tangent vectors to the path γ.*

Now we have all the information to introduce the concept of parallel section.

Def. 7.3.8 (Parallel section) *Let ∇ be a connection on the vector bundle E over M and let $\gamma : I \to M$ be a path in M. A section $s \in \Gamma(E, \gamma)$ is said to be parallel (along γ) if $Ds \equiv 0$.*

Instinctively, the request $Ds \equiv 0$ could leads us to think that the section s remains 'constant' along γ, but this is not the case and the word parallel is actually more adequate, let us see why.

As t runs in I, the corresponding point in M over the image of the path γ changes, thus when we apply the section s to t we obtain a sequence of vectors belonging to the fibers $E_{\gamma(t)}$, $t \in I$.

Now, since D measures the rate of variation of the section s along γ, the fact that $Ds \equiv 0$ is naturally interpreted as the fact that the vectors $s(t)$ are as similar as possible as we move to one point to another of the image of γ in M.

Since these vectors belong to different fibers and the fibers are not canonically isomorphic vector spaces, being as similar as possible cannot be translated to being constant, i.e. the same vector. It is thus more correct to use the word 'parallel' instead of constant.

Let us use eq. (7.2), i.e. $\nabla_X(s^k) = X(s^k) + \Gamma^k_{jh} X^j s^h$, to further analyze the consequences of the condition $Ds \equiv 0$. Thanks to property 3. of theorem 7.3.3, the action of X in $\gamma(t)$ is simply the derivative of the path γ in t, i.e. $X_{\gamma(t)} = \gamma'(t)$, thus:

$$X(s^k) + \Gamma^k_{jh} X^j s^h = 0 \qquad \Longleftrightarrow \qquad \frac{ds^k}{dt}(\gamma(t)) + \Gamma^k_{jh}\frac{d\gamma^j}{dt}(t) s^h = 0 \qquad \forall k = 1, \ldots, r,$$

where $\frac{ds^k}{dt}(\gamma(t))$ replaces $X(s^k)$ because this is the derivation of the function s^k in the direction given by X, but X is tangent to γ in every point. But, thanks to the point 6. of the flux theorem 5.2.3, computing the derivative of a s^k in the tangent direction to γ is the same as evaluating s^k on the points belonging to the curve $\gamma(t)$ and then computing the derivative w.r.t. the parameter t.

Thanks to these identifications, we have written explicitly eq. (7.2) as a system of ordinary differential equations, that, as we recall in the next theorem, always admits a unique solution.

Theorem 7.3.4 ($\exists!$ of solutions of a system of ODE) *Let $I \subseteq \mathbb{R}$ be an interval, $k \geqslant 1$, $t_0 \in I$, $x_0, \ldots, x_{k-1} \in \mathbb{R}^n$, and $A : I \times (\mathbb{R}^n)^k \to \mathbb{R}^n$ a $\mathscr{C}^\infty$ function, linear w.r.t. the variables in $(\mathbb{R}^n)^k$. Then, the Cauchy problem*

$$\begin{cases} \frac{d^k s}{dt^k}(t) = A(t, s(t), \ldots, \frac{d^{k-1}s}{dt^{k-1}}(t)) \\ s(t_0) = x_0 \\ \frac{ds}{dt}(t)(t_0) = x_1 \\ \quad\vdots \\ \frac{d^{k-1}s}{dt^{k-1}}(t_0) = x_{k-1}, \end{cases}$$

admits a unique $\mathscr{C}^\infty$ solution $s : I \to \mathbb{R}^n$.

Thanks to this result, given any point $p \in M$, we can extend any vector $v \in E_p$, keeping it 'parallel to itself', along a curve passing through p, as depicted in figure 7.3.2.

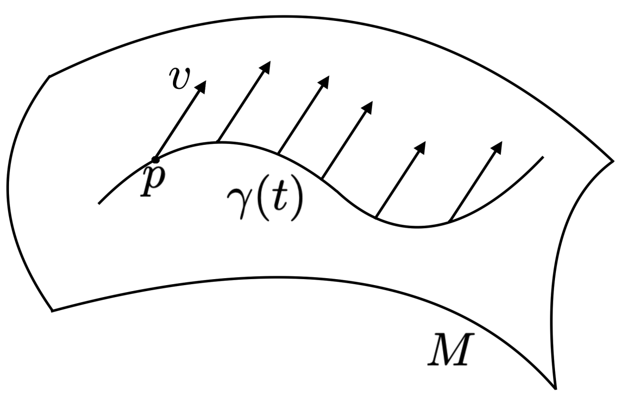


Figure 7.1: A graphic representation of the concept of parallel transport of a vector along a curve thanks to the presence of a connection.

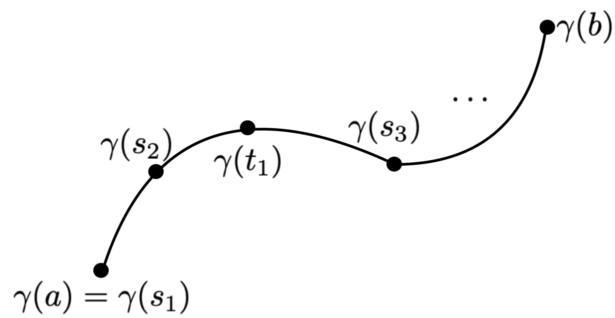
Theorem 7.3.5 *Let $\pi : E \to M$ a vector bundle on M, ∇ a connection on E and $\gamma : [a,b] \to M$ a path in M. If $p = \gamma(a)$, then, for all $v \in E_p$ there exists a unique parallel section $V \in \Gamma(E,\gamma)$ such that $V(a) = v$.*

Proof. The only slightly technical part of the proof consists in the fact that the curve, in general, is not contained in a chart domain. This problem can be fixed by using the compactness of the interval $[a,b]$: the fact that it exists a finite open covering of $[a,b] = \bigcup_{j=1}^{k} [s_j, t_j]$ implies that there is a finite number of charts $(U_1, \varphi_1), \dots, (U_k, \varphi_k)$, chosen from a local trivialization of E, that cover the image of γ.

Modulo a suitable choice of the covering, we can also suppose that $\gamma([s_j,t_j]) \subseteq \gamma([a,b]) \cap U_j$, for $j = 1, \dots, k$.

Then, the existence and uniqueness theorem for solutions of a system of ODE quoted before implies that it exists a unique parallel section V_1 along $\gamma|_{[s_1,t_1]}$ such that $V_1(a) = v$.

Thanks again to compactness, we have the freedom to chose the covering of $[a,b]$ as follows: $a = s_1 < s_2 < t_1 < s_3 < t_2 < \cdots < t_{k-1} < t_k = b$, i.e. the sub-intervals that cover $[a,b]$ are partially overlapping (see the picture below).



This trick serves our purposes because, when we solve the system of ODEs in the second open neighborhood, we obtain a unique parallel section V_2 along $\gamma|_{[s_2,t_2]}$ such that $V_2(t_1) = V_1(t_1)$.

By uniqueness, V_1 and V_2 must be equal on $[s_2, t_1]$, so, by gluing together V_1 and V_2, we get a unique parallel section along $\gamma|_{[s_1,t_2]}$.

Following this procedure until $t_k = b$, we obtain a unique parallel section V along γ such that $V(a) = v$. □

This result allows us the possibility to define the extremely useful concept of parallel transport.

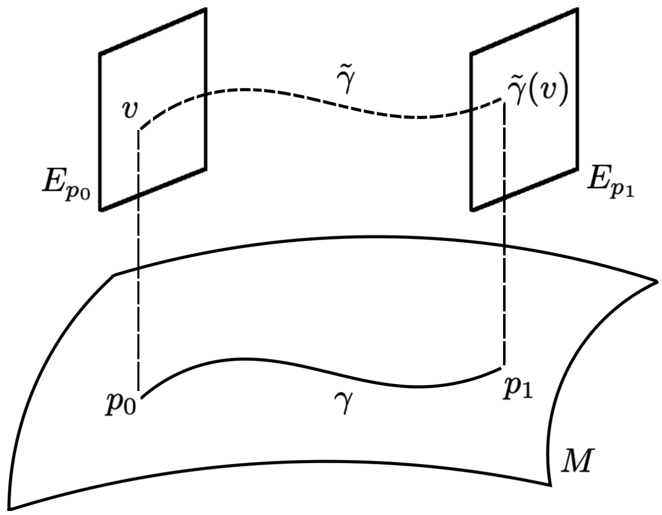
Def. 7.3.9 (Parallel transport) *Let $\pi : E \to M$ a vector bundle on M, ∇ a connection on E and $\gamma : [0,1] \to M$ a path in M with $\gamma(0) = p_0$ and $\gamma(1) = p_1$.*

Given $v \in E_{p_0}$, the only section $V \in \Gamma(E,\gamma)$ parallel along γ and such that $V(0) = v \in E_{p_0}$ is called the parallel extension of v along γ.

The parallel transport along γ is the function:

$$\tilde{\gamma} : E_{p_0} \to E_{p_1}$$

defined by $\tilde{\gamma}(v) = V(1)$, $V \in \Gamma(E,\gamma)$ being the parallel extension of $v \in E_{p_0}$.



The picture above shows the action of the parallel transport.

The most important property of the parallel transport is expressed by the following result.

Theorem 7.3.6 *The parallel transport along γ is a linear isomorphism between the vector spaces E_{p_0} and E_{p_1}, the inverse of $\tilde{\gamma}$ being the parallel transport along γ_-, where $\gamma_-(t) := \gamma(1-t)$ is the path that describes that same curve as γ, but traveled in reverse, so: $\tilde{\gamma}^{-1} = \tilde{\gamma}_-$.*

Proof. We have seen that the condition $Dv \equiv 0$, which characterizes parallel sections to a curve, is locally equivalent to:

$$\frac{dV^k}{dt} + \Gamma^k_{jh}(V^j)' s^h = 0 \qquad \forall k = 1, \ldots, r,$$

which is a *linear* system of ODEs. A classical results of the theory of ODEs guarantees that linearity implies that the solution $V(t)$ depends linearly on the initial conditions. This fact is translated in the linearity of the function $\tilde{\gamma} : E_{p_0} \to E_{p_1}$.

Let us now prove that $\tilde{\gamma}^{-1} = \tilde{\gamma}_-$. We denote with D^- the covariant derivative along γ_-. For all section $V \in \Gamma(E, \gamma)$, we set $V^-(t) := V(1-t)$ in such a way that $V^- \in \Gamma(E, \gamma_-)$.

Since $\gamma'_-(t) = \gamma'(1-t) \cdot (1-t)' = -\gamma'(1-t)$, a direct calculation gives $D^-_t V^- = -D_{1-t}V$. Since the only difference between the two covariant derivatives is the sign, it follows that V^- is parallel along γ_- if and only if V is parallel along γ. But then, if V is the parallel extension of $v \in E_{p_0}$ along γ, then V^- is the parallel extension of $V(1) = \tilde{\gamma}(v) \in E_{p_1}$ along γ_-.

This implies that $\tilde{\gamma}_- = \tilde{\gamma}^{-1}$, so $\tilde{\gamma}$ is an isomorphism. □

The *parallel transport is defined also along piece-wise smooth paths*: it is enough to compose the parallel transport along the smooth pieces and use the final value of a piece as the initial condition for the following piece.

7.4 Relationship between connections and differential forms

It is possible to give an alternative definition of a connection, which is more suitable to be used than the previous definition in certain situations. This alternative formulation reveal a strong link between connections and differential forms.

We recall from def. 4.4.3 that a k-form on a manifold M is a section of $\Lambda^k(T^*M)$, i.e. a smooth assignment of an alternating tensor on T^*M and that the vector space of all k-forms on M is written either $\mathbb{A}^k(M)$ or $\Omega^k(M)$.

Consider now a vector bundle $\pi : E \to M$.

Def. 7.4.1 *A k-form with values in E is a section of $\Lambda^k(T^*M) \otimes E$. The vector space of all k-forms with values in E is denoted with either $\mathbb{A}^k(E)$ or $\Omega^k(E)$.*

In local coordinates, the general element of $\mathbb{A}^k(E)$ can be written as

$$\sum_i \omega_i \otimes s_i, \tag{7.3}$$

where s_i are sections of E, i.e. elements of $\mathbb{A}^0(E) \equiv \Gamma(E, M)$, while $\omega_i \in \mathbb{A}^k(M)$ are k-forms on M.

With these definitions and notations, an alternative definition of connection on E can be given as follows.

Def. 7.4.2 (Alternative definition of connection) *A connection on E is a $\mathbb{R}$-linear operator*

$$\nabla : \mathbb{A}^0(E) \equiv \Gamma(E, M) \to \mathbb{A}^1(E)$$

such that

$$\boxed{\nabla(fs) = f\nabla s + df \otimes s} \qquad \forall f \in \mathscr{C}^\infty(M),\ \forall s \in \Gamma(E, M). \tag{7.4}$$

The request expressed in (7.4) is the equivalent of the Leibniz rule in the present context. The first term of (7.4), i.e. $f\nabla s$ is immediate to understand: it is the function f not derived multiplied by the derivative of s, which is provided by ∇ itself.

To comprehend the reason underlying the second term, i.e. $df \otimes s$, notice that we expect s not derived 'multiplied by a derivative of f', and this derivative must provide a 1-form on M. Thanks to (7.3) we see that the only intrinsic way to achieve this is by taking as multiplication the tensor product and as 'derivative' of $f : M \to \mathbb{R}$ its differential, which, as we know, is a 1-form on M.

At first glance, this definition of connection, apart the request of a Leibniz-like behavior just discussed, seems quite unrelated to the original definition 7.3.1 because no vector field enters into play here. To understand the link between the two definitions we must consider the following pairing (which acts on E-values 1-forms and vector fields on M and gives back sections of E):

$$\begin{array}{rccc} \langle\ ,\ \rangle : & \mathbb{A}^1(E) \times \mathfrak{X}(M) & \longrightarrow & \mathbb{A}^0(E) \equiv \Gamma(E, M) \\ & \left(\alpha \equiv \sum\limits_i \omega_i \otimes s_i, X\right) & \longmapsto & \langle \alpha, X\rangle := \sum\limits_i \omega_i(X) s_i, \end{array}$$

perfectly well-defined because ω_i and X are dual objects, one belongs to the tangent and the other to the cotangent space to M, so that $\omega_i(X) \in \mathscr{C}^\infty(M)$.

The relationship between ∇_X and ∇ is then:

$$\nabla_X s = \langle \nabla s, X \rangle. \tag{7.5}$$

Once establish this, let us see how the novel definition of connection can be written in local coordinates. Let $(e_1, \ldots, e_r)$ a local frame for E on an open $U \subseteq M$, i.e. a set of r sections of E that, in every point $p \in U$, form a basis of the fiber E_p, then

$$\nabla e_j = \omega_j^k \otimes e_k, \qquad k = 1, \ldots, r,$$

where ω_j^k are 1-forms defined on the open U.

What just said is true for every open U, in the particular case when U is a chart domain for M, we have at disposal a local coordinate system $(U, \varphi = (x^1, \ldots, x^n))$ and the 1-forms $dx^1, \ldots, dx^n$ are a local basis of T^*M, hence we can represent the 1-forms ω_j^k as follows:

$$\omega_j^k = \sum_{i=1}^n \Gamma_{ij}^k dx^i$$

for suitable functions $\Gamma_{ij}^k \in \mathscr{C}^\infty(U)$. They are denoted like this because, as we shall see in a moment, they agree with the connection coefficients defined in 7.3.4. To verify this, we select as vector field $X = \partial_i$ and we compute the covariant derivative of e_j w.r.t. X by means of eq. (7.5):

$$\begin{aligned}
\nabla_{\partial_i} e_j &= \langle \nabla e_j, \partial_i \rangle = \langle \omega_j^k \otimes e_k, \partial_i \rangle \\
&= \langle \omega_j^k \otimes e_k, \partial_i \rangle \\
&\quad \text{definition of } \langle\ ,\ \rangle \\
&= \omega_j^k(\partial_i) e_k = \sum_{k=1}^r \Gamma_{hj}^k dx^h(\partial_i) e_k = \sum_{k=1}^r \Gamma_{hj}^k \delta_i^h e_k \\
&= \Gamma_{ij}^k e_k, \qquad k = 1, \ldots, r,\ i, j = 1, \ldots, n,
\end{aligned}$$

but then the functions Γ_{ij}^k satisfy eq. (7.1), i.e. the definition of connection coefficients. These considerations justify the following definition.

Def. 7.4.3 (Connection 1-form) *The matrix of 1-forms $\omega = (\omega_j^k)$, $j = 1, \ldots, n$, $k = 1, \ldots, r$, where*

$$\omega_j^k = \Gamma_{ij}^k dx^i$$

are 1-forms defined on the chart domain $(U, (x^1, \ldots, x^n))$, is called the connection 1-form associated to the connection ∇ w.r.t. the local frame selected.

As always, it is important to establish how the expression of ω changes when we change the local reference frame. Since a local frame is a basis of a vector space, if $(\tilde{e}_1, \ldots, \tilde{e}_r)$ is another local frame for E on the same chart domain U, there exists an invertible matrix $A = (a_h^k)$ of functions $a_h^k \in \mathscr{C}^\infty(U)$ such that:

$$\tilde{e}_h = a_h^k e_k,$$

where the a_h^k are smooth functions of the point $p \in U$ because the dependence of the fiber E_p on p is smooth.

Let $\tilde{\omega} = (\tilde{\omega}_i^h)$ the connection 1-form of ∇ w.r.t. the local frame $(\tilde{e}_1, \ldots, \tilde{e}_r)$, then, by using the multi-linearity of the tensor product, we have:

$$\nabla \tilde{e}_i = \tilde{\omega}_i^h \otimes \tilde{e}_h = \tilde{\omega}_i^h \otimes a_h^k e_k = a_h^k \tilde{\omega}_i^h \otimes e_k,$$

on the other side, thanks to Leibniz's rule, we also have:

$$\begin{aligned}\nabla \tilde{e}_i &= \nabla(a_i^k e_k) = a_i^k \nabla e_k + da_i^k \otimes e_k = a_i^k \omega_k^\ell \otimes e_\ell + da_i^k \otimes e_k \\ &\quad \text{indices change :} k \leftrightarrow j,\ \ell \leftrightarrow k \\ &= a_i^j \omega_j^k \otimes e_k + da_i^k \otimes e_k \\ &= (a_i^j \omega_j^k + da_i^k) \otimes e_k.\end{aligned}$$

Since the vector basis (e_k) of the two expressions of $\nabla \tilde{e}_i$ that we have determined are the same, the coefficients must agree, this implies that:

$$a_h^k \tilde{\omega}_i^h = a_i^j \omega_j^k + da_i^k, \qquad \forall k = 1, \ldots, r,\ i = 1, \ldots, n.$$

We notice that $a_h^k \tilde{\omega}_i^h$ is nothing but the matrix product between A and $\tilde{\omega}$, while (notice the indices position) $a_i^j \omega_j^k$ is the matrix product between ω and A, so, in matrix notation, the previous transformation law can be written as follows:

$$A\tilde{\omega} = \omega A + dA,$$

or

$$\boxed{\tilde{\omega} = A^{-1}\omega A + A^{-1} dA}, \tag{7.6}$$

an expression that has a fundamental importance in gauge field theory.

Example of computation of covariant derivative: let us consider the simple case of a vector bundle of rank 1, i.e. a line bundle (each fiber is a straight line). In this case, the matrix $\omega = (\omega_j^k)$ is a 1×1 matrix of 1-forms, i.e. simply a 1-form

$$\omega = \omega_1^1 = \Gamma_{i1}^1 dx^i = \Gamma_i dx^i.$$

Hence, the connection 1-form in this case is simply a differential form, or covector:

$$\omega = (\Gamma_i) = (\Gamma_1, \ldots, \Gamma_n).$$

If $X = X^j \partial_j \in \mathfrak{X}(U)$ and $s = s^1 e_1 \in \Gamma(E, M)$, where $s^1 \in \mathscr{C}^\infty(U)$ and e_1 is a local basis of E in U, we have:

$$\nabla_X s = (X(s^1) + \Gamma_{j1}^1 X^j s^1) e_1 \equiv (X(s^1) + \Gamma_j X^j s^1) e_1,$$

since 1 is fixed, so the only running index for Γ is indeed j.

If we avoid the specification of the basis element e_1 and we simplify the expression by writing simply s instead of s^1, we get:

$$\nabla_X s = X(s) + \Gamma_j X^j s.$$

Finally, if we choose as particular vector field $X = \partial_i$, then the covariant derivative takes the following form:

$$\nabla_{\partial_i} s = \partial_i s + \Gamma_i s,$$

which shows that, for line bundles, the covariant derivative is simply the partial derivative plus an extra term proportional to the section itself by the (only) connection coefficient.

7.5 Induced connection on tensor bundles

In the same way as we have extended the concept of Lie derivative from vector to tensor fields by forcing the Leibniz rule to be satisfied, we can extend the concept of connection to tensor bundles. The following proposition state this rigorously.

Theorem 7.5.1 *Let M be a smooth manifold and ∇ a connection on TM. Then, it exists a unique way to define a connection ∇ on $T^p_q M$, $\forall p, q$, that satisfies the following properties:*

1. *∇ coincides with the given connection on TM (i.e. it is an actual extension of ∇, this is why we keep the same symbol)*

2. *on $T^0 M \equiv \mathscr{C}^\infty(M)$ the action of ∇ is simply the usual derivation implemented by a vector field, i.e. $\nabla_X(f) = X(f)$, $\forall X \in \mathfrak{X}(M)$*

3. *if $t_j \in T^{h_j}_{k_j}(M)$, $j = 1, 2$, and $X \in \mathfrak{X}(M)$, the following Leibniz rule holds:*

$$\nabla_X(t_1 \otimes t_2) = (\nabla_X t_1) \otimes t_2 + t_1 \otimes (\nabla_X t_2)$$

4. *∇ commutes with contractions.*

Moreover, if $\eta \in T_1(M) \equiv \mathbb{A}^1(M)$ and $X, Y \in \mathfrak{X}(M)$, the following Leibniz rule holds[2]*:*

$$X(\eta(Y)) = (\nabla_X \eta)(Y) + \eta(\nabla_X Y), \tag{7.7}$$

which gives a formula to compute ***the covariant derivative of a 1-form****:*

$$\boxed{(\nabla_X \eta)(Y) = X(\eta(Y)) - \eta(\nabla_X Y)}\ . \tag{7.8}$$

Proof. Let us verify the uniqueness. Suppose that ∇ satisfies the properties 1. $-$ 4. Then, given $\eta \in \mathbb{A}^1(M)$ and $X, Y \in \mathfrak{X}(M)$, we have that $\eta(Y)$ is a function belonging to $\mathscr{C}^\infty(M)$, thus, thanks to 2., $\nabla_X(\eta(Y)) = X(\eta(Y))$.

Now, using the Leibniz rule satisfied by ∇_X, we get eq. (7.7). This shows that the connection on TM determines uniquely the connection on T^*M.

Property 3. determines uniquely the connection on all the tensor bundles $T^h_k M$:

$$\begin{aligned}(\nabla_X t)(\omega^1, \dots, \omega^h, Y_1, \dots, Y_k) = {} & X(t(\omega^1, \dots, \omega^h, Y_1, \dots, Y_k)) \\ & - \sum_{r=1}^{h} t(\omega^1, \dots, \nabla_X \omega^r, \dots, \omega^h, Y_1, \dots, Y_k) \\ & - \sum_{s=1}^{k} t(\omega^1, \dots, \omega^h, Y_1, \dots, \nabla_X Y_s, \dots, Y_k).\end{aligned}$$

To show the existence, it is enough to define ∇ on T^*M and $T^h_k M$ as above, the fact that it is a connection is tautological because we have defined it by requiring the validity of the Leibniz rule (the other properties are automatically satisfied). $\square$

[2] The coupling $\eta(Y)$ must be thought as of product.

7.5.1 Explicit formulae for covariant derivatives of tensors relatives to linear connections

Given a smooth manifold M of dimension n, let ∇ be a linear connection on M, i.e. a connection on TM, and let $(U, \varphi \equiv (x^1, \dots, x^n))$ a local chart on M. We know that $(\partial_1, \dots, \partial_n)$ is a local frame for TM on the open set U and we can write

$$\boxed{\nabla_{\partial_i}\partial_j = \Gamma^h_{ij}\partial_h},$$

where Γ^h_{ij} are the Christoffel symbols.

Our aim is to find explicit formulae to compute the covariant derivative of any tensor in the case of a linear connection.

We have seen that, if $Y \in \mathfrak{X}(M) = T^1_0(M)$, $Y = Y^j\partial_j$, then, by Leibniz's rule:

$$\nabla_{\partial_i}Y = \nabla_{\partial_i}(Y^j\partial_j) = (\partial_i Y^j)\partial_j + Y^j\nabla_{\partial_i}\partial_j = (\partial_i Y^h)\partial_h + Y^j\Gamma^h_{ij}\partial_h = (\partial_i Y^h + Y^j\Gamma^h_{ij})\partial_h,$$

thus, the **components of the covariant derivative of a vector field** $Y \in \mathfrak{X}(M)$ can be explicitly written as follows:

$$\boxed{\nabla_{\partial_i}(Y^h) = \partial_i Y^h + \Gamma^h_{ij}Y^j}, \tag{7.9}$$

i.e. **the sum of the usual derivative, plus an extra term containing the components of the vector field multiplied by the Christoffel symbols of the linear connection**.

Let us now repeat the computation by considering 1-forms, i.e. the cotangent space. Let $(dx^1, \dots, dx^n)$ be a local frame for T^*M on U, then $\nabla_{\partial_i}dx^j = \tilde{\Gamma}^j_{ih}dx^h$, where $\tilde{\Gamma}^j_{ih}$ is another set of Christoffel symbols. Notice that now the running index for the sum, h, is positioned below, while before it was positioned above.

The Christoffel symbols $\tilde{\Gamma}^j_{ih}$ and Γ^j_{ih} are of course related and to make their relation explicit we just have to recall that $\langle dx^j, \partial_h\rangle = \delta^j_h$ which is a constant (either 0 or 1), thus $\partial_i\langle dx^j, \partial_h\rangle = 0$. Recalling that the action of ∇_{∂_i} on a smooth scalar function is the same as the action of ∂_i, we get $0 = \partial_i\langle dx^j, \partial_h\rangle = \nabla_{\partial_i}\langle dx^j, \partial_h\rangle$, so, thanks to Leibniz's rule and the bilinearity of the pairing $\langle\ ,\ \rangle$ we have:

$$\begin{aligned} 0 &= \nabla_{\partial_i}\langle dx^j, \partial_h\rangle = \langle \nabla_{\partial_i}dx^j, \partial_h\rangle + \langle dx^j, \nabla_{\partial_i}\partial_h\rangle = \langle \tilde{\Gamma}^j_{i\ell}dx^\ell, \partial_h\rangle + \langle dx^j, \Gamma^k_{ih}\partial_k\rangle \\ &= \tilde{\Gamma}^j_{i\ell}\langle dx^\ell, \partial_h\rangle + \Gamma^k_{ih}\langle dx^j, \partial_k\rangle = \tilde{\Gamma}^j_{i\ell}\delta^\ell_h + \Gamma^k_{ih}\delta^j_k = \tilde{\Gamma}^j_{ih} + \Gamma^j_{ih}, \end{aligned}$$

which implies $\tilde{\Gamma}^j_{ih} = -\Gamma^j_{ih}$, i.e. **the Christoffel symbols that appear in the covariant derivative of the differential form dx^j are exactly the opposites of those appearing in the covariant derivative of the vector field ∂_j**. This implies that:

$$\boxed{\nabla_{\partial_i}dx^j = -\Gamma^j_{ih}dx^h}.$$

As a consequence, if $\omega = \omega_j dx^j$ is a 1-form, we have:

$$\begin{aligned} \nabla_{\partial_i}\omega &= \nabla_{\partial_i}(\omega_j dx^j) \underset{\nabla_{\partial_i}(\omega_j) = \partial_i(\omega_j)}{=} (\partial_i\omega_j)dx^j + \omega_j\nabla_{\partial_i}dx^j \\ &= (\partial_i\omega_h)dx^h + \omega_j(-\Gamma^j_{ih}dx^h) = (\partial_i\omega_h - \Gamma^j_{ih}\omega_j)dx^h, \end{aligned}$$

thus, the **components of the covariant derivative of a 1-form** $\omega \in \mathbb{A}^1(M)$ can be explicitly written as follows:

$$\boxed{\nabla_{\partial_i}(\omega_h) = \partial_i \omega_h - \Gamma^j_{ih}\omega_j}\,, \tag{7.10}$$

i.e. **the sum of the usual derivative, minus an extra term containing the components of the 1-form multiplied by the Christoffel symbols of the linear connection**.

Since a vector field is a tensor field $X \in T^1_0(M)$ and a 1-form is a tensor field $\omega \in T^0_1(M)$, by comparing eqs. (7.9) and (7.10) it is not difficult to imagine, by multilinearity of the tensor, that the explicit formula for the components of the covariant derivative of a tensor field $t \in T^1_1(M)$ is just the usual derivative with two extra terms proportional to the tensor field, with coefficients given by the Christoffel symbols of the connections with plus and minus sign.

To verify this guess, we write $t = t^h_k \partial_h \otimes dx^k$, where the coefficient functions t^h_k are smooth on U. Then we have:

$$\begin{aligned}
\nabla_{\partial_i} t = \nabla_{\partial_i}(t^h_k \partial_h \otimes dx^k) &= (\partial_i t^h_k)\partial_h \otimes dx^k + t^h_k(\nabla_{\partial_i}\partial_h) \otimes dx^k + t^h_k \partial_h \otimes (\nabla_{\partial_i} dx^k) \\
&\quad \text{(thanks to (7.9), (7.10))} \\
&= (\partial_i t^h_k)\partial_h \otimes dx^k + t^h_k \Gamma^\ell_{ih}\partial_\ell \otimes dx^k + t^h_k \partial_h \otimes (-\Gamma^k_{i\ell} dx^\ell) \\
&\quad \text{(exchanging } k \leftrightarrow \ell\text{)} \\
&= (\partial_i t^h_k)\partial_h \otimes dx^k + \Gamma^h_{i\ell} t^\ell_k \partial_h \otimes dx^k - \Gamma^\ell_{ik} t^h_\ell \partial_h \otimes dx^k \\
&= (\partial_i t^h_k + \Gamma^h_{i\ell} t^\ell_k - \Gamma^\ell_{ik} t^h_\ell)\partial_h \otimes dx^k
\end{aligned}$$

thus, in components:

$$\boxed{\nabla_{\partial_i}(t^h_k) = \partial_i t^h_k + \Gamma^h_{i\ell} t^\ell_k - \Gamma^\ell_{ik} t^h_\ell}\,, \tag{7.11}$$

which shows that the covariant derivative of a tensor field of type $\binom{1}{1}$ is the usual derivative plus two extra terms involving linear combinations of the tensor components with the Christoffel symbols, notice the difference of sign w.r.t. the position, above or below, of the running index for the sum.

By repeating this same computations for a tensor field of type $\binom{p}{q}$ we get the following explicit formula for the covariant derivative of the components:

$$\boxed{\begin{aligned}
\nabla_{\partial_i}(t^{h_1 h_2 \ldots h_p}_{k_1 k_2 \ldots k_q}) = {} & \partial_i t^{h_1 h_2 \ldots h_p}_{k_1 k_2 \ldots k_q} \\
& + \Gamma^{h_1}_{i\ell} t^{\ell h_2 \ldots h_p}_{k_1 k_2 \ldots k_q} + \Gamma^{h_2}_{i\ell} t^{h_1 \ell h_3 \ldots h_p}_{k_1 k_2 \ldots k_q} + \ldots \Gamma^{h_p}_{i\ell} t^{h_1 h_2 \ldots h_{p-1}\ell}_{k_1 k_2 \ldots k_q} \\
& - \Gamma^{\ell}_{ik_1} - t^{h_1 h_2 \ldots h_p}_{\ell k_2 \ldots k_q} - \Gamma^{\ell}_{ik_2} t^{h_1 h_2 \ldots h_p}_{k_1 \ell k_3 \ldots k_q} - \cdots - \Gamma^{\ell}_{ik_q} t^{h_1 h_2 \ldots h_p}_{k_1 k_2 \ldots k_{q-1}\ell}.
\end{aligned}}$$

To simplify the heavy notation, in literature we find also the symbol $t_{,i}$ to denote $\partial_i t$ and $t_{;i}$ to denote $\nabla_{\partial_i} t$, so that, for example for a vector field $Y = Y^h \partial_h$ we find the formula:

$$Y^h{}_{;i} = Y^h{}_{,i} + \Gamma^h_{ij} Y^j.$$

7.5.2 Covariant differential, hessian and divergence

Given a linear connection on M and a tensor field $t \in T^h_k(M)$, we define the covariant version of the differential as follows.

Def. 7.5.1 (Covariant differential) *The covariant differential or* ***total covariant derivative*** *is the operator:*

$$\begin{array}{rccc} \nabla : & T^h_k & \longrightarrow & T^h_{k+1} \\ & t & \longmapsto & \nabla t, \end{array}$$

$$\boxed{(\nabla t)(\omega^1, \ldots, \omega^h, Y_1, \ldots, Y_k, Y_{k+1}) := (\nabla_{Y_{k+1}} t)(\omega^1, \ldots, \omega^h, Y_1, \ldots, Y_k)}\ , \tag{7.12}$$

i.e. the covariant derivative w.r.t. the last vector field. If $\nabla t \equiv 0$, t is said to be a ***parallel tensor field****.*

Thanks to the covariant differential it is possible to define a parallel transport for tensors in the exactly analogous way that we introduced before for vector fields.

Let us now see how it is possible to extend two important objects of calculus in $\mathbb{R}^n$: the hessian and the divergence. In $\mathbb{R}^n$ the hessian is the square matrix that contains the second order partial derivatives of a scalar function; in the case of a smooth scalar function f on a manifold, its covariant derivative coincides with its differential, i.e. $\nabla f = df$ which is not a function anymore, but a differential form, thus, if we want to differentiate a second time, we must necessarily apply the covariant differential! These observations motivates the following definition of hessian.

Def. 7.5.2 (Hessian of a smooth scalar function) *Given $f \in \mathscr{C}^\infty(M)$ and a linear connection on M, the tensor field of type $\binom{0}{2}$ defined as:*

$$\nabla(\nabla f) = \nabla(df)$$

is called the hessian of f.

Let us provide a more explicit expression of the hessian. First of all, since $\nabla(\nabla f)$ is a 2-times covariant tensor, a bilinear form that must be applied to a couple of vector fields $X, Y \in \mathfrak{X}(M)$. Then,

$$\nabla(\nabla f)(X, Y) = \nabla_Y(\nabla(f))(X),$$

having used the definition of covariant differential, eq. (7.12), Y playing the role of the last vector field Y_{k+1}. Since $\nabla(f) = df$, we can rewrite the previous formula as

$$\nabla(\nabla f)(X, Y) = (\nabla_Y(df))(X),$$

but the formula to compute the covariant derivative of a 1-form is provided by eq. (7.8), which gives:

$$\nabla(\nabla f)(X, Y) = Y(df(X)) - df(\nabla_Y X),$$

but, by definition of differential, $df(X) = X(f)$ and $df(\nabla_Y X) = (\nabla_Y X)(f)$, so:

$$\boxed{\nabla(\nabla f)(X, Y) = Y(X(f)) - (\nabla_Y X)(f)}\ ,$$

which shows that the hessian is not simply the composition of the directional derivative of f w.r.t. to X and then w.r.t. Y, as provided by the first term, but there is also an extra term where the covariant derivative w.r.t. Y appears.

The expression of this explicit formula in coordinates will show us the link with the classical expression of the hessian. If $(U, \varphi \equiv (x^1, \ldots, x^n))$ is a local coordinate system in M, then, if we fix the basis $(\partial_1, \dots, \partial_n)$ of $TM|_U$, and take $X = \partial_i$ and $Y = \partial_j$, then we can associate to $\nabla(\nabla(f))$ a matrix whose (i, j) entry is given by:

$$\nabla(\nabla(f))(\partial_i, \partial_j) = \partial_j(\partial_i f) - (\nabla_{\partial_j}\partial_i)(f),$$

i.e.

$$\boxed{\nabla(\nabla(f))(\partial_i, \partial_j) = \partial^2_{ji} f - \Gamma^h_{ji}\partial_h f},$$

which shows that, if $M = \mathbb{R}^n$ with the classical flat connection $\nabla = d$ characterized by $\Gamma^h_{ji} \equiv 0$, we have that the hessian of $f \in \mathscr{C}^\infty(\mathbb{R}^n)$ is the matrix $\left(\frac{\partial^2 f}{\partial x^j \partial x^i}\right)_{ij}$. Instead, for a non-trivial manifold with a non-flat connection, an extra term involving the Christoffel symbols appears.

Let us now pass to the divergence: if X is a vector field on M and ∇ is a linear connection on M, then $\nabla X \in T^1_1(M)$, thus it is perfectly well-defined to contract this tensor of type $\binom{1}{1}$ w.r.t. its only covariant and contravariant index. What we obtain turns out to be the generalization of the classical divergence of a vector field on $\mathbb{R}^n$ to the case of a smooth manifold.

Def. 7.5.3 (Divergence of a vector field) *Given $X \in \mathfrak{X}(M)$ and a linear connection on M, the divergence of X is the smooth scalar function defined as follows:*

$$div(X) = C^1_1(\nabla X),$$

where C is the contraction operator.

As before, let us make this formula explicit by considering a vector field $X = X^k\partial_k$, then we know that

$$\nabla_{\partial_j} X = (\partial_j X^k + \Gamma^k_{jh} X^h)\partial_k,$$

from this it follows that the covariant differential can be written in every chart domain $(U, \varphi \equiv (x^1, \dots, x^n))$ as:

$$\nabla X = (dX^k + \Gamma^k_{jh} dx^j) \otimes \partial_k \in T^1_1(U).$$

We get the divergence of X by contracting the upper and bottom index, i.e. k and j, respectively, which can be done by renaming both of them as k and considering the implicit sum over k:

$$\boxed{\text{div}(X) = C^1_1(\nabla X) = \partial_k X^k + \Gamma^k_{kh} X^h}.$$

In the trivial case $M = \mathbb{R}^n$ with the flat connection, the Christoffel symbols are identically 0 and we obtain the classical formula of the divergence of a vector field, i.e.

$$\text{div}(X) = \sum_{k=1}^n \frac{\partial X^k}{\partial x^k}.$$

7.6 Compatibility between a linear connection and a (pseudo)-Riemannian metric

In this section we discuss the issue of compatibility between the definition of a linear connection on a manifold M, i.e. a connection defined on the tangent bundle TM of M and the Riemannian metric defined on M itself. We first need to formalize this concept.

Def. 7.6.1 *Let (M, g) be a (pseudo)-Riemannian manifold. A linear connection on M is compatible with the Riemannian metric g if, for all vector field $X, Y, Z \in \mathfrak{X}(M)$, it holds that:*

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{7.13}$$

The compatibility equation (7.13) is simply the request that a Leibniz-like behavior holds when ∇_X is applied to the scalar product of vector fields induced by the (pseudo)-Riemannian metric g.

Notice also that, since g is a (bilinear) smooth function, $\nabla_X g(Y, Z) = X(g(Y, Z))$.

The compatibility between a linear connection and a (pseudo)-Riemannian metric can be characterized in six other ways, which are listed in the following result.

Theorem 7.6.1 (Characterizations of compatibility connection-metric) *Let (M, g) be a (pseudo)-Riemannian manifold of dimension n and ∇ a linear connection on M. Then, the following assertions are equivalent.*

1. *∇ is compatible with g.*

2. *$\nabla g \equiv 0$, i.e. g is parallel w.r.t. ∇.*

3. *In all local coordinate system $(x^1, \ldots, x^n)$ it holds that:*

$$\partial_k g_{ij} = g_{\ell j}\Gamma^\ell_{ki} + g_{i\ell}\Gamma^\ell_{kj} \ . \tag{7.14}$$

4. *For every couple of vector fields V, W along the curve γ in M, it holds that[3]:*

$$\frac{d}{dt} g(V, W) = g(DV, W) + g(V, DW) \ . \tag{7.15}$$

5. *For all couple of vector field V, W parallel along γ, $g(V, W)$ is constant along γ.*

6. *The parallel transport defined by ∇ along each curve is an isometry, i.e. it is not only an isomorphism between all tangent spaces on the point traveled by the curve, but it also preserved the norms of tangent vectors and the distances between them.*

Proof. The strategy of the proof to demonstrate the equivalences is the following:

$$1. \iff 2. \quad 2. \iff 3. \quad 1. \implies 4. \quad 4. \implies 5. \quad 5. \implies 6. \quad 6. \implies 1.$$

$\boxed{1. \iff 2.}$

$\boxed{2. \iff 3.}$

$\boxed{1. \implies 4.}$

$\boxed{4. \implies 5.}$

$\boxed{5. \implies 6.}$

$\boxed{6. \implies 1.}$

□

[3]We recall that DV denoted the covariant derivative of V along the direction tangent to the curve γ, and analogously for DW.

7.7 The Levi-Civita connection

Chapter 8

Principal fiber bundles and applications to field theory (Dylan Russon)

Inspirational epithap wanted...

...

Fibre bundles play a major role in modern theoretical physics, whereas it is in general relativity or in the standard model of particle physics. This chapter will discuss largely fibre bundles and connections. In the first part of this chapter, we will define the notion of fibre bundle, beginning with the general fibre bundles and moving into the specific case of principal bundle where the notions of Lie groups defined in the previous chapter will play a central role. Then we will discuss the specific case (but important) of associated vector bundles. The second part will treat the notion of connections and covariant derivatives that are necessaries tools in gauge theories such as Yang-Mills theory.

8.1 Fibre Bundles

We already have encountered vector bundles, namely the tangent bundle TM and the cotangent bundle T^*M of a differential manifold M. For example, for the tangent space TM there was a natural projection $\pi : TM \to M$ that associate to each vector the point p in M at which it is tangent. The inverse image of any point p of M under π (called the fibre over p) was nothing more that the tangent space T_pM and vector fields could be defined as smooth cross-section of TM. We will generalize these notions in this section, going from the general definition of bundles to the specific case of vector bundles associated to principal bundles.

8.1.1 First definitions

First, let's give the proper definition of a bundle.

Def. 8.1.1 (Bundle) *A C^∞-bundle is the data of a surjective projection $\pi : E \to M$, where E and M are smooth manifolds and π is a C^∞-map. E is called the total space, M the base space and for every $p \in M$, $F_p := \pi^{-1}(\{p\})$ is called the fibre over p.*

In the following, we will sometimes denote a bundle by a greek letter like ξ. In this case, $E(\xi)$ will be the total space of the bundle and $M(\xi)$ will be its base space.

In the bundles we will treat, every fibre over $p \in M$ will be diffeomorphic to the same space F, in which case, we will talk about fibre bundle and F will be called the fibre of the bundle. This motivates the more specific definition.

Def. 8.1.2 (Fibre bundle) *Let F be a smooth manifold. The bundle $\pi : E \to M$ is said to be a fibre bundle if, for each $p \in M$, there is an open neighborhood $U \subset M$ and a diffeomorphism $h : U \times F \to E|_U := \pi^{-1}(U)$, called local trivialization of E such that we have the commutation*

$$\begin{tikzcd} E|_U \arrow[d, "\pi"] & U \times F \arrow[l, "h"'] \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

where pr_1 is the projection on U, or say differently $\pi(h(x, y)) = x$, for all $x \in U$ and $y \in F$.

A collection $\{(U_i, h_i)\}_i$ of local trivialization such that the open $\{U_i\}_i$ are covering M is called an atlas of the bundle. We have that if $U_i \cap U_j \neq \varnothing$, then for p in this intersection and $f \in F$, we can defined diffeomorphisms $\psi_{ij}(p) : F \to F$, called the bundle transition functions, such that $h_i^{-1} \circ h_j(p, f) = (p, \psi_{ij}(p)(f))$. These maps satisfy

1. $\psi_{ii}(p) = id_F$
2. $\psi_{ij}(p) = (\psi_{ji}(p))^{-1}$
3. $\psi_{ij}(p) \circ \psi_{jk}(p) = \psi_{ik}(p)$ for all U_i, U_j, U_k such that $U_i \cap U_j \cap U_k \neq \varnothing$.

We recognize what we had for TM where, to each $p \in M$, we had a local coordinate chart (U, ϕ) and we could define the map

$$\begin{array}{rcl} TM|_U & \longrightarrow & \phi(U) \times \mathbb{R}^m \\ v & \longmapsto & (x^1, \cdots x^p, v(x^1), \cdots, v(x^p)) \end{array}$$

The fact that the fibre of TM is $\mathbb{R}^m$, a vector space, tells us that we are in a special fibre bundle : a vector bundle.

Def. 8.1.3 (Vector bundle) *A vector bundle of rank k of base manifold M is a fibre bundle where the fibre is $\mathbb{R}^k$. More specifically, to each $p \in M$, $F_p = \pi^{-1}(\{p\})$ is a vector space of size k and there is an open neighborhood $U \subset M$ of p on which we have the local trivialization h that satisfies $\pi \circ h = pr_1$*

$$\begin{tikzcd} E|_U \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[l, "h"'] \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

We have in particular that, for each $p \in M$, by the local trivialization, the map $h_p : \{p\} \times \mathbb{R}^k \to \pi^{-1}(\{p\})$ is an isomorphism of vector spaces.

Now let's give some particular example of fibre bundle.

1. One of the most known fibre bundle is the Möbius strip where the base space is the circle S^1, and the fibre can be seen as a closed interval of $\mathbb{R}$. The total space can be regarded as a rectangle where the edge must be identified but in identifying opposite vertices. We can construct in the same spirit the Klein bottle.

   PUT THE FIGURE HERE

2. These two examples are in the case that the total space is "twisted" in some sense. A more simpler example is just considering the total space as the product of the base space with the fibre, and the projection map is just the projection on the base space, i.e. the bundle $pr_1 : M \times F \to M$.

3. If G is a Lie group and H is a Lie subgroup of G, then the bundle $\pi : G \to G/H$, where, $\forall\, g \in G$, we have $\pi(g) := gH$ is a fibre bundle with fibre H.

It is sometimes useful to see a bundle as a subspace of a bundle of reference.

Def. 8.1.4 (Sub-bundle) *We say that a bundle $\tilde{\pi} : \tilde{E} \to \tilde{M}$ is a sub-bundle of a bundle $\pi : E \to M$ if we have $\tilde{E} \subset E$, $\tilde{M} \subset M$ and if $\tilde{\pi}$ is the restriction of π to $\tilde{E}$.*

Now, as vector fields can be seen as cross-section of the tangent bundle TM, let's give the proper definition of a cross-section :

Def. 8.1.5 (Cross-section) *Let $\pi : E \to M$ be a bundle. A cross-section of the bundle is a map $\sigma : M \to E$ such that*

$$\begin{tikzcd} E \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "\sigma"] \end{tikzcd} \qquad \pi \circ \sigma = id_M$$

i.e. that for each point $p \in M$, its image $\sigma(p)$ is in the fibre $F_p = \pi^{-1}(\{p\})$

We can note that in the specific case to a product bundle $\pi : M \times F \to M$, by construction a cross-section σ gives rise to a unique function $\tilde{\sigma} : M \to F$ such that $\forall\, p \in M$, $\sigma(p) = (p, \tilde{\sigma}(p))$.

To end this part, let us give the definition of a bundle map :

Def. 8.1.6 (Bundle map) *Let $\pi_E : E \to M$ and $\pi_{E'} : E' \to N$ two bundles. A homomorphism of bundle is a pair of smooth maps (u, f) with $u : E \to E'$ and $f : M \to N$ such that we have the commutative diagram*

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi_E"] & E' \arrow[d, "\pi_{E'}"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

i.e. that we have $\pi_{E'} \circ u = f \circ \pi_E$.

In the case of vector bundles, we require in addition that the restriction of u *on the fibres to be linear, i.e. that for each* $p \in M$, $u_p : \pi_E^{-1}(\{p\}) \to \pi_{E'}^{-1}(\{f(p)\})$ *is an homomorphism of vector space.*

We can remark that the commutation $\pi_{E'} \circ u = f \circ \pi_E$ tells us that for all $p \in M$, $u(\pi_E^{-1}(\{p\})) \subset \pi_{E'}^{-1}(\{f(p)\})$ i.e. that the bundle maps sends fibers into fibers.

Now that we have defined bundle maps, a question arise whether it is possible or not to define the pull-back of a bundle. This is given by the following

Def. 8.1.7 (Pull-back) *Let* $\pi : E \to M$ *be a fibre bundle that we will denote by* β *and let* $f : M' \to M$ *be a map, where* M' *is another manifold. We define the pull-back of* β *to be the bundle* $\pi' : E' \to M'$, *denoted by* $f^*(\beta)$, *where*

1. M' *is the base space*
2. $E' := \{(x', e) \in M' \times E / f(x') = \pi(e)\}$
3. $\forall\, (x', e) \in E'$, $\pi'(x', e) = x'$

This gives rise to a bundle map (f_β, f) between the bundle $f^*(\beta)$ and β, where, for all $(x', e) \in E'$, $f_\beta(x', e) = e$. We can note that each fibre of $f^*(\beta)$ is diffeomorphic to the fibre of β so $f^*(\beta)$ is a fibre bundle of fibre F.

8.1.2 Principal bundles

There are special fibre bundles where the fibre is a Lie group G. These bundles have the particularity that we can associate to them, in a way that is to define, general bundles. But first, let's define what is a G-bundle.

Def. 8.1.8 (G-bundle) *Let* G *be a Lie group. We say that* $\pi : E \to M$ *is a* G*-bundle if* G *has a right action on* E *and if* $\pi : E \to M$ *is isomorphic to the bundle* $\rho : E \to E/G$ *where* E/G *is the space of the orbit given by the action of* G *on* E *and* ρ *is the canonical projection on the space of orbits.*

A principal bundle is thus a particular G-bundle in the following sense :

Def. 8.1.9 (Principal bundle) *A principal* G*-bundle is a* G*-bundle where the action of* G *on* E *is free.*

For the rest of this chapter, to emphasize that we have a principal map, we will denote the total space by P instead of E.

Note that in a principal G-bundle $\pi : P \to M$, we have a fibre bundle with fibre G. Indeed, if $x \in M$, and $p \in \pi^{-1}(x)$, $\pi^{-1}(x)$ is the orbit of p under the action of G. By the freedom of its action and by theorem **??**, we get that $\pi^{-1}(x)$ is isomorphic to G.

Now let's give a simple example of principal bundle.

If we consider the product bundle $pr_1 : M \times G \to M$, where the right action of G is simply the right multiplication : $\forall\, p \in M$ and $\forall\, x_0 \in G$, $(p, g_0)g := (p, g_0 g)$. This bundle is called the trivial principal bundle.

We would like to define principal bundle map as bundle maps that would preserve the group action. This is satisfied by requiring the map to be equivariant

Def. 8.1.10 (Principal bundle map) *Let $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to \tilde{M}$ be two principal G-bundles and let (u, f) be a bundle map. Then (u, f) is said to be a principal bundle map if $u : P \to \tilde{P}$ is G-equivariant as stated in definition **??**, i.e. we have, for all $p \in P$ and $g \in G$.*

$$u(pg) = u(p)g \tag{8.1}$$

*As for equation (**??**) we can generalize this in the case where $\pi : P \to M$ is a principal G-bundle, $\tilde{\pi} : \tilde{P} \to \tilde{M}$ a principal $\tilde{G}$-bundle and $\rho : G \to \tilde{G}$ a group homomorphism. Then the bundle map (u, f) is a principal bundle map if we have, for all $p \in P$ and $g \in G$*

$$u(pg) = u(p)\rho(g) \tag{8.2}$$

There is a particular case where (u, id_M) is a principal map between a pair of principal G-bundle $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to M$. Then in this case, u is an isomorphism.

By these principal bundle map, we can define a trivial principal G-bundle.

Def. 8.1.11 (Trivial principal bundle) *A principal G-bundle $\pi : P \to M$ is trivial if there is a principal bundle map from $\pi : P \to M$ to the product bundle $pr_1 : M \times G \to M$.*

There is a special characterization of trivial principal G-bundle when looking at cross-section. Mainly

Theorem 8.1.1 *A principal G-bundle is trivial if, and only if, it possesses a continuous cross-section.*

Proof. To do... □

8.1.3 Associated vector bundles

In this last part of this section, we will see how to associate a general bundle to a principal G-bundle by extended the action of the group on another manifold. First, we will define the G-product.

Def. 8.1.12 (G-product) *Let G be a Lie group and let X and Y two spaces on which G has a right-action given respectively by*

$$\begin{array}{rccl} \rho : & X \times G & \longrightarrow & X \\ & (x, g) & \longmapsto & \rho(x, g) = \rho_g(x) := xg \end{array}, \qquad \begin{array}{rccl} \theta : & Y \times G & \longrightarrow & Y \\ & (y, g) & \longmapsto & \theta(y, g) = \theta_g(y) := yg. \end{array}$$

Then, we can define the right action of G on the product space $X \times Y$ by the map

$$\begin{array}{rccl} \Theta_g : & X \times Y & \longrightarrow & X \times Y \\ & (x, y) & \longmapsto & \Theta_g(x, y) := (\rho_g(x), \theta_g(y)) = (xg, yg) \end{array}, \quad \forall\, g \in G$$

The G-product of X and Y is then the quotient of the product $X \times Y$ on the space of orbits of the action of Θ, i.e. that two elements (x, y) and (x', y') belongs to the same equivalence class if there exists $g \in G$ such that $x' = xg$ and $y' = yg$. We denote the G-product by $X \times_G Y$ and the equivalence class of $(x, y) \in X \times Y$ is written $[x, y]$.

In the case one of the space is G itself, then it can be shown that there is an diffeomorphism between $G \times_G Y$ and Y.

Now we have the key ingredients to define associated bundles.

Def. 8.1.13 (Associated bundle) *Let $\pi : P \to M$ a principal G-bundle and F a smooth manifold on which G acts on the left. We define its associated bundle through the action of G on F by the fibre bundle $\pi_F : P_F \to M$ with fibre F where*

- *$P_F := P \times_G F$ where the right action on this space is defined by*
$$(p, v)g := (pg, g^{-1}v) \tag{8.3}$$
- *π_F is defined by*
$$\pi_F([p, v]) = \pi(p) \tag{8.4}$$

We need to check that this bundle defined in this way is indeed a fibre bundle.

First, let's notice that π_F is well defined. If we take another representent $[p', v']$ of $[p, v]$, we have that there exists $g \in G$ such that $(p', v') = (pg, g^{-1}v)$ hence

$$\pi_F([p', v']) = \pi(p') = \pi(pg) = \pi(p) = \pi_F([p, v])$$

because p and pg belongs to the same orbit hence to the same fibre.

To see that $\pi_F : P_F \to M$ is indeed a fibre bundle. We need to find a local trivialization of P_F. Let's consider the atlas of the principal bundle $\pi : P \to M$ given by $\{(U_i, h_i\})_i$. Then, for every $x \in M$, there exists an open U such that we have the diffeomorphism $h : U \times G \to \pi^{-1}(U)$. We have thus the identification $U \times G \cong \pi^{-1}(U)$. Now let's consider $\pi_F^{-1}(U)$, we have :

$$\pi_F^{-1}(U) = \pi^{-1}(U) \times_G F \cong (U \times G) \times_G F = (U \times G \times F)/G = U \times (G \times_G F)$$

But we have the further identification that $G \times_G F \cong F$. Indeed, there is a diffeomorphism between $G \times_G F$ and F given by the map

$$\begin{array}{rccl} \iota : & G \times_G F & \longrightarrow & F \\ & [g, v] & \longmapsto & vg^{-1} \end{array}$$

- this map is well defined since, given two different representative $[g_1, v_1] = [g_2, v_2]$, then there exists $g \in G$ such that $g_2 = g_1 g$ and $v_2 = v_1 g$. Therefore, $v_2 g_2^{-1} = v_1 g (g_1 g)^{-1} = v_1 g g^{-1} g_1^{-1} = v_1 g_1^{-1}$.

- it is injective since, if $\iota([g,v]) = \iota([g',v'])$ then $vg^{-1} = v'g'^{-1}$. It follows that, applying the element $g^{-1}g'$ to g and v : $[g,v] = [gg^{-1}g', vg^{-1}g'] = [g', v'g'^{-1}g'] = [g', v']$. Hence we get injectivity

- the map is clearly surjective since, for all $v \in F$, we have $\iota([e,v]) = v$.

We finally get that $\pi_F^{-1}(U) \cong U \times F$ so the open covering of M defines also a local trivialization of P_F and the fibre at $x \in M$, $\pi_F^{-1}(x)$, is diffeomorphic to F. Hence $\pi_F : P_F \to M$ is a fibre bundle.

As we did for fibre bundle in general and principal bundle, let's define what is an associated bundle map.

Def. 8.1.14 (Associated bundle map) *Let $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to \tilde{M}$ two principal G-bundle with associated bundle $\pi_F : P \times_G F \to M$ and $\tilde{\pi}_F : \tilde{P} \times_G F \to \tilde{M}$ respectively and let (u,f) be a principal bundle map. An associated bundle map (u_F, f) between the pair of associated bundle is defined by*

$$u_F([p,v]) := [u(p), v] \tag{8.5}$$

This is well-defined since

$$u_F([pg, g^{-1}v]) = [u(pg), g^{-1}v] = [u(p)g, g^{-1}v] = [u(p), v] = u_F([p,v])$$

because u is equivariant as a principal bundle map. And it is a bundle map because we have, for all $[p,v] \in P \times_G F$

$$f \circ \pi_F([p,v]) = f \circ \pi(p)$$

$$\tilde{\pi}_F \circ u_F([p,v]) = \tilde{\pi}_F([u(p),v]) = \tilde{\pi}(u(p))$$

and $f \circ \pi(p) = \tilde{\pi}(u(p))$ since (u,f) is a bundle map.

Finally, we will get interest in the particular case of associated vector bundle where we replace the space F by a vector space V and requiring that the action of G on V is linear. Hence we require that the action

$$\begin{array}{rccc} \theta : & G & \longrightarrow & GL(V) \\ & g & \longmapsto & \theta_g \end{array}$$

is a representation of G in V.

More formally, if $\pi : P \to M$ is a principal G-bundle and V is a vector of dimension n on which G acts linearly, then the associated bundle $\pi_V : P \times_G V \to M$ can be given the structure of an n-dimensional real vector bundle. Indeed, if $x \in M$, let $p \in \pi^{-1}(\{x\})$ and define the homeomorphism

$$\begin{array}{rccc} \iota_p : & V & \longrightarrow & \pi_V^{-1}(\{x\}) \\ & v & \longmapsto & \iota_p(v) := [p,v] \end{array} \tag{8.6}$$

Then we define the operations

1. $\iota_p(v_1) + \iota_p(v_2) := \iota_p(v_1 + v_2), \quad \forall\, v_1, v_2 \in V$
2. $\lambda \iota_p(v) := \iota_p(\lambda v), \quad \forall \lambda \in \mathbb{R}, \forall\, v \in V$

This is well defined thanks to the linearity of the action of G on V, since if we take another element $p' \in \pi^{-1}(\{x\})$ such that $\iota'_p(v') = \iota_p(v)$, then we get

$$\begin{aligned}
[p', v'_1] + [p', v'_2] &= [p', v'_1 + v'_2] \\
&= [pg, g^{-1}v_1 + g^{-1}v_2] \\
&= [pg, g^{-1}(v_1 + v_2)] \\
&= [p, v_1 + v_2] \\
&= [p, v_1] + [p, v_2]
\end{aligned}$$

And let's end this section with an important example of principal bundle associated to a vector bundle : the bundle of frames.

Let $\pi : E \to M$ be a vector bundle of rank n. A frame at a point $p \in M$ is an ordered set of basis vectors for the vector space E_p. If we define by $\mathcal{F}_p$ the set of all frames at the point p, the bundle of frames $\mathcal{F}(E)$ is defined to be the disjoint union of all such spaces i.e. that a point in $\mathcal{F}(E)$ is a pair (p, b) where $p \in M$ and $b \in \mathcal{F}_p$ and the projection map $\pi_{\mathcal{F}} : \mathcal{F}(E) \to M$ is the function that takes a frame into the point in M to which it is attached.

In fact, since for each $p \in M$, there is an isomorphism between E_p and $\mathbb{R}^n$, a frame can be seen as a linear isomorphism. Indeed, let $\mathcal{B}$ be the canonical basis for $\mathbb{R}^n$, a frame b of E_p is uniquely determined by the image vectors of the vectors of $\mathcal{B}$ through a suitable isomorphism $\lambda : \mathbb{R}^n \to E_p$ which represents the base change.

We can define a natural free action of $GL(n, \mathbb{R})$ on $\mathcal{F}(E)$. If $\lambda \in \mathcal{F}_p$ represented by a matrix Λ and if α is an automorphism of $\mathbb{R}^n$ represented by a matrix A then we define the right action of $GL(n, \mathbb{R})$ on the fibres $\mathcal{F}_p$ by

$$\begin{array}{rrcl}
\theta : & \mathcal{F}_p \times GL(n, \mathbb{R}) & \longrightarrow & \mathcal{F}_p \\
& (\lambda, \alpha) & \longmapsto & \lambda \circ \alpha = \Lambda A
\end{array}$$

This action is transitive by unicity of base change and can be extended to the bundle $\mathcal{F}(E)$ by

$$\begin{array}{rrcl}
\Theta : & \mathcal{F}(E) \times GL(n, \mathbb{R}) & \longmapsto & \mathcal{F}(E) \\
& ((p, \lambda), A) & \longmapsto & (p, \theta(\lambda, A))
\end{array}$$

We can therefore see that $\pi_{\mathcal{F}} : \mathcal{F}(E) \to M$ is a $GL(n, \mathbb{R})$-principal bundle.

Indeed, the right action is free and the orbits coincide with the fibers. It remains to see that we have a local trivialization. Let $\{(U_i, \varphi_i)\}_i$ be a local trivialization for E where we have the diffeomorphisms $\varphi_i : E|_{U_i} \to U_i \times \mathbb{R}^n$ with restrictions on the fibers $\varphi_{i,p} : E_p \to \{p\} \times \mathbb{R}^n$. Then we can define the maps :

$$\begin{array}{rrcl}
\psi_i : & \mathcal{F}(E)|_{U_i} = \tilde{\pi}^{-1}(U_i) & \longmapsto & U_i \times GL(n, \mathbb{R}) \\
& (p, \lambda) & \longmapsto & (p, \varphi_{i,p} \circ \lambda)
\end{array}$$

These maps are invertible and differentiable with inverse differentiable so they are diffeomorphisms and $\{(U_i, \psi_i)_i\}$ is a local trivialization for $\mathcal{F}(E)$.

Consider a point p of M and U its open neighborhood, and $q$an element of $\mathcal{F}(E)|_U$. We can write $q = (p, \lambda)$. Then we have $\psi(q) = (p, \varphi_p \circ \lambda) = (\pi(q), h(q))$ for a certain function $h : \mathcal{F} \to GL(n, \mathbb{R})$. It remains to see that this map is equivariant w.r.t. the right action of the group :

$$\begin{aligned} h(\Theta(q, A)) &= h(\Theta((p, \lambda), A) = h(p, \theta(\lambda, A)) = h(p, \lambda \circ \alpha) \\ &= (p, \varphi_p \circ \lambda \circ \alpha) = (p, \theta(\varphi_p \circ \lambda, A)) = \Theta((p, \varphi_p \circ \lambda), A) \\ &= \Theta(h(p, \lambda), A) = \Theta(h(q), A) \end{aligned}$$

By local trivialization, we have that $\mathcal{F}/GL(n, \mathbb{R}) \cong M$ and each fibre is diffeormorphic to $GL(n, \mathbb{R})$. Hence $\tilde{\pi} : \mathcal{F}(E) \to M$ is a principal fibre of structure group $GL(n, \mathbb{R})$.

A special bundle of frames is the tangent frame bundle (called also the frame bundle of the manifold M) where the vector bundle in consideration is the tangent bundle. In this case, a local section is called a smooth local frame. One important example is that given a local coordinate chart $(U, \varphi = (x^1, \ldots, x^m))$ around a point $p \in M$, we have a basis of T_pM given by $(\partial_1|_p, \ldots, \partial_m|_p)$ so we can define a local section of TM by

$$\begin{array}{rccl} \partial_i : & U & \longrightarrow & TM \\ & p & \longmapsto & \partial_i(p) := \partial_i|_p \end{array}$$

The same can be done for the cotangent bundle.

8.2 Connection and parallel transport

The purpose of connections is to compare points belonging to different fibres in a way that is independent of a local trivialization. Hence, we are looking for vector fields that go for one fibre to another. In this section, we will define a connection in two ways, one as a collection of tangent spaces, the other as a differential one-form. We will first give the definition for general bundles then restrict ourselves to principal and associated bundle to conclude with parallel transport and curvature.

8.2.1 Connection of Ehresmann

Connection in a general bundle

First, let give the definition of a vertical subspace.

Def. 8.2.1 (Vertical subspace) *Let $\pi : E \to M$ be a bundle, and let $e \in E$. The vertical subspace V_eE of the tangent space T_eE is defined to be the kernel of the push-forward of π at e i.e.*

$$V_eE := \ker(\pi_*) = \{v \in T_eE, \pi_*(v) = 0\} \tag{8.7}$$

The elements of V_eE are called vertical. As $e \in E$ changes, these subspaces form a C^∞-subbundle VE of the bundle TE.

If E is an n-manifold and M a m-manifold, since π is a projective surjection, it is of constant rank m (i.e. that for all $e \in E$, $\pi_* : T_eE \to T_{\pi(e)}M$ is of rank m), then we have $\dim V_eE = n - m$.

Remark that V_eE can then be seen as the tangent space to the fiber $\pi^{-1}(\{\pi(e)\})$.

Now, as mentioned above, we want to look at vector fields that points away to the fibres, not tangent to it. This motivates the following definition :

Def. 8.2.2 (Ehresmann connection) *A general connection (or Ehresmann connection) on the bundle $\pi : E \to M$ is a smooth assignment to each point $e \in E$ of a vector subspace H_eE of T_eE such that*

$$T_eE = V_eE \bigoplus H_eE \tag{8.8}$$

The subbundle HE of TE associated to it is called the horizontal subbundle of TE and elements of H_eE are called horizontal.

Hence, this definition means that each vector $w \in T_eE$ can be written in a unique way as $w = v + h$ where $v \in V_eE$ and $h \in H_eE$. To emphasize this, we will sometimes write by $v = \mathrm{ver}(w)$ and $h = \mathrm{hor}(w)$ for respectively the vertical and horizontal components of w.

Also, by the definition of the vertical subspace V_eE of T_eE, we get that the restriction of π_* to H_eE is an isomorphism of vector space and we have $\dim H_eE = \dim T_{\pi(e)}M = m$.

We can give another definition of the connection in terms of a differential one-form. More precisely, we define a connection 1-form on a bundle $\pi : E \to M$ as a linear map

$$\Phi : TE \to VE \tag{8.9}$$

that satisfies

1. $\Phi \circ \Phi = \Phi$ (idempotent)
2. $\mathrm{Im}\Phi = VE$ (surjectivity)

This definition means that we can see the connection one-form as a projection of TE onto VE. In particular, if $e \in E$, then $\Phi_e : T_eE \to V_eE$ is the projection of T_eE into V_eE. The horizontal sub-bundle is thus defined by $HE := \ker \Phi$. These two different definitions are in fact equivalent and the decomposition (8.8) say that for all $e \in E$, Φ_e is simply the projection of T_eE on V_eE parallel to H_eE.

We can remark that, using the notation of ver and hor, for all $w \in T_eE$, we have $\Phi_e(w) = \mathrm{ver}(w)$.

Connection in a principal bundle

These definitions of vertical subspace and connection need to be slightly adapted in the framework of principal bundle to guarantee the action of the Lie group.

Since a principal bundle is a bundle, we have the same definition for the vertical subspace :

Def. 8.2.3 (Principal vectical subsapce) *Let $\pi : P \to M$ be a principal bundle of structure group G. For each $p \in P$, the vertical subspace V_pP of T_pP is defined to be the kernel of the linear push-forward π_* at p*

$$V_pP := \ker \pi_* = \{\tau \in T_pP / \pi_* \tau = 0\}$$

The particularity here is that V_pP can be identified with the Lie algebra $\mathfrak{g}$ of G. Let's introduced the necessary tools to show this.

Let $\Delta : P \times G \to P$ the usual right action of G on P given by $\Delta(p, g) = \delta_g(p) = pg$ where $\delta_g : P \to P$ is the usual diffeomorphism define in the previous chapter. Then, taking $\xi \in \mathfrak{g}$, we can define a curve in G locally : $t \mapsto \exp(t\xi) \in G$, for $t \in \mathbb{R}$ small enough, that passes though e the neutral element of G at $t = 0$ and tangent to ξ. Now letting this curve acts on $p \in P$, we obtain a curve on P given by $t \mapsto \delta_{\exp(t\xi)}(p) = p\exp(t\xi)$. This curve passes through p at $t = 0$ and so it is tangent to a vector of T_pP. This thus associates to each vector of $\mathfrak{g}$ a vector in T_pP by the map

$$\begin{array}{rccl} u_p : & \mathfrak{g} & \longrightarrow & T_pP \\ & \xi & \longmapsto & u_p(\xi) := \left.\dfrac{\mathrm{d}}{\mathrm{d}t} p\exp(t\xi)\right|_{t=0} \end{array} \tag{8.10}$$

By varying $p \in P$, we can define a map that associate to each vector $\xi \in \mathfrak{g}$ a vector field on P denoted by X^ξ whose value at $p \in P$ is given by $X^\xi_p := u_p(\xi)$.

$$\begin{array}{rccl} u : & \mathfrak{g} & \longrightarrow & \mathfrak{X}(P) \\ & \xi & \longmapsto & X^\xi \end{array} \tag{8.11}$$

In other words, we associate to the vector fields on G, whose integral curve is $\sigma_\xi : t \mapsto \exp(t\xi)$, the vector field on P whose integral curve is given by $t \mapsto p\exp(t\xi)$.

As $\mathfrak{g}$ and $\mathfrak{X}(P)$ are Lie algebras, we can notice that this u is a morphism of Lie algebras since it satisfies, for all $\xi, \eta \in \mathfrak{g}$

$$X^{[\xi,\eta]} = u([\xi, \eta]) = [u(\xi), u(\eta)] = [X^\xi, X^\eta]$$

We can thus show the identification mentioned above :

Theorem 8.2.1 *Let $\pi : P \to M$ be a principal bundle of group structure G and let $\mathfrak{g}$ be the Lie algebra of G. Then the map u_p defined above is an isomorphism between $\mathfrak{g}$ and V_pP.*

Proof. First notice that u_p is linear since for each $\xi \in \mathfrak{g}$, $u_p(\xi)$ is a derivative. Now let's prove injectivity and surjectivity.

- For injectivity, suppose that $\xi \in \ker(u_p)$, $\xi \neq 0$. Since the action of G on P is free, the only element that fixes the point p is the neutral element e. Hence, for $t \neq 0$, we will have $p\exp(t\xi) \neq p$ so the curve on P induced by $t \mapsto \exp(t\xi)$ is not constantly equal to p and the vector $u_p(\xi)$ tangent to this curve in p at $t = 0$ will be non-zero. This contradicts the fact that $\xi \in \ker(u_p)$ so we get that $\xi = 0$ and u_p is injective.

- The surjectivity follows immediately. Indeed, by a local trivialization of the principal bundle $\pi : P \to M$, $p \in P$ will correspond to a point $(x, g) \in M \times G$, hence the fibre $\pi^{-1}(\{x\})$ is diffeomorphic to G. By the dimension analysis seen for general bundle, we get that $\dim V_pP = \dim P - \dim M = \dim G$. Hence the restriction of u_p to the codomain V_pP, since u_p is linear, injective and $\dim V_pP = \dim G = \dim \mathfrak{g}$ guarantees that u_p is surjective.

□

Now, let's define the notion of a connection in the special case of a principal bundle.

Def. 8.2.4 (Connection in a principal bundle) *Let $\pi : P \to M$ a principal bundle with group structure G. A connection is a smooth assignment to each point $p \in P$ of the total space of a subspace H_pP of T_pP such that*

1. *$T_pP = V_pP \oplus H_pP$*
2. *the subspace H_pP is invariant by the action of the group G i.e.*

$$\delta_{g*}(H_pP) = H_{\delta_g(p)}P, \text{ for all } p \in P \text{ and } g \in G. \tag{8.12}$$

Another way of defining a connection is as a one-form

Def. 8.2.5 (Connection one-form) *Let $u_p^{-1} : V_pP \to \mathfrak{g}$ being the inverse of the isomorphism between $\mathfrak{g}$ and V_pP. We defined the connection one-form $\omega : TP \to P \times \mathfrak{g}$ of the principal G-bundle $\pi : P \to M$ as a $\mathfrak{g}$ valued one-form defined by, $\forall\, p \in P$:*

$$\begin{array}{rccl} \omega_p : & T_pP & \longrightarrow & \mathfrak{g} \\ & \tau & \longmapsto & \omega_p(\tau) := u_p^{-1}(\Phi_p(\tau)) \end{array} \tag{8.13}$$

where $\Phi_p : T_pP \to V_pP$ is the projection of T_pP on V_pP parallel to H_pP associated to the differential one-form $\Phi : TP \to VP$ defined in the same way as in (8.9).

This one-form satisfies several properties of which

Proposition 8.2.1

1. *If X^ξ is the vector field induced by u on ξ then, for all $p \in P$*

$$\omega_p(X_p^\xi) = \xi \tag{8.14}$$

2. *For all $g \in G$, $p \in P$ and $\tau \in T_pP$, we have*

$$(\delta_g^*\omega)_p(\tau) = Ad_{g^{-1}}(\omega_p(\tau)) \tag{8.15}$$

3. *$h \in H_pP$ if, and only if, $\omega_p(h) = 0$.*

Proof.

1. Since, for all $p \in P$, $X^\xi_p = u_p(\xi) \in V_pP$ by theorem 8.2.1, we have that

$$\omega_p(X^\xi_p) = u_p^{-1}(\mathrm{ver}(X^\xi_p)) = u_p^{-1}(X^\xi_p) = u_p^{-1}(u_p(\xi)) = \xi.$$

2. First, let's see the link between the action on V_pP and the action on $\mathfrak{g}$. Let $\xi \in \mathfrak{g}$ and let $g \in G$, we have

$$\delta_{g*}(u_p(\xi)) = \delta_{g*}\left(\frac{\mathrm{d}}{\mathrm{d}t} p\exp(t\xi)\right)\Bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\delta_g(p\exp(t\xi))\Bigg|_{t=0}$$

But we have

$$\begin{aligned}\delta_g(p\exp(t\xi)) &= p\exp(t\xi)g = pg(g^{-1}\exp(t\xi)g)\\ &= pg(C_{g^{-1}}(\exp(t\xi)) = pg\exp(t\mathrm{Ad}_{g^{-1}}(\xi))\end{aligned}$$

by equation (**??**). Hence

$$\delta_{g*}(u_p(\xi)) = \frac{\mathrm{d}}{\mathrm{d}t} pg\exp(t\mathrm{Ad}_{g^{-1}}(\xi))\Bigg|_{t=0} = u_{pg}(\mathrm{Ad}_{g^{-1}}(\xi)) \tag{8.16}$$

Furthermore, we have, for all $g \in G$, $p \in P$ and $\tau \in T_pP$

$$\begin{aligned}(\delta_g^*\omega)_p(\tau) &= \omega_{\delta_g(p)}(\delta_{g*}\tau)\\ &= u_{pg}^{-1} \circ \Phi_{pg}(\delta_{g*}\tau)\\ &= u_{pg}^{-1} \circ \delta_{g*} \circ \Phi_p(\tau)\end{aligned}$$

Because by (8.12) we have the commutation $\delta_{g*} \circ \Phi_p = \Phi_{pg} \circ \delta_{g*}$. Hence

$$\begin{aligned}(\delta_g^*\omega)_p(\tau) &= u_{pg}^{-1} \circ \delta_{g*} \circ u_p \circ u_p^{-1} \circ \Phi_p(\tau)\\ &= u_{pg}^{-1} \circ \delta_{g*} \circ u_p(\omega_p(\tau))\\ &= u_{pg}^{-1} \circ u_{pg}(\mathrm{Ad}_{g^{-1}}(\omega_p(\tau)))\\ &= \mathrm{Ad}_{g^{-1}}(\omega_p(\tau)).\end{aligned}$$

where in the penultima equality we use equation (8.16)

3. By definition, one of the implication is straightforward. The only if part is also direct because since u_p is an isomorphism between V_pP and $\mathfrak{g}$, if $\omega_p(h) = 0$ then this means that $\Phi(h) = 0$ hence $h \in H_pP$.

□

8.2.2 Covariant derivative and parallel transport

An important tool to compare points in is the use of parallel transport, that is a way to transport points from one fibre to another.

Parallel transport in a principal bundle

Let $\pi : P \to M$ be a principal G-bundle where G is a Lie group equipped with a connection ω. We have seen the basic definitions of vertical fields and horizontal fields. Vertical fields can be easily constructed but there is also a way to generate more explicitly horizontal fields from a given vector field : this is the operation of horizontal lifting. Indeed we have seen that for each $p \in P$, π_* is an isomorphism from H_pP to $T_{\pi(p)}M$. This yields to the following

Def. 8.2.6 (Horizontal lift) *Let X be a vector field on M. Then there exists a unique vector field, called the horizontal lift of X and that we denote by $X^\uparrow$ such that, for all $p \in P$*

1. $\pi_*(X_p^\uparrow) = X_{\pi(g)}$
2. $\Phi_p(X_p^\uparrow) = 0$

We can remark that since, for all $p \in P$, $X_p^\uparrow \in H_pP$, then we have that $\delta_{g*}(X_p^\uparrow) = X_{pg}^\uparrow$, i.e. the operation of horizontal lift is G-equivariant. In fact, this property and the point ii) of the definition guarantee that a vector field on P is the horizontal lifting of a vector field defined on M.

This operation of horizontal lifting satisfied several properties

1. $X^\uparrow + Y^\uparrow = (X+Y)^\uparrow$
2. $(fX)^\uparrow = f \circ \pi X^\uparrow$ for all $f \in C^\infty(M)$
3. $[X,Y]^\uparrow = \mathrm{hor}([X^\uparrow, Y^\uparrow])$

An analogue procedure of horizontal lifting can be applied to curves on the base manifold M.

Def. 8.2.7 (Horizontal lift of a curve) *Let $\sigma : [a,b] \to M$ be a smooth curve. We define the horizontal lifting of σ by the curve $\sigma^\uparrow : [a,b] \to P$ satisfying*

1. $\Phi_p([\sigma^\uparrow]) = 0$ *i.e that the curve is horizontal,*
2. $\pi(\sigma^\uparrow(t)) = \sigma(t)$ *for all $t \in [a,b]$.*

Here the last point is to satisfy the fact that applying the projection π to this curve on P will give us back the curve on M.

As for vector fields, we have seen that to each vector field on M we could associate a unique horizontal lift on P, we have also existence and uniqueness of horizontal lift of a curve

Theorem 8.2.2 *Let $\sigma : [a,b] \to M$ be a smooth curve on M. Then for all point $p \in \pi^{-1}(\{\sigma(a)\})$ of the fibre over $\sigma(a) \in M$, there exists a unique horizontal lift of σ such that $\sigma^\uparrow(a) = p$*

Proof. To do... □

This theorem shows us the spirit of our goal : we want to compare a point $p \in P$ of one fibre with another point $q \in P$ of another fibre. If we call $\pi(p) = m \in M$ and $\pi(q) = n \in M$, having a smooth curve on M, $\sigma : [a, b] \to M$ such that $\sigma(a) = m$ and $\sigma(b) = n$, then the horizontal lift of this curve $\sigma^{\uparrow}$ will allow us going from the fibre $\pi^{-1}(m)$ to the fibre $\pi^{-1}(n)$.

Def. 8.2.8 (Parallele translation) *Let $\sigma : [a, b] \to M$ be a smooth curve of M. We define the parallel translation along σ the map*

$$\begin{array}{rrcl} \tau : & \pi^{-1}(\{\sigma(a)\}) & \longrightarrow & \pi^{-1}(\{\sigma(b)\}) \\ & p & \longmapsto & \sigma^{\uparrow}(b) \end{array} \tag{8.17}$$

where $\sigma^{\uparrow}$ is the horizontal lift of σ that passes through p at $t = a$.

Connection and covariant derivative in an associated vector bundle

These notions of connection and parallel transport can be extended in the case of associated vector bundles. Let's see how the definitions of vertical and horizontal subspace (hence of connection) can be induced from the ones on a principal bundle.

Def. 8.2.9 *Let $\pi : P \to M$ be a principal G-bundle equipped with a connection ω and let $\pi_F : P_F \to M$ be its associated bundle through the action of G on F. Let $[p, v] \in P_F = P \times_G F$, we define*

1. *the vertical subspace $V_{[p,v]}(P_F)$ of the tangent space $T_{[p,v]}(P_F)$ by*

$$T_{[p,v]}(P_F) := \ker \pi_{F*} = \{\tau \in T_{[p,v]}(P_F) / \pi_{F*}\tau = 0\} \tag{8.18}$$

2. *the horizontal subspace $H_{[p,v]}(P_F)$ of the tangent space $T_{[p,v]}(P_F)$ by*

$$H_{[p,v]}(P_F) := k_{v*}(H_p P) \tag{8.19}$$

where

$$\begin{array}{rrcl} k_v : & P & \longrightarrow & P \times_G F \\ & p & \longmapsto & [p, v] \end{array} \tag{8.20}$$

The horizontal subspace is well defined. Indeed, we can notice that, for all $p \in P$, we have

$$k_{g^{-1}v} \circ \delta_g(p) = k_{g^{-1}v}(pg) = [pg, g^{-1}v] = [p, v] = k_v(p)$$

Hence $k_{g^{-1}v} \circ \delta_g = k_v$. Now if we choose $[p', v']$ another representative of the class $[p, v]$ (i.e. that there exists $g \in G$ such that $(p', v') = (pg, g^{-1}v)$) then, by the characterization of a horizontal subspace in the principal bundle, we have

$$\begin{aligned} k_{v'_*}(H_{p'}P) &= k_{g^{-1}v*}(H_{pg}P) \\ &= k_{g^{-1}v*}(\delta_{g*}(H_p P)) \\ &= (k_{g^{-1}v} \circ \delta_g)_*(H_p P) \\ &= k_{v*}(H_p P) \end{aligned}$$

Having a connection, we can also define the notions of lifting a curve and parallel transporting.

Let $\sigma : [a, b] \to M$ be a smooth curve on M and $[p, v]$ a point in the fibre $\pi_F^{-1}(\{\sigma(a)\})$. By theorem 8.2.2 there is a unique horizontal lift of σ on P such that $\sigma^{\uparrow}(a) = p$. We then define the horizontal lift of σ on $P \times_G F$ that passes to $[p, v]$ at $t = a$ to be the curve

$$\sigma_F^{\uparrow} := [\sigma^{\uparrow}, v] \tag{8.21}$$

Then the parallel translation along σ in the bundle $P \times_G F$ is simply the map

$$\begin{array}{rrcl} \tau_F : & \pi_F^{-1}(\{\sigma(a)\}) & \longrightarrow & \pi_F^{-1}(\sigma(b)\}) \\ & [p, v] & \longmapsto & \sigma_F^{\uparrow}(b) = [\sigma^{\uparrow}(b), v] \end{array} \tag{8.22}$$

where $\sigma^{\uparrow}$ is such that $\sigma^{\uparrow}(a) = p$.

In the case of a vector bundle, this parallel transport allows to define a derivative of a cross-section in a way that is independent of any choice of a local trivialization : this is the covariant derivative.

Def. 8.2.10 (Covariant derivative) *Let $\pi : P \to M$ be a principal bundle with structure group G and let V be a vector space on which G acts. Consider a smooth curve $\sigma : [0, \varepsilon] \to M$ in M such that $\sigma(0) = x_0$ and let τ_V^t be the parallel translation map going from the fibre $\pi_V^{-1}(\{\sigma(t)\})$ to the fibre $\pi_V^{-1}(\{x_0\})$. Then if $\psi : M \to P \times_G V$ is a cross-section, we define the covariant derivative of ψ at x_0 in the direction σ by*

$$\nabla_\sigma \psi := \lim_{t \to 0} \left(\frac{\tau_V^t \psi(\alpha(t)) - \psi(x_0)}{t} \right) \tag{8.23}$$

Considering two curves σ_1 and σ_2 tangent at x_0, we have that $\nabla_{\sigma_1}\psi = \nabla_{\sigma_2}\psi$. Hence, we can extend the definition of the covariant derivative to tangent vectors in $T_{x_0}M$ by defining that if $v \in T_{x_0}M$ is a tangent vector and σ is one of these representative curve, then

$$\nabla_v \psi := \nabla_\sigma \psi$$

and going further, we define the covariant derivative along a vector field X on M by

$$(\nabla_X \psi)(x_0) := \nabla_{X_{x_0}} \psi$$

∇_X is a linear operator. It also satisfies the following properties

1. $\nabla_X(f\psi) = f\nabla_X\psi + X(f)\psi$
2. $\nabla_{fX+Y}\psi = f\nabla_X\psi + \nabla_Y\psi$

The covariant derivative of a cross-section gives a cross-section. By expressing it in local coordinates, we will make appear the Christoffel symbols.

Let $\pi : E \to M$ be a vector bundle of rank r and consider $(U, \varphi = x^1, \ldots, x^m)$ a local coordinate chart such that there is a local trivialization $h : U \times \mathbb{R}^r \to E|_U$. We can define a local basis of r local section defined by

$$\begin{array}{rccl} e_k: & U & \longrightarrow & E|_U \\ & p & \longmapsto & h(p,(0,\ldots,0,1,0,\ldots,0)) \end{array}$$

Thus, for each $p \in U$, $(e_1(p), \ldots, e_r(p))$ forms a basis for the fibre E_p.

Now we want to apply the covariant derivative to these sections along the vector fields formed by the standard local frame of TM $(\partial_1, \ldots, \partial_m)$ given by the local chart. The new sections obtained can be expressed in the local basis $(e_k)_k$ so we define

$$\nabla_{\partial_j} e_h = \Gamma^k_{jh} e_k$$

where $\Gamma^k_{jh} \in C^\infty(U)$. These functions are called the connection coefficients or, in the case of the tangent bundle, they are known as the Christoffel symbols.

We can then express the covariant derivative in local coordinates : let $s \in \Gamma(U)$ a local section and $X \in \mathfrak{X}(M)$ a vector field. Then we have the decomposition $X = X^j \partial_j$ and $s = s^h e_h$. The covariant derivative of s along X is then

$$\begin{aligned} \nabla_X s = \nabla_X (s^h e_h) &= X(s^h) e_h + s^h \nabla_X e_h \\ &= X(s^h) e_h + s^h \nabla_{X^j \partial_j} e_h \\ &= X(s^h) e_h + s^h X^j \nabla_{\partial_j} e_h \\ &= X(s^h) e_h + \Gamma^k_{jh} s^h X^j e_k \\ &= X(s^k) e_k + \Gamma^k_{jh} s^h X^j e_k \\ &= (X(s^k) + \Gamma^k_{jh} s^h X^j) e_k \end{aligned}$$

Actually, there is another standard definition of these connections in terms of connection one-forms that we can write in a matrix. The connection one-form defined in this way is a matrix of one-forms $\omega = [\omega^k_k]$ where

$$\omega^k_j = \Gamma^k_{ij} \dot{x}^i$$

are one-forms defined on the coordinate chart $(U, (x^1, \ldots, x^m))$ which is associated to ∇ w.r.t. the local frame.

Curvature

Let $\pi : P \to M$ a principal G-bundle and let $HP = \{H_pP / p \in P\}$ its connection. We have already defined $\Phi : TP \to VP$ the vertical projection. In the same spirit, let's define $h : TP \to HP$ the horizontal projection, i.e. for all $p \in P$, $h_p : T_pP \to H_pP$ is the projection on H_pP parallel to V_pP. By this, we can define the exterior covariant derivative of a form.

Def. 8.2.11 (Exterior covariant derivative) *If ω is a k-form on P, we define the exterior covariant derivative $D\omega$ to be horizontal $(k+1)$-form defined by*

$$D\omega(X_1, \ldots, X_{k+1}) = \mathrm{d}\omega(hX_1, \ldots, hX_{k+1}) \tag{8.24}$$

where $X_1, \ldots, X_{k+1}$ are vector fields on P.

and the curvature

Def. 8.2.12 (Curvature) *Let ω be a connection one-form defined on the principal G-bundle $\pi : P \to M$. We define the curvature two-form Ω to be the exterior covariant derivative of the connection, i.e.*

$$\Omega = D\omega \tag{8.25}$$

We can give an explicit formula of this curvature two-form through the Cartan structure equation :

Theorem 8.2.3 (Cartan structure equation) *Let ω be a connection one-form and let $\Omega = d\omega$ be its curvature two-form. If X and Y are vector fields on P, we have, for all $p \in P$:*

$$\Omega_p(X_p, Y_p) = \mathrm{d}\omega_p(X_p, Y_p) + [\omega_p(X_p), \omega_p(Y_p)] \tag{8.26}$$

Proof. Since we have the direct sum $TP = VP \oplus HP$, and since we have linear functions in the equation, it suffices to show the equation in three simple case.

1. X and Y are horizontal. This case is the easiest because we have then that $\omega(X) = 0$ and $\omega(Y) = 0$. For the remaining term, we have by definition :

   $$\Omega_p(X_p, Y_p) = D\omega(X_p, Y_p) = \mathrm{d}\omega(h_p(X_p), h_p(Y_p)) = \mathrm{d}\omega(X_p, Y_p)$$

   since X and Y are horizontal.

2. X and Y are vertical. By (8.11), there exist $\xi, \eta \in \mathfrak{g}$ such that $X = X^\xi$ and $Y = Y^\eta$. Now by equation (**??**) we have

   $$\mathrm{d}\omega(X^\xi, Y^\eta) = X^\xi(\omega(Y^\eta)) - Y^\eta(\omega(X^\xi)) - \omega([X^\xi, Y^\eta])$$

   But the point iii) of proposition 8.2.1 guarantees that $\omega_p(X_p^\xi) = \xi$ and $\omega_p(Y_p^\eta) = \eta$ so they are constants and applying a vector fields annihilate them. For the second term, we have that $\omega([X^\xi, Y^\eta]) = \omega(X^{[\xi,\eta]}) = [\xi, \eta]$ and hence the right hand side of equation (8.26) vanishes. On the other hand, the left hand side is automatically 0 since X and Y are vertical.

3. X is horizontal and Y is vertical. Since X is horizontal, then $\omega(X) = 0$ so the commutator vanishes and $\Omega(X, Y) = 0$ because Y is vertical. It remains to show that $\mathrm{d}\omega(X, Y) = 0$. Now, doing the same procedure as in the previous case, there exists $\eta \in \mathfrak{g}$ such that $Y = X^\eta$ and we can write

   $$\mathrm{d}\omega(X, X^\eta) = X(\omega(X^\eta)) - X^\eta(\omega(X)) - \omega([X, X^\eta]) = -\omega([X, X^\eta])$$

   because $\omega(X^\eta) = \eta$ is constant and $\omega(X) = 0$ because X is horizontal.

   But we have the Lie derivative $[X, X^\eta] = -\mathfrak{L}_{X^\eta} X = \lim\limits_{t\to 0} \dfrac{\delta_{\exp(t\eta)*}(X) - X}{t}$. Hence if X is horizontal, the right action on it gives also a horizontal vector field, and so the difference by vector space structure. This proves that $[X, X^\eta]$ is horizontal and then $\omega([X, X^\eta]) = 0$

□

To finish this chapter, let's rewrite this formula in a local way.

If we denote by $\mathcal{B} = \{v_1, \cdots, v_n\}$ a basis of the Lie algebra $\mathfrak{g}$. Then equation (8.26) can be rewritten as :

$$\Omega^a = \mathrm{d}\omega^a + \frac{1}{2}c^a_{bc}\omega^b \wedge \omega^c \tag{8.27}$$

Indeed, let write in the basis $\mathcal{B}$, $\omega = \omega^a v_a$ and $\Omega = \Omega^a v_a$. Hence, we have

$$\mathrm{d}\omega(X,Y) = \mathrm{d}\omega^a(X,Y)v_a \tag{8.28}$$

And

$$\omega(X), \omega(Y)] = [\omega^b(X)v_b, \omega^c(Y)v_c] = \omega^b(X)\omega^c(Y)[v_b, v_c] = \omega^b(X)\omega^c(Y)c^a_{bc}v_a$$

But

$$\begin{aligned} c^a_{bc}\omega^b(X)\omega^c(Y) &= \frac{1}{2}\left[c^a_{bc}\omega^b(X)\omega^c(Y) + c^a_{cb}\omega^c(X)\omega^b(Y)\right] \\ &= \frac{1}{2}\left[c^a_{bc}\omega^b(X)\omega^c(Y) - c^a_{bc}\omega^c(X)\omega^b(Y)\right] \\ &= \frac{1}{2}c^a_{bc}\omega^b(X) \wedge \omega^c(Y) \end{aligned}$$

Finally

$$[\omega(X), \omega(Y)] = \frac{1}{2}c^a_{bc}\omega^b(X) \wedge \omega^c(Y)v_a \tag{8.29}$$

And adding (8.28) and (8.29) we get the result.

We also have the famous Bianchi identity

$$D\Omega = 0.$$

An example : connection in a straight bundle

Let $\pi : E \to M$ a complex vector bundle of rank 1 (called a straight bundle where each fiber is diffeomorphic at $\mathbb{C}$) on a base manifold M and place a connection on E

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E).$$

If $U \subset M$ is an open, any local section of E will be $e_1 \in \Gamma(E)$ such that $\pi \circ e_1 = id_U$; in particular if e_1 it is not canceled out on U, then it constitutes a local frame for E: we have that every other section $s \in \Gamma(E)$ will be written in the form $s = s^1 e_1$ with $s^1 \in C^\infty(U)$ with values in $\mathbb{C}$. If we then identify U with the domain of a local chart (U, φ) associated to the local coordinates $x^1, \ldots, x^m$ on M, then the local description of the connection will be

$$\nabla_{\partial_i} e_1 = \Gamma_i e_1, \tag{8.30}$$

If we take $X = X^j\partial_j \in \mathfrak{X}(U)$ and $s = s^1 e_1 \in \Gamma(E)$ we find that

$$\begin{aligned} \nabla_X s = \nabla_X(s^1 e_1) &= X(s^1)e_1 + s^1\nabla_{X^j\partial_j}e_1 = X(s^1)e_1 + s^1(X^j\nabla_{\partial_j}e_1) \\ &= X(s^1)e_1 + s^1X^j\Gamma_j e_1 = (X(s^1) + X^j\Gamma_j s^1)e_1 \end{aligned}$$

and, without indicating the local frame, we get the simpler formula, $\nabla_X s^1 = X(s^1) + X^j\Gamma_j s^1$; if in addition $X = \partial_i$, then

$$\nabla_{\partial_i} s^1 = \partial_i s^1 + \Gamma_i s^1. \tag{8.31}$$

What is also interesting is how the connection change when we change the local frame. If $e_1, \tilde{e}_1$ are two different local frame of E on the open U we can express one in terms of the other as $\tilde{e}_1 = he_1$ with $h \in C^\infty(U)$ with complex values, then we will have $\nabla_{\partial_i} e_1 = \Gamma_i e_1$ and $\nabla_{\partial_i}\tilde{e}_1 = \tilde{\Gamma}_i e_1$; in particular the connection coefficients transform according to the following law:

$$\begin{aligned}\nabla_{\partial_i}\tilde{e}_1 &= \nabla_{\partial_i}(he_1) = (\partial_i h)e_1 + h\Gamma_i e_1 = (\partial_i h + h\Gamma_i)e_1\\ &= (\partial_i h + h\Gamma_i)h^{-1}\tilde{e}_1 = (\Gamma_i + h^{-1}\partial_i h)\tilde{e}_1,\end{aligned}$$

whence it is obtained

$$\Gamma_i' = \Gamma_i + h^{-1}\partial_i h.$$

We also know that a connection one-form is a matrix $\omega = \left[\omega_i^j\right]$ where ω_i^j are one-forms. In this case ω is a matrix of order 1, therefore it will be

$$\omega = \omega_1^1 = \Gamma_{i1}^1 \underset{\cdot}{\mathrm{x}}^i = \Gamma_i \underset{\cdot}{\mathrm{x}}^i.$$

And using $\Omega = D\omega$, we can find the curvature 2-form $\Omega = \Omega_{ij}\underset{\cdot}{\mathrm{x}}^i \wedge \underset{\cdot}{\mathrm{x}}^j$:

$$\begin{aligned}\Omega &= (\partial_j\Gamma_i\underset{\cdot}{\mathrm{x}}^j) \wedge \underset{\cdot}{\mathrm{x}}^i = (\partial_j\Gamma_i - \partial_i\Gamma_j)\underset{\cdot}{\mathrm{x}}^j \wedge \underset{\cdot}{\mathrm{x}}^i\\ &= (\partial_i\Gamma_j - \partial_j\Gamma_i)\underset{\cdot}{\mathrm{x}}^i \wedge \underset{\cdot}{\mathrm{x}}^j,\end{aligned}$$

from which we get

$$\Gamma_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i. \tag{8.32}$$

8.3 An application in Physics : the case of electromagnetism

In this section, we will treat a remarkable application of the fibre bundles and connections theory in a physical context: gauge theories. We will see that the Lagrangian of a free relativistic particle is invariant under the global action of group $U_1(\mathbb{C})$; by introducing a principal bundle with this structural group on the $\mathbb{R}^4$ manifold - spacetime - it will be possible to introduce a covariant derivative (i.e. a connection) that safeguards its shape even for a local type of action and subsequently defines a curvature. In this way it will be possible to reinterpret the force exerted by the electromagnetic field as the physical manifestation of this curvature.

8.3.1 Some recall on electromagnetism

Electromagnetism is the study of the electromagnetic force, an interaction between electrically charged particles, one of the four fundamental interactions. Originally, the electricity and magnetism were seen as two different forces but works done in particular by Maxwell and Faraday show that these two forces can be seen as two faces of a same interaction : this yields to electromagnetism. One of its particularities is that it is compatible with special relativity.

In the following, we will place ourselves in the spacetime $\mathbb{R}^4$ seen as a (pseudo)Riemannian manifold equipped with the Minkowski metric η with signature $(+, -, -, -)$.

The spacetime components $\mathbf{x} \in \mathbb{R}^4$ will be denoted x^μ with $x^0 = ct$ where c is the speed of light.We will boldly indicate the points of spacetime while we will mark the vectors in ordinary three-dimensional space with arrows. The Greek indices (as in x^μ) will vary from 0 to 3, indicating the spacetime components and in particular the variable $x^0 = ct$ - denoting c the speed of light.

The electric field and magnetic field will be indicating by $\vec{E} = \vec{E}(t, \vec{x}) = (E^1, E^2, E^3)$ and $\vec{B} = \vec{B}(t, \vec{x}) = (B^1, B^2, B^3)$. As for the charge density and current density $\rho = \rho(t, \vec{x})$ and $\vec{\jmath} = \vec{\jmath}(t, \vec{x}) = (j^1, j^2, j^3)$ can be put together in the quadrivector $\daleth = j^\mu = (c\rho, \vec{\jmath})$ called quadricurrent.

Now let us recall Maxwell's equations in their classical differential form that show the link between the magnetic field and electric field :

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{8.33a}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{8.33b}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{8.33c}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{\jmath} \tag{8.33d}$$

Since we have the identities of the analysis $\vec{\nabla} \times (\vec{\nabla} F) = 0$ and $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$, then by the last identity, equation (8.33b) suggests that we can introduce a vector field $\vec{A} = \vec{A}(t, \vec{x}) = (A^1, A^2, A^3)$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$, called potential magnetic field vector; in this way we have

$$\vec{B} = \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix} = \begin{bmatrix} \partial_2 A^3 - \partial_3 A^2 \\ \partial_3 A^1 - \partial_1 A^3 \\ \partial_1 A^2 - \partial_2 A^1 \end{bmatrix} = \vec{\nabla} \times \vec{A}. \tag{8.34}$$

Now expressing the magnetic field in terms of this potential magnetic field vector in equation (8.33c) we obtain

$$0 = \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right),$$

and by the first identity, we can introduce a scalar field $\varphi = \varphi(t, \vec{x})$, called scalar potential such that

$$\vec{E} = \begin{bmatrix} E^1 \\ E^2 \\ E^3 \end{bmatrix} = \begin{bmatrix} -\partial_t A^1 - \partial_1 \varphi \\ -\partial_t A^2 - \partial_2 \varphi \\ -\partial_t A^3 - \partial_3 \varphi \end{bmatrix} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi. \tag{8.35}$$

Now we can combine these two potentials into a single potential quadrivector of the electromagnetic field :

$$\mathbf{A} = A^\mu = (A^0, \vec{A}) = \left(\frac{\varphi}{c}, \vec{A}\right).$$

This potential quadrivector is not univocally determined : indeed is we consider the quadigradient of a scalar function $\Lambda : \mathbb{R}^4 \to \mathbb{R}$ there is an invariance (called gauge invariance) under the following transformation:

$$\left(\frac{\phi}{c}, \vec{A}\right) \longrightarrow \left(\frac{\phi'}{c}, \vec{A}'\right) = \left(\frac{\phi}{c} - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \vec{A} + \vec{\nabla}\Lambda\right).$$

The gauge consists in the choice of the Λ function. Two of the main choice of gauge is the Coulomb's gauge where we impose

$$\vec{\nabla} \cdot \vec{A} = 0$$

The other (the one that we will take here) is the Lorentz's gauge where we choose a Λ such that $\partial_\mu A^\mu = 0$, i.e.

$$\frac{1}{c^2}\frac{\partial \phi'}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$$

In this case, Maxwell's equations become

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\phi}{c} - \Delta\phi = \Box\phi = \mu_0 c\rho = \mu_0 J^0$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \Delta\vec{A} = \Box\vec{A} = \mu\vec{\jmath}.$$

where Δ indicates the Laplacian in the spatial components and $\Box$ indicates the Dalembertian operator. These two equations can be put together in the form

$$\Box A^\mu = \mu_0 J^\mu.$$

8.3.2 Elements of analytical mechanics

The dynamic of a physical quantity can be described by a physical quantity, called the action, on which we do a variational principle.

$$S_{A\to B} = \int_A^B \mathcal{L}(s)\mathrm{d}s$$

where s represent the position of the system in phase space. The least action principle states that, on all possible trajectories, the one taken effectively is the one making the action extremal. Usually, for a material point, the phase space is constituted with the position $\vec{x}$ and the speed $\dot{\vec{x}}$ of the material point and the action is written

$$S\left[\vec{x}(\cdot)\right] = \int_{t_0}^{t_1} L\left(\vec{x}(t), \dot{\vec{x}}(t)\right)\mathrm{d}t,$$

where $L : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is an appropriate Lagrangian function.

In classical mechanics, the Lagrangian is given as the difference between kinetic energy and potential energy:

$$L\left(\vec{x}, \dot{\vec{x}}\right) = \frac{1}{2}m|\dot{\vec{x}}|^2 - V(\vec{x}),$$

The Hamilton's variational principle states that the evolution of a system minimizes the action : this principle leads to Euler-Lagrange equations that describe the motion of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0.$$

In the case of field theory, the notion of particle transforms to the concept of fields that can be seen as functions on space time with complex value. A particle is then an expression of an excited state of the physical field. In this case, we take into account the interaction of the particle in the Lagrangian by terms of interaction with the fields. If the fields are denoted by $\psi_k : \mathbb{R}^4 \to \mathbb{C}$, then we have

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \int_\Omega \mathcal{L}\left(\psi_k(\mathbf{x}), \partial_\mu \psi_k(\mathbf{x})\right) \mathrm{d}\mathbf{x}$$

where $\mathcal{L}(\psi_k(\mathbf{x}), \partial_\mu \psi_k(\mathbf{x}))$ is the Lagrangian density and Ω is a subset of spacetime.

As before, we can define the action by the integral of the Lagrangian between to instant and doing Hamilton's principle, we derive the Euler-Lagrange equations in this case

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_k)} = \frac{\partial \mathcal{L}}{\partial \psi_k}. \tag{8.37}$$

Now, the importance of investigating the action is that we can deduce conservation laws by looking at invariance of the action. This is the famous Noether's theorem

Theorem 8.3.1 *When the equations of motion (or, equivalently, the action S) are invariant under a continuous symmetry, there is a conserved current when the equation of motions are satisfied.*

Note that we can have two types of symmetries : symmetries of the Lagrangian under a transformation of the coordinates and internal symmetries of the Lagrangian that do not correspond to transformations on spacetime coordinates but characteristic of a field. In the following development, a field will be seen as a section of a bundle on $\mathbb{R}^4$ and an internal symmetry will be when the group action will be on the fiber points and not the point of the base space.

In our case of interest, we will look at the particles of the electromagnetic field : a photon γ. To specify its state, we need its spacetime position on the manifold $\mathbb{R}^4$ but also the polarization of the wave. If you do a rotation on the direction of oscillation, then the photon will still be at the same coordinates : an internal symmetry is then given by the action of the group of rotation $U_1(\mathbb{C})$ acting on the polarization plane.

8.3.3 $U_1(\mathbb{C})$ gauge theory

We now begin to introduce the Abelian gauge theory with gauge group $U_1(\mathbb{C})$ which describe electromagnetism.

We consider the Lagrangian density of a free relativistic particle which do not interact with other fields or other particles. In this case, $\mathcal{L}$ will depend only in the field of the particle and its derivatives. If we call $\psi : \mathbb{R}^4 \to \mathbb{C}$ the complex scalar field of the particle of mass m, then the Lagrangian is given by

$$\mathcal{L}(\psi, \overline{\psi}, \partial_\mu \psi, \partial_\mu \overline{\psi}) = \frac{1}{2}\lambda \left[\eta^{\mu\nu} \partial_\mu \psi \partial_\nu \overline{\psi} - \frac{m^2 c^2}{\hbar^2} \psi \overline{\psi} \right], \tag{8.38}$$

This Lagrangian has the particularity that it is invariant if we act on the field by an element of the group $U_1(\mathbb{C})$. Indeed, let $\alpha \in \mathbb{R}$ and let do the transformations :

$$\psi \longrightarrow \psi' = \mathrm{e}^{\mathrm{i}\alpha} \psi, \qquad \overline{\psi} \longrightarrow \overline{\psi'} = \mathrm{e}^{-\mathrm{i}\alpha} \overline{\psi}, \tag{8.39}$$

then the Lagrangian $\mathcal{L}' = \mathcal{L}(\psi', \overline{\psi'}, \partial_\mu \psi', \partial_\mu \overline{\psi'})$ transforms as follows

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2} \left[\eta^{\mu\nu} \partial_\mu \psi' \partial_\nu \overline{\psi'} - \frac{m^2 c^2}{\hbar^2} \psi' \overline{\psi'} \right] \\ &= \frac{1}{2} \left[\eta^{\mu\nu} \partial_\mu (\mathrm{e}^{\mathrm{i}\alpha} \psi) \partial_\nu (\mathrm{e}^{-\mathrm{i}\alpha} \overline{\psi}) - \frac{m^2 c^2}{\hbar^2} (\mathrm{e}^{\mathrm{i}\alpha} \psi)(\mathrm{e}^{-\mathrm{i}\alpha} \overline{\psi}) \right] \\ &= \frac{1}{2} \left[\mathrm{e}^{\mathrm{i}\alpha} \mathrm{e}^{-\mathrm{i}\alpha} \eta^{\mu\nu} \partial_\mu \psi \partial_\nu \overline{\psi} - \frac{m^2 c^2}{\hbar^2} \psi \overline{\psi} \right] = \mathcal{L}. \end{aligned}$$

Now one of the problem in our model is that we have to take into account the locality principle i.e. that distant objects cannot have instantaneous mutual influence. In this case, we can ask how to compare physically the field in two distinct points $\psi(\mathbf{x})$, $\psi(\mathbf{y})$.

Mathematical description of the problem

One way to solve this problem mathematically is to attach to each point $\mathbf{x}$ of spacetime a copy of $\mathbb{C}$ to which belong the value of $\psi(\mathbf{x})$. To do so, we introduce a fibre bundle structure of base manifold $\mathbb{R}^4$: in particular, we introduce a $U_1(\mathbb{C})$-principal bundle $\pi : P \to \mathbb{R}$ to which we associate a vector bundle $\pi_L : L \to \mathbb{R}^4$ where $L = P \times_{U_1(\mathbb{C})} \mathbb{C}$ which is a line bundle, on which $U_1(\mathbb{C})$ acts with the classical multiplication.

In this case, the field ψ can be reinterpreted as a differential section of L. More precisely, to each open U of $\mathbb{R}^4$, there is $s = \psi : U \to L|_U = \pi^{-1}(U)$ such that $\pi_L \circ s = id_U$ and for all $\mathbf{x}$, $\psi(\mathbf{x}) \in L_{\mathbf{x}} = \pi^{-1}(\{x\}) \cong \mathbb{C}$.

As for the example made in the previous section, if we have a local frame $e_1 \in \Gamma(L)$, the section s will be written $s = \psi^1 e_1$, with $\psi^1 \in C^\infty(U)$ with value in $\mathbb{C}$. In all the following, we will identify the section with the field.

Now, a difference about what was done before is that it is reasonable to assume that the group action $U_1(\mathbb{C})$ depends on the point where it is applied. In fact, mathematically, this dependence is to take into account that the value $\psi(\mathbf{x})$ belong to the fibre $L_{\mathbf{x}}$. Then the problem of comparing to field at two distinct points can be overcome by defining a connection on L and then do a parallel translation to identify the 2 fibers $L_{\mathbf{x}}$ and $L_{\mathbf{y}}$.

Hence here, we replace the multiplication $\mathrm{e}^{\mathrm{i}\alpha}$ by $\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}$ where $\alpha : \mathbb{R}^4 \to \mathbb{R}$. The transformations of the field are then

$$\psi \longrightarrow \psi' = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\psi, \qquad \overline{\psi} \longrightarrow \overline{\psi'} = \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\overline{\psi}. \tag{8.40}$$

Now we need to see if the Lagrangian keeps its invariance if we apply this new transformation. For the derivatives of the field, we have the following transformations :

$$\begin{aligned} \partial_\mu \psi' &= \partial_\mu \left(\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\psi\right) = \mathrm{i}\partial_\mu \alpha(\mathbf{x})\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\psi + \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\partial_\mu \psi, \\ \partial_\mu \overline{\psi'} &= \partial_\mu \left(\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\overline{\psi}\right) = -\mathrm{i}\partial_\mu \alpha(\mathbf{x})\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\overline{\psi} + \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\partial_\mu \overline{\psi}; \end{aligned}$$

The invariance of the Lagrangian is lost !

In fact, the lost of the invariance can be understood as the fact that the standard directional derivative is no longer well defined on the field ψ seen as a section. The solution is to introduce a connection on the bundle L that will give us a covariant derivative ∇_μ to replace ∂_μ. This will give a properly derivative of the field ψ.

To do so, we introduce a field, called gauge field, which transforms in a way to preserve the invariance of the Lagrangian. We denote $\mathbf{A} = A_\mu$ (in fact, we are looking at the coefficients of the differential form corresponding to the field A^μ by $A_\mu = \eta_{\mu\nu}A^\nu$).

The transformation that we ask (and that we can retrieve, see below) is the following

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{q}\partial_\mu \alpha(\mathbf{x}). \tag{8.41}$$

where here q is the charge of the particle[1]. Then the connection is $\nabla : \mathfrak{X}(\mathbb{R}^4) \times \Gamma(L) \to \Gamma(L)$ which maps $(\partial_\mu, s) \longmapsto \nabla_\mu s$, where the operator ∇_μ is defined by the position

$$\nabla_\mu := \partial_\mu + \mathrm{i}qA_\mu.$$

Since we are in the case of a bundle of rank 1, we have the identification

$$\Gamma_\mu = \mathrm{i}qA_\mu$$

We can thus replace ∂_μ in the Lagrangian by ∇_μ to get

$$\mathcal{L}(\psi, \overline{\psi}, \partial_\mu\psi, \partial_\mu\overline{\psi}) \to \mathcal{L}(\psi, \overline{\psi}, \partial_\mu\psi, \partial_\mu\overline{\psi}, A_\mu) = \mathcal{L}(\psi, \overline{\psi}, \nabla_\mu\psi, \nabla_\mu\overline{\psi}).$$

We can remark that the action of $U(1)$ on ψ induce in fact an action on the local frame e_1. Indeed, if e_1 and $\tilde{e}_1$ are two local bases of L, then any section $s \in \Gamma(L)$ can be written $s = \psi^1 e_1 = \tilde{\psi}^1\tilde{e}_1$. Hence, we have

$$\psi^1 e_1 = s = \tilde{\psi}^1\tilde{e}_1 = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\psi\tilde{e}_1 \iff e_1 = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\tilde{e}_1,$$

[1] In general, if $\sigma : U \to P$ is a local section of a G-principal bundle $P \to M$, and if we define $A := \sigma^*(\omega)$ where ω is a connection of the principal bundle, then for every principal automorphism ϕ, there exists some $\Omega : U \to G$ such that for all $x \in U$, $\sigma(x) = \phi \circ \sigma(x)\Omega(x)$. In the case, the transformation of the local representative A can be written :

$$A_\mu(x) \mapsto \Omega(x)A_\mu(x)\Omega^{-1}(x) + \Omega(x)\partial_\mu\Omega^{-1}(x).$$

i.e. that the e_1 transform with the action of $U_1(\mathbb{C})$ as

$$e_1 \mapsto \tilde{e}_1 = \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} e_1.$$

We can also note that $U(1)$ also acts on the gauge potential, and this will allow us to get the transformation given by (8.41). Indeed, if we have two local bases of L, e_1 and $\tilde{e}_1$ such that $\tilde{e}_1 = \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}$ then we have $\Gamma_\mu = \mathrm{i}qA_\mu$ and $\Gamma'_\mu = \mathrm{i}qA'_\mu$. Let's see how Γ_μ and Γ'_μ are related :

$$\begin{aligned}
\Gamma'_\mu \tilde{e}_1 = \nabla_\mu \tilde{e}_1 = \nabla_\mu (\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} e_1) &= \partial_\mu (\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}) e_1 + \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} \nabla_\mu e_1 \\
&= -\mathrm{i}\partial_\mu \alpha(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} e_1 + \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} \Gamma_\mu e_1 \\
&= -\mathrm{i}\partial_\mu \alpha(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \tilde{e}_1 + \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} \Gamma_\mu \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \tilde{e}_1 \\
&= (\Gamma_\mu - \mathrm{i}\partial_\mu \alpha(\mathbf{x}))\, \tilde{e}_1,
\end{aligned}$$

hence we have $\Gamma'_\mu = \Gamma_\mu - \mathrm{i}\partial_\mu \alpha(\mathbf{x})$.

Now, it remains to see if indeed the new Lagrangian is invariant by the action of $U_1(\mathbb{C})$.

Since we have introduced a new field, we need to take it into account in the Lagrangian. We need to add another term $\mathcal{L}_{\mathbf{A}}$ that depends only on the gauge potential $\mathbf{A}$ and its derivatives and we construct it so that it is invariant under the transformation of the gauge potential. The new Lagrangian of electrodynamics is the following

$$\mathcal{L}_{ED}(\psi, \overline{\psi}, \nabla_\mu \psi, \overline{\nabla_\mu \psi}, \mathbf{A}, \partial_\mu \mathbf{A}) = \mathcal{L}(\psi, \overline{\psi}, \nabla_\mu \psi, \overline{\nabla_\mu \psi}) + \mathcal{L}_{\mathbf{A}}(\mathbf{A}, \partial_\mu \mathbf{A}).$$

Let's see if the first term of the Lagrangian is now invariant by the action under the transformations (8.40) and (8.41).

The new Lagrangian is written

$$\mathcal{L}(\psi, \overline{\psi}, \nabla_\mu \psi, \overline{\nabla_\mu \psi}) = \frac{1}{2} \left[\eta^{\mu\nu} \nabla_\mu \psi \overline{\nabla_\nu \psi} - \frac{m^2 c^2}{\hbar^2} \psi \overline{\psi} \right]$$

Let's see how the terms $\nabla_\mu \psi$ transforms by the action of the group :

$$\begin{aligned}
\nabla'_\mu \psi' = (\partial_\mu + \mathrm{i}qA'_\mu)\psi' &= (\partial_\mu + \mathrm{i}qA_\mu - \mathrm{i}\partial_\mu \alpha(\mathbf{x})) \left(\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \psi \right) \\
&= \mathrm{i}\partial_\mu \alpha(\mathbf{x}) \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \psi + \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \partial_\mu \psi + \mathrm{i}qA_\mu \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \psi - \mathrm{i}\partial_\mu \alpha(\mathbf{x}) \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \psi \\
&= \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \left(\partial_\mu + \mathrm{i}qA_\mu \right) \psi = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})} \nabla_\mu \psi
\end{aligned}$$

Similarly, the term $\overline{\nabla_\mu \psi}$ is transformed according to the law

$$\overline{\nabla_\mu \psi} \longrightarrow \overline{\nabla'_\mu \psi'} = \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})} \overline{\nabla_\mu \psi}.$$

Finally, we have

$$\begin{aligned}
\mathcal{L}' &= \mathcal{L}(\psi', \overline{\psi'}, \nabla'_\mu \psi', \overline{\nabla'_\mu \psi'}) \\
&= \frac{1}{2}\left[\eta_{\mu\nu}\nabla'_\mu \psi' \overline{\nabla'_\nu \psi'} - \frac{m^2c^2}{\hbar^2}\psi'\overline{\psi'}\right] \\
&= \frac{1}{2}\left[\eta^{\mu\nu}\left(\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\nabla_\mu\psi\right)\left(\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\overline{\nabla_\mu\psi}\right) - \frac{m^2c^2}{\hbar^2}\left(\mathrm{e}^{\mathrm{i}\alpha(\mathbf{x})}\psi\right)\left(\mathrm{e}^{-\mathrm{i}\alpha(\mathbf{x})}\overline{\psi}\right)\right] \\
&= \frac{1}{2}\left[\eta^{\mu\nu}\nabla_\mu\psi\overline{\nabla_\nu\psi} - \frac{m^2c^2}{\hbar^2}\psi\overline{\psi}\right] = \mathcal{L},
\end{aligned}$$

Hence the new Lagrangian is now invariant by the action of the group $U(1)$.

Curvature and electromagnetic field

Now that we have a connection, we can look at the corresponding curvature 2-form $\Omega = \Omega_{\mu\nu}\underset{\cdot}{\mathrm{x}}^\mu \wedge \underset{\cdot}{\mathrm{x}}^\nu$ given by the formula (8.32)

$$\Omega_{\mu\nu} = \partial_\mu(\mathrm{i}qA_\nu) - \partial_\nu(\mathrm{i}qA_\mu) = \mathrm{i}q(\partial_\mu A_\nu - \partial_\nu A_\mu) = \mathrm{i}qF_{\mu\nu},$$

where we have set $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$[2]. It is the coefficients of an antisymmetric 2-covariant tensor. Hence $F = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is a differential 2-form. It is also an exact form because it can be written as the exterior derivative of the 1-form $A = A_\mu \mathrm{d}\mu$

$$\begin{aligned}
F &= \mathrm{d}A = (\partial_\nu A_\mu \mathrm{d}x^\nu) \wedge \mathrm{d}x^\mu \\
&= (\partial_\nu A_\mu - \partial_\mu A_\nu)\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.
\end{aligned}$$

We can remark that this tensor is invariant under the action (8.41) of the group $U(1)$

$$\begin{aligned}
F_{\mu\nu} \longrightarrow F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu\left(A_\nu - \frac{1}{q}\partial_\nu\alpha(\mathbf{x})\right) - \partial_\nu\left(A_\mu - \frac{1}{q}\partial_\mu\alpha(\mathbf{x})\right) \\
&= \partial_\mu A_\nu - \frac{1}{q}\partial_\mu\partial_\nu\alpha(\mathbf{x}) - \partial_\nu A_\mu + \frac{1}{q}\partial_\nu\partial_\mu\alpha(\mathbf{x}) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.
\end{aligned}$$

In terms of matrix representation, $F = \left[F_{\mu\nu}\right]$, can be written in the form

$$F = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix}.$$

[2] In the case the group action is an non-abelian group, we would have additional terms with the commutator on the field. Mainly, we would have

$$F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - \mathrm{i}q[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

.

Now seeing the components A_μ as the components of the potential quadrivector of the electromagnetic field $\mathbf{A} = A^\mu = (\phi/c, \overrightarrow{A})$, and remembering that we can identify $\partial_0 = \frac{1}{c}\partial_t$, this matrix can be expressed in terms of the electric and magnetic fields. We have in particular

$$\begin{aligned}
F_{01} &= \partial_0 A_1 - \partial_1 A_0 = -\frac{1}{c}\partial_t A^1 - \partial_1\left(\frac{\phi}{c}\right) = \frac{1}{c}E^1 \\
F_{02} &= \partial_0 A_2 - \partial_2 A_0 = -\frac{1}{c}\partial_t A^2 - \partial_2\left(\frac{\phi}{c}\right) = \frac{1}{c}E^2 \\
F_{03} &= \partial_0 A_3 - \partial_1 A_0 = -\frac{1}{c}\partial_t A^3 - \partial_3\left(\frac{\phi}{c}\right) = \frac{1}{c}E^3 \\
F_{12} &= \partial_1 A_2 - \partial_2 A_1 = -\partial_1 A^2 + \partial_2 A^1 = -B^3 \\
F_{13} &= \partial_1 A_3 - \partial_3 A_1 = -\partial_1 A^3 + \partial_3 A^1 = B^2 \\
F_{23} &= \partial_2 A^3 - \partial_3 A_2 = -\partial_2 A^3 + \partial_3 A^2 = -B^1,
\end{aligned}$$

Finally we get the tensor

$$F = \begin{bmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^2 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{bmatrix}$$

which is the Faraday tensor. Hence the electromagnetic fields is the manifestation of the curvature associated to the connection on the fibre bundle $\pi : L \to \mathbb{R}^4$. [3]

Maxwell's equations

Finally, let's go back to Lagrangian and see how we can retrieve Maxwell's equations.

In the Lagrangian, we have still one unknown in the choice of the added term of the Lagrangian $\mathcal{L}_\mathbf{A}$. This term must be a scalar and should be invariant under the action of the group $U(1)$. A solution is to put

$$\mathcal{L}_\mathbf{A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{8.42}$$

Finally, the total expression of the Lagrangian is

$$\begin{aligned}
\mathcal{L}_{ED} = \mathcal{L} + \mathcal{L}_\mathbf{A} &= \frac{1}{2}\left[\eta^{\mu\nu}\nabla_\mu\psi\overline{\nabla_\nu\psi} - \frac{m^2c^2}{\hbar^2}\psi\overline{\psi}\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \frac{1}{2}\left[\eta^{\mu\nu}(\partial_\mu + \mathrm{i}qA_\mu)\psi(\partial_\nu - \mathrm{i}qA_\nu)\overline{\psi} - \frac{m^2c^2}{\hbar^2}\psi\overline{\psi}\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \frac{1}{2}\eta^{\mu\nu}[\partial_\mu\psi\partial_\nu\overline{\psi}] + \frac{1}{2}\eta^{\mu\nu}\mathrm{i}q[A_\mu\psi\partial_\nu\overline{\psi} - A_\nu\overline{\psi}\partial_\mu\psi - \mathrm{i}qA_\mu A_\nu\psi\overline{\psi}] - \frac{m^2c^2}{\hbar^2}\psi\overline{\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.
\end{aligned}$$

Now, the choice of $\mathcal{L}_\mathbf{A}$ was made so that with this Lagrangian, we could retrieve Maxwell's equation. Indeed, let go in vacuum where there is no particle. In this case, the Lagrangian reduce to the term $\mathcal{L}_\mathbf{A}$. First we get

[3] The curvature is physically expressed as the force of the electromagnetic field.

$$\begin{aligned}\mathcal{L}_{ED} = \mathcal{L}_{\mathbf{A}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu)\\ &= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu).\end{aligned}$$

And the Euler-Lagrange equations associated with this Lagrangian are :

$$\partial_\nu \frac{\partial \mathcal{L}_{\mathbf{A}}}{\partial(\partial_\nu A_\mu)} = \frac{\partial \mathcal{L}_{\mathbf{A}}}{\partial A_\mu}. \tag{8.43}$$

Since the Lagrangian does not dependent on the components of the gauge potential, the second term vanishes. For the first term, we have

$$\frac{\partial \mathcal{L}_{\mathbf{A}}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{2}(\partial^\nu A^\mu - \partial^\mu A^\nu) = -\frac{1}{2}F^{\nu\mu} = \frac{1}{2}F^{\mu\nu},$$

Hence, Euler-Lagrange equations are finally given by

$$\partial_\nu F^{\mu\nu} = 0. \tag{8.44}$$

This equation in fact is a covariant form of two of the Maxwell's equations.

Indeed, take $\mu = 0$. In this case we obtain Maxwell-Gauss equation :

$$0 = \partial_\nu F^{0\nu} = \partial_0 F^{00} + \partial_i F^{0i} = -\frac{1}{c}\partial_i E^i \iff \vec{\nabla}\cdot\vec{E} = 0;$$

Now taking $\mu = i$, this will lead to Maxwell-Ampere equation :

For example, if $i = 1$, we find

$$0 = \partial_\nu F^{1\nu} = \partial_0 F^{10} + \partial_j F^{1j} = \frac{1}{c^2}\partial_t E^1 + \partial_2(-B^3) + \partial_3 B^2,$$

which is equivalent to $\partial_2 B^3 - \partial_3 B^2 - \frac{1}{c^2}\partial_t E^1 = 0$ and is the first component of the vector equation

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = 0;$$

The other components are obtained by taking $i = 2, 3$.

It remains to find the two last equations. This will be done thanks to Bianchi's identity. From the curvature 2-form $\Omega = \mathrm{i}qF_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, we have

$$\begin{aligned}0 = D\Omega = \mathrm{d}\Omega &= (\partial_\alpha \Omega_{\beta\gamma}\mathrm{d}x^\alpha) \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma\\ &= (\partial_\alpha\Omega_{\beta\gamma} - \partial_\alpha\Omega_{\gamma\beta} + \partial_\beta\Omega_{\gamma\alpha} - \partial_\beta\Omega_{\alpha\gamma} + \partial_\gamma\Omega_{\alpha\beta} - \partial_\gamma\Omega_{\beta\alpha})\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma\\ &= 2(\partial_\alpha\Omega_{\beta\gamma} + \partial_\beta\Omega_{\gamma\alpha} + \partial_\gamma\Omega_{\alpha\beta})\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma,\end{aligned}$$

Hence we get $\partial_\alpha\Omega_{\beta\gamma} + \partial_\beta\Omega_{\gamma\alpha} + \partial_\gamma\Omega_{\alpha\beta} = 0$, and this leads ultimately to

$$\partial_\alpha F_{\beta_\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{8.45}$$

We can then derive the two last equations :

- Maxwell-Flux is obtained by only looking at space variables i.e. taking $\alpha\beta\gamma = ijk$. We get
$$0 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1(-B^1) + \partial_2(-B^2) + \partial_3(-B^3),$$
which is nothing less than
$$\vec{\nabla} \cdot \vec{B} = 0;$$
- For Maxwell-Faraday, we take $\alpha\beta\gamma = 0ij$. In this case, setting for example $i = 2$ and $j = 3$, we have
$$0 = \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = \frac{1}{c}\partial_t(-B^1) + \frac{1}{c}\partial_2(-E^3) + \frac{1}{c}\partial_3(E^2),$$
i.e. $\partial_3 E^2 - \partial_2 E^3 + \frac{1}{c}\partial_t B^1$, which is the first component of the vector equation
$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0.$$
and the two other components are obtained by taking $\alpha\beta\gamma = 0\ 1\ 2$ and $\alpha\beta\gamma = 0\ 1\ 3$.

Part II:

Homogeneous spaces and hyperbolic geometry

The 'uproar of the Boeotians'.
(Attributed to) Carl Friedrich Gauss

Chapter 9

Homogeneous spaces (Antoine Guennec and Edoardo Provenzi)

A homogeneous space X is to be understood as a space that remains stable under a group of transformations G and such that its points are all 'connected' by the transformations of G. In the theory of homogeneous spaces, *the main attention is concentrated on the transformations of G and not on the elements of X*, the reason underlying this is is given by the stabilizer-orbit theorem, which says that X *can be reconstructed via a suitable quotient of G*. This result will allow us to exhibit extremely important examples of homogeneous spaces.

9.1 Preliminaries : group actions and linear transformation groups

9.1.1 Group actions

In this section we shall consider G to be a group and X a non-empty set. 1_G denotes the neutral element of G.

Def. 9.1.1 *The action of a group G on X is given by an operation*

$$\begin{array}{rccl} \eta : & G \times X & \longrightarrow & X \\ & (g,x) & \longmapsto & \eta(g,x) := g \cdot x \end{array}$$

which verifies, for all $x \in X$ and $g, h \in G$:

1. $1_G \cdot x = x$
2. $g \cdot (h \cdot x) = (gh) \cdot x$.

If we fix any element $g \in G$, the group action η induces a bijective function on X by $\eta_g : X \to X$, $x \mapsto \eta_g(x) = \eta(g,x)$, its inverse being obviously $\eta_{g^{-1}}$. This remark shows that, if G acts on X, then it can be seen as a subgroup of $\mathrm{Sym}(X)$, the group of all bijective functions on X and the action η can be equivalently characterized by the group homomorphism $\tilde{\eta} : G \to \mathrm{Sym}(X)$, $g \mapsto \tilde{\eta}(g) := \eta_g$. In fact, requiring $\tilde{\eta}$ to be a group homomorphism we assure that $\tilde{\eta}(1_G) = Id_X$, hence $\tilde{\eta}(1_G)(x) = x \ \forall x \in X$, and $\tilde{\eta}(gh) = \tilde{\eta}(g) \circ \tilde{\eta}(h) = \eta_g \circ \eta_h$, so $\tilde{\eta}(gh)(x) = (gh) \cdot x \ \forall x \in X$.

Example 9.1.1 *Some basic examples of group actions are listed below.*

1. The usual multiplication by a scalar belonging to the field $\mathbb{K}$ on which a vector space V is defined is a group action by interpreting $\mathbb{K}$ as G and V as X:

$$\lambda \cdot \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} \lambda v^1 \\ \vdots \\ \lambda v^n \end{pmatrix}, \qquad \lambda \in \mathbb{K},\ v = (v^1, \ldots, v^n)^t \in V.$$

2. The usual matrix multiplication of $GL(n, \mathbb{R})$ on the vector space $\mathbb{R}^n$ is a group action.

3. $X = \{1, 2, 3\}$. Then, any subgroup of S_3, the group of all permutations of X, for example $A_3 = \{Id, (123), (132)\}$, operates as a group action on X.

4. $X = D_{\mathbb{R}}(0, 1) = \{(x, y)^t \in \mathbb{R}^2 :\ x^2 + y^2 \leqslant 1\}$, the unit disk in $\mathbb{R}^2$, and

$$G = SO(2) = \left\{ \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix},\ \vartheta \in [0, 2\pi) \right\},$$

   the group of rotations in $\mathbb{R}^2$. Then, G operates on $D_{\mathbb{R}}(0, 1)$ by matrix multiplication.

5. $X = D_{\mathbb{C}}(0, 1) = \{z \in \mathbb{C} :\ |z| \leqslant 1\}$, the unit disk in $\mathbb{C}$, and

$$G = U(1) := \{e^{i\theta} :\ \theta \in [0, 2\pi)\},$$

   the group of rotations in $\mathbb{C}$. Then, G operates on $D_{\mathbb{C}}(0, 1)$ by matrix multiplication.

We now define the most important subspaces of X and G associated to the action of a group: the orbit and the stabilizer, respectively.

Def. 9.1.2 *Let the group G act on the set X and fix any $x \in X$.*

1. *The G-**orbit of** x is the subset of X given by:*

$$\mathrm{Orb(x)} = \{g \cdot x : g \in G\} \subset X,$$

   i.e. all the elements $y \in X$ that can be connected to x by a transformation $g \in G$: $y = g \cdot x$.

2. *The **stabilizer group of** x (or **isotropy subgroup**, or **little group** of x) is given by:*

$$\mathrm{Stab}(x) = \mathrm{G}_x = \{g \in G : g \cdot x = x\} \subset G,$$

   i.e. the set of transformations of G that act as the identity on x, leaving it unaltered.

The use of the word group for the stabilizer of $x \in X$ is not accidental, one can easily prove that $\mathrm{Stab}(x)$ is a subgroup of G.

Def. 9.1.3 (G-homogeneous space) *We say that X is a G-homogeneous space (or that G operates transitively on X) if it exists at least one $x \in X$ such that $X = \mathrm{Orb}(x)$, i.e for all $y \in X$ there exists an element $g \in G$ such that $g \cdot x = y$.*

It is easy to see that the request of existence of at least one element x of X whose G-orbit is the whole X is equivalent to the fact that the G-orbits of all the elements of X are the whole X. In fact, consider two arbitrary elements $y, \overline{x} \in X$, then there exist $g, \overline{g} \in G$ such that $g \cdot x = y$ and $\bar{g} \cdot x = \bar{x}$, i.e. $x = \bar{g}^{-1} \cdot \bar{x}$, so $(g\bar{g}^{-1}) \cdot \bar{x} = y$ thus also $\text{Orb}(\bar{x}) = X$. Hence, **a G-homogeneous space has only one G-orbit: X itself**!

Often, a G-homogeneous space X is defined by requiring that, for any couple $x, y \in X$, there exists at least an element $g \in G$ such that $g \cdot x = y$. The two definitions are of course equivalent.

Consequently, a homogeneous space is fully 'connected' by the group that operates upon it: any point of X reaches any other point via a group transformation. This property is often popularized by saying that, set-theoretically speaking, *in a homogeneous space, no point is more important than other*, which explains the adjective 'homogeneous'.

Example 9.1.2 Consider again the group $U(1)$, the unit complex disk $D_{\mathbb{C}}(0,1)$ and its contour $\partial D_{\mathbb{C}}(0,1) = \{z \in \mathbb{C} : |z| = 1\}$. Then, $\partial D_{\mathbb{C}}(0,1)$ **is trivially $U(1)$-homogeneous** because for any couple of points z, w on the unit circle in $\mathbb{C}$ separated by the angle θ, we have that $w = e^{i\theta} z$.

However, $D_{\mathbb{C}}(0,1)$ **is not $U(1)$-homogeneous**, in fact for any $z, w \in D_{\mathbb{C}}(0,1)$ and any $\theta \in [0, 2\pi)$, if we write $w = e^{i\theta} z$ then $|w| = |z|$, thus it is enough to consider two elements inside the unit disk with different modulus, e.g. $z = \frac{1}{2}$ and $w = \frac{2}{3} i$, to exhibit a couple of points of $D_{\mathbb{C}}(0,1)$ that cannot be connected by a transformation of $U(1)$.

The same considerations can be repeated in the real case to prove that the contour of the real unit disk, i.e. $\partial D_{\mathbb{R}}(0,1) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cong S^1$**, is $SO(2)$-homogeneous**, and, of course, also $O(2)$-homogeneous, but the disk itself $D_{\mathbb{R}}(0,1)$ **is not $SO(2)$-homogeneous**. In spite of the fact that $D_{\mathbb{R}}(0,1)$ is not homogeneous w.r.t. rotations, we will see that **it is homogeneous w.r.t. hyperbolic rotations**.

Example 9.1.3 The subgroup H of a group G is always a G-homogeneous space. In fact, the unit element 1_G of G belongs to H and it is connected with all the other elements of H. To see this, take any $h \in H$, then h belongs also to G, so $h = 1_G \cdot h$, which shows the transitivity of G on H.

We now come to the most important result of this section. To introduce it, we first recall that, given a group G, a fixed element $g \in G$ and a subgroup H of G, the **left coset** of H in G relative to g is the set:

$$gH := \{gh \ : \ h \in H\}.$$

For all fixed $g \in G$, belonging to the g-left coset of H is an equivalence relationship on G, thus, as g varies in G, we subdivide G into disjoint subsets, the cosets gH. The union of these classes is the **quotient space** G/H:

$$G/H := \{gH, \ g \in G\} \equiv \{\{gh \ : \ h \in H\}, \ g \in G\}.$$

G/H is a group if and only if H is a **normal subgroup** of G, where H is called normal if it is stable under conjugation by elements of G, i.e. if $\forall h \in H$ and $\forall g \in G$ it holds $ghg^{-1} \in H$.

Clearly, the easiest case is represented by $H = \{1_G\}$, in this situation it is evident that $G/1_G \cong G$.

To introduce the paramount important orbit-stabilizer theorem, we first notice that, for every fixed $x \in X$, the map

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Orb}(x) \\ g & \longmapsto & g \cdot x \end{array}$$

is surjective by definition of orbit but, in general, is not injective. However, as the following result says, **if we quotient G on the stabilizer of x, then we remove all possible redundancy and we remain with a bijection**.

Theorem 9.1.1 (Orbit-stabilizer theorem) *Let $x, y \in X$ and G a group acting on X.*

1. *The map*

$$\begin{array}{ccc} G \,/\, \mathrm{Stab}(x) & \xrightarrow{\sim} & \mathrm{Orb}(x) \\ g\,\mathrm{Stab}(x) & \longmapsto & g \cdot x \end{array} \tag{9.1}$$

   is bijective.

2. $\mathrm{Orb}(x) = \mathrm{Orb}(y) \implies \exists g \in G$ *such that* $g\,\mathrm{Stab}(x)\,g^{-1} = \mathrm{Stab}(y)$, *i.e. if the orbits of two elements of X coincide, then their stabilizers are conjugated by an element g of G, and, as such, they are isomorphic to each other.*

Proof.

1. First of all, let us check that the application (9.1) is well-defined, i.e. it does not depend on the choice of the representative in the equivalence class $g\ \mathrm{Stab}(x)$. If $h \in g\,\mathrm{Stab}(x)$, there exists $k \in \mathrm{Stab}(x)$ such that $h = gk$, then $h \cdot x = g \cdot (k \cdot x) = g \cdot x$.

Injectivity of (9.1): let $g, h \in G$ such that $g \cdot x = h \cdot x$, we must prove that this implies $g\,\mathrm{Stab}(x) = h\,\mathrm{Stab}(x)$. To do this, notice that $(h^{-1}g) \cdot x = x$, so $h^{-1}g \in \mathrm{Stab}(x)$, i.e. $g \in h\,\mathrm{Stab}(x)$. However, g belongs also to $g\,\mathrm{Stab}(x)$ because $1_G \in \mathrm{Stab}(x)$, hence g belongs to the intersection of the equivalence classes $h\,\mathrm{Stab}(x)$ and $g\,\mathrm{Stab}(x)$, which, however, are disjoint. Thus, the only possibility that remains valid is that $g\,\mathrm{Stab}(x) = h\,\mathrm{Stab}(x)$.

Surjectivity of (9.1): any $y \in \mathrm{Orb}(x)$ is written as $g \cdot x = y$ for some $g \in G$, but then it is the image of (9.1) because any element of $g\,\mathrm{Stab}(x)$ can be written as gk, with $k \in \mathrm{Stab}(x)$, so $(gk) \cdot x = g \cdot (k \cdot x) = g \cdot x = y$.

2. We assume $\mathrm{Orb}(x) = \mathrm{Orb}(y)$, then there is a $g \in G$ such as $y = g \cdot x \iff g^{-1}y = x$. Now suppose $h \in \mathrm{Stab}(x)$ and observe that:

$$(ghg^{-1}) \cdot y = (gh) \cdot (g^{-1} \cdot y) = (gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = y.$$

Consequently, $g\,\mathrm{Stab(x)}g^{-1} \in \mathrm{Stab}(y)$, i.e. $g\,\mathrm{Stab(x)}g^{-1} \subseteq \mathrm{Stab}(y)$. By interchanging the roles of x and y, we find the opposite inclusion $\mathrm{Stab}(y) \subseteq g\,\mathrm{Stab}(x)\,g^{-1}$, so $g\,\mathrm{Stab}(x)\,g^{-1} = \mathrm{Stab}(y)$. □

If X is G-homogeneous, then $\mathrm{Orb}(x) = X$ for all $x \in X$, thus the orbit-stabilizer theorem implies the following, fundamental, result.

Corollary 9.1.1 *If X is a G-homogeneous space, then, for any fixed $x \in X$:*

1. *the map*
$$\begin{array}{ccc} G \diagup_{\mathrm{Stab}(x)} & \xrightarrow{\sim} & X \\ g\,\mathrm{Stab}(x) & \longmapsto & g \cdot x \end{array} \tag{9.2}$$
is bijective, i.e.
$$X \cong G \diagup_{\mathrm{Stab}(x)}, \tag{9.3}$$
so, ***every G-homogeneous space can be identified with a suitable set of transformations.***

2. *the stabilizers of all elements of X are conjugated, and thus isomorphic, to each other.*

In Figure 9.1.1 we provide a graphical interpretation of a homogeneous space.

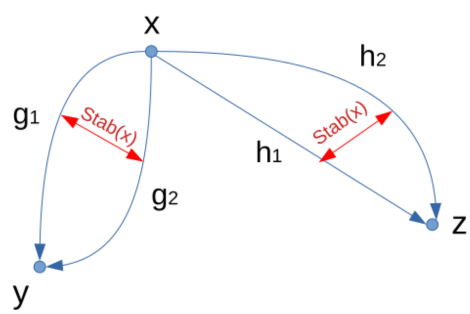


Figure 9.1: Fixed $x \in X$, every other element in X can be viewed as a transformation acting on x, i.e we identify y with all the transformations of G that allow us to pass from x to y modulo the transformations of the stabilizer in x. In the picture $g_2 = g_1 k$, with $k \in \mathrm{Stab}(x)$.

Example 9.1.4 *The straight lines in $\mathbb{R}^n$ are $\mathbb{R}$-homogeneous spaces.*

Let $L = \{u_0 + \lambda v : \lambda \in \mathbb{R}\}$ be the straight line in $\mathbb{R}^n$ passing through u_0 with direction v, u_0 and v are fixed in $\mathbb{R}^n$. Then the group $(\mathbb{R}, +)$ operates transitively on the set L via the action
$$\begin{array}{cccc} \eta_v : & \mathbb{R} \times L & \longrightarrow & L \\ & (\lambda, u) & \longmapsto & \eta_v(\lambda, u) := u + \lambda v. \end{array}$$
Of course, the stabilizer at any point of L is $\{0\}$ because any other $\lambda \neq 0$ will modify the vector u on L. The orbit-stabilizer theorem gives us the bijection $L \cong \mathbb{R} \diagup_{\{0\}} \cong \mathbb{R}$.

Remark 9.1.1 If X a G-homogeneous space, it is often interesting to search for a subgroup H of G whose action on X is still transitive and whose stabilizer is reduced to the unit element 1_G. If such a subgroup exists, then all the equivalence classes that compose the quotient group w.r.t. H are reduced to a single representative and so the orbit-stabilizer group implies that
$$H \cong X, \quad H \text{ transitive on } X \text{ with trivial stabilizer.}$$

9.2 Linear transformation groups and spheres

We remind the definitions of the real and complex linear transformation groups. The symbol $\langle\ ,\ \rangle$ will denote the real or complex Euclidean scalar product, respectively.

Def. 9.2.1 ***(Real matrix groups)***

- $GL(n,\mathbb{R}) = \{n \times n \text{ real invertible matrix}\}$ *(general linear group)*
- $GL^{+}(2,\mathbb{R}) = \{g \in GL(2,\mathbb{R})\ :\ \det(g) > 0\}$
- $SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : \det(g) = 1\}$ *(special linear group)*
- $O(n) = \{g \in GL(n,\mathbb{R}) : \forall x, y \in \mathbb{R}^n,\ \langle gx, gy\rangle = \langle x, y\rangle\}$ *(orthogonal group)*
- $SO(n) = \{g \in O(n) : \det(g) = 1\}$ *(special orthogonal group).*

Def. 9.2.2 ***(Complex matrix groups)***

- $U(n) = \{g \in GL(n,\mathbb{C}) : \forall x, y \in \mathbb{C}^n,\ \langle gx, gy\rangle = \langle x, y\rangle\}$ *(unitary group)*
- $SU(n) = \{g \in U(n) : \det(g) = 1\}$ *(special unitary group).*

We also remind that $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional sphere in $\mathbb{R}^n$ and $S^{2n-1} = \{z \in \mathbb{C}^n : \|z\| = 1\} \subset \mathbb{C}^n$ is the $(2n-1)$-dimensional real sphere in $\mathbb{C}^n$.

Remark 9.2.1 In finite dimension, a $n \times n$ matrix (real or complex) corresponds to a linear applications $f : E \longmapsto E$ with $E = \mathbb{R}^n$ or $\mathbb{C}^n$. However even if $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, one should not mix up $\mathbb{R}$-linear and $\mathbb{C}$-linear maps. A classical counter example is provided by $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, which is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

In the case of orthogonal and unitary group, we have the equivalent definitions :

$$\begin{aligned} g \in O(n,\mathbb{R}) &\iff g^t g = Id_n \\ g \in U(n) &\iff g^{\dagger} g = Id_n, \end{aligned}$$

where $g^{\dagger} = \overline{g}^t$ is the adjoint of g. This is easily shown by noticing that

$$\begin{aligned} \langle g^t g x, y\rangle = \langle gx, gy\rangle = \langle x, y\rangle \quad \forall x, y \in \mathbb{R}^n &\iff \langle g^t g x - x, y\rangle = 0 \quad \forall x, y \in \mathbb{R}^n \\ &\iff g^t g x = x \quad \forall x \in \mathbb{R}^n \\ &\iff g^t g = I_n, \end{aligned}$$

and equivalently in the case of the unitary group.

We introduce next some non-Euclidean transformation groups based on the Lorentzian product.

Def. 9.2.3 *We define the* **Lorentzian (or Minkowski) scalar product** *on* $\mathbb{R}^{n+1}$ *and* $\mathbb{C}^{n+1}$ *as :*

$$\begin{aligned} \langle x, y\rangle_L &= \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} = \langle \tilde{x}, \tilde{y}\rangle - x_{n+1} y_{n+1} \qquad x, y \in \mathbb{R}^{n+1} \\ &= \sum_{i=1}^{n} x_i \overline{y_i} - x_{n+1} \overline{y_{n+1}} = \langle \tilde{x}, \tilde{y}\rangle - x_{n+1} y_{n+1} \qquad x, y \in \mathbb{C}^{n+1}, \end{aligned}$$

where $\tilde{x} = (x_1, \ldots, x_n)^t$, $\tilde{y} = (y_1, \ldots, y_n)^t$ *and* $\langle\ ,\ \rangle$ *is the Euclidean product in* $\mathbb{R}^n$ *or* $\mathbb{C}^n$.

The linear groups of signature $(n,1)$*, also called* ***Lorentzian signature****, are the following:*

- $O(n,1) = \{g \in GL(n+1,\mathbb{R}) : \forall x, y \in \mathbb{R}^n,\ \ \langle gx, gy\rangle_L = \langle x, y\rangle_L\}$
- $SO(n,1) = \{g \in O(n,1) :\ \ \det(g) = 1\}$
- $U(n,1) = \{g \in GL(n+1,\mathbb{C}) : \forall x, y \in \mathbb{C}^n,\ \ \langle gx, gy\rangle_L = \langle x, y\rangle_L\}$
- $SU(n,1) = \{g \in U(n,1) :\ \ \det(g) = 1\}$.

The Lorentzian scalar product can be defined by using the Euclidean scalar product by noticing that it holds:

$$\langle x, y\rangle_L = \langle \eta x, y\rangle \qquad \text{with} \qquad \eta = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.4}$$

Similarly, the orthogonal and unitary group of signature $(n,1)$ can be re-defined through the conditions below:

$$g \in M(n+1,\mathbb{R}),\ \ g \in O(n,1) \quad \Longleftrightarrow \quad g^t \eta g = \eta \tag{9.5}$$

$$g \in M(n+1,\mathbb{C}),\ \ g \in U(n,1) \quad \Longleftrightarrow \quad g^\dagger \eta g = \eta. \tag{9.6}$$

Thanks to Binet's theorem, for all matrices $g \in O(n)$, $O(n,1)$, $U(n)$ or $U(n,1)$, it holds that $|\det(g)| = 1$.

9.3 Homogeneity of spheres under the group of rotations

The simplest and most intuitive homogeneous spaces are represented by spheres. We have already seen that the circle S^1 is homogeneous under the action of rotations, $SO(2)$ in the real case, $U(1)$ in the complex one. In what follows, we shall see that this result can be extended to higher dimensions.

Notation: in the whole section $(e_j)_{j=1,..,n}$ will denote the canonical basis of $\mathbb{R}^n$ or $\mathbb{C}^n$. Clearly, each e_j belongs to the sphere S^{n-1} since their Euclidean norm is 1.

9.3.1 Spheres in $\mathbb{R}^n$

Before starting, it is worth mentioning that the action of $GL(n,\mathbb{R})$ and its subgroups on $\mathbb{R}^n$ will be the usual matrix multiplication in this section. This will not always be the case in homogeneous spaces, as we will see later on.

Theorem 9.3.1 *Let* $n \geqslant 2$.

1. S^{n-1} *is* $SO(n)$*-homogeneous*
2. $\mathrm{Stab}(e_n) = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} : h \in SO(n-1) \right\} \cong SO(n-1)$

By the orbit-stabilizer theorem, we get:

$$S^{n-1} \cong SO(n) \big/_{SO(n-1)} \Longleftrightarrow S^{n-1} \cong \left\{ \left\{ g \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} : h \in SO(n-1) \right\},\ g \in SO(n) \right\}.$$

Proof.

1. To prove that $SO(n)$ operates transitively on S^{n-1}, we have to show that it exists at least one element of S^{n-1} that can be connected to all the other elements of S^{n-1} via transformations of $SO(n)$. We are going to show that this element is e_1, i.e. that $\forall x \in S^{n-1}$ $\exists g \in SO(n)$ such that $ge_1 = x$.

Fixed $x \in S^{n-1} \subset \mathbb{R}^n$, thanks to the Gram-Schmidt orthonormalization procedure, we can find $x_2, x_3, \ldots, x_n \in S^{n-1} \subset \mathbb{R}^n$ such that $(x, x_2, \ldots, x_n)$ is an orthonormal basis for $\mathbb{R}^n$.

If we use the vectors $(x, x_2, \ldots, x_n)$ as columns of a matrix A, then we know that $A \in O(n)$ and that $\det(A) = \pm 1$. To guarantee a determinant equal to 1, we slightly modify A by considering the matrix

$$g = \begin{pmatrix} | & | & & | \\ x & x_2 & \ldots & \epsilon x_n \\ | & | & & | \end{pmatrix},$$

with $\epsilon = \pm 1$ chosen such that $\det(g) = 1$, in this way $g \in SO(n)$. By direct computation we get $ge_1 = x$, but x was arbitrarily chosen in S^{n-1}, so the action of $SO(n)$ is transitive on S^{n-1}. Notice that the matrix g depends on x because the Gram-Schmidt orthonormalization is initiated by x itself.

2. Let us search under which conditions it is possible to build a matrix such that

$$g = \begin{pmatrix} h & b \\ c & d \end{pmatrix} \in SO(n),$$

with $h \in M(n-1, \mathbb{R})$, $b \in M((n-1) \times 1, \mathbb{R})$, $c \in M(1 \times (n-1), \mathbb{R})$ and $d \in \mathbb{R}$ that satisfies $ge_n = e_n$. The set of these matrices will give $\mathrm{Stab}(e_n)$.

First of all we notice that $g \in SO(n) \iff g^{-1} = g^t$, thus $ge_n = e_n \implies e_n = g^t e_n$. By direct computation we have:

- $ge_n = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ d \end{pmatrix}$, so $ge_n = e_n \iff \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \iff b = 0_{M((n-1)\times 1)}$ and $d = 1$

- $g^t e_n = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ d \end{pmatrix}$, so $e_n = g^t e_n \iff \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ d \end{pmatrix} \iff c = 0_{M(1\times(n-1))}$ and $d = 1$.

Hence, the desired matrix is $g = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$, which belongs to $SO(n)$ if and only if $h \in SO(n-1)$, i.e. $\det(h) = 1$ and $h^t h = I_{n-1}$. This shows that $\mathrm{Stab}(e_n) \cong SO(n-1)$ and, since all stabilizers of a homogeneous space are isomorphic to each other, $S^{n-1} \cong SO(n)/SO(n-1)$. □

9.3.2 Spheres in $\mathbb{C}^n$

The results and proofs are the nearly identical for spheres in $\mathbb{C}^n$ as for those in $\mathbb{R}^n$. The only difference is that we need take some precautions with the determinant of $\mathbb{C}$-linear applications.

Theorem 9.3.2 *Let $n \geqslant 2$.*

1. *S^{2n-1} is $SU(n)$-homogeneous*

2. *$Stab(e_n) = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} : h \in SU(n-1) \right\} \cong SU(n-1)$.*

By the orbit-stabilizer theorem, we get:

$$S^{2n-1} \cong SU(n) \diagup_{SU(n-1)} \iff S^{2n-1} \cong \left\{ \left\{ g \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} : h \in SU(n-1) \right\},\ g \in SU(n) \right\}.$$

Proof.

1. Fix an arbitrary $z \in S^{2n-1} \subset \mathbb{C}^n$ and apply again the Gram-Schmidt orthonormalization procedure to find $z_2, \ldots, z_n \in S^{2n-1} \subset \mathbb{C}^n$ such that $(z, z_2, \ldots, z_n)$ is an orthonormal basis for $\mathbb{C}^n$. Also, let $g_\theta \in U(n)$,

$$g_\theta = \begin{pmatrix} | & | & \ldots & | \\ z & z_2 & \ldots & e^{i\theta} z_n \\ | & | & \ldots & | \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Then we have again $g_\theta e_1 = z$, for all $\theta \in [0, 2\pi)$ and, thanks to the properties of the determinant, $\det(g_\theta) = e^{i\theta} \det(g_0) \in S^1$, where $\det(g_0) = e^{i\varphi}$. We finish by choosing $\theta = -\varphi$ so that $\det(g_\theta) = e^{-i\varphi} e^{i\varphi} = 1$, in order to have $g_\theta \in SU(n)$.

2. Exactly the same proof as in the real case, we simply need to replace the transpose matrix by the adjoint matrix and $SO(n)$ by $SU(n)$. □

Remark 9.3.1 Since all n-spheres of different radius are isomorphic, we have exactly the same results for spheres of positive radius, $S^n_R = \{x \in \mathbb{R}^{n+1} : \|x\| = R\}$. This fact will be useful later.

9.4 Homogeneity of the open unit ball: relationship between projective spaces and hyperbolic rotations

We have seen that the contour of the unit disk in $\mathbb{R}^2$ and $\mathbb{C}$ is a homogeneous space under the rotation group $SO(2)$ and $U(1)$, respectively, but that the unit disks $D_{\mathbb{R}}(0,1)$ and $D_{\mathbb{C}}(0,1)$ are not homogeneous under the action of these groups.

In this section we are going to show that $D_{\mathbb{R}}(0,1)$ and $D_{\mathbb{C}}(0,1)$ and, more generally, the open unit ball in $\mathbb{R}^n$ and $\mathbb{C}^n$, are homogeneous spaces w.r.t. the groups $SO(n,1)$ and $SU(n,1)$, the Lorentzian analogues of $SO(n)$ and $SU(n)$, whose action is implemented by *hyperbolic rotations*. In order to show this, it is useful to embed $\mathbb{R}^n$ and $\mathbb{C}^n$ in the real or complex projective space, respectively. For this reason, we begin by discussing the action of the general linear group on projective spaces.

9.4.1 The action of group $GL(n+1,\mathbb{R})$ on $\mathbb{RP}^n$

We have already seen in chapter 1 the real projective space[1]

$$\mathbb{RP}^n = \mathbb{R}^{n+1}\backslash\{0\} \ \diagup_{\mathbb{R}^\times} \equiv \left\{ \left\{ \lambda \begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix}, \ \lambda \neq 0 \right\}, \ \begin{pmatrix} u_1 \\ \vdots \\ u_{n+1} \end{pmatrix} \neq 0 \right\},$$

its twin brother is the complex projective space:

$$\mathbb{CP}^n = \mathbb{C}^{n+1}\backslash\{0\} \ \diagup_{\mathbb{C}^\times},$$

$\mathbb{R}^\times$ and $\mathbb{C}^\times$ being $\mathbb{R}$ and $\mathbb{C}$ without their 0 element. Here we are going analyze more thoroughly the projective space, for the sake of a smoother reading, we will fix our attention only on the real projective space, knowing that everything we will write in this subsection also holds true for the complex projective space, simply by replacing $\mathbb{R}$ with $\mathbb{C}$ and $\mathbb{R}^\times$ with $\mathbb{C}^\times$.

Notation: in this section, the equivalence class $\mathbb{R}^\times \cdot u \in \mathbb{RP}^n$, $u \in \mathbb{R}^{n+1}\backslash\{0\}$, will be denoted by :

$$[u] = \begin{bmatrix} u_1 \\ \vdots \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ \vdots \\ \lambda u_{n+1} \end{bmatrix}.$$

Notice now that, for every $u \in \mathbb{R}^n\backslash\{0\}$, the following map[2] is clearly an injection of $\mathbb{R}^n$ into $\mathbb{RP}^n$:

$$\begin{array}{ccc} \mathbb{R}^n & \hookrightarrow & \mathbb{RP}^n \\ u & \mapsto & \begin{bmatrix} u \\ 1 \end{bmatrix}. \end{array} \tag{9.7}$$

Since every element $v \in \mathbb{RP}^n$ can be written as $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ v_{n+1} \end{bmatrix}$, we have either $v_{n+1} \neq 0$, and so $v = \begin{bmatrix} u \\ 1 \end{bmatrix}$, with $u \in \mathbb{R}^n$, or $v_{n+1} = 0$, and so $v = \begin{bmatrix} u \\ 0 \end{bmatrix}$, with $u \in \mathbb{R}^n\backslash\{0\}$. Thus, the injection (9.7) becomes a bijection between $\mathbb{R}^n$ and the set $\left\{ \begin{bmatrix} u \\ 1 \end{bmatrix} : u \in \mathbb{R}^n \right\}$. As a consequence, we can split the real projective space in the following disjoint union:

$$\begin{aligned} \mathbb{RP}^n &= \left\{ \begin{bmatrix} u \\ 1 \end{bmatrix} : u \in \mathbb{R}^n \right\} \sqcup \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in \mathbb{R}^n\backslash\{0\} \right\} \\ &\underset{(9.7)}{\cong} \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}. \end{aligned}$$

Of course, we can iterate the splitting on the second set, obtaining:

$$\mathbb{RP}^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \mathbb{RP}^0.$$

[1] Geometrically speaking, we have seen in chapter 1 that the projective spaces is isomorphic to the set of straight lines passing through 0 in $\mathbb{R}^n$.

[2] The choice of setting to 1 the last coordinate is an arbitrary, yet usual, choice. Changing the position of 1 leads to an isomorphic decomposition in the following.

The description of $\mathbb{RP}^0$ deserves a special discussion: $\mathbb{RP}^0$ is the quotient space $\mathbb{R}\backslash\{0\} \diagup_{\mathbb{R}^\times}$, i.e.

$$\mathbb{RP}^0 = \{\{\lambda u,\ \lambda \neq 0\},\ u \neq 0\} \cong \{[1]\}$$

a set containing a single $\mathbb{R}^\times$-equivalence class, canonically chosen to be [1]. In the projective geometry literature, [1] is denoted with ∞ and called **the point at the infinite**. So, to resume:

$$\mathbb{RP}^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \{\infty\}. \tag{9.8}$$

We can now start with the definition of **the action of** $GL(n+1,\mathbb{R})$ **on** $\mathbb{RP}^n$:

$$\begin{array}{ccc} GL(n+1,\mathbb{R}) \times \mathbb{RP}^n & \longrightarrow & \mathbb{RP}^n \\ (g,[u]) & \mapsto & g \cdot [u] := [gu], \end{array}$$

which is well-defined because, thanks to the $\mathbb{R}$-linearity of g, for all $\lambda \in \mathbb{R}^\times$ we have:

$$g \cdot [\lambda u] = [g(\lambda u)] = [\lambda(gu)] = [gu] = g \cdot [u],$$

so the choice of the representative u in a class in $\mathbb{RP}^n$ does not impact the action.

However, we notice that this action is not stable when restricted on $\mathbb{R}^n$, interpreted as a subset of $\mathbb{RP}^n$ via the injection (9.7). To see this, take

$$g = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in GL(n+1,\mathbb{R})$$

with $A \in GL(n,\mathbb{R})$, $c \in M(1 \times n,\mathbb{R})$, $b \in M(n \times 1,\mathbb{R})$ and $d \in \mathbb{R}$, then, by direct computation[3]:

$$g \cdot u \underset{(9.7)}{\cong} g \cdot \begin{bmatrix} u \\ 1 \end{bmatrix} = \begin{bmatrix} Au + b \\ cu + d \end{bmatrix} = \begin{bmatrix} \frac{Au+b}{cu+d} \\ 1 \end{bmatrix} \underset{(9.7)}{\cong} \frac{Au + b}{cu + d} \in \mathbb{R}^n \iff cu + d \neq 0,$$

however, not all the matrices of $GL(n+1,\mathbb{R})$ satisfy the constraint $cu + d \neq 0$, e.g. for all $u \in \mathbb{R}^n\backslash\{0\}$, the matrix $g = \begin{pmatrix} A & 0 \\ \frac{u^t}{\|u\|^2} & -1 \end{pmatrix}$ with $A \in GL(n,\mathbb{R})$ belongs to $GL(n+1,\mathbb{R})$ but:

$$g \cdot u \underset{(9.7)}{\cong} g \cdot \begin{bmatrix} u \\ 1 \end{bmatrix} = \left[\begin{pmatrix} A & 0 \\ \frac{u^t}{\|u\|^2} & -1 \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix}\right] = \begin{bmatrix} Au \\ \frac{u^t u}{\|u\|^2} - 1 \end{bmatrix} = \begin{bmatrix} Au \\ 0 \end{bmatrix} \in \mathbb{RP}^{n-1} \neq \mathbb{R}^n.$$

9.4.2 Homogeneity of the open unit ball in $\mathbb{R}^n$

Even if the action $GL(n+1,\mathbb{R})$ is not stable when operating on $\mathbb{R}^n$, its subgroup $SO(n,1)$ acts in a stable way on the unit ball

$$B := B_{\mathbb{R}}(0,1) = \{x \in \mathbb{R}^n \ :\ \langle x, x\rangle = \|x\|^2 < 1\} \subset \mathbb{R}^n.$$

Even more, the action of $SO(n,1)$ is transitive on B. To prove this result, it is useful to show that we can find a copy of the unit ball in $\mathbb{RP}^n$.

[3] $cu \in \mathbb{R}$ because it is the matrix product of $c \in M(1 \times n,\mathbb{R})$ and u interpreted as an element of $M(n \times 1,\mathbb{R})$, so cu is nothing but the Euclidean scalar product $\langle c, u\rangle$ if we interpret both c and u as column vectors of $\mathbb{R}^n$.

Theorem 9.4.1 *The unit ball B in $\mathbb{R}^n$ can be characterized as follows:*

$$B \cong B' := \{[u] \in \mathbb{RP}^n \ : \ \langle u, u\rangle_L < 0\} \subset \mathbb{RP}^n.$$

Proof. We start by noting that the constraint that defines B', i.e. $\langle u, u\rangle_L < 0$, is well-defined in $\mathbb{RP}^n$, in fact, for each $[u] \in \mathbb{RP}^n$ and $\lambda \in \mathbb{R}^\times$, $\langle u, u\rangle_L < 0 \iff \langle \lambda u, \lambda u\rangle_L = \lambda^2 \langle u, u\rangle_L < 0$.

Now, let $v \in B \subset \mathbb{R}^n$, i.e. $\langle v, v\rangle < 1$, and let $\begin{bmatrix} v \\ 1 \end{bmatrix}$ be its copy in $\mathbb{RP}^n$, then

$$\langle \begin{bmatrix} v \\ 1 \end{bmatrix}, \begin{bmatrix} v \\ 1 \end{bmatrix} \rangle_L := \langle v, v\rangle - 1 < 0.$$

Conversely, let $[u] = \begin{bmatrix} \bar{u} \\ u_{n+1} \end{bmatrix} \in B'$, i.e. $\langle u, u\rangle < 0$, with $\bar{u} \in \mathbb{R}^n$, then

$$\langle \begin{pmatrix} \bar{u} \\ u_{n+1} \end{pmatrix}, \begin{pmatrix} \bar{u} \\ u_{n+1} \end{pmatrix} \rangle_L = \langle \bar{u}, \bar{u}\rangle - u_{n+1}^2 < 0, \tag{9.9}$$

which implies that u_{n+1} must be different than 0, given that $\langle \bar{u}, \bar{u}\rangle = \|\bar{u}\| \geqslant 0$. So,

$$\begin{bmatrix} \bar{u} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\bar{u}}{u_{n+1}} \\ 1 \end{bmatrix} \underset{(9.7)}{\cong} \frac{\bar{u}}{u_{n+1}} \in \mathbb{R}^n.$$

To verify that $\frac{\bar{u}}{u_{n+1}} \in B$ we notice that (9.9) implies that $\|\bar{u}\|^2 < u_{n+1}^2$, i.e. $\|\bar{u}\| < |u_{n+1}|$ so $\left\| \frac{\bar{u}}{u_{n+1}} \right\| = \frac{\|\bar{u}\|}{|u_{n+1}|} < 1$, thus $\frac{\bar{u}}{u_{n+1}} \in B$. □

Theorem 9.4.1 implies that **the unit ball in $\mathbb{R}^n$ can be identified with the (double) cone in $\mathbb{R}^{n+1}$ obtained as the set of straight lines passing through the origin of $\mathbb{R}^{n+1}$ and with slope strictly smaller than** 1. Figure 9.2 gives a pictorial illustration of this cone.

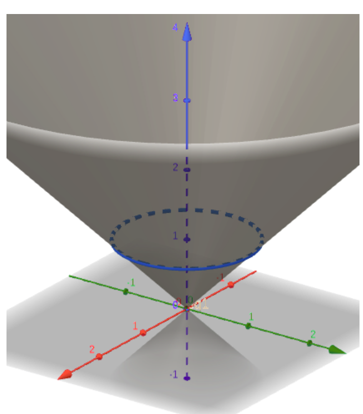


Figure 9.2: The double cone in $\mathbb{R}^{n+1}$ in bijection with the unit ball in $\mathbb{R}^n$.

Until the end of this section, the open unit ball B in $\mathbb{R}^n$ will be identified with its copy in $\mathbb{RP}^n$ as defined by the previous theorem.

The **action of $O(n,1)$ on B** is:

$$\begin{aligned} O(n,1) \times B &\longrightarrow B \\ (g,[u]) &\mapsto g \cdot [u] := [gu], \end{aligned}$$

well-defined because the action of $O(n,1)$ is stable on the elements of B since the matrices belonging to $O(n,1)$ preserve the Lorentzian product, so, for all $u \in B$ and $g \in O(n,1)$,

$$\langle gu, gu \rangle_L = \langle u, u \rangle_L < 0.$$

It turns out that the subgroup $SO(n,1)$ is enough to guarantee a transitive action on B. A couple of preliminary results will help us prove this result quite easily.

Lemma 9.4.1 *Let $a \in SO(n)$, then $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in SO(n,1)$.*

Proof. From eq. (9.5) we know that, given $\eta = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in SO(n,1)$ if and only if its determinant is 1, which is true, and if

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^t \eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \eta \iff \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^t \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^t a & 0 \\ 0 & -1 \end{pmatrix} \underset{a^t a = I_n}{=} \eta.$$

Therefore, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in SO(n,1)$. □

Lemma 9.4.2 *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(1,1)$, then $\begin{pmatrix} a & 0 & b \\ 0 & I_{n-1} & 0 \\ c & 0 & d \end{pmatrix} \in SO(n,1)$.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(1,1)$, then by eq. (9.5) it holds that

$$g^t \eta g = \eta \iff \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.10}$$

and so, if we set $h = \begin{pmatrix} a & 0 & b \\ 0 & I_{n-1} & 0 \\ c & 0 & d \end{pmatrix}$, then $\det(h) = \det(g) = 1$ and

$$h^t \eta h = \begin{pmatrix} a^2 - c^2 & 0 & ab - cd \\ 0 & I_{n-1} & 0 \\ ab - cd & 0 & b^2 - d^2 \end{pmatrix} \underset{(9.10)}{=} \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

so $h \in SO(n,1)$. □

Theorem 9.4.2 *Let $n \geqslant 2$.*

1. *The action of $SO(n,1)$ on B is transitive*

2. $\mathrm{Stab}(0) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \det(a) \end{pmatrix},\ a \in O(n) \right\} \cong O(n).$

Therefore, the stabilizer-orbit theorem implies:

$$B \cong SO(n,1) \diagup_{O(n)} \iff B \cong \left\{ \left\{ g \begin{pmatrix} a & 0 \\ 0 & \det(a) \end{pmatrix},\ a \in O(n) \right\},\ g \in SO(n,1) \right\}.$$

Proof.

1. Let $x \in B$ arbitrary, $x \neq 0$. We wish to show that there is a $g \in SO(n,1)$ such that $g \cdot x = 0$. We will do this following this path:

- we search for $g_1 \in SO(n,1)$ such that $g_1 \cdot x = \begin{pmatrix} |x| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

- we search for $g_2 \in SO(n,1)$ such that $g_2 \cdot \begin{pmatrix} |x| \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$

- finally, we set $g = g_2 g_1$ to get the wanted result : $g \cdot x = 0$.

Let $r = |x|$. Since $SO(n)$ is transitive on the sphere S_r^{n-1}, there exists $a \in SO(n)$ such that $a \cdot x = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. We then define $g_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, which belongs to $SO(n,1)$ thanks to Lemma 9.4.1, then:

$$g_1 \cdot x = \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \right] = \begin{bmatrix} r \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Next, it can be verified with straightforward computations that the matrix

$$\tilde{g}_2 = \frac{1}{\sqrt{1-r^2}} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \in SO(1,1)$$

verifies

$$\tilde{g}_2 \cdot \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{ar+b}{cr+b} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$$

We now use Lemma 9.4.2 to extend $\tilde{g}_2$ to $SO(n,1)$ with

$$g_2 = \frac{1}{\sqrt{1-r^2}} \begin{pmatrix} 1 & 0 & -r \\ 0 & I_{n-1} & 0 \\ -r & 0 & 1 \end{pmatrix},$$

with g_2 having the desired property: $g_2 \cdot \begin{pmatrix} r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$. Finally, if $g = g_2 g_1$, we get $g \cdot x = 0$.

2. Let $g = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SO(n,1)$, with A a $n \times n$ matrix such that $g \cdot 0 = 0$. Thus,

$$g \cdot 0 = 0 \iff \begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \frac{b}{d} = 0 \iff b = 0,$$

$d \neq 0$ since this would go against the already established stability of $SO(n,1)$ on B.

Moreover, $g \in O(n,1)$, thus:

$$\begin{aligned} g^t \eta g = \eta \quad &\iff \quad \begin{pmatrix} A^t A - c^t c & -c^t d \\ -dc & -d^2 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \\ &\iff \quad c = 0\,, \quad A^t A = I_n \quad \text{and} \quad d^2 = 1, \end{aligned}$$

therefore $A \in O(n)$. Finally, since $g \in SO(n,1)$, $\det(g) = d \det(A) = 1$, thus $d = \det(A)^{-1} = \det(A^{-1}) \underset{A \in O(n)}{=} \det(A^t) = \det(A)$. This concludes the proof since we have proven that:

$$\mathrm{Stab}(0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix} : A \in O(n) \right\} \cong O(n).$$

□

9.4.3 Homogeneity of the open unit ball in $\mathbb{C}^n$

To extend the previous results to the open unit ball in $\mathbb{C}^n$ we just need replace $O(n)$ by $U(n)$, the proofs are practically identical to the real case.

Once again, we can identify the complex open unit ball $B \subset \mathbb{C}^n$ with elements of $[u] \in \mathbb{CP}^n$ such that $\langle u, u \rangle_L < 0$ and the action of $U(n,1)$ on B is stable.

Theorem 9.4.3 *Let $n \geqslant 2$.*

1. *The action of $SU(n,1)$ is transitive on $B \subset \mathbb{C}^n$*
2. $\mathrm{Stab}(0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \frac{}{\det(A)} \end{pmatrix},\ A \in U(n) \right\} \cong U(n).$

Therefore, by the stabilizer-orbit theorem:

$$B \cong SU(n,1) \diagup_{U(n)} \iff B \cong \left\{ \left\{ g \begin{pmatrix} A & 0 \\ 0 & \frac{}{\det(A)} \end{pmatrix},\ A \in U(n) \right\},\ g \in SU(n,1) \right\}.$$

9.5 Homogeneity of the upper-half plane H

The upper half plane H is another very important example of homogeneous space. To prove this result we first need to introduce and discuss the Möbius transformations. The most general definition of **Möbius transformations** in two dimensions is given in the context of the action of $GL(2,\mathbb{C})$ on $\mathbb{CP}^1$, that is convenient to write through the splitting

$$\mathbb{CP}^1 \cong \mathbb{C} \sqcup \{\infty\} = \left\{ \begin{bmatrix} z \\ 1 \end{bmatrix} \ : \ z \in \mathbb{C} \right\} \sqcup \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} \ : \ z \in \mathbb{C}\backslash\{0\} \right\} \tag{9.11}$$

called the **Riemann sphere**. For all $z, w \in \mathbb{C}$, the action

$$\begin{array}{ccc} GL(2,\mathbb{C}) \times \mathbb{C} \sqcup \{\infty\} & \longrightarrow & \mathbb{C} \sqcup \{\infty\} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right) & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} z \\ w \end{bmatrix}, \end{array}$$

is defined by considering two cases corresponding to $w \neq 0$ and $w = 0$, respectively.
In the first case, i.e. for all $z \in \mathbb{C}$ and $w \neq 0$, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} z \\ 1 \end{bmatrix} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \right] = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{az+b}{cz+d} \\ 1 \end{bmatrix} & \text{if} \quad cz+d \neq 0 \\ \begin{bmatrix} az+b \\ 0 \end{bmatrix} & \text{if} \quad cz+d = 0 \end{cases}$$

$$\cong \begin{cases} \frac{az+b}{cz+d} \in \mathbb{C} & \text{if} \quad cz+d \neq 0 \\ \infty \in \{\infty\} & \text{if} \quad cz+d = 0 \end{cases}.$$

In the second case, i.e. for all $z \in \mathbb{C}$ and $w \neq 0$, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{cases} \frac{a}{c} \in \mathbb{C} & \text{if } c \neq 0 \\ \infty \in \{\infty\} & \text{if } c = 0 \end{cases}.$$

This very general definition of Möbius transformations is not needed to show that H is a homogeneous space, in fact, we can restrict our attention to the much simpler action of the group $\mathrm{SL}(2,\mathbb{R})$ on H to obtain this result, as we discuss in the next subsection.

9.5.1 Möbius transformations on the upper-half plane H

In this section, $H = \{z \in \mathbb{C} \ : \ \mathfrak{Im}(z) > 0\} = \{(x+iy) \in \mathbb{C} \ : \ y > 0\}$ will denote the upper half plane in $\mathbb{C}$. When we consider H, the Möbius transformations acquire a much simpler form as it is stated in the following result.

Lemma 9.5.1 *The* ***Möbius action***

$$\begin{array}{rccc} \mathscr{M}: & GL^+(2,\mathbb{R}) \times H & \longrightarrow & H \\ & \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d} \end{array},$$

is an actual group action on H.

Proof. First of all, we show that the operation is stable. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$, so that $\det(g) = ad - bc > 0$, and $z = x + iy \in H$, so that $y > 0$. Then,

- $cz + d = 0 \iff c(x + iy) = -d \iff cx = -d$ and $y = 0$, which cannot happen because $y > 0$, thus the denominator of the Möbius transformations is always different than 0 in the whole H.
- $\mathfrak{Im}(g \cdot z) = \mathfrak{Im}\left(\frac{az+b}{cz+d}\right) = \frac{1}{|cz+d|^2}\mathfrak{Im}((az+b)(c\overline{z}+d)) = \frac{1}{|cz+d|^2}\mathfrak{Im}(iy(ad-bc)) = \frac{y(ad-bc)}{|cz+d|^2} = \frac{y\det(g)}{|cz+d|^2} > 0$.

Hence, the Möbius transformation defined above is stable on H. We now need to verify the properties of group action.

1. If $g = I_n$, then $a = d = 1$, $b = c = 0$, so $I_n \cdot z = \frac{z+0}{0+1} = z$.
2. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbb{R})$ and $h = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in GL^+(2, \mathbb{R})$, then $gh = \begin{pmatrix} ak + bm & al + bn \\ ck + dm & cl + dn \end{pmatrix}$ and:
   - $(gh) \cdot z = \frac{(ak+bm)z+(al+bn)}{(ck+dm)z+(cl+dn)}$
   - $g \cdot (h \cdot z) = \frac{a\frac{kz+l}{mz+n}+b}{c\frac{kz+l}{mz+n}+d} = \frac{a(kz+l)+b(mz+n)}{c(kz+l)+d(mz+n)} = \frac{(ak+bm)z+(al+bn)}{(ck+dm)z+(cl+dn)}$.

   Hence, $(gh) \cdot z = g \cdot (h \cdot z)$ for all $z \in H$. □

$\mathrm{GL}^+(2, \mathbb{R})$ is the maximal stability group for the Möbius action on H, since $\mathrm{SL}(2, \mathbb{R})$ is a subgroup of $\mathrm{GL}^+(2, \mathbb{R})$, we get that also the Möbius action on H restricted to the matrices of $\mathrm{SL}(2, \mathbb{R})$ is an actual group action.

It turns out that the $\mathrm{SL}(2, \mathbb{R})$ Möbius action on H is enough to guarantee transitivity. Compared to the proof of other homogeneous spaces that we have discussed so far, the $\mathrm{SL}(2, \mathbb{R})$-homogeneity of H is relatively easy to demonstrate.

Theorem 9.5.1 *The following statements hold:*

1. *the upper-half plane H is $SL(2, \mathbb{R})$-homogeneous*
2. $\mathrm{Stab}(i) = SO(2)$.

Thus, the stabilizer-orbit theorem implies:

$$H \cong SL(2, \mathbb{R}) \diagup_{SO(2)} \iff H \cong \{\{gh,\ h \in SO(2)\},\ g \in SL(2, \mathbb{R})\}.$$

Proof.

1. Let $z = x + iy \in H$ arbitrary, so $y > 0$ and the matrix $g = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$ belongs to $SL(2, \mathbb{R})$.

Then,

$$g \cdot i = \frac{\sqrt{y}i + x/\sqrt{y}}{0 \cdot i + 1/\sqrt{y}} = \sqrt{y}\left(i\sqrt{y} + \frac{x}{\sqrt{y}}\right) = x + iy = z.$$

Thus, the action of $SL(2,\mathbb{R})$ on H is transitive because the whole H can reached by i via a Möbius transformations.

2. Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, let us explicitly write the stabilization condition on i:

$$g \cdot i = i \iff \frac{ai+b}{ci+d} = i \iff ai + b = di - c \iff a = d \text{ and } b = -c.$$

Furthermore, if we fuse these equalities with the fact that $\det(g) = ad - bc = 1$, we get the constraint $a^2 + b^2 = 1$. Consequently, every matrix of $\mathrm{Stab}(i)$ is written as $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with $a^2 + b^2 = 1$, which is the parameterization of a generic $SO(2,\mathbb{R})$ matrix, thus $\mathrm{Stab}(i) = SO(2)$. $\square$

Let $[g] = gSO(2,\mathbb{R})$ for a generic equivalence classes in $SL(2,\mathbb{R}) \diagup_{SO(2,\mathbb{R})}$ and let

$$\begin{array}{rccc} \varphi : & SL(2,\mathbb{R}) \diagup_{SO(2,\mathbb{R})} & \longrightarrow & H \\ & [g] & \longmapsto & g \cdot i \end{array}$$

be the bijection generated by the orbit-stabilizer theorem and $\mathscr{M}_g : H \to H$, $z \mapsto \mathscr{M}(g,z)$, for all $g \in SL(2,\mathbb{R})$, then the following diagram

$$\begin{tikzcd} SL(2,\mathbb{R}) \diagup_{SO(2,\mathbb{R})} \arrow[r, "\varphi"] \arrow[d, "Id"] & H \arrow[d, "\mathscr{M}_g"] \\ SL(2,\mathbb{R}) \diagup_{SO(2,\mathbb{R})} \arrow[r, "\varphi"] & H \end{tikzcd}$$

is commutative.

9.5.2 The isomorphism $H \cong \mathbf{Sym}_1^+(2,\mathbb{R})$

A very useful characterization of the upper half plane H is represented by the set (which is not a group) of matrices

$$\mathrm{Sym}_1^+(2,\mathbb{R}) = \{g \in SL(2,\mathbb{R}) : g = g^t,\ g \text{ positive definite: } u^t g u \geqslant 0\ \forall u \in \mathbb{R}^2\}.$$

To show this fact, we first need to recall a handy representation of the elements in $\mathrm{Sym}_1^+(2,\mathbb{R})$. In the proof of the theorem, (e_1, e_2) will denote the canonical basis of $\mathbb{R}^2$.

Lemma 9.5.2 $g = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathrm{Sym}_1^+(2,\mathbb{R})$ *if and only if* $\alpha > 0$ *and* $\det(g) = 1$.

Proof.

$\boxed{\implies}$: we assume $g \in \mathrm{Sym}_1^+(2,\mathbb{R})$. Then, $\det(g) = 1$ by definition and, since g is positive-definite, $\alpha = \langle ge_1, e_1\rangle > 0$, the inequality is strict because, if $\alpha = 0$, then $\det(g) = -\beta^2 \leqslant 0$, which contradicts the fact that $\det(g) = 1$.

$\boxed{\Longleftarrow}$: now we assume $g = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ such that $\alpha > 0$ and $\det(g) = 1$. First of all we notice that $\det(g) = 1 \iff \alpha\gamma = 1 + \beta^2$, which also implies that $\gamma > 0$.
If we write $u = xe_1 + ye_2 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, then:

$$\begin{aligned} \langle gu, u \rangle &= \langle g(xe_1 + ye_2), xe_1 + ye_2 \rangle = \alpha x^2 + \gamma y^2 + 2xy\beta \\ &= X^2 + Y^2 + 2XY \frac{\beta}{\sqrt{\alpha\gamma}} \quad \text{with} \quad X = x\sqrt{\alpha}\,,\, Y = y\sqrt{\gamma}. \end{aligned}$$

We remark that $\frac{\beta}{\sqrt{\alpha\gamma}} = \frac{\beta}{\sqrt{1+\beta^2}} \in (-1, 1)$ for all $\beta \in \mathbb{R}$. Therefore, if $\underline{XY \geqslant 0}$,

$$X^2 + Y^2 + 2XY \frac{\beta}{\sqrt{\alpha\gamma}} \geqslant X^2 + Y^2 - 2XY = (X - Y)^2 \geqslant 0$$

and in the other case, if $\underline{XY \leqslant 0}$,

$$X^2 + Y^2 + 2XY \frac{\beta}{\sqrt{\alpha\gamma}} \geqslant X^2 + Y^2 + 2XY = (X + Y)^2 \geqslant 0$$

Therefore, g is definite positive and $g \in \mathrm{Sym}_1^+(2, \mathbb{R})$. $\square$

By writing the determinant of g explicitly we get $\alpha\gamma - \beta^2 = 1$, solving w.r.t. γ we obtain $\gamma = \frac{1+\beta^2}{\alpha}$, so that the generic parameterization of a matrix in $\mathrm{Sym}_1^+(2, \mathbb{R})$ is:

$$\mathrm{Sym}_1^+(2, \mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \frac{1+\beta^2}{\alpha} \end{pmatrix},\ \alpha > 0,\ \beta \in \mathbb{R} \right\}.$$

Theorem 9.5.2 *The following assertions hold.*

1. *The function*

$$\begin{array}{rccc} F: & H & \xrightarrow{\sim} & \mathrm{Sym}_1^+(2, \mathbb{R}) \\ & z = x + iy & \longmapsto & \frac{1}{y} \begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix}, \end{array}$$

   is bijective with inverse given by:

$$\begin{array}{rccc} \omega: & \mathrm{Sym}_1^+(2, \mathbb{R}) & \xrightarrow{\sim} & H \\ & \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} & \longmapsto & \frac{1}{\alpha}(-\beta + i). \end{array}$$

2. *The map*

$$\begin{array}{rccc} *: & SL(2, \mathbb{R}) \times \mathrm{Sym}_1^+(2, \mathbb{R}) & \longrightarrow & \mathrm{Sym}_1^+(2, \mathbb{R}) \\ & (m, g) & \longmapsto & m * g = (m^{-1})^t g m^{-1} \end{array}$$

   is a group action.

Proof.

1. Both maps F and ω are well-defined thanks to the previous lemma. First we check that $\omega(F(z)) = z$ for all $z = x + iy \in H$:

$$F(z) = \frac{1}{y}\begin{pmatrix} 1 & -x \\ -x & x^2 + y^2 \end{pmatrix} \quad \Longrightarrow \quad \omega(F(z)) = y\left(\frac{x}{y} + i\right) = x + iy = z.$$

Now we check that $F(\omega(g)) = g$ for all $g \in \mathrm{Sym}_1^+(2, \mathbb{R})$. To this aim, let $g = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathrm{Sym}_1^+(2, \mathbb{R})$, then:

$$\omega(g) = \frac{-\beta}{\alpha} + i\frac{1}{\alpha} \quad \Longrightarrow \quad F(\omega(g)) = \alpha\begin{pmatrix} 1 & \frac{\beta}{\alpha} \\ \frac{\beta}{\alpha} & \frac{1+\beta^2}{\alpha^2} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \frac{1+\beta^2}{\alpha} \end{pmatrix} \underset{1=\det(g)=\alpha\gamma-\beta^2}{=} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

2. The axioms of group actions follow from direct computations. Let $g \in \mathrm{Sym}_1^+(2, \mathbb{R})$ and $m, n \in SL(2, \mathbb{R})$.

- $Id * g = Id\, g\, Id = g$
- $(mn) * g = ((mn)^{-1})^t\ g\ (mn)^{-1} = (m^{-1})^t(n^{-1})^t g\, n^{-1}m^{-1} = m * (n * g)$.

All that is left to prove is then the stability of the action on $Sym_1^+(2, \mathbb{R})$, i.e. that $m * g$ has unitary determinant, is symmetric and positive-definite for all $m \in SL(2, \mathbb{R})$ and $g \in Sym_1^+(2, \mathbb{R})$:

- $\det(m * g) = \det((m^{-1})^t) \det(g) \det(m) = \det(m) \det(g) \det(m) = 1 \implies m * g \in SL(2, \mathbb{R})$
- $\left((m^{-1})^t g m^{-1}\right)^t = (m^{-1})^t g m^{-1}$, thus $m * g$ is symmetric (which explains why we must consider $(m^{-1})^t$) in the definition of the action $*$ and not m^{-1})
- for all $u \in \mathbb{R}^2$, $\langle m * gu, u\rangle = \langle (m^{-1})^t g\, m^{-1}u, u\rangle = \langle g(m^{-1}u), (m^{-1}\, u)\rangle \geqslant 0$ since $g \in Sym_1^+(2, \mathbb{R})$.
  Moreover, $m^{-1}u = 0 \iff u = 0$ because m is invertible. Therefore, $m * g$ is positive-definite. □

To resume, we have determined the following isomorphisms:

$$\boxed{SL(2, \mathbb{R})/SO(2) \cong H \cong \mathrm{Sym}_1^+(2, \mathbb{R})}.$$

As we will see later, these are **three among the six prototypes of the hyperbolic plane** (also called hyperbolic models), the remaining three being the hyperboloid in $\mathbb{R}^3$, the Poincaré and the Klein disks.

9.5.3 The action of $\mathbf{SL}(2,\mathbb{R})$ on $\mathbf{Sym}_1^+(2,\mathbb{R})$

Finally, we show that the action of $SL(2,\mathbb{R})$ on $\mathrm{Sym}_1^+(2,\mathbb{R})$ is analogous to the action of $SL(2,\mathbb{R})$ on H by Möbius transformations. The proof of this result needs a lemma.

Lemma 9.5.3 *Let $g = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathrm{Sym}_1^+(2,\mathbb{R})$. Then, $\omega(g) = \frac{1}{\alpha}(-\beta+i)$ is the unique solution in H of the equation:*

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^t g \begin{pmatrix} z \\ 1 \end{pmatrix} = \alpha z^2 + 2\beta z + \gamma = 0.$$

Proof. The two complex solutions of the equation are:

$$z_{1,2} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\alpha\gamma}}{2\alpha} = \frac{-\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha},$$

but $1 = \det(g) = \alpha\gamma - \beta^2$, so $\beta^2 - \alpha\gamma = -1$, so

$$z_{1,2} = \frac{-\beta \pm i}{\alpha},$$

the only solution in H is the one corresponding to $+i$, i.e. the only solution in H is:

$$\frac{1}{\alpha}(-\beta + i) = \omega(g).$$

□

Theorem 9.5.3 *Let $m \in SL(2,\mathbb{R})$, $g \in \mathrm{Sym}_1^+(2,\mathbb{R})$ and $z \in H$. Then,*

1. *$\omega(m * g) = m \cdot \omega(g)$*
2. *$F(m \cdot z) = m * F(z)$,*

i.e. the following diagram

$$\begin{tikzcd} \mathrm{Sym}_1^+(2,\mathbb{R}) \arrow[r, shift left, "\omega"] \arrow[d, "m*"] & H \arrow[l, shift left, "F"] \arrow[d, "m\cdot"] \\ \mathrm{Sym}_1^+(2,\mathbb{R}) \arrow[r, shift left, "\omega"] & H \arrow[l, shift left, "F"] \end{tikzcd}$$

is commutative. Hence, the action of $SL(2,\mathbb{R})$ on $\mathrm{Sym}_1^+(2,\mathbb{R})$ is transitive.

Proof.

1. We start by fixing the notation:

- $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, so that $m^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- $\tilde{z} = \omega(m * g) \in H$
- $z = \omega(g) \in H$

- $\tilde{\omega} = m^{-1} \cdot \tilde{z} = \mathscr{M}(m^{-1}, \tilde{z}) \in H$.

We note that:

$$m^{-1}\begin{pmatrix}\tilde{z}\\1\end{pmatrix} = \begin{pmatrix}d\tilde{z}-b\\-c\tilde{z}+a\end{pmatrix} = \underbrace{(-c\tilde{z}+a)}_{\neq 0}\begin{pmatrix}\frac{d\tilde{z}-b}{-c\tilde{z}+a}\\1\end{pmatrix} = (-c\tilde{z}+a)\begin{pmatrix}m^{-1}\cdot\tilde{z}\\1\end{pmatrix} = (c\tilde{z}+a)\begin{pmatrix}\tilde{\omega}\\1\end{pmatrix}, \tag{9.12}$$

where $-c\tilde{z}+a \neq 0$ because, if we write $\tilde{z} \in H$ as $\tilde{z} = x + iy$, then $-c\tilde{z}+a = 0$ would be equivalent to $-cx + a - icy = 0$, but, since $y > 0$, this would imply $a = c = 0$, that cannot be because m^{-1} would have a null row and would not be invertible.

Thanks to Lemma 9.5.3, $\tilde{z} = \omega(m * g) = \omega((m^{-1})^t g m^{-1})$ verifies:

$$\begin{pmatrix}\tilde{z}\\1\end{pmatrix}^t (m^{-1})^t g\, m^{-1}\begin{pmatrix}\tilde{z}\\1\end{pmatrix} = 0, \tag{9.13}$$

but

$$\begin{pmatrix}\tilde{z}\\1\end{pmatrix}^t (m^{-1})^t = \left(m^{-1}\begin{pmatrix}\tilde{z}\\1\end{pmatrix}\right)^t \underset{(9.12)}{=} (-c\tilde{z}+a)\begin{pmatrix}\tilde{\omega}\\1\end{pmatrix}^t,$$

so:

$$\begin{aligned}\begin{pmatrix}\tilde{z}\\1\end{pmatrix}^t (m^{-1})^t g\, m^{-1}\begin{pmatrix}\tilde{z}\\1\end{pmatrix} = 0 \quad &\Longleftrightarrow \quad (-c\tilde{z}+a)^2\begin{pmatrix}\tilde{\omega}\\1\end{pmatrix}^t g\begin{pmatrix}\tilde{\omega}\\1\end{pmatrix} = 0\\ &\Longleftrightarrow \quad \begin{pmatrix}\tilde{\omega}\\1\end{pmatrix}^t g\begin{pmatrix}\tilde{\omega}\\1\end{pmatrix} = 0.\end{aligned}$$

Therefore, thanks to Lemma 9.5.3, $\tilde{\omega} = \omega(g)$ and so $\omega(m * g) = m \cdot \omega(g)$.

2. Having proven 1., i.e. $m \cdot \omega(g) = \omega(m * g)$, the proof of 2. is very easy. In fact, thanks to theorem 9.5.2, $z = \omega(g)$ so $F(z) = F(\omega(g)) = g$, thus:

$$F(m\cdot z) = F(m\cdot\omega(g)) = F(\omega(m * g)) = m * g = m * F(z).$$

$\square$

Chapter 10

Geometry of the Lorentz space and Lorentz transformations (Antoine Guennec and Edoardo Provenzi)

The geometry of the Lorentz space and its related Lorentz transformations are two of the three basic tools that will allow us to rigorously describe the different realizations (called models) of hyperbolic geometry, the third tool being represented by Möbius transformations, that will be analyzed in the following chapter.

10.1 A quick recap about the Euclidean scalar product

In this section, we recall very quickly just the basic facts of Euclidean geometry that will use in the rest of this chapter.

Def. 10.1.1 *The Euclidean scalar product on* $\mathbb{R}^n$ *is defined as:*

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n,$$

its associated norm[1] *is:*

$$|x| = \|x\|_E = \sqrt{\langle x, x\rangle} = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}},$$

and its associated metric is:

$$d_E(x, y) = |x - y| = \sqrt{\langle x - y, x - y\rangle}.$$

Lemma 10.1.1 (Cauchy-Schwarz inequality) *Let* $x, y \in \mathbb{R}^n$*. Then,*

$$|\langle x, y\rangle| \leqslant |x|\,|y|$$

and the equality holds if and only if x *and* y *are linearly dependent.*

Proof. Suppose x and y are linearly dependent. Then, it exists $t \neq 0$ such that $y = tx$, so

$$|\langle x, y\rangle| = |\langle x, tx\rangle| = |t\langle x, x\rangle| = |t||x|^2 = |x||t||x| = |x||y|.$$

[1] In this chapter, exceptionally, it is notationally more convenient to use the symbol $|\ |$ for the Euclidean norm and reserve $\|\ \|$ to the Lorentzian norm.

Conversely, assume x and y are linearly independent. Then for all $t \in \mathbb{R}$, $tx - y \neq 0$ and so

$$0 < |tx - y|^2 = |x|^2 t^2 - 2\langle x, y\rangle t + |y|^2 = f(t),$$

which implies that the discriminant of f is negative, i.e. $\Delta = 4\langle x, y\rangle^2 - 4|x|^2|y|^2 < 0$, hence

$$\langle x, y\rangle^2 < |x|^2|y|^2 \iff |\langle x, y\rangle| < |x||y|.$$

$\square$

We will denote by $E^n = (\mathbb{R}^n, d_E)$ the Euclidean metric n-space considering it as an affine space so that we can perform translations in E^n.

Def. 10.1.2 (Isometries and similarities in E^n) *The isometries of E^n are transformations that preserve distances:*

$$\mathcal{I}(E^n) = \left\{\phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ d_E(\phi(x), \phi(y)) = d_E(x, y) \ \ \forall x, y \in \mathbb{R}^n\right\}.$$

The similarities of E^n are transformations that preserve shapes:

$$\mathcal{S}(E^n) = \left\{\phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ \exists k > 0 \ : \ d_E(\phi(x), \phi(y)) = k d_E(x, y) \ \ \forall x, y \in \mathbb{R}^n\right\}.$$

The sets of isometries and similarities form a group under composition.

The most important set of transformations in Euclidean geometry are the orthogonal ones, which form the group $\mathrm{O}(n)$, defined as maps that preserve the scalar product:

$$\langle \phi(x), \phi(y)\rangle = \langle x, y\rangle \qquad \forall x, y \in \mathbb{R}^n.$$

The following lemma allows us to characterize the orthogonal transformations.

Lemma 10.1.2 *$\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if and only if it is linear and, given an orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$, $(\phi(u_1), \ldots, \phi(u_n))$ is an orthonormal basis of $\mathbb{R}^n$.*

Proof. Suppose ϕ is an orthogonal transformation and $(u_1, \ldots, u_n)$ is any orthonormal basis of $\mathbb{R}^n$. Then,

$$\langle \phi(u_i), \phi(u_j)\rangle = \langle u_i, u_j\rangle = \delta_{ij},$$

hence $(\phi(u_1), \cdots, \phi(u_n))$ is, by definition, an orthonormal basis of $\mathbb{R}^n$ and so, for all $x \in \mathbb{R}^n$,

$$\phi(x) = \sum_{i=1}^{n} \langle \phi(x), \phi(u_i)\rangle \phi(u_i) \underset{(\phi \text{ orthogonal})}{=} \sum_{i=1}^{n} \langle x, u_i\rangle \phi(u_i),$$

but we also have $x = \sum\limits_{i=1}^{n} \langle x, u_i\rangle u_i$, so, by writing $x_i = \langle x, u_i\rangle$, we get:

$$\phi\Big(\sum_{i=1}^{n} x_i u_i\Big) = \sum_{i=1}^{n} x_i \phi(u_i), \qquad \forall x \in \mathbb{R}^n. \tag{10.1}$$

Now, if we consider $\lambda x = \sum_{i=1}^{n} \lambda x_i u_i$, we have:

$$\phi(\lambda x) = \sum_{i=1}^{n} \lambda x_i \phi(u_i) = \lambda \sum_{i=1}^{n} x_i \phi(u_i) = \lambda \phi(x), \qquad \forall x \in \mathbb{R}^n.$$

Moreover, if we consider another vector $y = \sum_{i=1}^{n} y_i u_i \in \mathbb{R}^n$, thanks to (10.1) we get

$$\phi(\sum_{i=1}^{n} y_i u_i) = \sum_{i=1}^{n} y_i \phi(u_i),$$

thus

$$\begin{aligned}
\phi(x+y) &= \phi(\sum_{i=1}^{n} x_i u_i + \sum_{j=1}^{n} y_j u_j) = \phi(\sum_{k=1}^{n} (x_k + y_k) u_k) \underset{(10.1)}{=} \sum_{k=1}^{n} (x_k + y_k) \phi(u_k) \\
&= \sum_{i=1}^{n} x_i \phi(u_i) + \sum_{i=1}^{n} y_i \phi(u_i) \\
&= \phi(x) + \phi(y),
\end{aligned}$$

hence the linearity of ϕ.

Conversely, suppose that ϕ is linear and that, for any orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$, $(\phi(u_1), \ldots, \phi(u_n))$ is again an orthonormal basis of $\mathbb{R}^n$. Then,

$$\phi(x) = \phi(\sum_{i=1}^{n} x_i u_i) \underset{(\phi \text{ linear})}{=} \sum_{i=1}^{n} x_i \phi(u_i) \qquad \forall x \in \mathbb{R}^n,$$

thus

$$\begin{aligned}
\langle \phi(x), \phi(y) \rangle &= \left\langle \sum_{i=1}^{n} x_i \phi(u_i), \sum_{j=1}^{n} y_j \phi(u_j) \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \langle \phi(u_i), \phi(u_j) \rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \delta_{i,j} = \sum_{i=1}^{n} x_i y_i \\
&= \langle x, y \rangle .
\end{aligned}$$

□

Def. 10.1.3 *The function*

$$\begin{aligned}
q : \quad \mathbb{R}^n &\longrightarrow \mathbb{R} \\
x &\longmapsto q(x) := \langle x, x \rangle = \|x\|_E^2
\end{aligned}$$

is called the ***quadratic form*** *associated to the Euclidean scalar product.*

The following result shows how an orthogonal transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ can be characterized via the quadratic form q.

Lemma 10.1.3 *$\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if and only if is preserves the quadratic form q, i.e. $q(\phi(x)) = q(x)$ for all $x \in \mathbb{R}^n$.*

Proof. By direct computation we get

$$q(x-y) = q(x) - 2\langle x, y\rangle + q(y) \iff \langle x, y\rangle = \frac{q(x)+q(y)-q(x-y)}{2}, \qquad \forall x, y \in \mathbb{R}^n,$$

so, it also holds that

$$\langle \phi(x), \phi(y)\rangle = \frac{q(\phi(x)) + q(\phi(y)) - q(\phi(x)-\phi(y))}{2}, \qquad \forall x, y \in \mathbb{R}^n.$$

So, if $q(\phi(x)) = q(x)$ for all $x \in \mathbb{R}^n$, then:

$$\langle \phi(x), \phi(y)\rangle = \frac{q(\phi(x)) + q(\phi(y)) - q(\phi(x)-\phi(y))}{2} = \frac{q(x)+q(y)-q(x-y)}{2} = \langle x, y\rangle,$$

i.e. ϕ is orthogonal.

Vice-versa, if ϕ is orthogonal, then $q(\phi(x))+q(\phi(y))-q(\phi(x)-\phi(y)) = q(x)+q(y)-q(x-y)$ for all $x, y \in \mathbb{R}^n$, but this equality holds true no matter how x and y are chose only when $q(\phi(x)) = q(x)$ for all $x \in \mathbb{R}^n$. □

Finally, we come to the complete characterization of $\mathcal{I}(E^n)$ and $\mathcal{S}(E^n)$.

Theorem 10.1.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$.*

1. *$f \in \mathcal{I}(E^n)$ if and only if f is of the form $f(x) = a + \phi(x)$, with $a \in \mathbb{R}^n$ and $\phi \in O(n)$.*
2. *$f \in \mathcal{S}(E^n)$ if and only if f is of the form $f(x) = a + k\phi(x)$, with $a \in \mathbb{R}^n$, $k > 0$ and $\phi \in O(n)$.*

Proof. First of all, notice that 1. is simply a special case of 2. with $k = 1$, thus we will concentrate only on the proof of 2.

$\boxed{2. \implies}$: if $f(x) = a + \phi(x)$, then, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned} d(f(x), f(y)) &= \langle a + k\phi(x) - (a + k\phi(y)), a + k\phi(x) - (a + k\phi(y))\rangle^{\frac{1}{2}} \\ &= \langle k(\phi(x) - \phi(y)), k(\phi(x) - \phi(y))\rangle^{\frac{1}{2}} \\ &\underset{(\phi \text{ linear})}{=} k\langle(\phi(x-y)), \phi(x-y)\rangle^{\frac{1}{2}} \\ &\underset{(\phi \text{ orthogonal})}{=} k\langle x-y, x-y\rangle^{\frac{1}{2}} \\ &= kd(x, y). \end{aligned}$$

$\boxed{2. \impliedby}$: suppose $f \in \mathcal{S}(E^n)$ and let $a = f(0)$ and $\psi(x) = f(x) - a$. Since f is a similarity there is a $k > 0$ such that $|f(x) - f(y)| = k|x - y|$ for all $x, y \in \mathbb{R}^n$ and so

$$|\psi(x)| = |f(x) - f(0)| = k|x - 0| = k|x|.$$

Consequently, by setting $\phi = \frac{1}{k}\psi$ and using lemma 10.1.3, we have $\phi \in O(n)$ and

$$f(x) = a + k\phi(x).$$

$\square$

Corollary 10.1.1 *An affine function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = a + \lambda x$, with $a, x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}\backslash\{0\}$ is always a Euclidean similarity and it is a Euclidean isometry if and only if $\lambda = 1$.*

Proof. The proof consists simply in remarking that λ can be identified with a one-entry matrix, which is orthogonal if and only if $\lambda = 1/\lambda$, i.e. $\lambda = 1$. $\square$

10.2 The geometry of the Lorentz n-space

The main reference throughout this section is Ratcliffe's book [15].

Lorentzian geometry is founded on an alternative definition of the scalar product in $\mathbb{R}^n$ w.r.t.the Euclidean one for $n \geqslant 2$, when $n = 1$ the two products agree. For this reason, in this chapter we will always implicitly consider $n \geqslant 2$.

The Lorentz scalar product is actually a so-called pseudo-scalar product. The formal algebraic theory that allowed the modern definition of such a concept has been developed by E. Witt in [21]. Here we collect only the definitions and results that are needed to understand Lorentz's geometry, for a more thorough discussion see, e.g., [14].

Let V be a real vector space and x, y arbitrary vectors in V.

1. A bilinear form on V is an $\mathbb{R}$-bilinear function $b : V \times V \to \mathbb{R}$;

2. The quadratic form associated to b is the linear functional $q_b : V \to \mathbb{R}$ defined by $q_b(x) := b(x, x)$. It is often simpler to work with q_b than with b and no information is lost, since we can reconstruct b from q via the well-known polarization identity:

$$b(x, y) = \frac{1}{2}(q_b(x + y) - q_b(x) - q_b(y));$$

3. b is symmetric if $b(x, y) = b(y, x)$ for all x, y. In what follows, b will always be implicitly considered symmetric;

4. b is positive (negative) definite if $x \neq 0$ implies $q_b(x) > 0$ (< 0);

5. b is positive (negative) semi-definite if $x \neq 0$ implies $q_b(x) \geqslant 0$ ($\leqslant 0$);

6. Of course, if b is positive (negative) definite, then it is also positive (negative) semi-definite;

7. If b is neither positive nor negative semi-definite, b is called **indefinite**;

8. b is **nondegenerate** if $b(x, y) = 0 \ \forall y$ implies $x = 0$;

9. A **scalar product** g on V is a positive-definite nondegenerate symmetric bilinear form on V. (V, g) is called a scalar product space;

10. A **pseudo-scalar product** b on V is a nondegenerate symmetric bilinear form on V. Thus, the big difference between a pseudo- and a scalar product is the *lack of definite-positiveness* for the first. (V, b) is called a pseudo-scalar product space;

11. For any vector subspace $W \subset V$, we denote with $b|_W$ and $q_b|_W$ the restriction of b to $W \times W$ and of q_b to W, respectively. If b is a symmetric bilinear form, so is $b|_W$;

12. The **index** ν of a symmetric bilinear form b on V is the largest integer that coincides with the dimension of a subspace $W \subset V$ on which b_W is **negative definite**. Thus $0 \leqslant \nu \leqslant \dim(V)$, and $\nu = 0$ if and only if b is positive-semidefinite or positive-definite;

13. If $(u_1, \ldots, u_n)$ is a basis for V, the $n \times n$ matrix $B = (b_{ij}) = b(u_i, u_j)$ is called the **matrix of** b **relative to** $(u_1, \ldots, u_n)$. If b is symmetric, then B is a symmetric matrix. If $x = \sum_{i=1}^{n} x_i u_i$ and $y = \sum_{i=1}^{n} y_i u_i$, then, by bilinearity we have

$$b(x, y) = b(\sum_{i=1}^{n} x_i u_i, \sum_{i=1}^{n} y_i u_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j b(u_i, u_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j b_{ij} = \langle x, By \rangle = \langle Bx, y \rangle,$$

thus the action of b on the vectors of V is completely determined by B;

14. A symmetric bilinear form b on V is nondegenerate if and only if its matrix B relative to an arbitrary basis of V is invertible;

15. A vector $u \in (V, b)$ such that $q_b(u) = \pm 1$ is said to be a unit vector in (V, b). Two vectors $x, y \in (V, b)$ are orthogonal if $b(x, y) = 0$;

16. A set of $m \leqslant n$ mutually orthogonal unit vectors in (V, b) is said to be an orthonormal family. If $m = n$, then we talk about an orthonormal basis of (V, b);

17. The matrix of b associated to an orthonormal basis $(u_1, \ldots, u_n)$ is diagonal, in fact its entries are given by:

$$b(u_i, u_j) = \pm\delta_{i,j},$$

the ordered sequence of -1 and $+1$, repeated for all the time they appear in the diagonal of the matrix associated to b w.r.t.any orthonormal basis is called **signature** of b;

18. The signature appears in the orthonormal expansion of any $x \in (V, b)$ on an orthonormal basis $(u_1, \ldots, u_n)$ as follows:

$$x = \sum_{i=1}^{n} \varepsilon_i b(x, u_i) u_i \ , \tag{10.2}$$

where $\varepsilon_i = q_b(u_i) \in \{-1, +1\}$;

19. The orthogonal projection π of any $x \in (V, b)$ onto a subspace $W = \text{span}(u_1, \ldots, u_m)$, where $(u_1, \ldots, u_m)$ is an orthonormal family and $m < n$ is the following:

$$\pi(x) = \sum_{i=1}^{m} \varepsilon_i b(x, u_i) u_i \ ;$$

20. The **residual vector**

$$y := x - \pi(x) = x - \sum_{i=1}^{m} \varepsilon_i b(x, u_i) u_i \tag{10.3}$$

is orthogonal to all vectors of W: $b(y, w) = 0 \ \forall w \in W$;

21. The number of negative signs in the signature of b is constant for any orthonormal basis $(u_1, \ldots, u_m)$ of (V, b) and it coincides with the index ν. For the proof see [14], lemma 26 page 51.

Let us specify all this in the case of the Lorentz pseudo-scalar product in $\mathbb{R}^n$.

Def. 10.2.1 (Lorentz's pseudo-scalar product) *Let $x, y \in \mathbb{R}^n$. The Lorentz (or Lorentzian) pseudo-scalar product between x and y is defined as follows:*

$$\boxed{x \circ y = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}\ .$$

*$(\mathbb{R}^n, \circ)$, i.e. $\mathbb{R}^n$ interpreted as a vector space endowed with the Lorentzian pseudo-scalar product, is denoted by $\mathbb{R}^{1,n-1}$ and called **the Lorentzian n-space**.*

In literature, we find several other definitions of the Lorentzian scalar product. The first alternative definition that we discuss is the following:

$$x \circ y = x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n$$

and $\mathbb{R}^n$ endowed with this last Lorentzian scalar product is denoted by $\mathbb{R}^{n-1,1}$. Results in both cases are exactly the same and the choice of $\mathbb{R}^{n-1,1}$ or $\mathbb{R}^{1,n-1}$ depends on convenience or taste.

Instead, the following alternative and perfectly valid choice:

$$x \circ y = x_1 y_1 - \cdots - x_{n-1} y_{n-1} - x_n y_n$$

corresponds to the opposed signature w.r.t.the previous one.

The case of $n = 4$ is of particular importance in Physics, as it is the geometric setting of special relativity, with the coordinate $x_1 = t$ playing the role of time and $(x_2, x_3, x_4) = (x, y, z)$ the role of space coordinates. For this reason, it bears a special name.

Def. 10.2.2 *The Lorentz space $\mathbb{R}^{1,3}$ is called **Minkowski spacetime** $\mathcal{M}$.*

The bilinearity and symmetry of the Lorentz pseudo-scalar product is immediate to see, to prove its nondegeneracy it is enough to take as y all the vectors of the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$:

$$\begin{aligned} x \circ e_1 &= -x_1 \cdot 1 + x_2 \cdot 0 + \cdots + x_n \cdot 0 = -x_1 \\ &\vdots \\ x \circ e_n &= -x_1 \cdot 0 + x_2 \cdot 0 + \cdots + x_n \cdot 1 = x_n \end{aligned}$$

so, $x \circ e_i = 0$ for all $i = 1, \ldots, n$ implies $x = 0$.

Def. 10.2.3 *The quadratic form associated to the Lorentz pseudo-scalar product is:*

$$q(x) := x \circ x = -x_1^2 + x_2^2 + \cdots + x_n^2, \quad q(x) \in \mathbb{R},$$

and the Lorentz pseudo-norm is:

$$\|x\| := \sqrt{q(x)} = \sqrt{-x_1^2 + x_2^2 + \cdots + x_n^2}, \quad \|x\| \in \mathbb{R}^+ \cup \{0\} \cup i\mathbb{R}^+.$$

Remark 10.2.1 The Lorentz pseudo-scalar product can be rewritten using the Euclidean scalar product:

$$x \circ y = \langle x, \eta y \rangle = \langle \eta x, y \rangle, \ \text{ with } \eta = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} = \mathrm{diag}(-1, 1, \dots), \ \eta^t = \eta^{-1} = \eta, \ \eta^2 = I_n.$$

It is clear that *the Lorentz pseudo-scalar product is indefinite.* This fact allows us to separate the Lorentz n-space in three different subsets called time-like, light-like and space-like, a terminology taken from special relativity (see chapter 13 for the physical motivation of these names).

Def. 10.2.4 $x \in \mathbb{R}^{1,n-1}$ *is said to be*

- ***time-like*** *if* $x \circ x < 0 \iff q(x) < 0 \iff$ *its squared Lorentz pseudo-norm is negative;*
- ***light-like*** *if* $x \circ x = 0 \iff q(x) = 0 \iff$ *its squared Lorentz pseudo-norm is null;*
- ***space-like*** *if* $x \circ x > 0 \iff q(x) > 0 \iff$ *its squared Lorentz pseudo-norm is positive;*
- ***causal*** *if it is not space-like.*

One of the three options is called ***likeness*** *of* x.
Moreover, the ***orientation*** *(or* ***parity****) of a time-like or light-like vector is:*

- ***positive*** *if* $x_1 > 0$*;*
- ***negative*** *if* $x_1 < 0$.

Examples: if we consider the vectors $(e_1, \dots, e_n)$ of the canonical basis of $\mathbb{R}^n$, then $e_1 \circ e_1 = -1$ and $e_j \circ e_j = 1$ for all $2 \leqslant j \leqslant n$, so:

- e_1 is a time-like vector;
- e_j, for all $2 \leqslant j \leqslant n$, are space-like vectors;
- $e_1 + e_2 = (1, 1, 0, \dots, 0)^t$ is a light-like vector.

In the Minkowski space, we have the identification $(x_1, x_2, x_3, x_4) \equiv (t, x, y, z)$, where t is the time coordinate and (x, y, z) are the spatial ones. The Lorentz pseudo-norm in this case is called **Minkowski pseudo-norm**.

It is instructive to examine the geometric meaning of likeness in the case $n = 2$ for a generic $x \in \mathbb{R}^{1,1}$ starting from the light-like case.

- $u = (x, y)$ is light-like if and only if $x^2 = y^2 \iff |x| = |y|$, i.e. light-like vectors lie on the two $\pi/4$ degrees straight lines passing through the origin.
- The time-like case is characterized by $x^2 < y^2 \iff |x| < |y|$, i.e. time-like vectors belong to the interior of the upper and lower triangular regions in $\mathbb{R}^2$ delimited by the origin and the straight lines where light-like vectors live.
- The space-like case is obviously identified by the remaining areas. Figure 10.1 gives a graphical representation of this simple analysis.

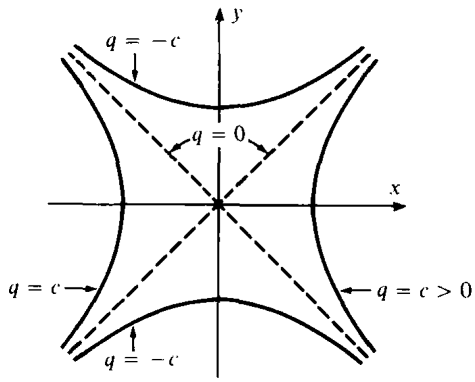


Figure 10.1: A graphical depiction of likeness regions in 2 dimensions, together with the level lines of the Lorentz pseudo-norm.

In the figure we can also see that the level lines of the quadratic form $q(x)$, i.e. the **vectors with same Lorentz pseudo-norm are hyperbolas** contained in either the time-like or space-like regions with asymptotes given by the light-like straight lines. In fact, for all $c \neq 0$, $q(x) = c \iff -x_1^2 + x_2^2 = c$, which is the equation of a hyperbola. If $c < 0$ the hyperbola belongs to the time-like region, if $c > 0$ to the space-like region.

More generally, the light-like equation $x \circ x = 0 \iff q(x) = 0 \iff \|x\| = 0$ defines a hypercone $\mathcal{C}^{n-1}$ in $\mathbb{R}^n$, called **light cone**. Time-like vectors belong to its interior, while space-like vectors belong to the external region. Figure 10.2 depicts the case $n = 3$.

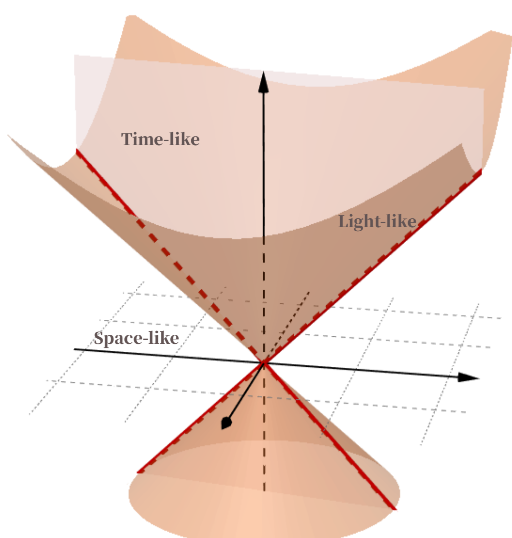


Figure 10.2: The Lorentz n-space is separated in three components: light-like vectors belong to the surface of the light cone, time-like vectors lie inside and space-like vectors lie outside.

The name light cone is an extension of the case $n = 4$, where $\mathcal{C}^{n-1}$ is the cone traveled by rays of light in the Minkowski spacetime of special relativity. The positive and negative time-like regions are called, respectively, **future** and **past light cone**.

In E^n, vectors with same Euclidean norm lie on spheres, while, in the n-Lorentz space, **vectors with same Lorentz pseudo-norm lie on hyperboloids** contained in either the time-like or space-like regions. This very simple consideration gives the first hint of why the Euclidean scalar product is related to spherical geometry while the Lorentzian pseudo-scalar product is related to hyperbolic geometry.

The following *notational convention* will simplify a lot future equations: whenever useful, we will write $x \in \mathbb{R}^n$ as

$$x = (x_1, x_2, \ldots, x_n)^t = (x_1, \bar{x})^t, \quad \text{i.e.} \quad \bar{x} = (x_2, x_3, \ldots, x_n)^t.$$

With this notation:

- Lorentz scalar product and norm:

$$x \circ y = \langle \bar{x}, \bar{y} \rangle - x_1 y_1, \quad \|x\|^2 = |\bar{x}|^2 - x_1^2 \ ;$$

- $\mathbb{R}^{1,n-1} \ni x = \begin{cases} \text{light-like:} & |\bar{x}| = |x_1| \quad (x \in \mathcal{C}^{n-1}) \\ \text{time-like:} & |\bar{x}| < |x_1| \quad (x \in \text{int}(\mathcal{C}^{n-1})) \\ \text{space-like:} & |\bar{x}| > |x_1| \quad (x \in \text{ext}(\mathcal{C}^{n-1})) \end{cases} .$

Just like in Euclidean geometry, orthogonality will play a major role.

Def. 10.2.5 *$x, y \in \mathbb{R}^{1,n-1}$ are called **Lorentz-orthogonal** if $x \circ y = 0$. They are **Lorentz-orthonormal** if they are Lorentz-orthogonal and the modulus of their pseudo-Lorentz norm is 1, i.e. $\|x\| = i$ if x is time-like and $\|x\| = 1$ if x is space-like.*

The Euclidean scalar product allows us to fully understand the Euclidean geometry, hence it is not surprising that many information about the Lorentz n-space geometry can be gathered by studying the Lorentz scalar product.

We start by first proving a very simple fact and then a result that will have important consequences.

Lemma 10.2.1 *For all $x \in \mathbb{R}^{1,n-1}$ and all $t > 0$, the vector tx has the same likeness and orientation as x.*

Proof. For all $t > 0$, $\|tx\| = t\,\|x\|$ and $(tx)_1 = tx_1$, thus tx and x have the same likeness and orientation. □

Theorem 10.2.1 *If $x, y \in \mathbb{R}^{1,n-1}$ are non-zero, equioriented and causal, then $x \circ y \leqslant 0$ and the equality holds if and only if x and y are linearly dependent light-like vectors.*

Proof. The case of x and y being both negatively oriented can be derived from the positive case by replacing x and y with $-x$ and $-y$, respectively, thus we can assume that both x and y are positively oriented, i.e. $x_1 > 0$ and $y_1 > 0$.

By hypothesis, x is time-like or light-like, so:

$$x \circ x \leqslant 0 \iff |\bar{x}|^2 \leqslant x_1^2 \underset{x_1 > 0}{\iff} |\bar{x}| \leqslant x_1,$$

and, similarly, $|\bar{y}| \leqslant y_1$. These inequalities, together with the Cauchy-Schwartz inequality 10.1.1 applied on the Euclidean scalar product $|\,|$, imply

$$\langle \bar{x}, \bar{y} \rangle \leqslant |\langle \bar{x}, \bar{y} \rangle| \leqslant |\bar{x}||\bar{y}| \leqslant x_1 y_1, \tag{10.4}$$

hence we come to the conclusion that $x \circ y = \langle \bar{x}, \bar{y} \rangle - x_1 y_1 \leqslant 0$.

Let us now examine when $x \circ y = \langle \bar{x}, \bar{y} \rangle - x_1 y_1 = 0$, i.e. $x_1 y_1 = \langle \bar{x}, \bar{y} \rangle$, thus we can replace $x_1 y_1$ on the rightmost part of inequality (10.4) with $\langle \bar{x}, \bar{y} \rangle$, this implies $\langle \bar{x}, \bar{y} \rangle \leqslant |\bar{x}||\bar{y}| \leqslant \langle \bar{x}, \bar{y} \rangle$, i.e. $\langle \bar{x}, \bar{y} \rangle = |\bar{x}||\bar{y}|$. Lemma 10.1.1 guarantees that this can happen if and only if $\bar{x}$ and $\bar{y}$ are linearly dependent, we can therefore set $\bar{y} = t\bar{x}$, with $t \neq 0$, and observe that

$$x \circ y = 0 \iff x_1 y_1 = t|\bar{x}|^2 \iff y_1 = \frac{t|\bar{x}|^2}{x_1}, \tag{10.5}$$

which implies two things: firstly $t > 0$ (x_1 and y_1 are both supposed to be positive), secondly, $y \circ y = |\bar{y}|^2 - y_1^2 \underset{\bar{y}=t\bar{x}}{=} t^2|\bar{x}|^2 - \frac{t^2|\bar{x}|^4}{x_1^2}$. Recalling that y is either time-like or light-like, we must have:

$$y \circ y \leqslant 0 \iff x_1^2 \leqslant |\bar{x}|^2 \iff 0 \leqslant |\bar{x}|^2 - x_1^2 \iff 0 \leqslant x \circ x.$$

If x is time-like, then $x \circ x < 0$ and the previous inequality is not verified, thus x must be light-like (i.e. $x \circ x = 0$). Being light-like and positively oriented, x satisfies $|\bar{x}| = x_1$ and so the central equation of formula (10.5) implies $x_1 y_1 = t x_1^2$, i.e. $y_1 = t x_1$. In conclusion, $\bar{y} = t\bar{x}$ and $y_1 = t x_1$ imply that $y = tx$ and also that x and y are both light-like vectors. □

So, **two light-like vectors are Lorentz-orthogonal if and only if they are scalar multiple of each other**. Instead, for time-like vectors it holds the following.

Corollary 10.2.1 *If $x, y \in \mathbb{R}^{1,n-1}$ are equioriented time-like vectors, then $x \circ y < 0$.*

A significant implication of this theorem is that, **in the Lorentzian geometry of $\mathbb{R}^{1,n-1}$, two orthogonal vectors are no longer characterized by a relative angle of** $\pi/2$: if one belongs to the interior and the other to the exterior of the light cone, they can be orthogonal even if their relative angle is different than $\pi/2$.

Corollary 10.2.2 *Let x, y be two non-zero Lorentz-orthogonal vectors, i.e. $x \circ y = y \circ x = 0$, then:*

$$x \text{ time-like} \implies y \text{ space-like}.$$

Proof. We can assume that y has the same orientation of x, in the opposite case it is sufficient to replace y with $-y$ to obtain the proof. If x is time-like then, by theorem 10.2.1, if y is time-like or light-like, we must have $x \circ y < 0$, where the inequality is strict because the equality can happen only with two linearly dependent light-like vectors. Since $x \circ y < 0$ is incompatible with the hypothesis $x \circ y = 0$, the only option that remains is that y must be space-like. □

Let us see an example using again the simple case of $n = 2$: $x \circ y = 0 \iff x_1 y_1 = x_2 y_2$, then of course $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are Lorentz orthogonal, but, for example, also any couple of vectors of the form $x = (1, a)$ and $y = (a, 1)$ is, $\forall a \in \mathbb{R}$. Figure 10.3 depicts the case $0 < c < 1$.

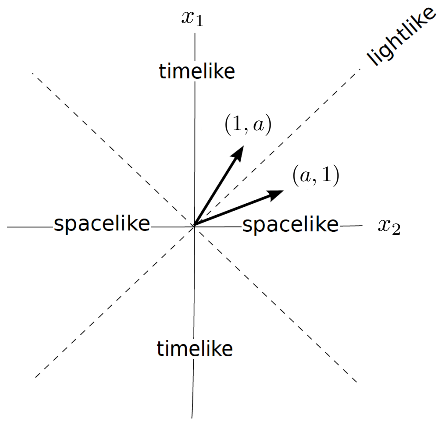


Figure 10.3: A graphical depiction of orthogonality in the 2 dimensional Lorentz space.

The relative position between the two vectors $(1, a)$ and $(a, 1)$ is not accidental, in fact, as we are going to prove, a time-like x and a space-like y vectors in $\mathbb{R}^{1,1}$ are Lorentz-orthogonal if and only if the angles α and β that they form w.r.t.the horizontal axis are complementary, i.e. they sum to $\pi/2$, modulo an integer multiple of π: $\alpha + \beta = \pi/2 + k\pi$, $k \in \mathbb{Z}$. To prove this, we just write down the polar coordinates $x = (r\cos\alpha, r\sin\alpha)^t$ and $y = (R\cos\beta, R\sin\beta)^t$ of x and y, $r, R > 0$, $\alpha, \beta \in [0, 2\pi)$, then, the Lorentz-orthogonality condition can be rewritten as

$$rR\cos\alpha\cos\beta - rR\sin\alpha\sin\beta = 0 \iff rR(\cos(\alpha+\beta)) = 0 \iff \cos(\alpha+\beta) = 0,$$

which implies $\alpha + \beta = \pi/2 + k\pi$.

Notice that *the reverse statement of the theorem is not true*: *given two non-zero Lorentz-orthogonal vectors $x, y \in \mathbb{R}^{1,n-1}$, if x is space-like then y can be both space-like and time-like.* A simple example is given by the vectors $e_2 = (0, 1, 0)^t$ and $e_3 = (0, 0, 1)^t$ of the canonical basis of $\mathbb{R}^{1,2}$: $\|e_2\|^2 = \|e_3\|^2 = 1$, so they are space-like, but $e_2 \circ e_3 = 0$.

The following result tell us, among other information, that the sum of two equioriented time-like vectors is still a time-like vector with the same orientation.

Corollary 10.2.3 *If x and y are non-zero, equioriented, causal vectors, then $x + y$ has the same orientation as x and y. Moreover, $x + y$ is light-like if and only if x and y are linearly dependent light-like vectors, otherwise $x + y$ is time-like.*

Proof. For the same reason given in the proof of theorem 10.2.1, we consider only the positively oriented case. In this case we have $x_1, y_1 > 0$, so $(x + y)_1 = x_1 + y_1 > 0$ and so $x + y$ is also positively oriented. Additionally, by direct computation we have:

$$\|x + y\|^2 = \|x\|^2 + 2(x \circ y) + \|y\|^2 ,$$

which is $\leqslant 0$ as the sum of three terms $\leqslant 0$. So, $x + y$ is either light-like or space-like. Finally, thanks to the previous theorem,

$$\|x + y\|^2 = 0 \iff \|x\|^2 = \|y\|^2 = x \circ y = 0$$

which is true if and only if x and y are light-like and linearly dependent. $\square$

To fix the ideas, let us consider only vectors oriented towards the future, then the previous result can be re-written as follows:

- the sum of two linearly dependent light-like vectors will be a light-like vectors towards the future;
- the sum of two non-linearly dependent light-like vectors will be a time-like vector towards the future;
- the sum of two time-like vectors will be a time-like vector towards the future;
- the sum of a time-like vector and a light-like vector will be a time-like vector towards the future.

Before stating the following corollary, we recall some definitions about cones taken from [5].

Def. 10.2.6 *Let $\mathcal{C}$ be a subset of a vector space V, then:*

- *$\mathcal{C}$ is a **cone** if, for all $t > 0$, $x \in \mathcal{C} \implies tx \in \mathcal{C}$;*
- *a cone $\mathcal{C}$ is **convex** if, for all $t \in [0, 1]$ and all couple of vectors $x, y \in \mathcal{C}$, $tx + (1-t)y \in \mathcal{C}$;*
- *a cone $\mathcal{C}$ is **proper** (or **regular**) if $\overline{\mathcal{C}} \cap -\overline{\mathcal{C}} = \{0\}$, where 0 is the zero vector of V, $\overline{\mathcal{C}}$ is the topological closure of $\mathcal{C}$ and $-\mathcal{C} := \{-x,\ x \in \mathcal{C}\}$.*

Corollary 10.2.4 *The set of all positively (respectively negatively) oriented time-like vectors forms an open connected proper convex cone in $\mathbb{R}^{1,n-1}$.*

Proof. Theorem 10.2.1 implies that the set of all positive (respectively negative) oriented time-like vectors forms a cone in $\mathbb{R}^{1,n-1}$. This cone is either the upper or the bottom part of the open set given by the interior of $\mathcal{C}^{n-1}$, thus it is connected, open and proper. Finally, let x and y two positively (resp. negatively) oriented time-like vectors.

The convexity of the proper cone they form follows from the combination of theorem 10.2.1 with theorem 10.2.1: for all $t \in (0, 1)$ set $\tilde{x} := tx$ and $\tilde{y} := (1-t)y$, then $\tilde{x}$ and $\tilde{y}$ belong to the same cone as x and y by theorem 10.2.1, thus their sum $\tilde{x} + \tilde{y} = tx + (1-t)y$ belongs to the same cone too for all $t \in (0, 1)$ by theorem 10.2.1. Since $x = (tx + (1-t)y)|_{t=1}$ and $y = (tx + (1-t)y)|_{t=0}$, we have that the convex combination $tx + (1-t)y$ belongs to the same cone as x and y for all $t \in [0, 1]$. □

Def. 10.2.7 (Time-like cone) *We call the cone of all positively (respectively, negatively) oriented time-like vectors in $\mathbb{R}^{1,n-1}$ the future (respectively, the past) time-like cone.*

These results explain why the concept of orientation is defined only for causal vectors: **the future and the past light-cones and time-cones are the connected components of two disjoint sets**, thus orientation allows us to single out which connected component we are dealing with.

Instead, **for all $n \geqslant 3$ the space-like region is connected** (but not convex because antipodal points w.r.t.the origin cannot communicate via a straight line segment), so specifying an orientation of a space-like vector does not single out any particular connected component of the space-like region. The only exception is represented by the case $n = 2$, but in the literature this special case is simply treated separately from the others without introducing a particular nomenclature for space-like vectors even for this exception.

10.2.1 Orthogonality and orthonormality in the Lorentz n-space

Motivated by the central importance of the concept of orthonormal basis in Euclidean geometry, we give the equivalent definition in Lorentzian geometry in $\mathbb{R}^{1,n-1}$.

As a preliminary remark, we notice that, since light-like vectors have null Lorentz pseudo-norm, **only time-like or space-like vectors can be unit vectors** in $\mathbb{R}^{1,n-1}$.

Def. 10.2.8 (Orthonormality in $\mathbb{R}^{1,n-1}$) *A set of $m \leqslant n$ mutually Lorentz-orthogonal unit vectors in $\mathbb{R}^{1,n-1}$ is said to be an orthonormal family. If $m = n$, then we have an orthonormal basis of $\mathbb{R}^{1,n-1}$.*

We now note that the matrix of the Lorentz pseudo-scalar product relative to an orthonormal basis $u_1, \ldots, u_n$ coincides with the diagonal matrix $\eta = \operatorname{diag}(-1, +1, +1, \ldots, +1)$.

Thus, the signature of the Lorentz pseudo-scalar product is $(-, +, +, \ldots, +)$, so, thanks to property 21 of the pseudo-scalar product previously quoted, its index ν is 1.

However, it is clear that **the index ν of the Lorentz pseudo-scalar product is the maximal number of linearly independent time-like vectors**. In fact, such vectors generate the subspace of $\mathbb{R}^{1,n-1}$ with highest dimension on which the Lorentz pseudo-scalar product is, by definition of time-likeness, negative-definite.

The consequence of this line of reasoning is that **in every Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$ there is exactly one time-like vector and $n-1$ space-like vectors**. By convention, the time-like vector is set to be the first basis vector. This justifies the following, more explicit, definition of Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$.

Def. 10.2.9 *A set of n vectors $\mathscr{B} = (u_1, \ldots, u_n)$ is a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$ if*

$$u_i \circ u_j = \begin{cases} -1 & \text{if} \quad i = j = 1 \\ 1 & \text{if} \quad i, j \geqslant 2,\ i = j \\ 0 & \text{if} \quad i \neq j \end{cases} .$$

Moreover, we say the basis is positive if u_1 is a positively oriented time-like vector.

By direct computation, it can be verified that **the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ is a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$.**

A Lorentz-orthonormal basis $\mathscr{B} = (u_1, \ldots, u_n)$ of $\mathbb{R}^{1,n-1}$ is an actual basis of $\mathbb{R}^n$, i.e. it is a family of n linearly independent vectors. To verify this, consider an arbitrary linear combination of the vectors of $\mathscr{B}$, i.e. the vector $\tilde{u} = \alpha_1 u_1 + \cdots + \alpha_n u_n$, with $\alpha_i \in \mathbb{R}$ for all i. Then, by using Lorentz-orthonormaly in the direct computation of the Lorentz pseudo-scalar products between $\tilde{u}$ and each vector of $\mathscr{B}$, we have:

$$\tilde{u} \circ u_i = \alpha_i (u_i \circ u_i) = \begin{cases} -\alpha_1 & \text{if} \quad i = 1 \\ \alpha_i & \text{if} \quad i \geqslant 2 \end{cases},$$

$$\tilde{u} = 0 \implies 0 = 0 \circ u_i = \begin{cases} -\alpha_1 & \text{if} \quad i = 1 \\ \alpha_i & \text{if} \quad i \geqslant 2 \end{cases} \implies \alpha_1 = \cdots = \alpha_n = 0.$$

The existence of Lorentz-orthonormal bases is guaranteed by the following result.

Theorem 10.2.2 (Gram-Schmidt Lorentz-orthonormalisation) *Let $\mathscr{B} = (u_1, \ldots, u_n)$ be a vector basis of $\mathbb{R}^{1,n-1}$ with u_1 a time-like vector. Then we can extract a basis $\mathscr{B}_L = (w_1, \cdots, w_n)$ from $\mathscr{B}$ such that:*

1. *$\mathscr{B}_L$ is a positive Lorentz-orthonormal basis.*
2. $\operatorname{span}(w_1, \ldots, w_k) = \operatorname{span}(u_1, \cdots, u_k)$ *for all $k \in \{1, \ldots, n\}$.*

Proof. This proof follows a path very similar to the proof for the usual Gram-Schmidt process, but with some minor tweaks. We start by setting

$$w_1 := \begin{cases} \frac{u_1}{|||u_1|||} & \text{if } u_1 \text{ is positive} \\ -\frac{u_1}{|||u_1|||} & \text{if } u_1 \text{ is negative} \end{cases},$$

and then we set

$$\begin{cases} v_2 := u_2 + (u_2 \circ w_1) w_1 \\ w_2 := \frac{v_2}{\|v_2\|} \end{cases}$$

and

$$\begin{cases} v_k := u_k + (u_k \circ w_1) w_1 - \sum_{i=2}^{k-1} (u_k \circ w_k) w_k \\ w_k := \frac{v_k}{\|v_k\|} \end{cases} \qquad \text{for} \quad k \geqslant 3.$$

With such a construction, $\mathscr{B}_L = \{w_1, \cdots, w_n\}$ is a positive Lorentz orthonormal basis that verifies the wanted conditions. □

Remark 10.2.2

Similarly, we can apply a similar process when $(u_1, \ldots, u_m)$ is a set of linearly independent, space-like vectors to extract a set of space-like vectors $(w_1, \ldots, w_m)$ such that

$$w_i \circ w_j = \delta_{ij} \quad \text{and} \quad \operatorname{span}(w_1, \ldots, w_k) = \operatorname{span}(u_1, \ldots, u_k),$$

for all $k \in \{1, \ldots, m\}$.

Def. 10.2.10 *Let V be a vector subspace of $\mathbb{R}^n$. The Lorentz-orthogonal complement of V is:*

$$V^L = \{x \in \mathbb{R}^n : x \circ y = 0 \;\; \forall y \in V\}.$$

The properties of the Lorentz-orthogonal complement are given in the next result, where we use the symbol $V^\perp$ to denote the Euclidean orthogonal complement of V.

Lemma 10.2.2 *Let V be a vector subspace of $\mathbb{R}^n$ and write $\eta(V^\perp) := \{\eta x,\ x \in V^\perp\}$, then:*

1. *$V^L = \eta(V^\perp)$;*
2. *$(V^L)^L = V$, i.e. the Lorentz-orthogonalization is an involutive operation.*

Proof.

1. : it is a consequence of remark 10.2.1, i.e. $x \circ y = \langle x, \eta y \rangle = \langle \eta x, y \rangle \ \forall x, y \in \mathbb{R}^n$, and of the fact that $\eta = \text{diag}(-1, 1, \dots, 1)$ verifies $\eta^2 = I_n$.

$\boxed{V^L \subseteq \eta(V^\perp)}$: let $\underline{x \in V^L}$, then $x \circ y = 0 \ \forall y \in V$ and so $\langle \eta x, y \rangle = 0 \ \forall y \in V$, i.e. $\eta x \in V^\perp$ but then $\underline{x = \eta(\eta x) \in \eta(V^\perp)}$.

$\boxed{\eta(V^\perp) \subseteq V^L}$: let $\underline{x \in \eta(V^\perp)}$, then $\eta x \in \eta(\eta(V^\perp)) = V^\perp$, so $\langle \eta x, y \rangle = 0 \ \forall y \in V$, but then $x \circ y = 0 \ \forall y \in V$, i.e. $\underline{x \in V^L}$.

2. : we have

$$x \in (V^L)^L \iff x \circ y = 0 \ \ \forall y \in V^L \underset{(1.)}{\iff} x \circ \eta z = 0 \ \ \forall z \in V^\perp,$$

which is equivalent to $\langle x, \eta(\eta z) \rangle = \langle x, z \rangle = 0 \ \forall z \in V^\perp$, i.e. $x \in (V^\perp)^\perp = V$. □

1. is clearly the consequence of the fact that $x \circ y = \langle x, \eta y \rangle = \langle \eta x, y \rangle$, hence Lorentz-orthogonality between two vectors x and y of $\mathbb{R}^{1,n-1}$ can be interpreted as the Euclidean orthogonality between one vector and the Euclidean orthogonal reflection of the other along its first coordinate.

We end this section with the classification of vector subspaces in $\mathbb{R}^{1,n-1}$ in 3 categories.

Def. 10.2.11 *Let V be a vector subspace of $\mathbb{R}^{1,n-1}$.*

- *V is **time-like** if it contains at least a time-like vector;*
- *V is **space-like** if every $x \in V \setminus \{0\}$ is space-like;*
- *V is **light-like** otherwise.*

$\mathbb{R}^n$, as improper subset of $\mathbb{R}^{1,n-1}$, is a time-like vector subspace, because $\mathbb{R}^n = \text{span}(e_1, \dots, e_n)$ and e_1 is time-like.

It might be a little surprising that a time-like vector subspace V of $\mathbb{R}^{1,n-1}$ is defined just by requiring the existence of a time-like vector in V, while, on the contrary, we demand all non-null vectors of a space-like vector subspace to be space-like. The difference is justified by corollary 10.2.2 which imposes a strong constraint on the number of time-like vectors that can appear in a Lorentz-orthonormal basis of a vector subspace: either one or zero! Even if a basis of a vector subspace V of $\mathbb{R}^{1,n-1}$ is composed only by time-like vectors, after orthonormalization, only one vector will remain time-like and the others will become space-like. Moreover, as a consequence of corollary 10.2.3, only **one light-like vector can appear in a basis of a light-like vector subspace and all the other basis vectors must be space-like**.

Figure 10.4 gives a graphical depiction of vector subspaces of $\mathbb{R}^{1,2}$ (which are necessarily either hyperplanes or straight lines passing through the origin).

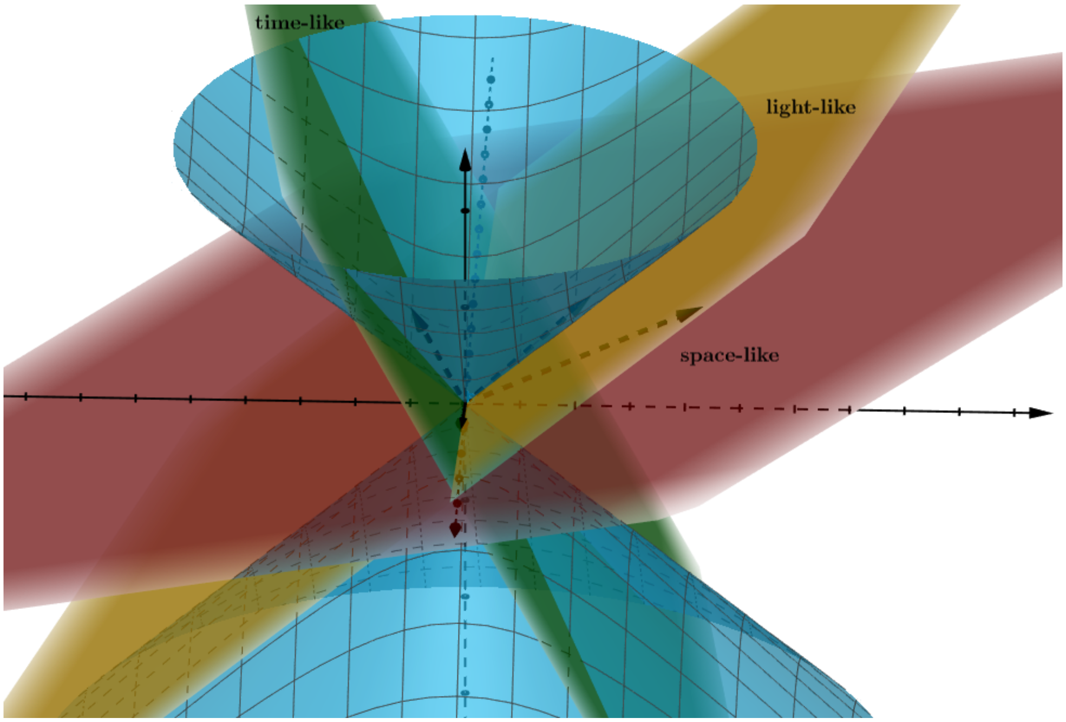


Figure 10.4: A graphical representation of vector subspaces in $\mathbb{R}^{1,2}$.

We note that:

- space-like vector subspaces intersect the double light-cone only in the origin, either perpendicularly w.r.t.the cone axis, or not, but without intersecting it in other points;
- time-like vector subspaces intersect the double light-cone not trivially;
- light-like vector subspaces are either straight lines defined by light-like vectors, or hyperplanes generated by a light-like vector and a space-like vector such that the hyperplane is tangent to the light-cone.

The relationship between time-like and space-like vector subspaces of $\mathbb{R}^{1,n-1}$ is established by the following result.

Theorem 10.2.3 *A vector subspace $V \subseteq \mathbb{R}^{1,n-1}$ is time-like if and only if V^L is space-like.*

Proof.

$\boxed{\Longrightarrow}$: suppose V is time-like and $x \in V$ a time-like vector. If $y \in V^L \backslash \{0\}$ then, by corollary 10.2.2, y must be space-like since x and y are Lorentz-orthogonal. Thus, V^L is space-like.

$\boxed{\Longleftarrow}$: we now assume V^L to be space-like. Since $(V^L)^L = V$, to prove that V is time-like it is enough to exhibit a time-like vector y Lorentz-orthogonal to V^L. A good candidate for this role is provided by the residual vector of the Lorentz-orthogonal projection on V^L of a vector $x \notin V^L$, as reported in eq. (10.3).

To this aim, we need to consider an orthonormal basis $\mathscr{B} = (u_1, \ldots, u_m)$, $m = \dim(V^L) \leqslant n$, of V^L, which we know to exist thanks to remark 10.2.2.

Let $x \in \mathbb{R}^n$ be a time-like vector, then $x \notin V^L$ because V^L is space-like, so, by definition, all its vectors are space-like. The residual vector of the Lorentz-orthogonal projection of x on V^L is:

$$y = x - \sum_{i=1}^{m} (x \circ u_i) u_i,$$

notice that the coefficients ϵ_i appearing in eq. (10.3) quoted before are all $+1$ because V^L is space-like.

Eq. (10.3) assures us that y is Lorentz-orthogonal to every vector in V^L (in particular, to every u_i) and so, to finish the proof, we just have to check if y is a time-like vector. By the bilinearity of $\circ$ we have:

$$\begin{aligned} y \circ y &= \left(x - \sum_{i=1}^{m} (x \circ u_i) u_i \right) \circ y = x \circ y - \sum_{i=1}^{m} (x \circ u_i) \underbrace{(u_i \circ y)}_{=0} = x \circ y \\ &= x \circ \left(x - \sum_{i=1}^{m} (x \circ u_i) u_i \right) = x \circ x - \sum_{i=1}^{m} (x \circ u_i)(x \circ u_i) = x \circ x - \sum_{i=1}^{m} (x \circ u_i)^2 \\ &\leqslant x \circ x, \end{aligned}$$

but x is time-like, so $x \circ x < 0$ and so $y \circ y < 0$ and y is time-like. □

If we interchange the role of V with that of V^L and we use the fact that $(V^L)^L = V$, we get the following corollary.

Corollary 10.2.5 *A vector subspace $V \subseteq \mathbb{R}^{1,n-1}$ is space-like if and only if V^L is time-like.*

Finally, if $V \subseteq \mathbb{R}^{1,n-1}$ is a light-like vector subspace, since it cannot be neither time-like nor space-like due to the previous results, we get the set of light-like vector subspaces is stable w.r.t.Lorentz-orthogonalization.

Corollary 10.2.6 *A vector subspace $V \subseteq \mathbb{R}^{1,n-1}$ is light-like if and only if V^L is light-like.*

10.3 Lorentz transformations

The analogue of orthogonal linear transformations of the group $\mathrm{O}(n)$ for the Euclidean n-space are the Lorentz transformations for the Lorentz n-space.

Def. 10.3.1 *A Lorentz transformation on $\mathbb{R}^{1,n-1}$ is a map $\phi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$ that preserves the Lorentz pseudo-scalar product, i.e.*

$$\boxed{\phi(x) \circ \phi(y) = x \circ y} \qquad \forall x, y \in \mathbb{R}^{1,n-1}.$$

It is simple to prove that the set of Lorentz transformation on $\mathbb{R}^{1,n-1}$ form a group under composition.

Def. 10.3.2 (The Lorentz group) *The group of Lorentz transformation on $\mathbb{R}^{1,n-1}$ is called the Lorentz group and it is denoted with the symbol $\mathrm{O}(1, n-1)$ or $\mathscr{L}$.*

The following result is the equivalent of theorem 10.1.2 for the Lorentz n-space.

Theorem 10.3.1 *$\phi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$ is a Lorentz transformation if and only if it is linear and, given a Lorentz-orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^{1,n-1}$, $(\phi(u_1), \ldots, \phi(u_n))$ is a Lorentz-orthonormal basis.*

Proof.

$\boxed{\Longrightarrow}$: we start by assuming that $\phi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$ preserves the Lorentz pseudo-scalar product. Then, for all Lorentz-orthonormal basis $(u_1, \ldots, u_n)$,

$$\phi(u_i) \circ \phi(u_j) = u_i \circ u_j = \begin{cases} -1 & \text{if} \quad i = j = 1 \\ \delta_{ij} & \text{otherwise} \end{cases},$$

so $(\phi(u_1), \cdots, \phi(u_n))$ is a Lorentz-orthonormal basis. Hence, for all $x \in \mathbb{R}^{1,n-1}$, thanks to eq. (10.2) we have:

$$\phi(x) = \sum_{i=1}^{n} \varepsilon_i(\phi(x) \circ \phi(u_i))\phi(u_i) \underset{(\phi \text{ Lorentz transf.})}{=} \sum_{i=1}^{n} \varepsilon_i(x \circ u_i)\phi(u_i),$$

with $\varepsilon_1 = -1$ and $\varepsilon_i = +1$ for all $2 \leqslant i \leqslant n$. Eq. (10.2) implies also that $x = \sum_{i=1}^{n} \varepsilon_i(x \circ u_i)u_i$, so, by writing $x_i = \varepsilon_i(x \circ u_i)$, we get:

$$\phi(\sum_{i=1}^{n} x_i u_i) = \sum_{i=1}^{n} x_i \phi(u_i), \qquad \forall x \in \mathbb{R}^{1,n-1}. \tag{10.6}$$

Following exactly the same line of reasoning used in the proof of theorem 10.1.2, we obtain the linearity of the Lorentz transformation.

$\boxed{\Longleftarrow}$: conversely, suppose that ϕ is linear and that, for any Lorentz-orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^{1,n-1}$, $(\phi(u_1), \ldots, \phi(u_n))$ is again a-Lorentz orthonormal basis of $\mathbb{R}^{1,n-1}$. Then,

$$\phi(x) = \phi(\sum_{i=1}^{n} x_i u_i) \underset{(\phi \text{ linear})}{=} \sum_{i=1}^{n} x_i \phi(u_i) \qquad \forall x \in \mathbb{R}^{1,n-1},$$

thus

$$\phi(x) \circ \phi(y) = \left(\sum_{i=1}^{n} x_i \phi(u_i)\right) \circ \left(\sum_{j=1}^{n} y_j \phi(u_j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \phi(u_i) \circ \phi(u_j),$$

but $\phi(u_1) \circ \phi(u_1) = -1$ and $\phi(u_i) \circ \phi(u_j) = \delta_{ij}$ otherwise, so

$$\phi(x) \circ \phi(y) = \sum_{i=1}^{n} x_i y_i - x_1 y_1 = x \circ y,$$

i.e. ϕ is a Lorentz transformation. $\square$

Being a linear transformation in $\mathbb{R}^n$, a Lorentz transformation ϕ can be written as a matrix, usually denoted with $\Lambda \in \mathrm{M}(n, \mathbb{R})$. More precisely, we have the following definition.

Def. 10.3.3 (Lorentzian matrix) *A matrix $\Lambda \in \mathrm{M}(n, \mathbb{R})$ is called Lorentzian if the function $\phi_\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi_\Lambda(x) := \Lambda x$, for all $x \in \mathbb{R}^n$, is a Lorentz transformation.*

Remark 10.3.1 *Λ can be interpreted as the matrix associated to ϕ_Λ w.r.t.the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$.*

As it happens for orthogonal transformations, Lorentzian matrices and Lorentz transformations can be identified, moreover, the algebraic properties of a Lorentzian matrix characterize completely the associated Lorentz transformation, as stated in the following theorem.

Theorem 10.3.2 *Let Λ be a $n \times n$ real matrix and $\eta = \mathrm{diag}(-1, 1, \dots, 1)$. The following statements are equivalent.*

1. *Λ is a Lorentzian matrix*
2. *Λ^t is a Lorentzian matrix*
3. *The columns of Λ form a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$*
4. *The rows of Λ form a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$*
5. *Λ verifies $\Lambda^t \eta \Lambda = \eta$*
6. *Λ verifies $\Lambda \eta \Lambda^t = \eta$*
7. *Λ preserves the quadratic form $q(x) = -x_1^2 + x_2^2 + \cdots + x_n^2 = \|x\|^2$ associated to the Lorentz pseudo-scalar product, i.e. $q(\Lambda x) = q(x) \iff \|\Lambda x\|^2 = \|x\|^2$.*

Proof.

$\boxed{1. \iff 3.}$: thanks to remark 10.3.1, Λ has on the columns the vectors $(\phi_\Lambda(e_1), \dots, \phi_\Lambda(e_n))$ of $\mathbb{R}^n$, which is a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$. Lemma 10.3.1 can then be used to guarantee the equivalence between 1. and 3.

$\boxed{1. \iff 7.}$: the proof is exactly the same as the one of lemma 10.1.3, the only difference being that the Euclidean scalar product $\langle \, , \, \rangle$ as to be replaced by the Lorentz pseudo-scalar product $\circ$.

$\boxed{1. \iff 5.}$: by remark 10.2.1 we get

$$\Lambda x \circ \Lambda y = \langle \Lambda x, \eta \Lambda y \rangle = \langle x, \Lambda^t \eta \Lambda y \rangle, \qquad \forall x, y \in \mathbb{R}^n,$$

so, $\forall x, y \in \mathbb{R}^n$ it holds that:

$$\begin{aligned} \Lambda \text{ Lorentzian} \quad &\iff \quad \Lambda x \circ \Lambda y = x \circ y \\ &\iff \quad \langle x, \Lambda^t \eta \Lambda y \rangle = \langle x, \eta y \rangle \\ &\iff \quad \Lambda^t \eta \Lambda = \eta \end{aligned}$$

which shows that 1. is equivalent to 5.

$\boxed{1. \iff 6.}$: since $\eta = \eta^{-1}$, property 5. can be restated by saying that Λ is Lorentzian if and only if $\eta \Lambda^t \eta \Lambda = I_n$, or that $\eta \Lambda^t \eta$ is the left inverse of Λ and so, by elementary linear algebra in $\mathbb{R}^n$, Λ is invertible with inverse $\Lambda^{-1} = \eta\Lambda^t\eta$. Actually also Λ^{-1} is a Lorentzian matrix, in fact, if the associated transformation is $\phi_\Lambda^{-1}(x) := \Lambda^{-1}x$ for all $x \in \mathbb{R}^n$, then we have

$$x \circ y = ((\phi_\Lambda\phi_{\Lambda^{-1}})x) \circ ((\phi_\Lambda\phi_{\Lambda^{-1}})y) = (\phi_\Lambda(\phi_{\Lambda^{-1}}(x)) \circ (\phi_\Lambda(\phi_{\Lambda^{-1}}(y)) \qquad \forall x, y \in \mathbb{R}^{1,n-1},$$

but ϕ_Λ is a Lorentz transformation, so

$$x \circ y = (\phi_{\Lambda^{-1}}(x)) \circ (\phi_{\Lambda^{-1}}(y)) \qquad \forall x, y \in \mathbb{R}^{1,n-1},$$

hence $\Lambda^{-1} = \eta\Lambda^t\eta$ is Lorentzian and we can use property 5. on Λ^{-1} to write

$$(\eta\Lambda^t\eta)^t\eta(\eta\Lambda^t\eta) = \eta \iff \eta\Lambda\eta\eta^2\Lambda^t\eta = \eta \iff \eta^{-1}\eta\Lambda\eta\Lambda^t\eta\eta = \eta^{-1}\eta\eta \iff \Lambda\eta\Lambda^t = \eta,$$

having used the fact that $\eta^2 = I_n$ and $\eta = \eta^t = \eta^{-1}$.

$\boxed{1. \iff 2.}$: immediate consequence of the equivalence between 1. and 5. and 1. and 6. In fact, by the first equivalence we have that Λ is Lorentizan if and only if $\Lambda^t\eta\Lambda = \eta$, by the latter this is equivalent to $\Lambda\eta\Lambda^t = \eta$, which is nothing by the first equivalence written for Λ^t, thus implying that Λ is Lorentzian if and only if Λ^t is.

$\boxed{1. \iff 4.}$: Λ is Lorentzian if and only if Λ^t is, if and only if (by 1. $\iff$ 3.) Λ^t has a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$ on its columns, which is equivalent to say that Λ has a Lorentz-orthonormal basis of $\mathbb{R}^{1,n-1}$ on its rows. □

Despite the extreme simplicity of its proof, this theorem has important consequences. The first one is an immediate consequence of property 7.

Corollary 10.3.1 *A Lorentz transformation preserves the likeness of a vector $x \in \mathbb{R}^{1,n-1}$, i.e. even if vectors can be modified by a Lorentz transformation,*

- *Lorentz-transformed time-like vectors still belong to the interior of the light-cone;*
- *Lorentz-transformed space-like vectors still belong to the exterior of the light-cone;*
- *Lorentz-transformed light-like vectors still belong to the light-cone.*

*Thus, **as a whole, the light cone, the time-like and the space-like regions remain unaltered after a Lorentz transformation.***

The light cone is characterized by the equation $q(x) = 0$, i.e. is the 0-level set of the quadratic form q associated to the Lorentz pseudo-scalar product. Of course, there is nothing special about the value 0, as underlined by the next corollary.

Corollary 10.3.2 *The level sets of the quadratic form q associated to the Lorentz pseudo-scalar product are preserved by a Lorentz transformation, i.e. if $x \in \mathbb{R}^{1,n-1}$ belongs to the hyperboloid defined by $q(x) = c$, then also its Lorentz transformed (which is, in general, another vector) belongs to the same hyperboloid. In other words, **as a whole, the hyperboloid** $q(x) = c$, $c \in \mathbb{R}$, **remains unaltered after a Lorentz transformation.***

As we have seen in the proof, a Lorentzian matrix Λ is invertible with inverse $\Lambda^{-1} = \eta\Lambda^t\eta$ which is a Lorentzian matrix too, thus, if we identify Lorentz transformations ϕ with their Lorentzian matrices Λ, we can identify $\mathrm{O}(1, n-1)$ with a subgroup of $\mathrm{GL}(n, \mathbb{R})$ as follows:

$$\begin{aligned}\mathrm{O}(1, n-1) &= \left\{\phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ \phi(x) \circ \phi(y) = x \circ y, \ \ \forall x, y \in \mathbb{R}^{1,n-1}\right\} \\ &= \left\{\Lambda \in \mathrm{GL}(n, \mathbb{R}) \ : \ \Lambda^t\eta\Lambda = \eta = \Lambda\eta\Lambda^t, \ \ \eta = \mathrm{diag}(-1, 1, \ldots, 1)\right\}.\end{aligned}$$

Notice that if $\circ$ is replaced by the Euclidean scalar product $\langle \, , \, \rangle$ and η by I_n, then $\mathrm{O}(1, n-1)$ becomes the group $\mathrm{O}(n)$:

$$\begin{aligned}\mathrm{O}(n) &= \left\{\phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ \langle\phi(x), \phi(y)\rangle = \langle x, y\rangle, \ \ \forall x, y \in \mathbb{R}^n\right\} \\ &= \left\{A \in \mathrm{GL}(n, \mathbb{R}) \ : \ A^t I_n A = I_n = A I_n A^t\right\}.\end{aligned}$$

By Binet's theorem, $\det(\eta) = \det(\Lambda^t\eta\Lambda) = \det(\Lambda^t)\det(\eta)\det(\Lambda)$, i.e. $\det(\Lambda^t)\det(\Lambda) = 1$, but $\det(\Lambda^t) = \det(\Lambda)$, thus

$$\boxed{\det(\Lambda) = \pm 1},$$

as it happens for an orthogonal matrix. As usual, we denote with

$$\mathrm{SO}(1, n-1) \equiv \mathscr{L}_+ := \left\{\Lambda \in \mathrm{O}(1, n-1) \ : \ \det(\Lambda) = 1\right\},$$

the **special** (or **proper**) **Lorentz group**.

Another important subgroup of $\mathrm{O}(1, n-1)$ is defined below.

Def. 10.3.4 (Positive Lorentz transformations) *A Lorentz transformation $\Lambda \in \mathrm{O}(1, n-1)$ is called positive (or positively oriented) if it preserves the orientation of the light cone, i.e. if, for all $x \in \mathbb{R}^{1,n-1}$, x time-like,*

$$x_1 > 0 \implies (\Lambda x)_1 > 0.$$

The subgroup of $\mathrm{O}(1, n-1)$ given by positive Lorentz transformations is denote as follows:

$$\mathrm{PO}(1, n-1) \equiv \mathscr{L}^{\uparrow} := \left\{\Lambda \in \mathrm{O}(1, n-1) \ : \ x \text{ time-like}, \ \ x_1 > 0 \implies (\Lambda x)_1 > 0\right\},$$

and called either ***positive*** *or* ***orthochronous*** *Lorentz group.*

We stress that, if $\Lambda \in \mathrm{PO}(1, n-1)$ and $x \in \mathbb{R}^{1,n-1}$ is a time-like vector such that $x_1 < 0$, then $-x_1 > 0$ and $-(\Lambda x)_1 = (\Lambda(-x))_1 > 0$, hence $(\Lambda x)_1 < 0$. In other words, a positive Lorentz transformation preserves the orientation of time-like vectors in the interior of both the upper and the lower parts of the light-cone.

In relativistic theories, where the coordinate x_1 is identified with time t, the previous remark is translated by saying that positive Lorentz transformations preserve the orientation of time-like vectors both in the interior of both the future and the past light-cone.

The subgroup

$$\mathrm{SPO}(1, n-1) \equiv \mathrm{SO}^+(1, n-1) \equiv \mathscr{L}_+^{\uparrow} := \mathrm{SO}(1, n-1) \cap \mathrm{PO}(1, n-1),$$

is called **special positive** or **proper orthochronous** or **restricted Lorentz group**. It can be proven to be the *connected component to the identity* of the Lorentz group.

More insights about the structure of the subgroups of the Lorentz group just defined are provided by the next theorems. In particular, it is quite remarkable that *the positivity of a Lorentz transformation is fully encoded in the first entry of the Lorentzian matrix* Λ.

Theorem 10.3.3 *Let $\Lambda = (\Lambda_{ij})_{1\leqslant i,j\leqslant n} \in \mathrm{O}(1, n-1)$.*

1. *$\Lambda \in \mathrm{PO}(1, n-1)$ if and only if $\Lambda_{11} \geqslant 1$.*
2. *$\Lambda \in \mathrm{O}(n) \cap \mathrm{PO}(1, n-1)$ if and only if $\Lambda_{11} = 1$.*

Proof.

1. : we start by showing that Λ is positive if and only if $\Lambda_{11} > 0$. Since Λ has a Lorentz-orthonormal basis on its rows and since the first vector of this basis is time-like, the entries of the first row of Λ form a the time-like vector that we denote with v_1.

Fixed a generic positive time-like vector $x \in \mathbb{R}^n$, Λ is positive if and only if Λx is a positive time-like vector, i.e. if the first entry of Λx is positive, but this is nothing but $\langle v_1, x\rangle$, so Λ is positive if and only if $\langle v_1, x\rangle > 0$.

However $\langle v_1, x\rangle = \langle v_1, \eta^2 x\rangle = \langle v_1, \eta(\eta x)\rangle = v_1 \circ \eta x$, thus Λ is positive if and only if $v_1 \circ \eta x > 0$. Notice that ηx is a negative vector because x is positive, so $x_1 > 0$, but $\eta = \mathrm{diag}(-1, 1, \dots, 1)$, so $(\eta x)_1 < 0$. By theorem 10.2.1, if v_1 is a negative vector too, then $v_1 \circ \eta x \leqslant 0$, so the fact that $v_1 \circ \eta x > 0$ implies that v_1 must be positive, i.e. $\Lambda_{11} > 0$.

Moreover, since v_1 a unit norm time-like vector, we have $\|v_1\|^2 = -1$, but $\|v_1\|^2 = -{\Lambda_{11}}^2 + |\bar{v}_1|^2$, thus

$$\Lambda_{11}^2 = 1 + |\bar{v}_1|^2 \geqslant 1,$$

which implies $\Lambda_{11} \geqslant 1$ since we have shown that Λ_{11} is positive.

2. : first of all we remark that 1. implies the following equivalence $\Lambda^t \in \mathrm{PO}(1, n-1) \iff \Lambda_{11} \geqslant 1$, in fact Λ^t is a Lorentzian matrix too an it shares the first entry, Λ_{11}, with Λ.

Now, if $\Lambda_{11} = 1$ then $\Lambda_{11}^2 = 1 + |\bar{v}_1|^2$ implies $|\bar{v}_1|^2 = 0$, so $v_1 = (1, 0, \dots, 0) \equiv {e_1}^t$ (the transposed of the first vector of the canonical basis of $\mathbb{R}^n$, recalling that we always work under the assumption that vectors in $\mathbb{R}^n$ are column vectors). It follows that the first column c_1 of Λ is $c_1 = v_1^t = e_1$. Therefore, Λ has the form

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

$A \in \mathrm{GL}(n-1, \mathbb{R})$ and, by theorem 10.3.2,

$$\Lambda^t \eta \Lambda = \eta \iff \begin{pmatrix} 1 & 0 \\ 0 & A^t \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \implies A^t A = I_{n-1},$$

but then $\Lambda^t \Lambda = I_n$ and so $\Lambda \in \mathrm{O}(n)$.

It remains only to prove that if $\Lambda \in \mathrm{O}(n) \cap \mathrm{PO}(1, n-1)$, then, necessarily, its first entry is equal to 1. To this aim, notice that, in this case, every column of Λ is an orthonormal vector w.r.t.both the Euclidean and the Lorentz scalar product, this implies that c_1, the first column of Λ, satisfies $\|c_1\|^2 = -1$ and $|c_1|^2 = 1$, so $\|c_1\|^2 + |c_1|^2 = 0$. But, since the first element of c_1 is actually Λ_{11}, we also have $\|c_1\|^2 = -{\Lambda_{11}}^2 + |\bar{c}_1|^2$ and $|c_1|^2 = {\Lambda_{11}}^2 + |\bar{c}_1|^2$, so

$$0 = \|c_1\|^2 + |c_1|^2 = -{\Lambda_{11}}^2 + |\bar{c}_1|^2 + {\Lambda_{11}}^2 + |\bar{c}_1|^2 = 2|\bar{c}_1|^2,$$

i.e. $\bar{c}_1 = 0$. Since $c_1 = (\Lambda_{11}, \bar{c}_1)$, this implies $c_1 = (\Lambda_{11}, 0, \dots, 0)$ and the only what that c_1 has unit Euclidean norm is that $\Lambda_{11} = \pm 1$, but $\Lambda_{11} \geqslant 1$ so only the option $\Lambda_{11} = 1$ is valid. □

During the proof of the last theorem, we have also proven this result, which says, geometrically, that the only orthogonal transformations that preserve the light-cone are the orthogonal transformations that leave the first axis invariant.

Corollary 10.3.3

$$\begin{aligned} \mathrm{O}(n) \cap \mathrm{PO}(1, n-1) &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{O}(n-1) \right\} \\ &\cong \mathrm{O}(n-1). \end{aligned}$$

We end this section by showing that the group of Lorentz transformations acts transitively on the vector subspace of $\mathbb{R}^{1,n-1}$ of different likeness, which become, then, homogeneous spaces w.r.t.the action of $\mathscr{L}$. In fact, we can prove an even stronger result: transitivity is guaranteed already by the subgroup of positive Lorentz transformations.

Theorem 10.3.4 *Let $\mathcal{V}_m^T$, $\mathcal{V}_m^S$ and $\mathcal{V}_m^L$ be the set of m-dimensional time-like, space-like and light-like vector subspace of $\mathbb{R}^{1,n-1}$, respectively. Then, the action of $\mathrm{PO}(1, n-1)$ is transitive on each of them.*

Proof.

Transitivity of $\mathrm{PO}(1, n-1)$ on $\mathcal{V}_m^T$. We start by observing that $\overline{V}_m := \mathrm{span}(e_1, \ldots, e_m) \cong \mathbb{R}^m$ is a time-like vector subspace of $\mathbb{R}^{1,n-1}$ because e_1 is time-like. We will prove the transitivity of $\mathrm{PO}(1, n-1)$ on $\mathcal{V}_m^T$ by showing that for a given time-like m-dimensional vector subspace V belonging to the set $\mathcal{V}_m^T$ there is a $\Lambda \in \mathrm{PO}(1, n-1)$ such that $\Lambda\big(V_m\big) = V$.

By theorem 10.2.2 we can guarantee the existence of a positive Lorentz-orthonormal basis $\mathscr{B} = (w_1, \ldots, w_m)$ of V. We use the vectors of $\mathscr{B}$ to define the matrix

$$\Lambda := \begin{pmatrix} | & \ldots & | \\ w_1 & \ldots & w_n \\ | & \ldots & | \end{pmatrix},$$

which, thanks to theorem 10.3.2, belongs to $\mathrm{PO}(1, n-1)$ because its columns form a positive Lorentz-orthonormal basis. By direct computation we have that $\Lambda(e_i) = w_i \ \forall i = 1, \ldots, m$, so, by linearity, $\Lambda(V_m) = V$.

Transitivity of $\mathrm{PO}(1, n-1)$ on $\mathcal{V}_m^S$. This time, we set $\overline{W}_m := \mathrm{span}(e_2, \ldots, e_{m+1})$ to be our m-dimensional space-time vector subspace of reference (note that the dimension m of a space-like vector subspace must be strictly less than n since it cannot contain any time-like vector by definition). Let $W \in \mathcal{V}_m^S$ and $u \in W^L$, where W^L is the Lorentz-orthogonal of W, a positive time-like vector such that $\|u\|^2 = -1$. Then, $\tilde{V} := \mathrm{span}(u, W)$ is a time-like vector subspace of $\mathbb{R}^{1,n-1}$ and so, by what we have just proven, it is connected to $\overline{V}_{m+1}$ by a positive Lorentz transformation, that we indicate again with $\Lambda \in \mathrm{PO}(1, n-1)$ for simplicity. So: $\Lambda(\tilde{V}) = \overline{V}_{m+1}$ and $\Lambda(u) = e_1$ (the time-like vectors are connected by Λ). If we prove that $\Lambda(W) = \overline{W}_m$, then, being Λ invertible, we have $\Lambda^{-1}(\overline{W}_m) = W$, thus proving the transitivity. To this aim, let $w \in W \subset \tilde{V}$. Then $\Lambda(w) \in \mathrm{span}(e_1)^L$ because

$$\Lambda(w) \circ e_1 = \Lambda(w) \circ \Lambda(u) = w \circ u = 0,$$

and so, since the Lorentz-orthogonal of $\mathrm{span}(e_1)$ is $\mathrm{span}(e_2, \ldots, e_n) = \overline{W}_{n-1}$,

$$\Lambda(w) \in \mathrm{span}(e_1)^L \cap \Lambda(\tilde{V}) = \overline{W}_{n-1} \cap \overline{V}_{m+1} = \overline{W}_m,$$

which allows us to conclude that $\Lambda(W) \subset \overline{W}_m$. Finally, if $\mathscr{B} = (w_1, \ldots, w_m)$ is a Lorentz-orthonormal basis of W, then $w_i \circ w_j = \delta_{ij}$ and so $\Lambda(w_i) \circ \Lambda(w_j) = \delta_{ij}$, which means that also $(\Lambda(w_i))_{1 \leqslant i \leqslant m}$ is a basis of $\overline{W_m}$. Hence, by linearity, we have $\Lambda(W) = \overline{W_m}$.

We leave the transitivity of $\mathrm{PO}(1, n-1)$ on $\mathcal{V}_m^L$ as an exercise. □

The previous theorem has an important consequence: the transitivity of Lorentz transformations on the hyperboloids in $\mathbb{R}^{1,n-1}$ defined as set-level surfaces of the quadratic form associated to the Lorentz pseudo-scalar product. To understand how this is possible, it is sufficient to consider the particular case of $m = 1$: the elements of $\mathcal{V}_1^T$ and $\mathcal{V}_1^S$ are straight lines passing through the origin and belonging to the interior or the exterior of the light-cone, respectively.

Each one of these straight lines intersects the hyperboloid defined by the equation $\|x\|^2 = \alpha$, $\alpha \in \mathbb{R}\backslash\{0\}$, in two antipodal points w.r.t.the origin and belonging to the two disconnected hyperboloid sheets. Thus, the transitivity of positive Lorentz transformations on $\mathcal{V}_1^T$ and $\mathcal{V}_1^S$ implies that every couple of vectors belonging to same sheet of the hyperboloid can be connected by a positive Lorentz transformation.

The following result summarizes the previous arguments.

Corollary 10.3.4 *Let $\alpha \in \mathbb{R}\backslash\{0\}$, fixed.* $\mathrm{O}(1, n-1)$ *acts transitively on the hyperboloid*

$$\mathcal{H}_\alpha^{n-1} = \{x \in \mathbb{R}^{1,n-1} : \|x\|^2 = \alpha\}.$$

Proof. We start by remarking that given $x \in \mathcal{H}_\alpha^{n-1}$, we have

$$V_x \cap \mathcal{H}_\alpha^{n-1} = \{x, -x\}\ , \quad \text{where} \quad V_x = \mathrm{span}\{x\}.$$

Let $x, y \in \mathcal{H}_\alpha^{n-1}$. Then by theorem 10.3.4, there is a transformation $\Lambda \in PO(1, n-1)$ such that $\Lambda(V_x) = V_y$. Because Λ preserves the Lorentz pseudo-scalar,

$$\Lambda(V_x \cap \mathcal{H}_\alpha^{n-1}) = V_y \cap \mathcal{H}_\alpha^{n-1}$$

and so we have either

$$\Lambda(x) = y \quad \text{or} \quad \Lambda(x) = -y.$$

In the second case, it suffices to take $-\Lambda \in O(1, n-1)$ as the transitive action from x to y. □

Chapter 11

Möbius transformations (Antoine Guennec, Nicoletta Prencipe and Edoardo Provenzi)

Möbius transformations are the main toolbox for the conformal model of hyperbolic geometry. Before discussing them rigorously, we give an intuitive introduction.

11.1 Introduction to Möbius transformations

The most natural setting for Möbius transformations is that of sphere, where a Möbius transformation is defined as a finite composition of basic geometric transformations called *inversions*, whose basic idea is depicted in Figure 11.1: by sliding continuously an elastic band on a ball, we can transform it into the equator, thus transforming the surface of the ball contained in the interior of the elastic band to half the surface of ball; in this situation the ball surface contained in the elastic band and the remaining one are isomorphic.

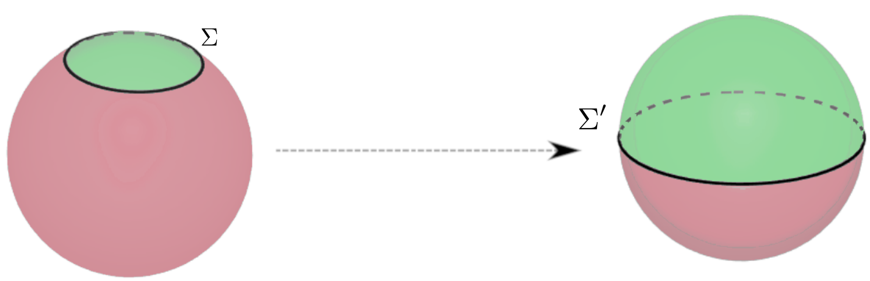


Figure 11.1: Inversion on a sphere: as the radius of the circle increased until reaching the diameter of the sphere, the 'interior' of the circle (in green) is diffeomorphic to the 'outside' (in pink).

Once written in mathematical terms, this continuous transformation that maps the spherical surface contained in a circle to the one left outside is called *inversion on a sphere*. By noticing that a circle can be obtained by cutting a sphere in $\mathbb{R}^3$ with a plane, it should not be surprising that, in the definition of inversion on a sphere of an arbitrary (finite) dimension n, a circle is replaced by a hypersphere, i.e. the intersection between a n-hyperplane and a n-sphere.

The group of Möbius transformations on the sphere, denoted by $\mathcal{M}(S^n)$ is the subgroup of $\mathrm{Aut}(S^n) = \{f : S^n \to S^n,\ f \text{ bijective}\}$ generated by inversions w.r.t. hyperspheres.

While it is true that handling Möbius transformations on the sphere is the most economical way to do it in terms of transformations involved (i.e. inversions), it is also true that it is more intuitive to analyze Möbius transformations in the Euclidean space. This can be done thanks to the stereographic projection introduced in chapter 1, which allows us to set up a bijection between the n-sphere minus the north pole and the hyperplane in $\mathbb{R}^{n+1}$ defined by $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, which can be identified with $\mathbb{R}^n$. To obtain a complete bijection we will need to introduce an artificial element, an abstract point denoted by ∞ and called the *point at infinity*.

Dealing with Möbius transformations in the Euclidean domain enlarged with the point at infinity comes with a price: we will see (corollary 11.4.1) that we no longer need only inversions to characterize them, but also other geometric operations, called reflections w.r.t. hyperplanes.

The importance of reflections and inversions motivates why we start the formal analysis of Möbius transformations by defining them in the next subsection.

11.2 Reflections and inversions

Consider a unit vector $a \in \mathbb{R}^n$, $|a| = 1$, then, by the orthogonal projection theorem, we have $\mathbb{R}^n = \text{span}(a) \oplus \text{span}(a)^\perp$ and so $\text{span}(a)^\perp := \{x \in \mathbb{R}^n \,:\, \langle x, a\rangle = 0\}$ is a $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$, i.e. a hyperplane in $\mathbb{R}^n$ passing through the origin. If we consider the affine structure of $\mathbb{R}^n$, the vectors of $\text{span}(a)^\perp$ can be rigidly translated away from 0 by a real quantity t via the transformation $x \mapsto x - ta$. This operation identifies an **affine space of dimension** $n-1$ whose algebraic expression can be obtained by replacing x with $x - ta$ in the equation $\langle x, a\rangle = 0$, i.e. $\langle x - ta, a\rangle = 0 \iff \langle x, a\rangle - t|a|^2 = 0 \underset{|a|=1}{\iff} \langle x, a\rangle = t$.

These considerations justify the following definition.

Def. 11.2.1 (Hyperplane in $\mathbb{R}^n$) *Given $a \in \mathbb{R}^n$, $|a| = 1$, and $t \geqslant 0$, the hyperplane associated to a and t is the set*

$$P(a,t) := \{x \in \mathbb{R}^n,\ \langle x, a\rangle = t\}.$$

Thus:

- a is the **normal vector** to $P(a,t)$
- t is the **distance** between $P(a,t)$ and 0, which can be taken non-negative because its possible negative sign can be incorporated in the vector a without changing its unit norm by redefining t and a as follows:

$$t \mapsto |t| \geqslant 0 \text{ and } a \mapsto \text{signum}(t)\, a.$$

Geometrically, the reflection w.r.t. $P(a,t)$ is the map ρ that takes any point $x \in \mathbb{R}^n$ at a distance d from $P(a,t)$ to a point $\rho(x)$ which lies specularly on the other side of the hyperplane at the same distance d. The 2D version of this operation is depicted in Figure 11.2.

To understand how to analytically define $\rho(x)$ notice that, if we perform the sum $x + \lambda a$, then we move x perpendicularly w.r.t. $P(a,t)$ and by a magnitude λ. Let λ^* be such that $x + \lambda^* a \in P(a,t)$, then clearly $\rho(x) = x + 2\lambda^* a$. To make λ^* explicit we have to write $\langle x + \lambda^* a, a\rangle = t \iff \langle x, a\rangle + \lambda^* \cancelto{1}{\|a\|^2} = t \iff \lambda^* = t - \langle x, a\rangle$.

We formalize this concept in the following definition.

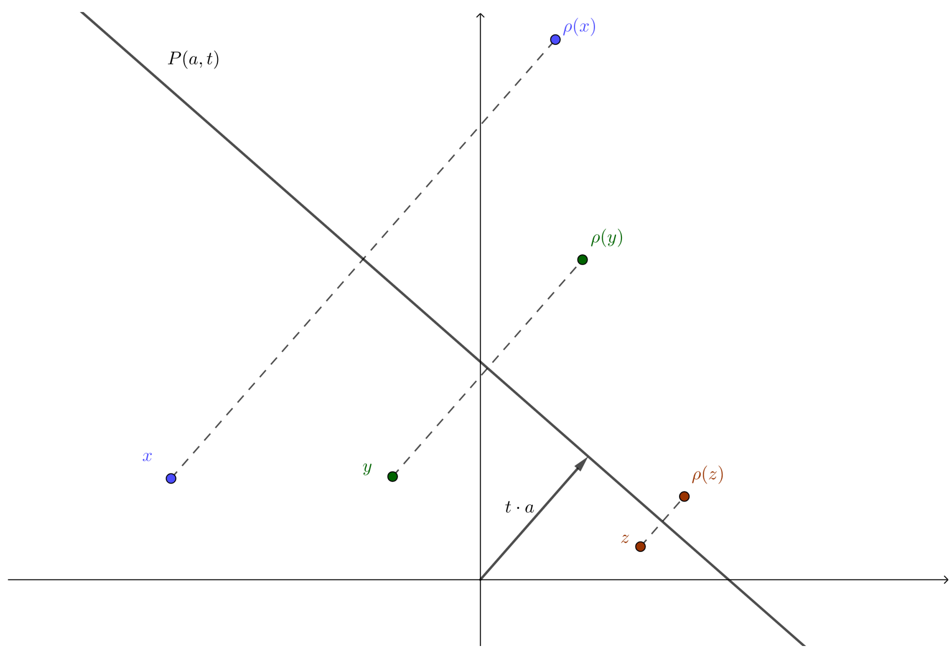


Figure 11.2: 2D graphical representation of a reflection w.r.t. a hyperplane, which is a straight line in two dimensions.

Def. 11.2.2 *A **reflection** in $\mathbb{R}^n$ w.r.t. the hyperplane $P(a,t)$ is the affine function:*

$$\begin{array}{rccl} \rho_{a,t}: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \mapsto & \rho_{a,t}(x) := x + 2(t - \langle x, a\rangle)a. \end{array} \tag{11.1}$$

$\rho_{a,t}(x)$ is said to be the reflection of x w.r.t. to the hyperplane $P(a,t)$.

If the dependence of ρ on the parameters a and t of the hyperplane $P(a,t)$ is not significant, we will simplify the notation and write ρ instead of $\rho_{a,t}$.

Figure 11.2 suggests some geometrical properties of $\rho_{a,t}$, e.g. the vectors belonging to $P(a,t)$ are unaffected by the action of $\rho_{a,t}$, if we apply it two times we come back to the original vector, so that the inverse of $\rho_{a,t}$ is itself and the Euclidean distance between any two reflected vectors is the same as the original distance.

These properties, and one more, are rigorously stated in the following theorem.

Theorem 11.2.1 *$\rho_{a,t}$ satisfies the following properties for all $x, y \in \mathbb{R}^n$:*

1. *$\rho_{a,t}(x) = x$ if and only if $x \in P(a,t)$*

2. *$\rho_{a,t}^2(x) = x$, i.e. $\rho_{a,t}$ is an involution, and so $\rho_{a,t}^2 = id_{\mathbb{R}^n}$, i.e. $\rho_{a,t}$ is a bijection with $\rho_{a,t}{}^{-1} = \rho_{a,t}$*

3. *$\rho_{a,t}$ is a Euclidean isometry: $|\rho_{a,t}(x) - \rho_{a,t}(y)| = |x - y|$*

4. *$\rho_{a,t} \in \mathrm{O}(n) \iff t = 0$.*

Proof. The proofs can be obtained by direct computation.

1. :

$$\begin{aligned}
\rho_{a,t}(x) = x \quad &\iff \quad x = x + 2(t - \langle x, a\rangle)a \quad (a \neq 0) \\
&\iff \quad \langle x, a\rangle = t \\
&\iff \quad x \in P(a,t).
\end{aligned}$$

2. :

$$\begin{aligned}
\rho^2_{a,t}(x) = \rho_{a,t}(\rho_{a,t}(x)) \quad &= \quad \rho_{a,t}(x) + 2(t - \langle \rho_{a,t}(x), a\rangle)a \\
& \quad\quad \text{(letting } s \equiv 2(t - \langle x, a\rangle)) \\
&= \quad x + sa + 2(t - \langle x + sa, a\rangle)a \\
&= \quad x + sa + 2(t - \langle x, a\rangle)a - 2sa \\
&= \quad x + sa + sa - 2sa \\
&= \quad x.
\end{aligned}$$

3. : first of all, we note that

$$\rho_{a,t}(x) - \rho_{a,t}(y) = x - y - 2\langle y - x, a\rangle a,$$

so

$$\begin{aligned}
|\rho_{a,t}(x) - \rho_{a,t}(y)|^2 \quad &= \quad |(x-y) - 2\langle y - x, a\rangle a|^2 \\
&= \quad |x - y|^2 - 4\langle y - x, a\rangle^2 + 4\langle y - x, a\rangle^2 \\
&= \quad |x - y|^2.
\end{aligned}$$

4. : if $t \neq 0$, then $\rho_{a,t}(0) = 2ta \neq 0$ since $a \in S^n$, thus $\rho_{a,t}$ is not linear and thus it cannot belong to $\mathrm{O}(n)$. If $t = 0$ then, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\langle \rho_{a,0}(x), \rho_{a,t}(y)\rangle &= \langle x - 2\langle x, a\rangle a, y - 2\langle y, a\rangle a\rangle \\
&= \langle x, y\rangle - 2\langle x, a\rangle\langle a, y\rangle - 2\langle y, a\rangle\langle x, a\rangle + 4\langle x, a\rangle\langle a, y\rangle \cancelto{1}{|a|^2} \\
&= \langle x, y\rangle.
\end{aligned}$$

□

As a consequence of property **2.**, $\rho_{a,t}$ is bijective, thus a reflection w.r.t. a hyperplane in $\mathbb{R}^n$ maps bicontinuously any point in $\mathbb{R}^n$ that lies on one side of the hyperplane to a unique point that lies on the other side.

The concept of inversion deals with the same problem, with one (major) difference: the hypersurface w.r.t. the inversion is performed is not a hyperplane but a (hyper)sphere. While a hyperplane extends towards the infinite, a sphere is bounded, this fact implies that it is impossible to continuously fill the whole outer space to the spherical surface simply by reflecting its interior points w.r.t. the tangent hyperplane to the sphere at a point, a different geometrical operation is needed.

As proven by G. Bellavitis in his 1836 paper [2], this operation consists in mapping any point x inside the sphere to the unique point $\sigma(x)$ outside the sphere characterized by the following two properties: firstly, $\sigma(x)$ lies on the same line joining x with the center of the sphere; secondly, the norm of $\sigma(x)$ is inverted w.r.t. that of x.

The easiest way to formalize this idea is by first considering the unit sphere S^{n-1} in $\mathbb{R}^n$ centered in 0: if $x \in \mathbb{R}^n$ is such that $|x| < 1$, then $\sigma_{0,1}(x) := \frac{1}{|x|}\frac{x}{|x|} = \frac{1}{|x|^2}x$ is the desired inverted point outside S^{n-1}.

If, instead of S^{n-1}, we consider S_r^{n-1}, $r > 0$, then we can turn back to the previous case by applying $\sigma_{0,1}$ to $\frac{x}{r}$ and then by restoring the correct radius via a multiplication by r, denoted with m_r. Mathematically, this corresponds to the composed function $\sigma_{0,r} := m_r \circ \sigma_{0,1} \circ m_{1/r}$, hence, given any $x \in \mathbb{R}^n$ such that $|x| < r \iff |x/r| < 1$ we have

$$\sigma_{0,r}(x) = (m_r \circ \sigma_{0,1} \circ m_{1/r})(x) = m_r(\sigma_{0,1}(x/r)) = m_r\left(\frac{r^{\cancel{2}}}{|x|^2}\frac{x}{\cancel{r}}\right) = m_r\left(\frac{r}{|x|^2}x\right) = \left(\frac{r}{|x|}\right)^2 x.$$

The most general case is that of $S_{a,r}^{n-1}$, the $(n-1)$-sphere centered in $a \in \mathbb{R}^n$ with radius $r > 0$, i.e.

$$S_{a,r}^{n-1} := \{x \in \mathbb{R}^n \ : \ |x - a| = r\}.$$

Following the same argument used above, the inversion $\sigma_{a,r}$ will be given by the composition $\tau_a \circ \sigma_{0,r} \circ \tau_{-a}$, τ being the translation operator. Thus, for all $x \in \mathbb{R}^n \backslash \{a\}$ satisfying $|x - a| < r \iff |x - a|/r < 1$:

$$\sigma_{a,r}(x) = \tau_a(\sigma_{0,r}(x-a)) = \tau_a\left(\left(\frac{r}{|x-a|}\right)^2 (x-a)\right) = a + \frac{r^2}{|x-a|^2}(x-a).$$

Def. 11.2.3 *Let $a \in \mathbb{R}^n$ and $r > 0$, then the **inversion** in $\mathbb{R}^n$ w.r.t. the sphere $S_{a,r}^{n-1}$ is the non-linear function*

$$\begin{array}{rccl} \sigma_{a,r}: & \mathbb{R}^n\backslash\{a\} & \longrightarrow & \mathbb{R}^n\backslash\{a\} \\ & x & \mapsto & \sigma_{a,r}(x) := a + \frac{r^2}{|x-a|^2}(x-a). \end{array}$$

*$\sigma_{a,r}(x)$ is said to be the **inverse of** x w.r.t. to the sphere $S_{a,r}^{n-1}$.*

If the specification of the parameters a and r is not significant, we will simply write σ instead of $\sigma_{a,r}$.

The following result shows that the conjugation that is needed to define a generic inversion starting from the inversion w.r.t. to unit sphere can be operated by fusing into an affine function the multiplication by r and the translation by a.

Lemma 11.2.1 *Let:*

- *$\sigma_{0,1}$ the inversion w.r.t. S^{n-1}, the unit sphere centered in 0 in $\mathbb{R}^n$*
- *$\sigma_{a,r}$ the inversion w.r.t. the sphere $S_{a,r}^{n-1}$, $a \in \mathbb{R}^n$, $r > 0$*
- *for all $x \in \mathbb{R}^n$, $\phi(x) = a + rx$, with a and r as above.*

Then,

$$\boxed{\sigma_{a,r} = \phi \circ \sigma_{0,1} \circ \phi^{-1}}.$$

Proof. For all $x \in \mathbb{R}^n$ we have, by definition, $\sigma_{a,r}(x) = a + \frac{r^2}{|x-a|^2}(x-a)$, but, if we replace x by $r\frac{(x-a)}{|x-a|^2}$ as argument of ϕ we get

$$\phi\left(r\frac{x-a}{|x-a|^2}\right) = a + \frac{r^2}{|x-a|^2}(x-a),$$

so $\sigma_{a,r}(x) = \phi\left(r\frac{x-a}{|x-a|^2}\right)$. Since $\sigma_{0,1}(x) = \frac{x}{|x|^2}$,

$$\sigma_{0,1}\left(\frac{x-a}{r}\right) = \frac{x-a}{r}\frac{r^2}{|x-a|^2} = r\frac{x-a}{|x-a|^2},$$

hence $\sigma_{a,r}(x) = \phi \circ \sigma_{0,1}\left(\frac{x-a}{r}\right)$ for all $x \in \mathbb{R}^n$.

Finally, by solving $\phi(x) = a + rx$ w.r.t. x we obtain $\phi^{-1}(x) = \frac{x-a}{r}$, so that $\sigma_{a,r}(x) = \phi \circ \sigma_{0,1} \circ \phi^{-1}(x)$ for all $x \in \mathbb{R}^n$. □

Remark 11.2.1 Both **reflection** w.r.t. a hyperplane **and inversion** w.r.t. a sphere **are**, essentially, **one-dimensional operations**, in the sense that all the points belonging to the same straight line orthogonal to the hyperplane involved in a reflection are left on this straight line; in the same way, all the points belonging to the straight line passing through the origin of the sphere involved in an inversion are left on that line.

Contrarily to a reflection w.r.t. a hyperplane, which is defined on the whole $\mathbb{R}^n$, an inversions w.r.t. a sphere $S^{n-1}_{a,r}$ is defined on $\mathbb{R}^n$ deprived of the sphere center a.

Notice also that $\sigma_{a,r}(x)$ is the only point verifying

$$|\sigma_{a,r}(x) - a||x-a| = r^2, \tag{11.2}$$

thus, **the closer x is to the center of the sphere a, the further apart $\sigma_{a,r}(x)$ is sent on the straight line connecting x to a.** Figure 11.3 gives a graphical representation of this phenomenon in two dimensions.

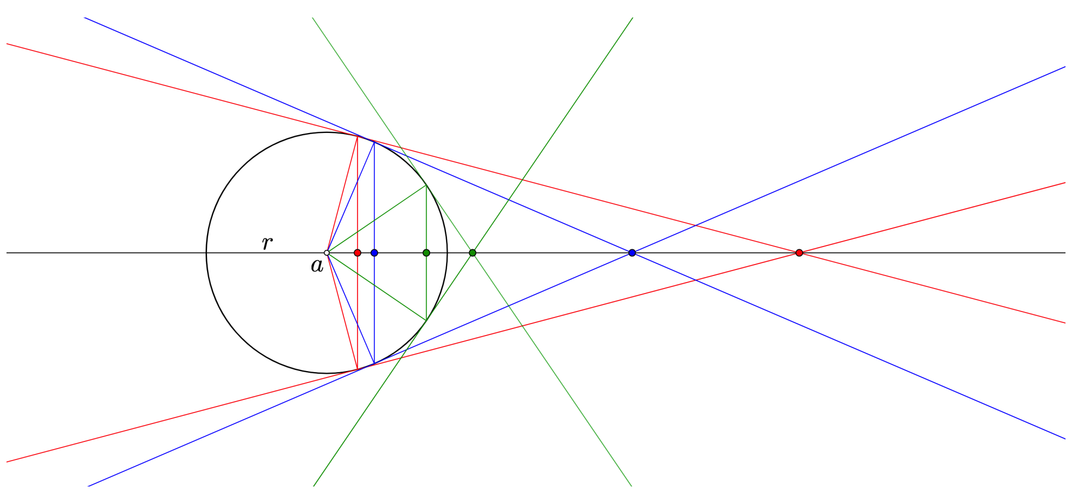


Figure 11.3: 2D graphical representation of an w.r.t the circle $S^1_{a,r}$.

As it can be seen, as we approach the center of the circle, the inverted point goes farther and farther. It is not difficult to imagine that, if we want to extend the concept of inversion to contemplate also the center of the sphere a, then we have to associate it to a point at infinite, that we will rigorously define later.

In the following theorem we prove the properties of inversions analogous to those of reflections. We put the accent on the fact that, by its own definition, an inversion cannot be an isometry, except for the points belonging to the sphere w.r.t. the inversion is performed.

Theorem 11.2.2 *Let $a \in \mathbb{R}^n$, $r > 0$ and σ be the inversion w.r.t. $S^{n-1}_{a,r}$. Then, for all $x, y \in \mathbb{R}^n \backslash \{a\}$:*

1. $\sigma_{a,r}(x) = x$ *if and only if* $x \in S^{n-1}_{a,r}$

2. $\sigma^2_{a,r}(x) = x$, *i.e.* $\sigma_{a,r}$ *is an involution, and so* $\sigma_{a,r}$ *is invertible with* ${\sigma_{a,r}}^{-1} = \sigma_{a,r}$

3. $|\sigma_{a,r}(x) - \sigma_{a,r}(y)| = \frac{r^2}{|x-a||y-a|}|x-y|$.

Proof. Let $x, y \in \mathbb{R}^n \backslash \{a\}$.

1. : the relationship $|\sigma_{a,r}(x) - a||x-a| = r^2$ always holds for $\sigma_{a,r}$, thus $\sigma_{a,r}(x) = x$ if and only if $|x-a|^2 = r^2$, i.e. $x \in S^n_{a,r}$.

2. :

$$\begin{aligned}
\sigma^2_{a,r}(x) = \sigma_{a,r}(\sigma_{a,r}(x)) &= a + \frac{r^2}{|\sigma_{a,r}(x) - a|^2}(\sigma_{a,r}(x) - a) \\
&= a + \frac{|x-a|^2}{r^2}\left(\frac{r^2}{|x-a|^2}(x-a)\right) = x.
\end{aligned}$$

3. :

$$\begin{aligned}
|\sigma_{a,r}(x) - \sigma_{a,r}(y)| &= \left|\frac{r^2}{|x-a|^2}(x-a) - \frac{r^2}{|y-a|^2}(y-a)\right| = r^2\left|\frac{x-a}{|x-a|^2} - \frac{y-a}{|y-a|^2}\right| \\
&= r^2\left\langle \frac{x-a}{|x-a|^2} - \frac{y-a}{|y-a|^2}, \frac{x-a}{|x-a|^2} - \frac{y-a}{|y-a|^2}\right\rangle^{\frac{1}{2}} \\
&= r^2\left|\frac{|x-a|^2}{|x-a|^4} - 2\frac{\langle x-a, y-a\rangle}{|x-a|^2|y-a|^2} + \frac{|y-a|^2}{|y-a|^4}\right|^{\frac{1}{2}} \\
&= r^2\left|\frac{1}{|x-a|^2} - 2\frac{\langle x-a, y-a\rangle}{|x-a|^2|y-a|^2} + \frac{1}{|y-a|^2}\right|^{\frac{1}{2}} \\
&= r^2\left|\frac{|y-a|^2 - 2\langle x-a, y-a\rangle + |x-a|^2}{|x-a|^2|y-a|^2}\right|^{\frac{1}{2}} \\
&= r^2\left|\frac{\langle (x-a)-(y-a), (x-a)-(y-a)\rangle}{|x-a|^2|y-a|^2}\right|^{\frac{1}{2}} \\
&= r^2\left|\frac{\langle x-y, x-y\rangle}{|x-a|^2|y-a|^2}\right|^{\frac{1}{2}} = r^2\frac{|x-y|}{|x-a||y-a|}.
\end{aligned}$$

□

Property **3.** of theorem 11.2.2 states that, if x and y do not belong to the sphere $S^{n-1}_{a,r}$, then their Euclidean distance after the application of $\sigma_{a,r}$ will be proportional to their original Euclidean distance inside the sphere, with a non-linear proportionality coefficient that depends on both x and y through the formula $r^2/|x-a||y-a|$.

This property is crucial to understand the profound link between Möbius transformations and the so-called *cross ratio*.

The following theorems underline the importance of reflections and inversions, by relating them to the Euclidean isometries and similarities.

Theorem 11.2.3 *Every Euclidean isometry of* $\mathbb{R}^n$ *is a composition of at most* $n+1$ *reflections.*

Proof. As a preliminary observation, we recall that, by theorem 10.1.1, all Euclidean isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ can be written as $f(x) = a + \phi(x)$ with $a \in \mathbb{R}^n$ and $\phi \in \mathrm{O}(n)$, for all $x \in \mathbb{R}^n$. Hence, an isometry is an orthogonal transformation if and only is it leaves 0 fixed.

The proof is constructive and it is based on the following strategy:

- we start by building the first reflection ρ_0 such that $\phi_0 := \rho_0 \circ f$ belongs to $\mathrm{O}(n)$;
- then we build *by induction* the other n reflections $\rho_1, \dots, \rho_n$ such that, for all $k \in \{1, \dots, n\}$, the transformation $\phi_k := \rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_0 \circ f$ belongs to $\mathrm{O}(n)$ and leaves all the first k vectors of the canonical basis $e_1, \dots, e_k$ of $\mathbb{R}^n$ fixed;
- when we arrive to $k = n$ we obtain an orthogonal (hence linear) transformation $\phi_n = \rho_n \circ \cdots \circ \rho_0 \circ f$ which leaves all the vectors of the canonical basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ fixed. The matrix associated to ϕ_n w.r.t. the canonical basis of $\mathbb{R}^n$ is of course I_n, so $\phi_n \equiv id_{\mathbb{R}^n}$;
- finally, we observe that

  $$\phi_n = \rho_n \circ \cdots \circ \rho_0 \circ f = id_{\mathbb{R}^n} \iff (\rho_0 \circ \cdots \circ \rho_n) \circ (\rho_n \circ \cdots \circ \rho_0) \circ f = \rho_0 \circ \cdots \circ \rho_n$$

  but the reflections ρ_i are involutions, i.e. $\rho_i^2 = id_{\mathbb{R}^n}$ for all $i = 0, 1, \dots, n$, so that $f = \rho_0 \circ \cdots \circ \rho_n$, which proves that we need at most $n+1$ reflections to represent any arbitrary isometry f.

Let us start by building the reflection ρ_0 such that $\phi_0 = \rho_0 \circ f \in \mathrm{O}(n)$. We write $x_0 := f(0) = a$ and we set

$$\rho_0 := \begin{cases} id_{\mathbb{R}^n} & \text{if } x_0 = 0 \\ \rho_{\frac{x_0}{|x_0|}, \frac{|x_0|}{2}} & \text{otherwise} \end{cases},$$

ϕ_0 is clearly an isometry as composition of two isometries, f and the reflection ρ_0. Let us verify if it leaves 0 fixed: by using definition (11.1) we have

$$\rho_0(x_0) = \begin{cases} id_{\mathbb{R}^n}(0) = 0 & \text{if } x_0 = 0 \\ \rho_{\frac{x_0}{|x_0|}, \frac{|x_0|}{2}}(x_0) = x_0 + 2\left(\frac{|x_0|}{2} - \left\langle x_0, \frac{x_0}{|x_0|} \right\rangle\right) \frac{x_0}{|x_0|} = x_0 - x_0 = 0 & \text{otherwise} \end{cases},$$

so $\phi_0(0) = \rho_0(f(0)) = \rho_0(x_0) = 0$, hence ϕ_0 is indeed an orthogonal transformation.

Let us pass to the construction of the remaining reflections $\rho_1, \dots, \rho_n$. As we have previously declared, we will use the induction technique, so we need to start by proving that there exists a reflection ρ_1 such that $\phi_1 := \rho_1 \circ \phi_0 = \rho_1 \circ \rho_0 \circ f$ is an orthogonal transformation that leaves e_1 fixed. We set $x_1 := \phi_0(e_1) - e_1$ and we define such a reflection as follows

$$\rho_1 := \begin{cases} id_{\mathbb{R}^n} & \text{if } \phi_0(e_1) = e_1 \\ \rho_{\frac{x_1}{|x_1|},0} & \text{otherwise} \end{cases},$$

ϕ_1 is either ϕ_0, which is an orthogonal transformation, or the composition of ϕ_0 with the reflection $\rho_{\frac{x_1}{|x_1|},0}$, which is orthogonal thanks to property **4.** of theorem 11.2.1, in both cases $\phi_1 \in \mathrm{O}(n)$.

We also observe that $\phi_1(e_1) = \rho_1(\phi_0(e_1))$, which is equal to $id_{\mathbb{R}^n}(e_1) = e_1$ if $\phi_0(e_1) = e_1$, otherwise:

$$\phi_1(e_1) = \rho_1 \circ \phi_0(e_1) = \phi_0(e_1) - 2\langle \phi_0(e_1), x_1 \rangle \frac{x_1}{|x_1|^2},$$

but $|\phi_0(e_1) - x_1|^2 = |\phi_0(e_1)|^2 - 2\langle \phi_0(e_1), x_1 \rangle + |x_1|^2$, so $-2\langle \phi_0(e_1), x_1 \rangle = |\phi_0(e_1) - x_1|^2 - |\phi_0(e_1)|^2 - |x_1|^2$, thus

$$\begin{aligned} \phi_1(e_1) &= \phi_0(e_1) + \big(\underbrace{|\phi_0(e_1) - x_1|^2}_{=1} - \underbrace{|\phi_0(e_1)|^2}_{=1} - |x_1|^2\big)\frac{x_1}{|x_1|^2} \\ &= \phi_0(e_1) - |x_1|^2 \frac{x_1}{|x_1|^2} = \phi_0(e_1) - x_1 \\ &= e_1, \end{aligned}$$

where we have used the fact that $|\phi_0(e_1) - x_1| = |e_1| = 1$ and $|\phi_0(e_1)| = 1$ because $\phi_0 \in \mathrm{O}(n)$ and $|e_1| = 1$. To resume, $\phi_1 \in \mathrm{O}(n)$ and it leaves e_1 fixed, thus the first induction step is fulfilled.

We now assume that, for all[1] $k \in \{3, \ldots, n\}$ there exists $\phi_{k-1} \in \mathrm{O}(n)$ that fixes $e_1, \ldots, e_{k-1}$. Let $x_k := \phi_{k-1}(e_k) - e_k$ and define

$$\rho_k := \begin{cases} id_{\mathbb{R}^n} & \text{if } \phi_{k-1}(e_k) = e_k \\ \rho_{\frac{x_k}{|x_k|},0} & \text{otherwise} \end{cases}.$$

By repeating exactly the same computations performed in the case of ϕ_1, it can be verified that $\phi_k := \rho_k \circ \phi_{k-1} \in \mathrm{O}(n)$, and that ϕ_k leaves e_k fixed. To verify that ϕ_k leaves also $e_1, \ldots, e_{k-1}$ fixed we write

$$\phi_k(e_i) = \rho_k(\phi_{k-1}(e_i)) = \rho_{\frac{x_k}{|x_k|},0}(\phi_{k-1}(e_i)) = \phi_{k-1}(e_i) - 2\langle x_k, \phi_{k-1}(e_i) \rangle \frac{x_k}{|x_k|^2},$$

but, for $1 \leqslant i < k \leqslant n-1$,

$$\phi_{k-1}(e_i) = e_i \tag{11.3}$$

by hypothesis of induction and so

$$\begin{aligned} \langle x_k, \phi_{k-1}(e_i) \rangle &\underset{(11.3)}{=} \langle \phi_{k-1}(e_k) - e_k, e_i \rangle = \langle \phi_{k-1}(e_k), e_i \rangle - \cancelto{0}{\langle e_k, e_i \rangle} = \langle \phi_{k-1}(e_k), e_i \rangle \\ &\underset{(11.3)}{=} \langle \phi_{k-1}(e_k), \phi_{k-1}(e_i) \rangle \underset{\phi_{k-1} \in \mathrm{O}(n)}{=} \cancelto{0}{\langle e_k, e_i \rangle} \\ &= 0, \end{aligned}$$

[1] notice that for $k = 1, 2$ we have already built ϕ_0 and ϕ_1, so we do not need to *assume* their existence.

which implies that

$$\phi_k(e_i) = \underbrace{\phi_{k-1}(e_i)}_{=e_i} - 2\underbrace{\langle x_k, \phi_{k-1}(e_i)\rangle}_{=0} \frac{x_k}{|x_k|^2} = e_i, \quad \forall i = 1 \leqslant i < k \leqslant n-1.$$

Hence, $\phi_k \in \mathrm{O}(n)$ and fixes $e_1, \dots, e_k$, which is what we had to verify in order to conclude the proof. □

We can easily extend the previous result to Euclidean similarities.

Corollary 11.2.1 *Every Euclidean similarity is a composition of at most $n+3$ reflections and inversions.*

Proof. First we treat the special case of the similarity $g(x) = kx$, $k > 0$. Let $\sigma_1 := \sigma_{0,1}$ and $\sigma_2 := \sigma_{0,\sqrt{k}}$. Then, by direct computation, we get

$$\sigma_2 \circ \sigma_1(x) = \sigma_2\left(\frac{x}{|x|^2}\right) = kx = g(x).$$

More generally, a similarity $f \in \mathcal{S}(\mathbb{R}^n)$, by theorem 10.1.1, can be written as $f(x) = a + k\phi(x)$ with $a \in \mathbb{R}^n$, $k > 0$ and $\phi \in \mathrm{O}(n)$. As we have seen in the proof of the previous theorem, by letting $x_0 = f(0) = a$ and $\rho_0 = \rho_{\frac{x_0}{|x_0|}, \frac{|x_0|}{2}}$, we have

$$\begin{aligned} f(x) &= \rho_0 \circ f(x) = k\phi(x) \\ &= \sigma_2 \circ \sigma_1 \circ \phi(x), \end{aligned}$$

and, as seen in proof of the previous result, ϕ can be decomposed into n reflections and so the corollary is proven. □

11.3 The stereographic projection as an inversion and the one point compactification of $\mathbb{R}^n$

In chapter 1 we have shown that the stereographic projection allows us to identify the n-sphere without the north (or south) pole with the hyperplane in $\mathbb{R}^{n+1}$ defined by:

$$P(e_{n+1}, 0) = \{x \in \mathbb{R}^{n+1} \ : \ \langle x, e_{n+1} \rangle = 0\} = \{(x_1, \ldots, x_n, 0), \ x_1, \ldots, x_n \in \mathbb{R}\} \equiv \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n.$$

Figure 11.4 gives a schematic depiction of the stereographic projection in 3D.

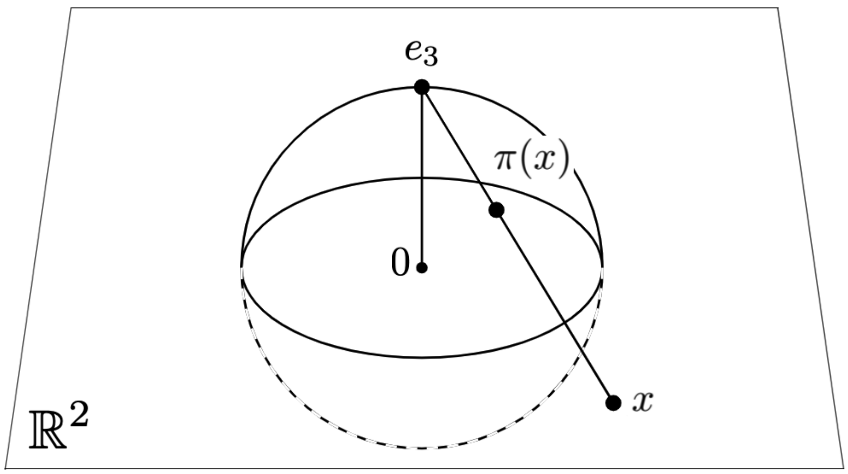


Figure 11.4: Stereographic projection in 3D.

Geometrically, **it is intuitive that the stereographic projection acts as an inversion**. In this section we are going to give a formalization of this fact.

In order to keep the analysis as simple as possible, we will implicitly identify $\mathbb{R}^n$ with the hyperplane in $\mathbb{R}^{n+1}$ passing through the origin and orthogonal to e_{n+1} whenever needed and we will also reverse the roles of $\mathbb{R}^n$ and the n-sphere, as specified in the following definition.

Def. 11.3.1 *The map*

$$\begin{array}{rccl} \pi: & \mathbb{R}^n & \xrightarrow{\sim} & S^n \backslash \{e_{n+1}\} \\ & x & \longmapsto & \pi(x) := \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2} \right), \end{array}$$

is called the stereographic projection from $\mathbb{R}^n$ to $S^n \backslash \{e_{n+1}\}$.

π coincides with the map φ_N^{-1} defined in eq. (1.6) with $R = 1$, i.e. the inverse stereographic chart relative to the north pole of S^n, that is a bijection between $\mathbb{R}^n$ and $S^n \backslash \{e_{n+1}\}$.

The following result gives an alternative geometric representation of π.

Theorem 11.3.1 *For all $x = (x_1, \ldots, x_n, 0) \in \mathbb{R}^n$, the stereographic projection $\pi(x)$ can be written as follows:*

$$\pi(x) = x + \frac{|x|^2 - 1}{1 + |x|^2}(e_{n+1} - x). \tag{11.4}$$

This means that the stereographically projected vector $\pi(x)$ is obtained through the vector sum of x with a modulated version of the difference $e_{n+1} - x$, where the modulation factor $s = \frac{|x|^2-1}{1+|x|^2} \neq 1$ is introduced to guarantee that $\pi(x)$ lies on the unit sphere, i.e. that $|\pi(x)| = 1$.

Proof. Let us write $\pi(x) = x + s(e_{n+1} - x) = (1-s)x + se_{n+1}$, then, by direct computation,

$$\begin{aligned} |\pi(x)| = 1 \quad &\iff \quad |x|^2(1-s)^2 + 2s(1-s)\underbrace{\langle x, e_{n+1}\rangle}_{=0} + s^2 \cancelto{1}{|e_{n+1}|^2} = 1 \\ &\iff \quad |x|^2 = \frac{1-s^2}{(1-s)^2} = \frac{1+s}{1-s} \iff s = \frac{|x|^2-1}{1+|x|^2}, \end{aligned}$$

if we introduce this expression of s in $\pi(x) = (1-s)x + se_{n+1}$ we get

$$\begin{aligned} \pi(x) &= \left(1 - \frac{|x|^2-1}{1+|x|^2}\right)(x_1, \ldots, x_n, 0) + \frac{|x|^2-1}{1+|x|^2}(0, \ldots, 0, 1) \\ &= \left(\frac{2}{1+|x|^2}\right)(x_1, \ldots, x_n, 0) + \frac{|x|^2-1}{1+|x|^2}(0, \ldots, 0, 1) = \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2}\right), \end{aligned}$$

which coincides with the definition of stereographic projection given in (11.3.1). □

Consider $S^n_{e_{n+1},\sqrt{2}}$, the sphere with radius $\sqrt{2}$ centered in e_{n+1}, then the inversion $\sigma_{e_{n+1},\sqrt{2}} : \mathbb{R}^{n+1}\backslash\{e_{n+1}\} \xrightarrow{\sim} \mathbb{R}^{n+1}\backslash\{e_{n+1}\}$, $\sigma_{e_{n+1},\sqrt{2}}(x) = e_{n+1} + \frac{2}{|x-e_{n+1}|^2}(x - e_{n+1})$, is a bijection. If we restrict this bijection to the hyperplane $P(e_{n+1}, 0) \cong \mathbb{R}^n$ we still get a bijection with its codomain. It turns out that the codomain of $\sigma_{e_{n+1},\sqrt{2}}\big|_{\mathbb{R}^n}$ is $S^n\backslash\{e_{n+1}\}$ and that its analytical form coincides with the one of the stereographic projection.

Theorem 11.3.2 *It holds that*

$$\boxed{\pi = \sigma_{e_{n+1},\sqrt{2}}\big|_{\mathbb{R}^n}}. \tag{11.5}$$

Proof. We simply have to apply $\sigma_{e_{n+1},\sqrt{2}}$ to $(x, 0)$, with $x \in \mathbb{R}^n$, to verify that we get $\pi(x)$. To this aim, we first remark that: $|x - e_{n+1}|^2 = |x|^2 - 2\cancelto{0}{\langle x, e_{n+1}\rangle} + \cancelto{1}{|e_{n+1}|^2} = 1 + |x|^2$, so

$$\begin{aligned} \sigma_{e_{n+1},\sqrt{2}}(x) \quad &:= \quad e_{n+1} + \frac{2}{|x-e_{n+1}|^2}(x - e_{n+1}) = e_{n+1} + \frac{2}{1+|x|^2}(x - e_{n+1}) \\ &= \quad (0, \ldots, 0, 1) + \frac{2}{1+|x|^2}(x_1, \ldots, x_n, -1) \\ &= \quad \left(\frac{2x_1}{1+|x|^2}, \ldots, \frac{2x_n}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2}\right) \\ &= \quad \pi(x). \end{aligned}$$

□

The so-called one point compactification of $\mathbb{R}^n$ is obtained by extending the stereographic projection $\pi : \mathbb{R}^n \xrightarrow{\sim} S^n\backslash\{e_{n+1}\}$ to a larger space, denoted with $\hat{\mathbb{R}}^n$, in such a way that the extended map $\hat{\pi} : \hat{\mathbb{R}}^n \to S^n$ is a bijection that encompasses also e_{n+1}. This is done by adding one single abstract point to $\mathbb{R}^n$ that, however, gets a very concrete representation in $\mathbb{R}^{n+1}$ thanks to $\hat{\pi}$.

Def. 11.3.2 (Point at infinity) *The abstract point denoted with ∞ and defined by*

$$\hat{\pi}(\infty) = e_{n+1} \iff \infty = \hat{\pi}^{-1}(e_{n+1}) \tag{11.6}$$

is called the point at infinity of $\mathbb{R}^n$.

The reason for this name is twofold: first, geometrically, the extension of the stereographic projection to e_{n+1} cannot give a finite value of $\mathbb{R}^n$, second, analytically, if we have a sequence $(x_n)_{n\geqslant 0} \subset \mathbb{R}^n$ such that $|x_n| \underset{n\to+\infty}{\longrightarrow} +\infty$, then $\pi(x_n) \underset{n\to+\infty}{\longrightarrow} e_{n+1} = \hat{\pi}(\infty)$. Since S^n is compact, the map $\hat{\pi} : \hat{\mathbb{R}}^n \xrightarrow{\sim} S^n$ creates a bijection between $\hat{\mathbb{R}}^n$ and a compact space, which motivates the name 'compactification'.

Def. 11.3.3 (One point compactification of $\mathbb{R}^n$) *Let ∞ be an abstract point not belonging to $\mathbb{R}^n$. The one point compactification of $\mathbb{R}^n$ is the set $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, the extension of the stereographic projection π to $\hat{\mathbb{R}}^n$ is the bijection*

$$\begin{array}{rccl} \hat{\pi} : & \hat{\mathbb{R}}^n & \xrightarrow{\sim} & S^n \\ & x & \longmapsto & \hat{\pi}(x) := \begin{cases} \pi(x) & \text{if } x \neq \infty \\ e_{n+1} & \text{if } x = \infty \end{cases} \end{array} .$$

The point at infinity of $\mathbb{R}^n$ can be identified, thanks to $\hat{\pi}$, with the north pole of the unit sphere S^n, which is a point that lives in the $(n+1)$-dimensional space $\mathbb{R}^{n+1}$. To better understand this fact, let us consider the cases $n = 1, 2$.

- The one-point compactification of $\mathbb{R}$ is

  $$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1 \underset{(9.8)}{=} \mathbb{RP}^1,$$

  i.e. the unit circle in $\mathbb{R}^2$ and the point at infinity of the real line, interpreted as a hyperplane in $\mathbb{R}^2$, can be identified with e_2.

- The one-point compactification of $\mathbb{R}^2$ is

  $$\hat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\} \cong S^2 \underset{(9.8)}{=} \mathbb{RP}^2,$$

  i.e. the unit sphere in $\mathbb{R}^3$ and the point at infinity of the real plane $\mathbb{R}^2$, interpreted as a hyperplane in $\mathbb{R}^3$, can be identified with e_3.

The one-point compactification of $\mathbb{C}$ has a special name.

Def. 11.3.4 (Riemann sphere) *The one-point compactification of the complex plane $\mathbb{C}$ is called Riemann sphere*

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \underset{(9.11)}{=} \mathbb{CP}^1.$$

Thanks to the bijection provided by $\hat{\pi}$, it is possible to endow $\hat{\mathbb{R}}^n$ with a metric.

Def. 11.3.5 (Chordal metric) *The chordal metric d_C on $\hat{\mathbb{R}}^n$ is:*

$$d_C(x, y) := |\hat{\pi}(x) - \hat{\pi}(y)|, \qquad \forall x, y \in \hat{\mathbb{R}}^n.$$

So, to compute the chordal metric, we stereographically project $x, y \in \hat{\mathbb{R}}^n$ on the sphere S^n and then we compute the Euclidean norm of the difference between the two projections, interpreted as points of $\mathbb{R}^{n+1}$. Of course, if $x = y = \infty$, $d_C(\infty, \infty) = |e_{n+1} - e_{n+1}| = 0$. The following result shows what are the values taken by the chordal metric in all the other cases.

Theorem 11.3.3 *Let $x, y \in \mathbb{R}^n$. Then, for all $x, y \in \mathbb{R}^n$*

1. $d_C(x, \infty) = \frac{2}{\sqrt{1+|x|^2}}$

2. $d_C(x, y) = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$.

Proof. First we remind that $\pi = \sigma_{e_{n+1},\sqrt{2}}$ and we observe that, for all $x \in \mathbb{R}^n$, we have $|x - e_{n+1}|^2 = |x|^2 - 2\cancelto{0}{\langle x, e_{n+1}\rangle} + \cancelto{1}{|e_{n+1}|^2} = 1 + |x|^2$.

1.

$$\begin{aligned} d_C(x, \infty) &= |\pi(x) - \pi(\infty)| = \left| \cancel{e_{n+1}} + \frac{2}{1+|x|^2}(x - e_{n+1}) - \cancel{e_{n+1}} \right| \\ &= \frac{2}{1+|x|^2}|x - e_{n+1}| = \frac{2}{1+|x|^2}\left(|x - e_{n+1}|^2\right)^{\frac{1}{2}} = \frac{2}{1+|x|^2}\sqrt{1+|x|^2} \\ &= \frac{2}{\sqrt{1+|x|^2}}. \end{aligned}$$

2. Since here $x, y \in \mathbb{R}^n$, we can write $d_C(x, y) = |\pi(x) - \pi(y)| = |\sigma_{e_{n+1},\sqrt{2}}(x) - \sigma_{e_{n+1},\sqrt{2}}(y)|$. Using property **3.** of theorem 11.2.2 we find

$$\begin{aligned} d_C(x, y) &= \frac{2|x-y|}{|x - e_{n+1}||y - e_{n+1}|} \\ &= \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}. \end{aligned}$$

□

Property **1.** says that the chordal distance between any point $x \in \mathbb{R}^n$ with the point at infinity is finite. Property **2.** of this last theorem implies that the metrical intuition that we have in Euclidean spaces can be transferred to $\hat{\mathbb{R}}^n$ for all the points different than ∞.

Corollary 11.3.1 *$f : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ is continuous in a point $x_0 \in \mathbb{R}^n$ w.r.t. the chordal metric if and only if f is continuous in x_0 w.r.t. the Euclidean metric.*

The following definitions formalize the quite intuitive extension of reflections and inversions to $\hat{\mathbb{R}}^n$ (for simplicity we keep the same symbols). In particular, note that the center of a sphere is mapped to the point at infinite by the corresponding inversion, and vice-versa.

Def. 11.3.6 *Let $\rho_{a,t}$ be a reflection and $\sigma_{a,r}$ an inversion in $\mathbb{R}^n$. The extension of $\rho_{a,t}$ in ∞ and of $\sigma_{a,r}$ in ∞ and a are defined as follows:*

$$\rho_{a,t}(\infty) := \infty \qquad \text{and} \qquad \begin{cases} \sigma_{a,r}(\infty) := a \\ \sigma_{a,r}(a) := \infty \end{cases}.$$

The properties listed in theorems 11.2.1 and 11.2.2 are valid also for their extended versions. The great advantage of considering the point at infinity is that **both reflections and inversions become bijections on $\hat{\mathbb{R}}^n$**.

We also extend isometries and similarities to $\hat{\mathbb{R}}^n$ as follows.

Def. 11.3.7 *The sets of isometries and similarities on $\hat{\mathbb{R}}^n$ are:*

$$\mathcal{I}(\hat{\mathbb{R}}^n) := \{\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n,\ \phi|_{\mathbb{R}^n} \text{ is an isometry and } \phi(\infty) = \infty\}$$

$$\mathcal{S}(\hat{\mathbb{R}}^n) := \{\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n,\ \phi|_{\mathbb{R}^n} \text{ is a similarity and } \phi(\infty) = \infty\}.$$

The request $\phi(\infty) = \infty$ is fully justified by theorem 11.2.3 for isometries: they are compositions of reflections, which fix ∞. Similarities instead are compositions of reflections and inversions, so the request to fix ∞ does not seems well-motivated. In fact, we will see soon, corollary 11.2.1 and theorem 11.4.2, that the action on ∞ of the inversions involved in the creation of a similarity cancel out, remaining with a map that fixes ∞ also in the case of similarities.

The final information that we need before passing to the definition and analysis of Möbius transformations is the concept of sphere in the one point compactification of $\mathbb{R}^n$.

In the same way as we can identify the hyperplane $P(e_{n+1}, 0) \cong \mathbb{R}^n$ united with $\{\infty\}$ with the sphere S^n by means of $\hat{\pi}$, we can identify the union of a hyperplane with the point at the infinity with a sphere. This consideration justifies the following definition.

Def. 11.3.8 *A sphere Σ in $\hat{\mathbb{R}}^n$ is either a Euclidean sphere $S^{n-1}_{a,r}$ or the union of a hyperplane with the point at infinity $\hat{P}(a,t) := P(a,t) \cup \{\infty\}$.*

11.4 Möbius transformations in the Euclidean space

Möbius transformations arise from the combinations of inversions and reflections of $\hat{\mathbb{R}}^n$, one of the main interest in combining them is that, when they are fused together, they form a group. Notice that this is not a trivial statement because neither the set of reflections nor the set inversions form a group: we do not have a identity element or any stability. However, theorem 11.2.3 tells us that by combining reflections and inversions we can obtain the identity function and the group of similarities.

Def. 11.4.1 *A* ***Möbius transformation*** *$\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ is a finite composition of reflections w.r.t. a hyperplane and inversions w.r.t. a sphere in $\hat{\mathbb{R}}^n$. The group of Möbius transformations is:*

$$\mathcal{M}(\hat{\mathbb{R}}^n) = \big\{\phi = \mu_1 \circ \cdots \circ \mu_m\ \ m \in \mathbb{N},\ \mu_i \text{ reflections or inversions of } \hat{\mathbb{R}}^n,\ i \in \{1, \dots, m\}\big\}.$$

It can be verified that $\mathcal{M}(\hat{\mathbb{R}}^n)$ is a group under composition. We underline that, by definition, **a Möbius transformation is a bijection in $\hat{\mathbb{R}}^n$**.

We shall see that Möbius transformation can be equivalently characterized in three different ways:

1. they are the only transformation that preserve the cross-ratio (see below)

2. they are the only transformations that map spheres of $\hat{\mathbb{R}}^n$ into other spheres of $\hat{\mathbb{R}}^n$

3. when $n \geqslant 3$ and we restrict them to $\mathbb{R}^n$ they are the only transformations of $\mathbb{R}^n$ that preserve angles, i.e they are the only conformal maps of $\mathbb{R}^n$.

Let us start by analyzing the relationship between Möbius transformations and the cross ratio.

Since every Euclidean isometry and similarity can be decomposed into a combination of reflections and inversions thanks to theorem 11.2.3 and corollary 11.2.1, we have the following chain of inclusions among groups:

$$\mathcal{I}(\hat{\mathbb{R}}^n) \subset \mathcal{S}(\hat{\mathbb{R}}^n) \subset \mathcal{M}(\hat{\mathbb{R}}^n).$$

11.4.1 Möbius transformations and the cross ratio

The cross ratio is the fundamental invariant of Möbius transformations.

Def. 11.4.2 *Let $u, v, x, y \in \hat{\mathbb{R}}^n$ such that $u \neq y, v \neq x$. The cross-ratio of (u, v, x, y) is the (continuous) function:*

$$\begin{array}{rccl} [\cdot,\cdot,\cdot,\cdot]: & \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n & \xrightarrow{\sim} & [0, +\infty) \\ & (u, v, x, y) & \longmapsto & [u, v, x, y] := \frac{d_C(u,x) d_C(v,y)}{d_C(u,y) d_C(v,x)}. \end{array}$$

In the special case that u, v, x, y belong to $\mathbb{R}^n$, then, thanks to property **2.** of theorem 11.3.3, we can re-write their cross ratio as follows:

$$[u, v, x, y] = \frac{|u - x||v - y|}{|u - y||v - x|}. \tag{11.7}$$

We remark that if one of the four points, say u, is ∞, that point can simply be 'dropped out' of the computation, in the sense that the factor in which it appears can be simply set to 1. The reason underlying this rule relies on theorem 11.3.3, in fact

$$[\infty, v, x, y] = \frac{d_C(\infty, x) d_C(v, y)}{d_C(\infty, x) d_C(v, y)} = \frac{\frac{2}{\sqrt{1+|x|^2}} \frac{2|v-y|}{\sqrt{1+|v|^2}\sqrt{1+|y|^2}}}{\frac{2}{\sqrt{1+|y|^2}} \frac{2|v-x|}{\sqrt{1+|v|^2}\sqrt{1+|x|^2}}} = \frac{|v - y|}{|v - x|}$$

and similarly if ∞ has any other place in the cross ratio. From now on, we will use the following formulae as definitions of cross ratio when ∞ is one of the points involved in its computation:

$$\begin{cases} [\infty, v, x, y] = \frac{|v-y|}{|v-x|} \\ [u, \infty, x, y] = \frac{|u-x|}{|u-y|} \\ [u, v, \infty, y] = \frac{|v-y|}{|u-y|} \\ [u, v, x, \infty] = \frac{|u-x|}{|v-x|} \end{cases}. \tag{11.8}$$

It must be stressed that there are several definitions of cross ratio in the literature, most of the time with little consequences since we can switch u, v, x, y in the cross-ratio in many different ways without changing the overall result. In particular, the definition that we gave is different than the one given by Ratcliffe in [15]. We chose the definition above because it will be the handier when we will deal with the conformal hyperbolic model.

Theorem 11.4.1 *A map $\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ is a Möbius transformation if and only if ϕ preserves the cross ratio.*

Proof.

$\boxed{\Longrightarrow}$: suppose ϕ is a Möbius transformation, then it is enough to show that any generic inversion and reflection preserves the cross ratio since, by definition, every Möbius transformation is a combination of inversions and reflections and thus the composition of cross ratio preserving functions will be overall cross ratio preserving.

First of all we suppose that the values taken by ϕ are finite, we will deal with the ∞ later. In this case, we can use formula (11.7) to compute the cross ratio.

If ϕ is a reflection $\rho_{a,t}$, then the cross ratio is preserved by the fact that reflections are Euclidean isometries.

This argument cannot be used if ϕ is an inversion $\sigma_{a,r}$, because inversions are not isometries. If we remove a form the possible values that the points u, v, x, y can take, then, by property **3.** of theorem 11.2.2, i.e.

$$|\sigma_{a,r}(x) - \sigma_{a,r}(y)| = \frac{r^2|x-y|}{|x-a||y-a|},$$

we have:

$$\begin{aligned}
[\sigma_{a,r}(u), \sigma_{a,r}(v), \sigma_{a,r}(x), \sigma_{a,r}(y)] &= \frac{|\sigma_{a,r}(u) - \sigma_{a,r}(x)||\sigma_{a,r}(v) - \sigma_{a,r}(y)|}{|\sigma_{a,r}(u) - \sigma_{a,r}(y)||\sigma_{a,r}(v) - \sigma_{a,r}(x)|} \\
&= \frac{r^2|u-x|}{r^2|u-y|}\frac{r^2|v-y|}{r^2|v-x|}\underbrace{\frac{|x-a||y-a||u-a||v-a|}{|x-a||y-a||u-a||v-a|}}_{=1} \\
&= [u, v, x, y].
\end{aligned}$$

Suppose now that[2] $\phi(u) = \infty$. By definition 11.3.6, if $\phi = \rho_{a,t}$, this can happen only if $u = \infty$ since $\rho_{a,t}(\infty) = \infty$, i.e. we must prove that

$$[\infty, \rho_{a,t}(v), \rho_{a,t}(x), \rho_{a,t}(y)] = [\infty, v, x, y] \underset{(11.8)}{=} \frac{|v-y|}{|v-x|},$$

which is very simple:

$$[\infty, \rho_{a,t}(v), \rho_{a,t}(x), \rho_{a,t}(y)] \underset{(11.8)}{=} \frac{|\rho_{a,t}(v) - \rho_{a,t}(y)|}{|\rho_{a,t}(v) - \rho_{a,t}(x)|} \underset{\rho_{a,t}\text{ isometry}}{=} \frac{|v-y|}{|v-x|} = [\infty, v, x, y].$$

Instead, if $\phi = \sigma_{a,r}$, then we know that $\phi(u) = \infty$ only if $u = a$. So, we must prove that

$$[\infty, \sigma_{a,r}(v), \sigma_{a,r}(x), \sigma_{a,r}(y)] = [a, v, x, y],$$

on the left-hand side we have

$$[\infty, \sigma_{a,r}(v), \sigma_{a,r}(x), \sigma_{a,r}(y)] \underset{(11.8)}{=} \frac{|\sigma_{a,r}(v) - \sigma_{a,r}(y)|}{|\sigma_{a,r}(v) - \sigma_{a,r}(x)|} \underset{(\mathbf{3.}\text{ of th. } 11.2.2)}{=} \frac{r^2\frac{|v-y|}{|v-a||y-a|}}{r^2\frac{|v-x|}{|v-a||x-a|}} = \frac{|v-y||x-a|}{|y-a||v-x|},$$

[2] This argument can be extended verbatim to v, x, y, so we will consider only u.

on the right-hand side we have

$$[a, v, x, y] = \frac{|v-y||x-a|}{|y-a||v-x|},$$

which verifies the preservation of the cross ratio also in this case. To resume, all Möbius transformations preserve the cross ratio.

$\boxed{\Longleftarrow}$: conversely, we assume that ϕ preserves the cross ratio. We analyze first the case when ϕ fixes ∞, i.e. $\phi(\infty) = \infty$, we will deal with the other option later. Let $u, v, x, y \in \mathbb{R}^n$ such that $u \neq y,\ v \neq x$ and $(u, v) \neq (x, y)$. If $u \neq x$, then

$$
\begin{aligned}
i)\ [\phi(u), \infty, \phi(x), \phi(y)] = [u, \infty, x, y] \quad & \underset{(11.8)}{\Longleftrightarrow} \quad \frac{|\phi(u)-\phi(x)|}{|\phi(u)-\phi(y)|} = \frac{|u-x|}{|u-y|} \\
& \Longleftrightarrow \quad \frac{|\phi(u)-\phi(x)|}{|u-x|} = \frac{|\phi(u)-\phi(y)|}{|u-y|}
\end{aligned}
$$

$$
\begin{aligned}
ii)\ [\phi(u), \phi(v), \phi(x), \infty] = [u, v, x, \infty] \quad & \underset{(11.8)}{\Longleftrightarrow} \quad \frac{|\phi(u)-\phi(x)|}{|\phi(v)-\phi(x)|} = \frac{|u-x|}{|v-x|} \\
& \Longleftrightarrow \quad \frac{|\phi(u)-\phi(x)|}{|u-x|} = \frac{|\phi(v)-\phi(x)|}{|v-x|}.
\end{aligned}
$$

Similarly, if $v \neq y$,

$$
\begin{aligned}
i)\ [\infty, \phi(v), \phi(x), \phi(y)] = [\infty, v, x, y] \quad & \underset{(11.8)}{\Longleftrightarrow} \quad \frac{|\phi(v)-\phi(y)|}{|\phi(v)-\phi(x)|} = \frac{|v-y|}{|v-x|} \\
& \Longleftrightarrow \quad \frac{|\phi(v)-\phi(y)|}{|v-y|} = \frac{|\phi(v)-\phi(x)|}{|v-x|}
\end{aligned}
$$

$$
\begin{aligned}
ii)\ [\phi(u), \phi(v), \infty, \phi(y)] = [u, v, \infty, y] \quad & \underset{(11.8)}{\Longleftrightarrow} \quad \frac{|\phi(v)-\phi(y)|}{|\phi(u)-\phi(y)|} = \frac{|v-y|}{|u-y|} \\
& \Longleftrightarrow \quad \frac{|\phi(v)-\phi(y)|}{|v-y|} = \frac{|\phi(u)-\phi(y)|}{|u-y|}.
\end{aligned}
$$

Hence, by combining $i)$ and $ii)$ in both cases we obtain that, for all $u, v, x, y \in \mathbb{R}^n$ such that $u \neq y$ and $v \neq x$,

$$\frac{|\phi(u)-\phi(y)|}{|u-y|} = \frac{|\phi(v)-\phi(x)|}{|v-x|},$$

if we set $k = |\phi(v) - \phi(x)|/|v - x|$, then $k > 0$ and it does not depend on u and y, which are two generic distinct elements of $\mathbb{R}^n$, so that we can write $|\phi(u) - \phi(y)| = k|u - y|$, which shows that ϕ is a Euclidean similarity and, hence a Möbius transformation.

Finally, if $a \neq \infty$ and $\phi(\infty) = a$, then we can combine ϕ with any inversion of the type $\sigma_{a,r}$, $r > 0$, obtaining $(\sigma_{a,r} \circ \phi)(\infty) = \infty$. Using the result obtained above, we have that $\sigma_{a,r} \circ \phi$ is a Möbius transformations, and so ϕ is also Möbius transformation by definition. $\square$

The following result gives important stuctural information about Möbius transformations.

Theorem 11.4.2 *Let $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$. Then:*

1. *$\phi(\infty) = \infty$ if and only if ϕ is a similarity of $\hat{\mathbb{R}}^n$*
2. *if $\phi(\infty) \neq \infty$, then, there exist:*
   - *a unique sphere Σ in $\mathbb{R}^n$ on which ϕ acts as a Euclidean isometry, i.e. for all $x, y \in \Sigma$, $|\phi(x) - \phi(y)| = |x - y|$*
   - *a unique inversion σ w.r.t. Σ and a unique Euclidean isometry $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$ such that ϕ can be decomposed as follows $\phi = \psi \circ \sigma$.*

Proof.

1. : the previous theorem implies directly that if $\phi(\infty) = \infty$ and ϕ is a Möbius transformation, then ϕ is a similarity on $\hat{\mathbb{R}}^n$. Vice-versa, a similarity on $\mathbb{R}^n$ is a Möbius transformation on $\mathbb{R}^n$; now, thanks to the proof of corollary 11.2.1, every similarity is the composition of at most $n+1$ reflections and two inversions w.r.t. the same center. This implies that, the only possible extension of ϕ to the point at infinity is the one that fixes ∞, in fact, reflections fix ∞ and also the composition of the two inversions will globally leave ∞ fixed. As previously said, this argument provides a full justification of the definition given in 11.3.7.

2. : first we prove the existence of the decomposition $\phi = \psi \circ \sigma$ and then its uniqueness. Notice that this automatically implies that we also have to exhibit the sphere Σ w.r.t. the inversion σ is defined.

<u>Existence</u>: since ϕ is a Möbius transformation that modifies the point at infinity, it is natural to set the center of the sphere Σ that we are looking for as $a := \phi^{-1}(\infty)$. Regarding the ray of the sphere, let us preliminarly set it to 1, i.e. let us consider the sphere $S^{n-1}_{a,1}$ and the inversion $\bar{\sigma}$ w.r.t. to it.

Clearly, $\phi \circ \bar{\sigma}$ fixes ∞ and so it is a Euclidean similarity thanks to point 1. Hence, it exists $k > 0$ such that, for all $x, y \in \mathbb{R}^n$, we have

$$|(\phi \circ \bar{\sigma})(x) - (\phi \circ \bar{\sigma})(y)| = k|x - y|. \tag{11.9}$$

Furthermore, $\bar{\sigma}$ is an inversion and so it is also an involution, $\bar{\sigma}^2 = id_{\mathbb{R}^n}$, so

$$\begin{aligned} |\phi(x) - \phi(y)| &= |(\phi \circ id_{\mathbb{R}^n})(x) - (\phi \circ id_{\mathbb{R}^n})(y)| = |(\phi \circ \bar{\sigma}^2)(x) - (\phi \circ \bar{\sigma}^2)(y)| \\ &= |(\phi \circ \bar{\sigma})(\bar{\sigma}(x)) - (\phi \circ \bar{\sigma})(\bar{\sigma}(y))| \\ &\underset{(11.9)}{=} k|\bar{\sigma}(x) - \bar{\sigma}(y)| \underset{\textbf{3.} \text{ of th.11.2.2}}{=} k\frac{|x-y|}{|x-a||y-a|}. \end{aligned} \tag{11.10}$$

Observe now that $x, y \in S^n_{a,r}$, then $|x - a| = |y - a| = r$, so, if we set $r := \sqrt{k}$, then

$$|\phi(x) - \phi(y)| = k\frac{|x-y|}{|x-a||y-a|} = k\frac{|x-y|}{k} = |x - y|,$$

and so ϕ is a Euclidean isometry on $\Sigma := S^{n-1}_{a,r}$ if and only if $r = \sqrt{k}$ is the radius of Σ, which implies its uniqueness. Finally, we can conclude by setting $\sigma = \sigma_{a,r}$ and $\psi = \phi\sigma$. The following computations show that $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$. Clearly $\psi(\infty) = \phi(\sigma(\infty)) = \phi(a) = \infty$. Moreover, if $x, y \in \mathbb{R}^n$, then the following chain of equalities holds

$$
\begin{aligned}
|\psi(x) - \psi(y)| = |\phi(\sigma(x)) - \phi(\sigma(y))| &\underset{(11.10)}{=} k\frac{|\sigma(x) - \sigma(y)|}{|\sigma(x) - a||\sigma(y) - a|} \\
&\underset{\mathbf{3.}\text{ of th.11.2.2}}{=} k\frac{k\frac{|x-y|}{|x-a||y-a|}}{|\sigma(x) - a||\sigma(y) - a|} = k^2\frac{|x-y|}{|x-a||y-a|}\frac{1}{|\sigma(x) - a||\sigma(y) - a|} \\
&\underset{(11.2)}{=} k^2\frac{|x-y|}{|x-a||y-a|}\frac{|x-a||y-a|}{k^2} = |x-y|.
\end{aligned}
$$

Uniqueness: suppose that we also have $\phi = \psi_0 \circ \sigma_0$ with ψ_0 a Euclidean isometry and σ_0 an inversion w.r.t. a sphere $S^{n-1}_{a_0,r_0}$. We start by proving that Σ and Σ_0 share their center: the decomposition $\phi = \psi \circ \sigma$ gives $\phi(a) = \infty$, while the decomposition $\phi = \psi_0 \circ \sigma_0$ gives $\phi(a_0) = \infty$, so $a = \phi^{-1}(\infty) = a_0$.

As proven above, ϕ is an isometry on both Σ and Σ_0 if and only if their radius has a specific, fixed, value, thus not only Σ and Σ_0 are concentric, but they also share their radius, i.e. $\Sigma = \Sigma_0$. This implies that also σ and σ_0 coincide, which, in turn, implies that $\psi = \psi_0$.

This shows that uniqueness of Σ and of the decomposition of ϕ.

Finally, let us ask if there exists another sphere S^{n-1} on which ϕ acts isometrically. For sure, this sphere S^{n-1} must have a center different than a, otherwise, as we have just proven, we fall back to the previous sphere Σ. Thus, let us suppose that $S^{n-1} = S^{n-1}_{b,s}$, with $b \in \mathbb{R}^n$, $b \neq a$, and $s > 0$. The idea to prove the uniqueness of Σ is to show that there exist two points $\bar{x}, \bar{y} \in S^{n-1}_{b,s}$ such that $|\phi(\bar{x}) - \phi(\bar{y})| \neq |\bar{x} - \bar{y}|$, and so ϕ does not act as an isometry on $S_{b,s}$.

To this aim, we observe that it exists $\alpha > 0$, $\alpha \neq 1$, such that $S^{n-1}_{a,\alpha r}$ and $S^{n-1}_{b,s}$ intersect in two points, that will constitute the two points $\bar{x}$ and $\bar{y}$ that we are searching for, in fact, recalling that $r^2 = k$ we have, by eq. (11.10) :

$$
|\phi(\bar{x}) - \phi(\bar{y})| = \frac{r^2|\bar{x} - \bar{y}|}{|\bar{x} - a||\bar{y} - a|} = \frac{r^2}{\alpha^2 r^2}|\bar{x} - \bar{y}| \neq |\bar{x} - \bar{y}|,
$$

hence ϕ cannot be an isometry on $S^{n-1}_{b,s}$. □

Remark that the same arguments used in the proof above can be used to assure it does not exist any hyperplane $P(a, t)$ on which ϕ acts as an isometry. Hence, Σ is the not only the unique sphere in $\mathbb{R}^n$ on which ϕ is isometric, but also on $\hat{\mathbb{R}}^n$. For this reason, the following definition is completely justified.

Def. 11.4.3 *For any Möbius transformation $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ such that $\phi(\infty) \neq \infty$, the unique sphere Σ on which ϕ acts as an isometry is called the **isometric sphere** of ϕ.*

We now arrive to the analogous of corollary 11.2.1 for Möbius transformations.

Corollary 11.4.1 *Every Möbius transformation on $\hat{\mathbb{R}}^n$ is at most the composition of $n + 3$ reflections or inversions.*

Proof. Let $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$. If $\phi(\infty) = \infty$ then ϕ is a similarity and so it can be decomposed into $n + 3$ reflections or inversions by corollary 11.2.1. Instead, if $\phi(\infty) \neq \infty$ then, by theorem

11.4.2, we have the following decomposition $\phi = \psi \circ \sigma$ with $\psi \in \mathcal{I}(\mathbb{R}^n)$ and σ an inversion. Since every isometry is the product of at most $n+1$ reflections, ϕ is the composition of at most $(n+2)$ reflections or inversions. $\square$

The importance of the last corollary lies not in the actual number of reflections or inversions that make up a Möbius transformation, but on the fact that there is a *finite* upper bound limit: this was not evident from the original definition of Möbius transformations. While it is still big, the Möbius group is still limited in size and it is only slightly bigger than the group of Euclidean similarities $\mathcal{S}(\mathbb{R}^n)$.

A very important consequence of what we have just proven is the possibility to connect the Möbius transformations geometrically defined as compositions of reflections and inversions, with the analytical formula used in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{C}$ that makes use of fractional linear transformations.

Corollary 11.4.2 *The Möbius transformations on $\mathbb{R}$, $\mathbb{R}^2$ or H are fractional linear transformations, i.e.*

$$\phi(x) = \frac{ax+b}{cx+d}, \quad x \in R$$

e la formula per $\mathbb{R}^2$ e $\mathbb{C}$...vedi anche capitolo 8 sezione 8.5...

Proof.

$\square$

11.4.2 The action of Möbius transformations on the set spheres in $\hat{\mathbb{R}}^n$

In the previous subsection we have seen how Möbius transformations and spheres of $\hat{\mathbb{R}}^n$ are linked. We now show a very powerful result: Möbius transformations acts transitively on the set of spheres of $\hat{\mathbb{R}}^n$.

We start by proving the *stability of the set of spheres in $\hat{\mathbb{R}}^n$ w.r.t. Möbius transformations.* We will do this by using several results that we underline in separated lemmas because of their stand-alone interest.

Lemma 11.4.1 *The following assertions hold:*

1. *isometries and similarities in $\mathbb{R}^n$ are stable on the set of hyperplanes and on the set of Euclidean spheres, i.e. they map hyperplanes into hyperplanes and Euclidean spheres into Euclidean spheres*

2. *the group of Euclidean isometries $\mathcal{I}(\mathbb{R}^n)$ (and so that of Euclidean similarities) acts transitively on the set of hyperplanes in $\mathbb{R}^n$ and the group of Euclidean similarities $\mathcal{S}(\mathbb{R}^n)$ acts transitively on the set of spheres in $\mathbb{R}^n$.*

Proof.

1. : since isometries are similarities, we will prove this result directly on the set of similarities. From theorem 10.1.1 we know that $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if it can be expressed in the form

$$f(x) = b + k\phi(x) \iff \phi(x) = \frac{f(x)-b}{k}, \qquad \forall x \in \mathbb{R}^n, \tag{11.11}$$

with $b \in \mathbb{R}^n$, $k > 0$ and $\phi \in \mathrm{O}(n)$. If $k = 1$, $f \in \mathcal{I}(\mathbb{R}^n)$.

Hyperplanes: given a hyperplane $P(a,t) = \{x \in \mathbb{R}^n \ : \ \langle x, a\rangle = t\}$, with $a \in \mathbb{R}^n$ such that $|a| = 1$ and $t \geqslant 0$, our aim is to show that $f(P(a,t))$ is a hyperplane.

We recall that, since ϕ belongs to $\mathrm{O}(n)$, it is linear, invertible and $\langle \phi(x), \phi(y)\rangle = \langle x, y\rangle$, $\forall x, y \in \mathbb{R}^n$. Hence $\forall x \in P(a,t)$ the following chain of equalities holds:

$$\begin{aligned}
\langle \phi(x), \phi(a)\rangle = \langle x, a\rangle = t \iff \langle k\phi(x), k\phi(a)\rangle = k^2 t &\underset{(11.11)}{\iff} \langle f(x) - b, k\phi(a)\rangle = k^2 t \\
&\iff \langle f(x), k\phi(a)\rangle - \langle b, k\phi(a)\rangle = k^2 t \\
&\iff \langle f(x), k\phi(a)\rangle = k^2 t + \langle b, k\phi(a)\rangle \\
&\iff \langle f(x), \frac{k\phi(a)}{|k\phi(a)|}\rangle = \frac{k^2 t + \langle b, k\phi(a)\rangle}{|k\phi(a)|},
\end{aligned}$$

note that we are allowed to divide by $k|\phi(a)|$ because $k > 0$ and $a \neq 0$, so $|\phi(a)| = |a| \neq 0$. Moreover $|k\phi(a)| = k|\phi(a)| = k|a| = k$, hence $\langle f(x), \phi(a)\rangle = kt + \langle b, \phi(a)\rangle$ $\forall x \in P(a,t)$.

This means that $f(x) \in P(\phi(a), kt + \langle b, \phi(a)\rangle)$, which is well defined as a hyperplane because $|\phi(a)| = 1$. As noticed in the definition of hyperplane, the positivity of $kt + \langle b, \phi(a)\rangle$ is not an issue because its possible negative sign we can integrated in the vector $\phi(a)$ without changing its norm. To avoid a cumbersome notation, we consider this as implicitly performed.

This proves that

$$\boxed{f(P(a,t)) = P(\phi(a), kt + \langle b, \phi(a)\rangle)} \qquad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{11.12}$$

i.e. the image of a hyperplane orthogonal to a and distant t from 0 is still a hyperplane orthogonal to $\phi(a)$ and distant $\langle b, \phi(a)\rangle$ from 0.

Euclidean spheres: Let $S^{n-1}_{a,r} = \{x \in \mathbb{R}^n \ : \ |x - a| = r\}$, with $a \in \mathbb{R}^n$ and $r > 0$. We want to show that $f(S^{n-1}_{a,r})$ is a sphere in $\mathbb{R}^n$. Let x be a point of $S^{n-1}_{a,r}$, then

$$r^2 = |x-a|^2 = \langle x-a, x-a\rangle = \langle \phi(x-a), \phi(x-a)\rangle = \langle \phi(x)-\phi(a), \phi(x)-\phi(a)\rangle = |\phi(x)-\phi(a)|^2.$$

Multiplying both sides by k^2 and using eq. (11.11) we obtain:

$$k^2 r^2 = |k\phi(x) - k\phi(a)|^2 = |f(x) - b - k\phi(a)|^2 = |f(x) - (b + k\phi(a))|^2 = |f(x) - f(a)|^2,$$

so, if $x \in S^{n-1}_{a,r}$, then $|f(x) - f(a)| = kr$, i.e.

$$\boxed{f(S^{n-1}_{a,r}) = S^{n-1}_{f(a),kr}} \qquad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{11.13}$$

i.e. the image of a Euclidean sphere of center a and radius r is still a Euclidean sphere of center $f(a)$ and radius kr.

2. : we first prove the thesis for hyperplanes and then for spheres.

Hyperplanes: let us fix two hyperplanes $P(a,t)$ and $P(a',t')$, with $a, a' \in \mathbb{R}^n$ such that $|a| = |a'| = 1$ and $t, t' \geqslant 0$. Our aim is to prove that it always exists an isometry $f \in \mathcal{I}(\mathbb{R}^n)$ such that $f(P(a,t)) = P(a',t')$. Thanks to eq. (11.12) with $k = 1$ because f is an isometry, we can rewrite the last equation as $P(\phi(a), t + \langle b, \phi(a)\rangle) = P(a', t')$ so that our problem is equivalent to showing that there exist $\phi \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$ such that the system

$$\begin{cases} \phi(a) = a' \\ t + \langle b, \phi(a)\rangle = t' \end{cases}$$

has at least one solution for all vectors $a, a' \in \mathbb{R}$ such that $|a| = |a'| = 1$, i.e. $a, a' \in S^{n-1}$. Thanks to the transitivity of $\mathrm{O}(n)$ on S^{n-1}, it surely exists $\phi \in \mathrm{O}(n)$ such that $a' = \phi(a)$, if we introduce this in the second equation we get $t + \langle b, a'\rangle = t'$, or $\langle b, a'\rangle = t' - t$, which is satisfied by all vectors $b \in \mathbb{R}^n$ such that $b \in P(a'\text{signum}(t' - t), |t' - t|)$.

Euclidean spheres: analogously to the previous case, once fixed any two spheres $S^{n-1}_{a,r}$ and $S^{n-1}_{a',r'}$, $a, a' \in \mathbb{R}^n$ and $r, r' > 0$, we must prove that it exists a similarity $f \in \mathcal{S}(\mathbb{R}^n)$ such that $f(S^{n-1}_{a,r}) = S^{n-1}_{a',r'}$. Thanks to eq. (11.13), we can rewrite the last equation as $S^{n-1}_{f(a),kr} = S^{n-1}_{a',r'}$, or $S^{n-1}_{b+k\phi(a),kr} = S^{n-1}_{a',r'}$, and so our problem is equivalent to the existence of $\phi \in \mathrm{O}(n)$, $k > 0$ and $b \in \mathbb{R}^n$ such that the system

$$\begin{cases} b + k\phi(a) = a' \\ kr = r' > 0 \end{cases}$$

has at least one solution for all $a, a' \in \mathbb{R}^n$ and $r, r' > 0$. We have immediately that $k = r'/r > 0$, which implies $b + \frac{r'}{r}\phi(a) = a'$. If we set $\phi = id_{\mathbb{R}^n} \in \mathrm{O}(n)$, then we get $b + \frac{r'}{r}a = a'$, which leads to $b = a' - \frac{r'}{r}a$. So, in conclusion, $k = r'/r$, $\phi = id_{\mathbb{R}^n}$ and $b = a' - \frac{r'}{r}a$ solve the system above, thus implying the transitivity of $\mathcal{S}(\mathbb{R}^n)$ on the set of Euclidean spheres in $\mathbb{R}^n$. $\square$

Note that if we consider similarities in $\hat{\mathbb{R}}^n$, since they fix ∞, this lemma states that similarities in $\hat{\mathbb{R}}^n$ map hyperplanes $\cup \{\infty\}$ into hyperplanes $\cup \{\infty\}$ and Euclidean spheres into Euclidean spheres. Hence, a weaker, but useful, reformulation of this assertion is that *isometries and similarities in $\hat{\mathbb{R}}^n$ are stable on the set of spheres in $\hat{\mathbb{R}}^n$.*

As a consequence, reflections, which are particular types of isometries, are stable on the set of spheres in $\hat{\mathbb{R}}^n$.

Lemma 11.4.2 *Let $a \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha\beta < |a|^2$. Then, the set of points defined by*

$$\Sigma_{\alpha,\beta} := \{x \in \mathbb{R}^n \ : \ \alpha|x|^2 + 2\langle x, a\rangle + \beta = 0\} \tag{11.14}$$

represents either a hyperplane or a sphere in $\mathbb{R}^n$.

Proof. First of all we note that if $a = 0$ we cannot hope to find the equation of a $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ simply because the orthogonal complement of the null vector of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. However, it is clear that when $a = 0$, eq. (11.14) represents the equation of a sphere centered in 0 and with radius $r = \sqrt{-\beta/\alpha}$. So, for $a = 0$, eq. (11.14) represents all the spheres centered in 0 and with arbitrary (strictly positive) radius provided that $\alpha \neq 0$ and $\alpha\beta < 0$.

Let us now consider the case $a \in \mathbb{R}^n \backslash \{0\}$. If $\alpha = 0$, eq. (11.14) represents the hyperplane $P(\frac{a}{|a|}, -\frac{\beta}{2|a|})$.

If $\alpha \neq 0$, eq. (11.14) represents (under a constraint that we will determine below) the sphere $S^{n-1}_{-\frac{a}{\alpha}, \frac{\sqrt{|a|^2-\alpha\beta}}{|\alpha|}}$. In fact, by dividing both sides of eq. (11.14) by $-\alpha$ we get

$$-|x|^2 - \frac{2}{\alpha}\langle x, a\rangle - \frac{\beta}{\alpha} = 0 \iff -|x|^2 + 2\left\langle x, -\frac{a}{\alpha}\right\rangle - \frac{\beta}{\alpha} = 0 \iff |x|^2 - 2\left\langle x, -\frac{a}{\alpha}\right\rangle + \frac{\beta}{\alpha} = 0,$$

which coincides with the equation of a sphere of radius r and center c, i.e.

$$|x-c|^2 = r^2 \iff |x|^2 - 2\langle x, c\rangle + |c|^2 - r^2 = 0,$$

if and only if the center is $c = -\frac{a}{\alpha}$ and the radius r satisfies $\frac{\beta}{\alpha} = \frac{|a|^2}{\alpha^2} - r^2$, i.e. $r^2 = \frac{|a|^2}{\alpha^2} - \frac{\beta}{\alpha}$, or $r = \frac{\sqrt{|a|^2-\alpha\beta}}{|\alpha|}$. Thus, the constraint that allows eq. (11.14) to represent any sphere of arbitrary radius centered in $a \neq 0$ is $\alpha\beta < |a|^2$.

We can resume our analysis by saying that eq. (11.14) represents any sphere or hyperplane in $\mathbb{R}^n$ provided that $\alpha\beta < |a|^2$ for all $a \in \mathbb{R}^n$, as it was to be proven. □

The next lemma shows that also inversions are stable on the set of spheres in $\hat{\mathbb{R}}^n$.

Lemma 11.4.3 *Inversions are stable on the set of spheres in* $\hat{\mathbb{R}}^n$*.*

Proof. Thanks to lemma 11.2.1, we can write any inversion $\sigma = \sigma_{a,r}$ as $\sigma = \phi \circ \sigma_{0,1} \circ \phi^{-1}$, with $\phi(x) = a + rx$ for all $x \in \mathbb{R}^n$. Thanks to corollary 10.1.1, ϕ and ϕ^{-1} are similarities, which are stable on the set of spheres in $\hat{\mathbb{R}}^n$ thanks to lemma 11.4.1. Hence we can reduce the proof to the case of $\sigma = \sigma_{0,1}$, i.e. from now on we will consider $\sigma(x) = \frac{x}{|x|^2}$, $x \neq 0$, and what we have to prove is that if we apply σ to either a sphere or a hyperplane we get back another sphere or hyperplane.

Since eq. (11.14) represents all possible hyperplane or sphere in $\mathbb{R}^n$ provided that $\alpha\beta < |a|^2$, to finish the proof of the theorem it is enough to show that σ preserves the structure of that equation. This turns out to be very easy: let x satisfy eq. (11.14), i.e. $\alpha|x|^2 + 2\langle x, a\rangle + \beta = 0$, which is equivalent to

$$\alpha|x|^2 + 2\langle x, a\rangle + \beta = 0 \iff \alpha + 2\left\langle \frac{x}{|x|^2}, a\right\rangle + \frac{\beta}{|x|^2} = 0 \iff \alpha + 2\langle \sigma(x), a\rangle + \frac{\beta}{|x|^2} = 0,$$

but $|\sigma(x)|^2 = \left|\frac{x}{|x|^2}\right|^2 = \frac{|x|^2}{|x|^4} = \frac{1}{|x|^2}$, so $\frac{\beta}{|x|^2} = \beta|\sigma(x)|^2$ and so we obtain that

$$\alpha|x|^2 + 2\langle x, a\rangle + \beta = 0 \iff \beta|\sigma(x)|^2 + 2\langle \sigma(x), a\rangle + \alpha = 0,$$

which shows that $\sigma(x)$ satisfies an equation of the same form as the one satisfied by x with the same constraint $\alpha\beta < |a|^2$. □

Theorem 11.4.3 *The Möbius transformations on* $\hat{\mathbb{R}}^n$ *are stable on the set of spheres in* $\hat{\mathbb{R}}^n$*.*

Proof. The proof will be just a sequence of considerations based on results that we have already proven that will allow us to greatly simplify the rest of the proof.

First of all, if $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ fixes ∞, then, by theorem 11.4.2, ϕ is a similarity on $\hat{\mathbb{R}}^n$ and so it is stable on the set of spheres in $\hat{\mathbb{R}}^n$ by lemma 11.4.1.

If $\phi(\infty) \neq \infty$, then, again thanks to theorem 11.4.2, we can decompose $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ as $\phi = \psi \circ \sigma$, where ψ is a Euclidean isometry and σ is an inversion w.r.t. a sphere. Again by lemma 11.4.1, ψ will be stable on the set of spheres in $\hat{\mathbb{R}}^n$, so what is left to prove is just that an inversion σ is stable on the set of spheres in $\hat{\mathbb{R}}^n$, which is guaranteed by lemma 11.4.3. $\square$

Thanks to this theorem, the natural action of the group of Möbius transformations on the set of spheres in $\hat{\mathbb{R}}^n$ defined by

$$\begin{array}{ccc} \mathcal{M}(\hat{\mathbb{R}}^n) \times \text{Spheres in } \hat{\mathbb{R}}^n & \longrightarrow & \text{Spheres in } \hat{\mathbb{R}}^n \\ (\phi, \Sigma) & \longmapsto & \phi(\Sigma) \end{array}$$

is well-defined.

Theorem 11.4.4 *The action of $\mathcal{M}(\hat{\mathbb{R}}^n)$ on the set of spheres of $\hat{\mathbb{R}}^n$ is transitive.*

Proof. Property 2. of Lemma 11.4.1 says that the group of similarities $\mathcal{S}(\hat{\mathbb{R}}^n) \subset \mathcal{M}(\hat{\mathbb{R}}^n)$ acts transitively on the set of hyperplanes united with $\{\infty\}$ of $\hat{\mathbb{R}}^n$ and on the set of Euclidean spheres in the following sense: for every fixed couple of spheres in $\hat{\mathbb{R}}^n$, Σ_1 and Σ_2 (*both* hyperplanes united with $\{\infty\}$ or *both* Euclidean spheres), there exists a similarity $\psi \in \mathcal{S}(\hat{\mathbb{R}}^n)$ such that $\psi(\Sigma_1) = \Sigma_2$.

However, the set of similarities of $\hat{\mathbb{R}}^n$ is not transitive on the whole set of generalized spheres in $\hat{\mathbb{R}}^n$. Indeed, it is not possible to map a hyperplane united with $\{\infty\}$ into a Euclidean sphere, or vice-versa, through a similarity. A simple explanation of this fact is that, clearly, ∞ belongs to the first category of sphere in $\hat{\mathbb{R}}^n$ but not to the second. Moreover, property 1. of theorem 11.4.2 says that similarities in $\hat{\mathbb{R}}^n$ leave the point ∞ fixed, thus they cannot map an object containing ∞ into an object not containing it, or vice-versa.

So, to conclude the proof, we must show that if we have two spheres Σ_1, Σ_2 in $\hat{\mathbb{R}}^n$, such that $\Sigma_1 = P(a,t) \cup \{\infty\}$ and $\Sigma_2 = S^{n-1}_{b,r}$, there exists a Möbius transformation ϕ, necessarily in $\mathcal{M}(\hat{\mathbb{R}}^n)\backslash\mathcal{S}(\hat{\mathbb{R}}^n)$ for what we have just observed, such that $\phi(\Sigma_1) = \Sigma_2$.

By a straightforward computation, it can be verified that $\sigma_{0,1}(P(e_1, \frac{1}{2}) \cup \{\infty\}) = S^{n-1}_{e_1,1}$, notice that $\sigma_{0,1}(\infty) = 0 \in S^{n-1}_{e_1,1}$. By property 2. of Lemma 11.4.1, there exist $\psi_1, \psi_2 \in \mathcal{S}(\hat{\mathbb{R}}^n)$ such that $\psi_1(\Sigma_1) = P(e_1, \frac{1}{2}) \cup \{\infty\}$ and $\psi_2(\Sigma_2) = S^{n-1}_{e_1,1}$. Hence $\phi \equiv \psi_2^{-1} \circ \sigma_{0,1} \circ \psi_1$ is a Möbius transformation, because composition of two similarities and an inversion, moreover, clearly, $\phi(\Sigma_1) = \Sigma_2$.

$\square$

Theorem 11.4.5 *Let $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ and let Σ be a sphere of $\hat{\mathbb{R}}^n$ such that $\phi(x) = x$ $\forall x \in \Sigma$. Then ϕ is either $id_{\hat{\mathbb{R}}^n}$ or the reflection or inversion w.r.t. Σ, depending on the fact that Σ is a hyperplane united with ∞ or a Euclidean sphere, respectively.*

Proof. Σ is either a hyperplane $\cup \{\infty\}$ or a $(n-1)$-dimensional sphere in $\mathbb{R}^n$. Thus an inversion w.r.t. Σ can be either a reflection w.r.t. a hyperplane (that fixes ∞) or an inversion w.r.t. a Euclidean sphere.

We start by assuming that $\Sigma = P(e_n, 0) \cup \{\infty\}$, but

$$P(e_n, 0) = \text{span}(e_n)^{\perp} = \text{span}(e_1, \ldots, e_{n-1}) = \mathbb{R}^{n-1},$$

hence $\Sigma = \hat{\mathbb{R}}^{n-1}$. By the hypothesis that ϕ fixes all the points of Σ we have, in particular:

- $\phi(\infty) = \infty \implies \phi$ is a Euclidean similarity by th. 11.4.2, i.e. $\phi(x) = a + kAx$, $k > 0$, $A \in \mathrm{O}(n)$, for all $x \in \mathbb{R}^n$
- $\phi(0) = 0 \implies \phi = kA$
- $\phi(e_1) = e_1 \implies |\phi(e_1) - \phi(0)| = |e_1 - 0| = |e_1| = 1$, but since $\phi = kA$ this is equivalent to $|kAe_1 - kA0| = k|Ae_1| \underset{A \in \mathrm{O}(n)}{=} k|e_1| = k$, which implies $k = 1$, so $A \in \mathrm{O}(n)$
- $\phi(e_j) = e_j$, $j = 2, \ldots, n-1$ implies that the matrix $A \in \mathrm{O}(n)$ associated to ϕ w.r.t. the canonical basis of $\mathbb{R}^n$ is either

$$A = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix} = id_{\mathbb{R}^n} \quad \text{or} \quad A = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

because these are the only possible options compatible with the fact that $\det(A) = \pm 1$.

So, either ϕ is the identity on $\mathbb{R}^n$, extended to the identity on $\hat{\mathbb{R}}^n$ because $\phi(\infty) = \infty$, or ϕ is the reflection w.r.t Σ. Hence the thesis is proven when $\Sigma = \hat{\mathbb{R}}^{n-1}$.

We now assume that Σ is an arbitrary sphere of $\hat{\mathbb{R}}^n$ and that ϕ fixes Σ. By the transitivity of $\mathcal{M}(\hat{\mathbb{R}}^n)$ on the set of spheres of $\hat{\mathbb{R}}^n$, there exists a Möbius transformation $\psi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ such that $\psi(\Sigma) = \hat{\mathbb{R}}^{n-1}$, i.e. $\psi(s) = x \in \hat{\mathbb{R}}^{n-1}$ for all $s \in \Sigma$. It follows that, for all $x \in \hat{\mathbb{R}}^{n-1}$,

$$(\psi \circ \phi \circ \psi^{-1})(x) = \psi(\phi(s)) = \psi(s) = x,$$

i.e. ψ fixes $\hat{\mathbb{R}}^{n-1}$ so, thanks to what proven above, $\psi \circ \phi \circ \psi^{-1} = id_{\hat{\mathbb{R}}^n}$ or $\psi \circ \phi \circ \psi^{-1} \equiv \rho$, the reflection w.r.t. $\hat{\mathbb{R}}^{n-1}$. By composing on the left both members by ψ^{-1} and on the right by ψ, we have that it is either $\phi = \psi^{-1} \circ \psi = id_{\hat{\mathbb{R}}^n}$ or $\phi = \psi^{-1} \circ \rho \circ \psi$.

We now want to understand what kind of transformation $\psi^{-1} \circ \rho \circ \psi$ is. To this scope, let us consider, instead of the generic $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$, a reflection or inversion σ w.r.t. Σ, which is not the identity. By repeating the argument above on σ, we obtain that $\psi \circ \sigma \circ \psi^{-1} = \rho$, the reflection or inversion w.r.t. Σ, i.e. $\sigma = \psi^{-1} \circ \rho \circ \psi = \phi$. $\square$

This result will be fundamental to prove theorem 11.5.1.

We know that reflections and inversions fix the points of the hyperplane or sphere w.r.t. they act, respectively. The theorem just proven tells us that *this condition is sufficient to determine if a Möbius transformation is a pure reflection or inversion*, provided that we have excluded the possibility that it is the identity on the whole $\hat{\mathbb{R}}^n$, which is particularly easy because it is sufficient to consider any point not belonging to the hyperplane or the sphere.

The last property of Möbius transformations that we prove here refers to inverse points.

Def. 11.4.4 *Let Σ be a sphere of $\hat{\mathbb{R}}^n$ and σ the reflection or inversion w.r.t. Σ. Two points $x, y \in \hat{\mathbb{R}}^n$ are said to be* ***inverse points*** *w.r.t. Σ if $y = \sigma(x)$.*

Theorem 11.4.6 *Let $\phi \in \mathcal{M}(\hat{\mathbb{R}}^n)$ and let Σ be a sphere of $\hat{\mathbb{R}}^n$. If x and y are inverse points w.r.t. Σ, then $\phi(x)$ and $\phi(y)$ are also inverse points w.r.t. $\Sigma' = \phi(\Sigma)$.*

Proof. The thesis of the theorem is trivially true if ϕ is the identity. So, let us assume that ϕ is not the identity and that σ is the reflection or inversion w.r.t. Σ. Then, $\phi \circ \sigma \circ \phi^{-1}$ fixes each point of $\Sigma' = \phi(\Sigma)$ and so $\phi \circ \sigma \circ \phi^{-1} = \rho$ is the reflection or the inversion w.r.t. Σ'. Finally, if x and y are inverse points w.r.t. Σ, i.e. $y = \sigma(x)$, then

$$\rho(\phi(x)) = (\rho \circ \phi)(x) = (\phi \circ \sigma \circ \phi^{-1} \circ \phi)(x) = (\phi \circ \sigma)(x) = \phi(\sigma(x)) = \phi(y),$$

i.e. $y = \sigma(x)$ implies $\phi(y) = \rho(\phi(x))$. □

11.4.3 The conformality of Möbius transformations

A conformal transformation is a map that maintains angles. In this section we show that conformal and Möbius transformations are tightly interconnected to the point of being confounded in a Euclidean vector space of dimension higher or equal to 3.

An intuitive idea behind this fact can be obtained by considering two intersecting spheres of $\hat{\mathbb{R}}^n$, Σ_1 and Σ_2: if ϕ is a Möbius transformation, then $\phi(\Sigma_1)$ and $\phi(\Sigma_2)$ are two other intersecting spheres Σ_1' and Σ_2' of $\hat{\mathbb{R}}^n$. It is natural to ask oneself how Σ_1' and Σ_2' are positioned to one another when compared to Σ_1 and Σ_2, since Möbius transformations are continuous functions and contain Euclidean similarities, intuitively we imagine that they are positioned more or less in the same way.

Furthermore, if this is the case, then the angle between the normal vectors n_1 of Σ_1 and n_2 of Σ_2 at an intersecting point $x \in \Sigma_1 \cap \Sigma_2$ should not change.

The path that we will follow to make this argument rigorous starts with a definition.

Def. 11.4.5 *Let $U \subseteq \mathbb{R}^n$ open and $\phi : U \to \mathbb{R}^n$, $f \in \mathscr{C}^1(U)$, i.e. all the partial derivatives $\frac{\partial \phi_i}{\partial x_j}$ exists and they are continuous functions on U. ϕ is said to be* ***conformal*** *if there is a function $\kappa : U \to \mathbb{R}^+$, called the* ***scale factor*** *of ϕ, such that*

$$\frac{1}{\kappa(x)} J_\phi(x) \in \mathrm{O}(n) \qquad \forall x \in U,$$

$J_\phi(x)$ being the Jacobian matrix of ϕ calculated in x.

In other words, a conformal function is a continuously differentiable map whose Jacobian matrix can be turned into an orthogonal one simply by re-scaling its coefficients with a positive factor that is allowed to change in every point of the function domain.

Def. 11.4.6 *Given $x, y \in \mathbb{R}^n$, $x, y \neq 0$, we denote with $\theta(x, y)$ the angle between them, i.e. the only angle in $[0, \pi]$ that verifies this equation:*

$$\cos(\theta(x, y)) = \frac{\langle x, y \rangle}{|x||y|}. \tag{11.15}$$

$f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the angle between non-zero vectors if $\theta(f(x), f(y)) = \theta(x, y)$ for all $x, y \in \mathbb{R}^n$.

It is clear that an orthogonal tranformation $f \in \mathrm{O}(n)$ preserves the angle between non-zero vectors because it preserves the scalar product between them and their norms. The next lemma says that, among linear transformations, the orthogonal ones are the only angle preserving maps modulo a scalar coefficient.

Lemma 11.4.4 *Let A a $n \times n$ real matrix. Then, there is a $k \in \mathbb{R}^+$ such that $k^{-1}A \in \mathrm{O}(n)$ if and only if A preserves the angle between non-zero vectors.*

Proof.

$\boxed{\Longrightarrow}$: we assume there is a k such that $k^{-1}A$ is an orthogonal matrix, then $k^{-1}A$ is non-singular and so, for all $x, y \in \mathbb{R}^n$, $x, y \neq 0$, also Ax and Ay are non-zero vectors and we can write:

$$\begin{aligned}\cos(\theta(Ax, Ay)) &= \frac{\langle Ax, Ay\rangle}{|Ax||Ay|} = \frac{\langle k^{-1}Ax, k^{-1}Ay\rangle}{|k^{-1}Ax||k^{-1}Ay|} \\ &\underset{k^{-1}A \in \mathrm{O}(n)}{=} \frac{\langle x, y\rangle}{|x||y|} = \cos(\theta(x, y)).\end{aligned}$$

$\boxed{\Longleftarrow}$: conversely, we suppose that A preserves the angle between non-zero vectors. Then, in particular,

$$\theta(Ae_i, Ae_j) = \theta(e_i, e_j) = \frac{\pi}{2} \qquad \forall i, j \in \{1, \ldots, n\},\ i \neq j.$$

Hence, $(Ae_1, \ldots, Ae_n)$ is an orthogonal basis of $\mathbb{R}^n$, so, if we normalize each vector and we set it as a column of a matrix B, i.e.

$$B = \begin{pmatrix} | & \cdots & | \\ \frac{Ae_1}{|Ae_1|} & \cdots & \frac{Ae_n}{|Ae_n|} \\ | & \cdots & | \end{pmatrix},$$

then B belongs to $\mathrm{O}(n)$ and so does B^{-1} because $\mathrm{O}(n)$ is a group.

By direct computation we get $Be_i = \frac{Ae_i}{|Ae_i|}$ for all $i = 1, \ldots, n$, so, if we multiply both members by $|Ae_i|$ and compose them with B^{-1} we get $B^{-1}Ae_i = |Ae_i|e_i \equiv c_i e_i$, with $c_i > 0$, for all $i = 1, \ldots, n$.

Finally, notice that $B^{-1}A$ preserves the angles between non-zero vectors because it is the composition of two angle-preserving operators, so, using definition (11.15) and the injectivity of the cosine function in $[0, \pi]$ we have that, for all $i, j = 1, \ldots, n$, $i \neq j$,

$$\begin{aligned}\theta(B^{-1}A(e_i + e_j), B^{-1}Ae_j) = \theta(e_i + e_j, e_j) &\iff \frac{\langle c_i e_i + c_j e_j, c_j e_j\rangle}{|c_j e_j||c_i e_i + c_j e_j|} = \frac{\langle e_i + e_j, e_j\rangle}{|e_i + e_j||e_j|} \\ &\iff \frac{c_j^2}{c_j\sqrt{c_i^2 + c_j^2}} = \frac{1}{\sqrt{2}} \\ &\iff \frac{c_j}{\sqrt{c_i^2 + c_j^2}} = \frac{1}{\sqrt{2}} \\ &\iff \frac{\sqrt{2c_j^2}}{\sqrt{c_i^2 + c_j^2}} = 1 \iff 2c_j^2 = c_i^2 + c_j^2 \\ &\iff c_j = c_i,\end{aligned}$$

thanks to the strict positivity of the coefficients c_i. Thus all the coefficients can be identified with a constant $k > 0$, which implies $B^{-1}Ae_i = ke_i$, $\forall i = 1, \ldots, n$, i.e., by direct computation,

$B^{-1}A = kI_n$, or $\frac{1}{k}A = B \in \mathrm{O}(n)$. □

We recall that, given a differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$, the tangent vector to γ at $\gamma(0)$ is the vector of $\mathbb{R}^n$ defined by the formula:

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}.$$

Def. 11.4.7 *Let $\alpha, \beta : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be two differentiable curves with $\alpha(0) = \beta(0)$ and $\alpha'(0), \beta'(0) \neq 0$. The angle between α and β is defined as the angle between the vectors of $\mathbb{R}^n$ given by $\alpha'(0)$ and $\beta'(0)$.*

We can now give a characterization of conformality that it is often used as an alternative definition of this property.

Theorem 11.4.7 *Let $U \subseteq \mathbb{R}^n$ be open, $\phi : U \to \mathbb{R}^n$, $\phi \in \mathscr{C}^1(U)$. Then, ϕ is conformal if and only if ϕ preserves the angle between curves.*

Proof.

$\boxed{\Longrightarrow}$: if ϕ is conformal, then there is a scale factor $\kappa : U \to \mathbb{R}^+$ such that $\kappa^{-1}(x) J_\phi(x) \in \mathrm{O}(n)$ for all $x \in U$. Let $\alpha, \beta : (-\varepsilon, \varepsilon) \to U$ be two $\mathscr{C}^1$ curves such that $\alpha(0) = \beta(0)$ and $\alpha'(0), \beta'(0) \neq 0$. Then, $\kappa(\alpha(0)) J_\phi(\alpha(0)) = \kappa(\beta(0)) J_\phi(\beta(0))$ is an orthogonal matrix and so, by Lemma 11.4.4, $J_\phi(\alpha(0)) = J_\phi(\beta(0))$ preserves angles between the non-zero (by hypothesis) vectors $\alpha'(0)$ and $\beta'(0)$. Hence

$$\begin{aligned} \theta\big((\phi \circ \alpha)'(0), (\phi \circ \beta)'(0)\big) &= \theta\big(J_\phi(\alpha(0))\alpha'(0), J_\phi(\beta(0))\beta'(0)\big) \\ &= \theta\big(\alpha'(0), \beta'(0)\big), \end{aligned}$$

which shows that the angle between α and β is the same as the one between $\phi \circ \alpha$ and $\phi \circ \beta$, i.e. ϕ preserves the angle between curves.

$\boxed{\Longleftarrow}$: conversely, by Lemma 11.4.4, if ϕ preserves angles, then $J_\phi(x)$ preserves angles between non-zero vectors for each fixed $x \in U$. Hence, there exists a $\kappa > 0$ such that $\kappa(x)^{-1} J_\phi(x)$ is orthogonal for all $x \in U$ and so ϕ is conformal on U. □

Def. 11.4.8 *Let $U \subseteq \mathbb{R}^n$ open and let $\phi : U \to \mathbb{R}^n$ be a differentiable function. ϕ preserves (resp. reverses) orientation at a point $x \in U$ if $\det J_\phi(x) > 0$ (resp. $\det J_\phi(x) < 0$).*

ϕ preserves (resp. reverses) orientation if ϕ preserves (resp. reverses) orientation at each point of its domain.

Theorem 11.4.8 *Every reflection and inversion in $\mathbb{R}^n$ is conformal and reverses orientation.*

Proof.

Reflections. Let ρ be a reflection w.r.t. a hyperplane. The easiest way to prove that ρ is conformal is by recalling that it is an isometry, hence there exist $b \in \mathbb{R}^n$ and $B \in \mathrm{O}(n)$ such that $\rho(x) = b + B(x)$ for all $x \in \mathbb{R}^n$, thus $J_\rho(x) = B$ and so ρ verifies the definition of conformality with $\kappa(x) = 1$ $\forall x \in \mathbb{R}^n$.

However, for later use in this proof, let us also verify the conformality of ρ by computing directly the Jacobian of the original expression of the reflection, i.e. $\rho(x) = x + 2(t - \langle a, x\rangle a) = id_{\mathbb{R}^n}(x) + 2t - 2\langle a, x\rangle a$, $|a| = 1$, $t \geqslant 0$. Thanks to eq. (B.11) we have

$$J_\rho(x) = I - 2A,$$

where A is the matrix $A = (a_i a_j)_{1\leqslant i,j\leqslant n}$. Notice that $J_\rho(x)$ does not depend on the parameter t, so we are allowed to set it to 0, but then ρ becomes an orthogonal (hence linear transformation), i.e. $\rho(x) = J_\rho(x)$, $\forall x \in \mathbb{R}^n$, hence, by property **4.** of theorem 11.2.1 this implies that

$$J_\rho(x) = I - 2A \in \mathrm{O}(n) \tag{11.16}$$

for all $A = (a_i a_j)_{1\leqslant i,j\leqslant n}$ with $a \in \mathbb{R}^n$, $|a| = 1$.

Let us now prove that ρ reverses orientation. By the transitivity of the action of $\mathrm{SO}(n)$ on S^{n-1}, there is a $\psi \in \mathrm{SO}(n)$ such that $\psi(a) = e_1$ and so for any $x \in \mathbb{R}^n$,

$$\begin{aligned} (\psi \circ \rho \circ \psi^t)(x) &= \psi(\rho(\psi^t(x))) = \psi\big(\psi^t(x) + 2(t - \big\langle a, \psi^t(x)\big\rangle)a\big) \\ &\underset{\psi^t=\psi^{-1}}{=} x + 2(t - \langle \psi(a), x\rangle)\psi(a) \\ &= x + 2(t - \langle e_1, x\rangle)e_1, \end{aligned}$$

but $\langle e_1, x\rangle = (x_1, 0, \dots, 0)^t$, so, by direct computation we get

$$x + 2(t - \langle e_1, x\rangle)e_1 = (-x_1 + 2t, x_2, \dots, x_n)^t = \eta x + 2te_1,$$

with $\eta = \mathrm{diag}(-1, 0, \dots, 0)$, hence $J_{\psi\circ\rho\circ\psi^t}(x) = \eta$ and so $\det(J_{\psi\circ\rho\circ\psi^t}(x)) = -1$ for all $x \in \mathbb{R}^n$. Furthermore, $J_{\psi\circ\rho\circ\psi^t}(x) = J_{\psi\circ\rho\circ\psi^{-1}}(x)$ and the functions ψ and ρ are linear and affine, respectively, so their Jacobian matrices do not depend of the evaluation point, which can be arbitrarily taken to be x. Thanks to these considerations and to the chain rule for Jacobian matrices we have

$$J_{\psi\circ\rho\circ\psi^t}(x) = J_\psi(x)J_\rho(x)J_{\psi^{-1}}(x) = J_\psi(x)J_\rho(x)J_\psi(x)^{-1},$$

and so, by Binet's theorem:

$$-1 = \det(J_{\psi\circ\rho\circ\psi^t}(x)) = \cancel{\det(J_\psi(x))}\det(J_\rho(x))\cancel{\det(J_\psi(x))^{-1}} = \det(J_\rho(x))$$

for all $x \in \mathbb{R}^n$, hence ρ reverses orientation.

Inversions. Let us start by considering an inversion w.r.t. a sphere centered in 0, i.e. $\sigma_{0,r}(x) = \frac{r^2}{|x|^2}x$, defined for $x \neq 0$, we will consider the generic case later. By theorem B.0.4, the computation of the Jacobian matrix of $\sigma_{0,r}$ gives:

$$J_{\sigma_{0,r}}(x) = \frac{r^2}{|x|^2}\left(I - 2\frac{x_i x_j}{|x|^2}\right) = \frac{r^2}{|x|^2}(I - 2A_x) \equiv \kappa(x)B_x,$$

where $A_x = \left(\frac{x_i}{|x|}\frac{x_j}{|x|}\right)_{1\leqslant i,j\leqslant n}$, $B_x = I - 2A_x$ and $\kappa(x) = \frac{r^2}{|x|^2} \in \mathbb{R}^+$ for all $x \neq 0$. Notice that the entries of the matrix A_x are the components of the normalized vector $\frac{x}{|x|}$, so, thanks to eq. (11.16), $B_x = I - 2A_x$ is orthogonal, for all $x \neq 0$. This proves that $\sigma_{0,r}$ is conformal.

Let us now prove that $\sigma_{0,r}$ reverses orientation: the properties of the determinant imply

$$\det(J_{\sigma_{0,r}}(x)) = \frac{r^{2n}}{|x|^{2n}}\det(I - 2A_x) = \frac{r^{2n}}{|x|^{2n}}\det(J_\rho(x)) \underset{\det(J_\rho(x))=-1}{=} -\frac{r^{2n}}{|x|^{2n}} < 0.$$

Note that $\det(I - 2A_x) = \det(I - 2A) = \det(J_\rho(x))$, because for every fixed $x \neq 0$ the matrix $J_{\sigma_{0,r}}(x) = I - 2A_x = J_\rho(x)$, with $\rho = \rho_{\frac{x}{|x|},0}$.

Finally, let us consider the generic inversion $\sigma_{a,r}(x) = a + \frac{r^2}{|x-a|^2}(x-a)$. If $\tau_a(x) = x + a$ is the translation operator by a, then it is clear that $\sigma_{a,r} = \tau_a \circ \sigma_{0,r} \circ \tau_a^{-1}$. So, since $J_{\tau_a}(x) = I$ for all x, the chain rule for the Jacobian gives:

$$J_{\sigma_{a,r}}(x) = J_{\sigma_{0,r}}(x-a),$$

which allows us to conclude that also $\sigma_{a,r}$ is conformal and reverses orientation for all $a \in \mathbb{R}^n$ thanks to the previous analysis of $\sigma_{0,r}$. The procedure is totally analogous, paying attention to impose the condition $x \neq a$ instead of $x \neq 0$. $\square$

Since Möbius transformations are finite compositions of reflections and inversions, they are conformal too.

Corollary 11.4.3 *Every Möbius transformation is conformal.*

In 2 dimensions, all holomorphic[3] and anti-holomorphic[4] functions with a non-vanishing Jacobian are conformal mappings.

However, as soon as we pass to the third dimensions, conformal mappings are completely determined by Möbius transformations. This result has been first proven by Liouville [12] in 1850 in the case of $\mathscr{C}^3$ mappings in $\mathbb{R}^3$ and then it has been quickly extended to higher dimensions. Nonetheless, it remained an open problem for over a century how to relax the hypothesis of this theorem by considering only $\mathscr{C}^1$ functions, until Hartman solved it [6], [19].

Theorem 11.4.9 (Liouville-Hartman theorem of conformal mappings) *Let $U \subseteq \mathbb{R}^n$ open, $n \geqslant 3$ and $f : U \to \mathbb{R}^n$ a $\mathscr{C}^1$ map. Then, f is conformal is and only if f is the restriction of a Möbius transformation on U.*

11.5 Möbius transformations in the upper half space $\mathcal{U}^n$ and the open unit ball $\mathcal{B}^n$

Up to now we have analyzed the set of Möbius transformations $\mathcal{M}(\hat{\mathbb{R}}^n)$ *on the whole space* $\hat{\mathbb{R}}^n$. In this section we will focus our attention on subgroups of $\mathcal{M}(\hat{\mathbb{R}}^n)$ given by *Möbius transformations that preserve proper subsets of* $\hat{\mathbb{R}}^n$ and on the relationship between them. The information that we will gather will prove to be of crucial importance in the analysis of the hyperbolic models that we will discuss in chapter 12.

The proper subsets of $\hat{\mathbb{R}}^n$ that we will consider are

$$\hat{\mathbb{R}}^{n-1} \cong P(e_n, 0) \cup \{\infty\} = \{x \in \mathbb{R}^n : \langle x, e_n\rangle = 0\} \cup \{\infty\},$$

[3] A function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = u(z) + iv(z)$, is said to be holomorphic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

[4] anti-holomorphic if $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

which separates $\hat{\mathbb{R}}^n$ into two disjoint, connected subsets of dimension n, the upper and the lower half plane, together with the open connected subsets of $\hat{\mathbb{R}}^n$ given by the unit ball and the complementary of its closure. The explicit definitions are given below:

- the upper-half space: $\mathcal{U}^n = \{x \in \mathbb{R}^n \ : \ x_n > 0\} = \{x \in \mathbb{R}^n : \langle x, e_n \rangle > 0\}$;
- the lower-half space: $\mathcal{L}^n = \{x \in \mathbb{R}^n \ : \ x_{n+1} < 0\} = \{x \in \mathbb{R}^n \ : \ \langle x, e_n \rangle < 0\}$;
- the open unit ball: $\mathcal{B}^n = \{x \in \mathbb{R}^n \ : \ |x| < 1\}$;
- the complementary in $\hat{\mathbb{R}}^n$ of the closed unit ball: $(\overline{\mathcal{B}^n})^c = \{x \in \mathbb{R}^n \ : \ |x| > 1\} \cup \{\infty\}$.

Clearly,

$$\hat{\mathbb{R}}^n = \mathcal{U}^n \sqcup P(e_n, 0) \sqcup \{\infty\} \sqcup \mathcal{L}^n = \mathcal{B}^n \sqcup S^{n-1} \sqcup (\overline{\mathcal{B}^n})^c.$$

The mathematical analysis starts with the stereographic projection relative to the sphere embedded in $\hat{\mathbb{R}}^n$, which is the isomorphism between $\hat{\mathbb{R}}^{n-1} \cong P(e_n, 0) \cup \{\infty\}$ and S^{n-1} given by

$$\begin{array}{rccl} \hat{\pi}: & \hat{\mathbb{R}}^{n-1} & \xrightarrow{\sim} & S^{n-1} \\ & x & \longmapsto & \hat{\pi}(x) := \begin{cases} \left(\frac{2x_1}{1+|x|^2}, \dots, \frac{2x_{n-1}}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2} \right) & \text{if } x \neq \infty \\ e_n \quad \text{if } x = \infty \end{cases} \end{array}.$$

For the aim of this section, it is fundamental to use theorem 11.3.2 which guarantees that we can interpret $\hat{\pi}$ as the restriction of an inversion w.r.t. a sphere in $\hat{\mathbb{R}}^n$, precisely $\hat{\pi} = \sigma_{e_n, \sqrt{2}}\Big|_{\hat{\mathbb{R}}^{n-1}}$, or

$$\begin{array}{rccl} \hat{\pi}: & \hat{\mathbb{R}}^{n-1} \subset \hat{\mathbb{R}}^n & \xrightarrow{\sim} & S^{n-1} \subset \hat{\mathbb{R}}^n \\ & x & \longmapsto & \begin{cases} e_n + \frac{2}{|x-e_n|^2}(x - e_n) & \text{if } x \neq \infty \\ e_n \quad \text{if } x = \infty \end{cases} \end{array}.$$

We have all the information we need in order to understand the action of both $\sigma_{e_n, \sqrt{2}}$ and $\hat{\pi}$, which is depicted in Figure 11.5 in the two-dimensional: by the properties of an inversion w.r.t. a sphere, $\sigma_{e_n, \sqrt{2}}$ will leave the sphere $S^{n-1}_{e_n, \sqrt{2}}$ fixed and it will inverse the points in its interior to points in its exterior. In particular, its center, given by e_n, will be sent to ∞ and the points on S^{n-1} will be mapped on $\hat{\mathbb{R}}^{n-1}$.

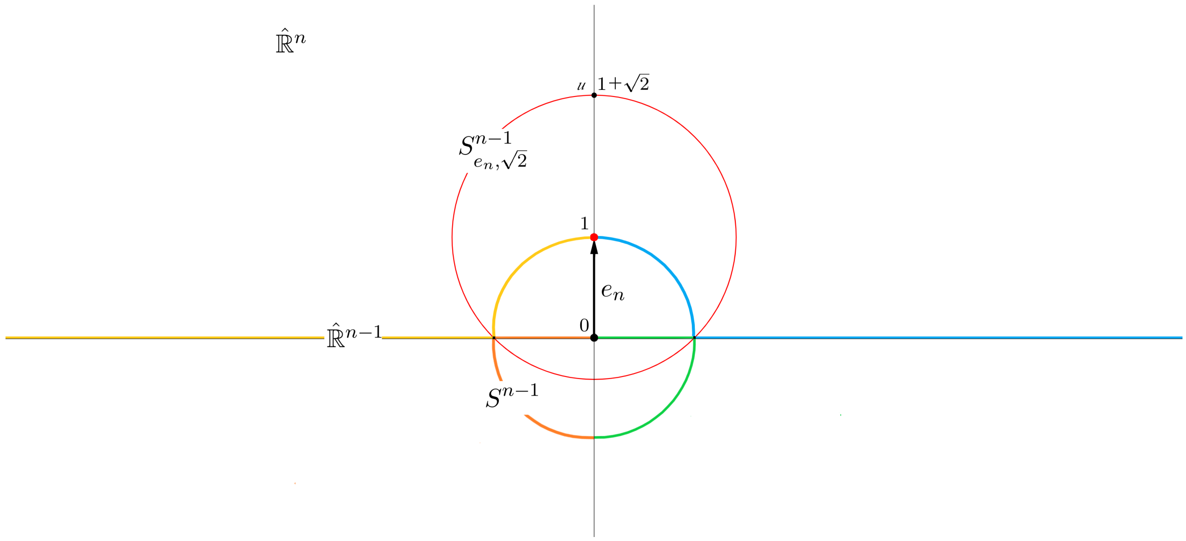


Figure 11.5: Two-dimensional representation of the geometric objects involved in the stereographic projection in $\hat{\mathbb{R}}^n$.

Let us now study the action of $\sigma_{e_n,\sqrt{2}}$ on $\mathcal{U}^n, \mathcal{L}^n, \mathcal{B}^n, (\overline{\mathcal{B}^n})^c$. The most important piece of information that we need to understand this action consists in recalling that $\sigma_{e_n,\sqrt{2}}$ is a homeomorphism, so it maps connected subsets of $\hat{\mathbb{R}}^n$ into connected subsets of $\hat{\mathbb{R}}^n$.

In order to single out the image of each connected subset via $\sigma_{e_n,\sqrt{2}}$ it is sufficient to think about the fact that $\hat{\mathbb{R}}^{n-1}$ splits $\hat{\mathbb{R}}^n$ into $\mathcal{U}^n$ and $\mathcal{L}^n$, and also to the fact that $\hat{\mathbb{R}}^n$ it is mapped to the spherical surface S^{n-1}, thus:

- either $\sigma_{e_n,\sqrt{2}}$ maps $\mathcal{U}^n$ to $\mathcal{B}^n$ and $\mathcal{L}^n$ to $(\overline{\mathcal{B}^n})^c$
- or, $\sigma_{e_n,\sqrt{2}}$ maps $\mathcal{U}^n$ to $(\overline{\mathcal{B}^n})^c$ and $\mathcal{L}^n$ to $\mathcal{B}^n$.

In order to choose between these two mutually exclusive options, it is enough to consider the image via $\sigma_{e_n,\sqrt{2}}$ of a wisely chosen point, i.e. $u = (1+\sqrt{2})e_n$. In fact, $u \in S^{n-1}_{e_n,\sqrt{2}} \cap \mathcal{U}^n \cap (\overline{\mathcal{B}^n})^c$ and u will remain fixed after the application of $\sigma_{e_n,\sqrt{2}}$ thanks to property 1. in 11.2.2, thus u, as all the other points belonging to the upper half space, will be mapped to $(\overline{\mathcal{B}^n})^c$, i.e.

$$\sigma_{e_n,\sqrt{2}}(\mathcal{U}^n) = \overline{\mathcal{B}^n}^c \quad \text{and} \quad \sigma_{e_n,\sqrt{2}}(\mathcal{L}^n) = \mathcal{B}^n.$$

We can say more, thanks to property 2. in 11.2.2, $\sigma^{-1}_{e_n,\sqrt{2}} = \sigma_{e_n,\sqrt{2}}$, so also the opposite is true, i.e.

$$\sigma_{e_n,\sqrt{2}}(\overline{\mathcal{B}^n}^c) = \mathcal{U}^n \quad \text{and} \quad \sigma_{e_n,\sqrt{2}}(\mathcal{B}^n) = \mathcal{L}^n.$$

The result that we have obtained can be reached in an alternative way. To do that, we need the following preliminary results, direct consequences of a straightforward computation:

$$|\sigma_{e_n,\sqrt{2}}(x)|^2 = \begin{cases} 1 + \frac{4x_n}{|x-e_n|^2} & \text{if } x \neq \infty \\ 1 & \text{if } x = \infty \end{cases}, \tag{11.17}$$

and

$$\langle \sigma_{e_n,\sqrt{2}}(x), e_n \rangle = \begin{cases} \frac{|x|^2-1}{|x-e_n|^2} & \text{if } x \neq \infty \\ 1 & \text{if } x = \infty \end{cases}. \tag{11.18}$$

Let $x \in \mathcal{L}^n$, i.e. $x_n = \langle x, e_n \rangle < 0$, then, using eq. (11.17) we obtain that $|\sigma_{e_n,\sqrt{2}}(x)|^2 < 1$, hence $\sigma_{e_n,\sqrt{2}}(x) \in \mathcal{B}^n$, i.e. $\mathcal{L}^n$ is mapped into $\mathcal{B}^n$. Furthermore, for all $x \in \mathcal{B}^n$, i.e. such that $|x| < 1$, by using eq. (11.18) we get $\langle \sigma_{e_n,\sqrt{2}}(x), e_n \rangle < 0$, i.e. $\sigma_{e_n,\sqrt{2}}(x) \in \mathcal{L}^n$, so also $\mathcal{B}^n$ is mapped into $\mathcal{L}^n$. With analogous arguments it is possible to verify that $\mathcal{U}^n$ is mapped into $\overline{\mathcal{B}^n}^c$ and vice-versa.

Historically, the upper half space and the interior of the unit ball have been, arbitrarily, privileged w.r.t. their counterparts. This explains why, in general, we prefer to identify $\mathcal{U}^n$ with $\mathcal{B}^n$ instead of $(\overline{\mathcal{B}^n})^c$. This can be achieved very easily by swapping $\mathcal{U}^n$ with $\mathcal{L}^n$ thanks to the reflection w.r.t. $\hat{\mathbb{R}}^{n-1}$, i.e. $\rho_{e_n,0}$.

Clearly, $\sigma_{e_n,\sqrt{2}} \circ \rho_{e_n,0} \in \mathcal{M}(\hat{\mathbb{R}}^n)$ and so it is an isomorphism between $\hat{\mathbb{R}}^n$ and itself. Thus, the transformation $\sigma_{e_n,\sqrt{2}} \circ \rho_{e_n,0}|_{\mathcal{U}^n} : \mathcal{U}^n \xrightarrow{\sim} \mathcal{B}^n$ is an isomorphism between the upper half space and the interior of the unit ball.

Def. 11.5.1 *The Möbius transformation $\eta \equiv \sigma_{e_n,\sqrt{2}} \circ \rho_{e_n,0} \in \mathcal{M}(\hat{\mathbb{R}}^n)$, whose restriction to $\mathcal{U}^n$ allows us to identify $\mathcal{U}^n$ and $\mathcal{B}^n$ is called* ***standard transformation****.*

Def. 11.5.2 *We call $\mathcal{M}(\mathcal{U}^n)$ and $\mathcal{M}(\mathcal{B}^n)$, the set of Möbius transformations stable on the upper-half space and the open unit ball, respectively, i.e.:*

$$\mathcal{M}(\mathcal{U}^n) = \{\phi \in \mathcal{M}(\hat{\mathbb{R}}^n) \ : \ \phi(\mathcal{U}^n) = \mathcal{U}^n\}; \tag{11.19}$$

$$\mathcal{M}(\mathcal{B}^n) = \{\phi \in \mathcal{M}(\hat{\mathbb{R}}^n) \ : \ \phi(\mathcal{B}^n) = \mathcal{B}^n\}. \tag{11.20}$$

It is possible to verify that both of them are subgroups of $\mathcal{M}(\hat{\mathbb{R}}^n)$.

Since $\mathcal{U}^n$ and $\mathcal{B}^n$ are identified through a Möbius transformation, it is possible to define in a natural way, via η, an isomorphism that permits to identify their Möbius subgroups as shown in the following commutative diagram:

$$\begin{tikzcd} \mathcal{U}^n \arrow[r, "\eta"] \arrow[d, "\phi"'] & \mathcal{B}^n \arrow[d, dashed, "\eta\circ\phi\circ\eta^{-1}"] \\ \mathcal{U}^n \arrow[r, "\eta"'] & \mathcal{B}^n. \end{tikzcd}$$

The function

$$\begin{array}{rccc} \iota: & \mathcal{M}(\mathcal{U}^n) & \xrightarrow{\sim} & \mathcal{M}(\mathcal{B}^n) \\ & \phi & \longmapsto & \iota(\phi) := \eta \circ \phi \circ \eta^{-1} \end{array}$$

is clearly an isomorphism of groups.

Let us now focus on the link between $\mathcal{M}(\hat{\mathbb{R}}^n)$ and $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$. In particular, the problem of extending an element of $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$ to the whole $\hat{\mathbb{R}}^n$ is related to the concept defined as follows.

Def. 11.5.3 *Let $t \geqslant 0$, $r > 0$, $a \in \mathbb{R}^{n-1}$, $|a| = 1$, and $\tilde{a} = (a, 0) \in \mathbb{R}^n$. The Poincaré extension $\tilde{\phi} \in \mathcal{M}(\hat{\mathbb{R}}^n)$ of $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$ is defined as follows:*

- *if $\phi = \rho_{a,t}$, then $\tilde{\phi} := \rho_{\tilde{a},t}$;*

- *if* $\phi = \sigma_{a,r}$*, then* $\tilde{\phi} = \sigma_{\tilde{a},r}$*;*

- *if* $\phi = \phi_1 \circ \cdots \circ \phi_m$*, then* $\tilde{\phi} := \tilde{\phi}_1 \circ \ldots \tilde{\phi}_m$*, where* ϕ_i *is a reflection on an inversion of* $\hat{\mathbb{R}}^{n-1}$*,* $\forall i \in \{1, \ldots, m\}$*.*

The following intermediate result will prove to be useful in the sequel.

Lemma 11.5.1 *Let* $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$*, then its Poincaré extension* $\tilde{\phi}$ *is stable on the hyperplane* $\hat{\mathbb{R}}^{n-1} = P(e_n, 0) \cup \{\infty\}$*, i.e.* $\tilde{\phi}(\hat{\mathbb{R}}^{n-1}) = \hat{\mathbb{R}}^{n-1}$*.*

Proof. By definition of Poincaré extension, to prove the statement it is sufficient to prove it for the simple cases of $\phi = \rho_{a,t}$ and $\phi = \sigma_{a,r}$.

- If $\phi = \rho_{a,t}$, then $\tilde{\phi} = \rho_{\tilde{a},t}$, with $\tilde{a} = (a, 0)$. Let us consider the hyperplane $\hat{\mathbb{R}}^{n-1} = P(e_n, 0) \cup \{\infty\}$. Clearly $\rho_{\tilde{a},t}(\infty) = \infty$. Let us consider $x \in P(e_n, 0)$, i.e. $\langle x, e_n\rangle = 0$. By definition $\rho_{\tilde{a},t}(x) = x + 2(t - \langle x, \tilde{a}\rangle)\tilde{a}$, since $\langle \tilde{a}, e_n\rangle = 0$, then

$$\langle \rho_{\tilde{a},t}(x), e_n\rangle = \langle x + 2(t - \langle x, \tilde{a}\rangle)\tilde{a}, e_n\rangle = \langle x, e_n\rangle + 2(t - \langle x, \tilde{a}\rangle)\langle \tilde{a}, e_n\rangle = 0,$$

  hence $\rho_{\tilde{a},t}(x) \in P(e_n, 0)$, so $\hat{\mathbb{R}}^{n-1}$ is globally fixed by $\tilde{\phi}$.

- If $\phi = \sigma_{a,r}$, then $\tilde{\phi} = \sigma_{\tilde{a},r}$, with $\tilde{a} = (a, 0)$ because $\langle \tilde{a}, e_n\rangle = 0$. Let us consider the hyperplane $\hat{\mathbb{R}}^{n-1} = P(e_n, 0) \cup \{\infty\}$. Thence $\sigma_{\tilde{a},t}(\infty) = \tilde{a} \in P(e_n, 0)$. Let us consider $x \in P(e_n, 0)$, i.e. $\langle x, e_n\rangle = 0$. By definition $\sigma_{\tilde{a},r}(x) = \tilde{a} + \frac{r^2}{|x-\tilde{a}|^2}(x - \tilde{a})$, since $\langle \tilde{a}, e_n\rangle = 0$, then

$$\langle \sigma_{\tilde{a},t}(x), e_n\rangle = \langle \tilde{a} + \frac{r^2}{|x - \tilde{a}|^2}(x - \tilde{a}), e_n\rangle = \langle \tilde{a}, e_n\rangle + \frac{r^2}{|x - \tilde{a}|^2}(\langle x, e_n\rangle - \langle \tilde{a}, e_n\rangle) = 0.$$

  hence $\sigma_{\tilde{a},r}(x) \in P(e_n, 0)$, so $\hat{\mathbb{R}}^{n-1}$ is globally fixed by $\tilde{\phi}$. $\square$

The following theorem gives a further link between the two subgroups of $\mathcal{M}(\hat{\mathbb{R}}^n)$, $\mathcal{M}(\mathcal{U}^n)$ or its isomorphic group $\mathcal{M}(\mathcal{B}^n)$, and $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$, through the Poincaré extension.

Theorem 11.5.1 *Let* $\tilde{\phi} \in \mathcal{M}(\hat{\mathbb{R}}^n)$*. Then,* $\tilde{\phi} \in \mathcal{M}(\mathcal{U}^n)$ *if and only if* $\tilde{\phi}$ *is the Poincaré extension of* $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$*.*

Proof.

$\boxed{\Longleftarrow}$: as we did in the previous lemma, it is sufficient to prove the statement for $\tilde{\phi}$ as the Poincaré extension of a reflection $\phi = \rho_{a,t}$, i.e. $\tilde{\phi} = \rho_{\tilde{a},t}$, or an inversion $\phi = \sigma_{a,r}$, i.e. $\tilde{\phi} = \sigma_{\tilde{a},r}$, with $\tilde{a} = (a, 0)$. Let $x \in \mathcal{U}^n$, i.e. $\langle x, e_n\rangle > 0$. Note that $\langle \tilde{a}, e_n\rangle = 0$.

- if $\tilde{\phi} = \rho_{\tilde{a},t}$, then $\rho_{\tilde{a},t}(x) = x + 2(t - \langle x, \tilde{a}\rangle)\tilde{a}$, then

$$\langle \rho_{\tilde{a},t}(x), e_n\rangle = \langle x, e_n\rangle + 2(t - \langle x, \tilde{a}\rangle)\langle \tilde{a}, e_n\rangle = \langle x, e_n\rangle > 0,$$

  thus $\rho_{\tilde{a},t}(x) \in \mathcal{U}^n$;

- if $\tilde{\phi} = \sigma_{\tilde{a},r}$, then $\sigma_{\tilde{a},r}(x) = \tilde{a} + \frac{r^2}{|x-\tilde{a}|^2}(x - \tilde{a})$, then

$$\langle \sigma_{\tilde{a},r}(x), e_n \rangle = \langle \tilde{a}, e_n \rangle + \frac{r^2}{|x - \tilde{a}|^2}(\langle x, e_n \rangle - \langle \tilde{a}, e_n \rangle) = \frac{r^2}{|x - \tilde{a}|^2}\langle x, e_n \rangle > 0,$$

thus $\sigma_{\tilde{a},r}(x) \in \mathcal{U}^n$.

Note that we could repeat an analogous procedure with $\mathcal{L}^n$ instead of $\mathcal{U}^n$, with the opposite inequality, obtaining that, if $\tilde{\phi}$ is the Poincaré extension of $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$, then it preserves also the lower-half space.

$\boxed{\Longrightarrow}$: suppose $\psi \in \mathcal{M}(\mathcal{U}^n) \subset \mathcal{M}(\hat{\mathbb{R}}^n)$, we must prove that it exists $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$ such that $\tilde{\phi} = \psi$. As a natural candidate we consider $\phi \equiv \psi|_{\hat{\mathbb{R}}^{n-1}}$.

First of all, let us check that our candidate is suitable, i.e. that its domain and image are $\hat{\mathbb{R}}^{n-1}$. This is an immediate consequence of the fact that ψ, as a Möbius transformation, is an homeomorphism, so it leaves $\partial\mathcal{U}^n \cong \hat{\mathbb{R}}^{n-1}$ fixed.

Moreover $\phi \equiv \psi|_{\hat{\mathbb{R}}^{n-1}}$ is an homeomorphism on $\hat{\mathbb{R}}^{n-1}$. Moreover, since ψ is a Möbius transformation on $\hat{\mathbb{R}}^n$, ψ preserves the cross ratios in $\hat{\mathbb{R}}^n$, so its restriction ϕ must preserve the cross ratios in $\hat{\mathbb{R}}^{n-1}$, hence, by theorem 11.4.1 $\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$.

The only thing that remains to be done is to prove that $\tilde{\phi}$, the Poincaré extension of ϕ, is ψ, or, analogously, that $\tilde{\phi} \circ \psi^{-1} = id_{\hat{\mathbb{R}}^n}$, in fact, this implies that ψ is the right inverse of $\tilde{\phi}$, which is invertible as an element of $\mathcal{M}(\hat{\mathbb{R}}^n)$, so it coincides with the inverse $\tilde{\phi}^{-1}$. In order to obtain this result, we need two preliminary facts:

1. $\tilde{\phi} \circ \psi^{-1}$ is stable on $\mathcal{U}^n$ as composition of functions that are stable on $\mathcal{U}^n$, indeed $\psi^{-1} \in \mathcal{M}(\mathcal{U}^n)$ by hypothesis and $\tilde{\phi} \in \mathcal{M}(\mathcal{U}^n)$ thanks to the first implication of this theorem;

2. $\tilde{\phi} \circ \psi^{-1}$ fixes $\hat{\mathbb{R}}^{n-1}$ pointwise, i.e. $\tilde{\phi} \circ \psi^{-1}(x) = x$ for all $x \in \hat{\mathbb{R}}^{n-1}$, in fact, by definition $\tilde{\phi}\Big|_{\hat{\mathbb{R}}^{n-1}} = \phi = \psi|_{\hat{\mathbb{R}}^{n-1}}$, so $\tilde{\phi} \circ \psi^{-1}\Big|_{\hat{\mathbb{R}}^{n-1}} = \tilde{\phi}\Big|_{\hat{\mathbb{R}}^{n-1}} \circ \psi^{-1}|_{\hat{\mathbb{R}}^{n-1}} = \psi|_{\hat{\mathbb{R}}^{n-1}} \circ \psi^{-1}|_{\hat{\mathbb{R}}^{n-1}} = id_{\hat{\mathbb{R}}^{n-1}}$.

The result 2. guarantees that we can apply theorem 11.4.5 with $\Sigma = \hat{\mathbb{R}}^{n-1}$. This implies that, either $\tilde{\phi} \circ \psi^{-1} = id_{\hat{\mathbb{R}}^n}$ or $\tilde{\phi} \circ \psi^{-1} = \rho_{e_n,0}$. However, this second option is not possible because, by 1. it is stable on $\mathcal{U}^n$ while $\rho_{e_n,0}(\mathcal{U}^n) = \mathcal{L}^n$, hence $\tilde{\phi} = \psi$. □

An immediate consequence of this last theorem is the following corollary:

Corollary 11.5.1 *$\mathcal{M}(\mathcal{U}^n)$ and $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$ are isomorphic (as subgroups of $\mathcal{M}(\hat{\mathbb{R}}^n)$).*

The isomorphism p between $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$ and $\mathcal{M}(\mathcal{U}^n)$ is given by the Poincaré extension as follows:

$$\begin{array}{rccc} p: & \mathcal{M}(\hat{\mathbb{R}}^{n-1}) & \xrightarrow{\sim} & \mathcal{M}(\mathcal{U}^n) \\ & \phi & \longmapsto & p(\phi) := \tilde{\phi} \end{array}$$

Indeed by the definition of the Poincaré extension it is clear that for all ϕ $\exists!$ Poincaré extension $\tilde{\phi}$ and $\tilde{\phi} \in \mathcal{M}(\mathcal{U}^n)$ because of the first implication we proved in the previous theorem. Moreover, thanks to the second implication of the theorem, for all $\psi \in \mathcal{M}(\mathcal{U}^n)$ $\exists!\phi \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$ such that $\tilde{\phi} = \psi$. By direct computation, it can be proven that p is also a group homomorphism.

Now we are going to analyze the link between $\mathcal{M}(\mathcal{U}^n)$ and $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$ from another perspective which involves angles. This is not surprising as we have already underlined the conformality of between Möbius transformations in subsection 11.4.3.

To proceed gradually we need to introduce the concept of orthogonality between (generalized) spheres in $\hat{\mathbb{R}}^n$.

Def. 11.5.4 *Two spheres Σ_1 and Σ_2 of $\hat{\mathbb{R}}^n$ are said to be orthogonal if $\Sigma_1 \cap \Sigma_2 \in \mathbb{R}^n$ and for all $x \in \Sigma_1 \cap \Sigma_2$ the two normal vectors at x to each sphere are orthogonal.*

The condition $\Sigma_1 \cap \Sigma_2 \in \mathbb{R}^n$ is introduced to guarantee that Σ_1 and Σ_2 actually intersect in at least a point in $\mathbb{R}^n$. The normal vector to a hyperplane has already been defined, while, here, we take as normal vector to a sphere in one of its points any vector that is normal to the tangent space to the sphere in the given point.

Since a generalized sphere in $\hat{\mathbb{R}}^n$ can be either a hyperplane $\cup\{\infty\}$ or a Euclidean sphere, there are three possible scenarios, depicted in Figure 11.6:

1. if $\Sigma_1 = P(a,t) \cup \{\infty\}$ and $\Sigma_2 = P(b,s) \cup \{\infty\}$, then they are orthogonal if and only if a and b are orthogonal vectors;
2. if $\Sigma_1 = P(a,t) \cup \{\infty\}$ and $\Sigma_2 = S^{n-1}_{b,r}$, then they are orthogonal if and only if $b \in P(a,t)$;
3. if $\Sigma_1 = S^{n-1}_{a,r}$ and $\Sigma_2 = S^{n-1}_{b,s}$, then they are orthogonal if and only if $|a-b|^2 = r^2 + s^2$.

Note that, by symmetry, in cases 2. and 3. it is sufficient to check the orthogonality condition of the normal vectors in just one of the two points of intersection between the spheres.

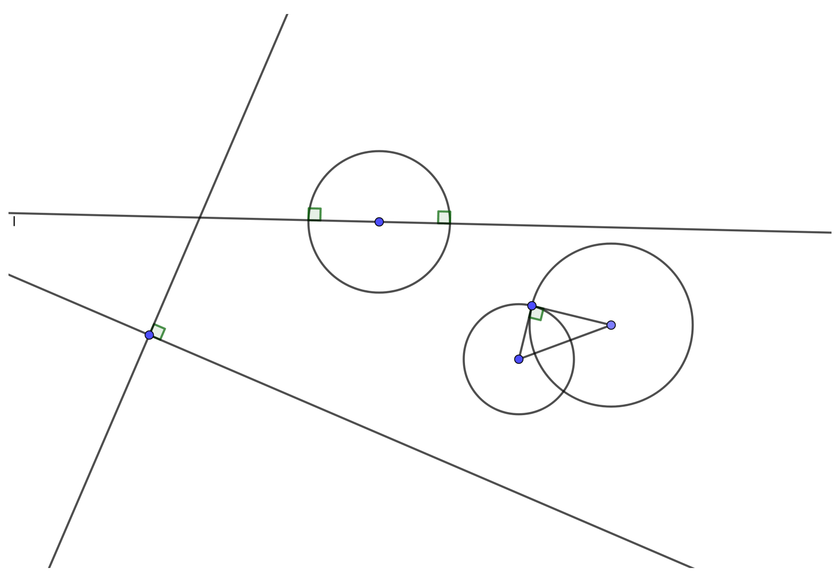


Figure 11.6: the three types of orthogonal spheres in $\hat{\mathbb{R}}^2$.

Theorem 11.5.2 *Let ϕ be a reflection or inversion with respect to a sphere Σ in $\hat{\mathbb{R}}^n$, then $\phi \in \mathcal{M}(\mathcal{U}^n)$ if and only if Σ is orthogonal to $\hat{\mathbb{R}}^{n-1}$.*

Proof. Let us recall that $\hat{\mathbb{R}}^{n-1} \cong P(e_n, 0) \cup \{\infty\}$. We will consider the cases of reflection and inversion separately.

1. Let $\phi = \rho_{\tilde{a},t}$ be the reflection with respect to $\Sigma = P(\tilde{a}, t) \cup \{\infty\}$, then the following chain of equivalent assertions holds:

$$\begin{aligned}
\rho_{\tilde{a},t} \in \mathcal{M}(\mathcal{U}^n) \quad &\overset{\text{th.11.5.1}}{\Longleftrightarrow} \quad \exists \rho_{a,t} \in \mathcal{M}(\hat{\mathbb{R}}^{n-1}) \text{ such that } \tilde{\rho}_{a,t} = \rho_{\tilde{a},t} \text{ and } \tilde{a} = (a, 0) \\
&\Longleftrightarrow \quad \langle \tilde{a}, e_n \rangle = 0 \\
&\Longleftrightarrow \quad \tilde{a} \text{ and } e_n \text{ are orthogonal vectors} \\
&\Longleftrightarrow \quad \Sigma = P(\tilde{a}, t) \cup \{\infty\} \text{ and } \hat{\mathbb{R}}^{n-1} = P(e_n, 0) \cup \{\infty\} \text{ are orthogonal.}
\end{aligned}$$

2. Let $\phi = \sigma_{\tilde{a},r}$ be the inversion with respect to $\Sigma = S^{n-1}_{\tilde{a},r}$, then the following chain of equivalent assertions holds:

$$\begin{aligned}
\sigma_{\tilde{a},r} \in \mathcal{M}(\mathcal{U}^n) \quad &\overset{\text{th.11.5.1}}{\Longleftrightarrow} \quad \exists \sigma_{a,r} \in \mathcal{M}(\hat{\mathbb{R}}^{n-1}) \text{ s.t. } \tilde{\sigma}_{a,r} = \sigma_{\tilde{a},r} \text{ and } \tilde{a} = (a, 0) \\
&\Longleftrightarrow \quad \langle \tilde{a}, e_n \rangle = 0 \\
&\Longleftrightarrow \quad \tilde{a} \in P(e_n, 0) \\
&\Longleftrightarrow \quad \Sigma = S^{n-1}_{\tilde{a},r} \text{ and } \hat{\mathbb{R}}^{n-1} = P(e_n, 0) \cup \{\infty\} \text{ are orthogonal.}
\end{aligned}$$

□

Since every Möbius transformation can be written as the composition of reflections and inversions, a direct consequence of the previous theorem is the following.

Corollary 11.5.2 *Every Möbius transformation $\phi \in \mathcal{M}(\mathcal{U}^n)$ is the composition of reflections and inversions with respect to spheres in $\hat{\mathbb{R}}^n$ which are orthogonal to $\hat{\mathbb{R}}^{n-1}$.*

For the following theorem we need to define the subgroup of $\mathcal{I}(\hat{\mathbb{R}}^n)$ stable on the upper half space:

$$\mathcal{I}(\mathcal{U}^n) = \{\psi \in \mathcal{I}(\hat{\mathbb{R}}^n) \ : \ \psi(\mathcal{U}^n) = \mathcal{U}^n\} = \mathcal{I}(\hat{\mathbb{R}}^n) \cap \mathcal{M}(\mathcal{U}^n). \tag{11.21}$$

Theorem 11.5.3 *Let $\phi \in \mathcal{M}(\mathcal{U}^n)$ such that $\phi(\infty) \neq \infty$. Let Σ be the isometric sphere of ϕ and $\phi = \psi \circ \sigma$ its decomposition (see theorem* 11.4.2*), with σ the reflection or inversion w.r.t. Σ and $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$. Then Σ is orthogonal to $\hat{\mathbb{R}}^{n-1}$ and $\psi \in \mathcal{I}(\mathcal{U}^n)$.*

Proof. From corollary 11.5.2, since $\phi \in \mathcal{M}(\mathcal{U}^n)$ and $\phi = \psi \circ \sigma$, Σ is orthogonal to $\hat{\mathbb{R}}^{n-1}$. What remains to be proven is that $\psi \in \mathcal{I}(\mathcal{U}^n)$. σ is the reflection or inversion with respect to Σ, which is orthogonal to $\hat{\mathbb{R}}^{n-1}$. From theorem 11.5.2 this implies that $\sigma \in \mathcal{M}(\mathcal{U}^n)$. Since $\sigma^{-1} = \sigma$, then $\psi = \sigma \circ \phi$. Now $\sigma \in \mathcal{M}(\mathcal{U}^n)$ and $\phi \in \mathcal{M}(\mathcal{U}^n)$, hence $\psi \in \mathcal{M}(\mathcal{U}^n)$ as composition of elements of $\mathcal{M}(\mathcal{U}^n)$. Moreover $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$, so we can conclude that $\psi \in \mathcal{I}(\mathcal{U}^n)$. □

Now we are going to analyze the properties of the last Möbius subgroup: $\mathcal{M}(\mathcal{B}^n)$.

As we already know it is possible to identify $\mathcal{U}^n$ and $\mathcal{B}^n$ through the standard transformation η defined in 11.5.1 as $\eta = \sigma_{e_n,\sqrt{2}} \circ \rho_{e_n,0}$. Moreover, it permits to define the isomorphism of subgroups ι as follows:

$$\begin{aligned}
\iota : \quad \mathcal{M}(\mathcal{U}^n) \quad &\xrightarrow{\sim} \quad \mathcal{M}(\mathcal{B}^n) \\
\phi \quad &\longmapsto \quad \iota(\phi) := \eta \circ \phi \circ \eta^{-1}.
\end{aligned}$$

Up to now we have analyzed in detail the properties of $\mathcal{M}(\mathcal{U}^n)$. Because of the isomorphism between $\mathcal{M}(\mathcal{U}^n)$ and $\mathcal{M}(\mathcal{B}^n)$ it is reasonable to think that analogous properties should hold for $\mathcal{M}(\mathcal{B}^n)$. This is actually the case and it order to prove it, we will make large use of the isomorphism between the two subgroups.

As we have seen in corollary 11.5.1 the Poincaré extension induces the isomorphism p between $\mathcal{M}(\mathcal{U}^n)$ and $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$.

$$\begin{array}{rccc} p: & \mathcal{M}(\hat{\mathbb{R}}^{n-1}) & \xrightarrow{\sim} & \mathcal{M}(\mathcal{U}^n) \\ & \phi & \longmapsto & p(\phi) := \tilde{\phi}. \end{array}$$

Note that $\hat{\mathbb{R}}^{n-1} = \partial\mathcal{U}^n$, hence the Poincaré extension gives a correspondence between the Möbius subgroup of $\mathcal{U}^n$ and the Möbius subgroup of its border $\partial\mathcal{U}^n$. Analogously we would like to define a Poincaré extension which links the Möbius subgroups associated to $\mathcal{B}^n$ and its border $\partial\mathcal{B}^n = S^{n-1}$, respectively.

First of all we need to identify $\mathcal{M}(S^{n-1})$. For that, we need to search for an analogous version of p for $\mathcal{B}^n$, to do that we will clearly make use of p, which connects $\mathcal{M}(\mathcal{U}^n)$ with $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$. Thence it is important to define $\mathcal{M}(S^{n-1})$ as something related to $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$.

Before giving this definition let us recall that the extended stereographic projection $\hat{\pi} : \hat{\mathbb{R}}^{n-1} \longrightarrow S^{n-1}$ maps bijectively $\hat{\mathbb{R}}^{n-1}$ onto S^{n-1}, $\hat{\pi}(\hat{\mathbb{R}}^{n-1}) = S^{n-1}$ and $\hat{\pi}^{-1}(S^{n-1}) = \mathbb{R}^{n-1}$.

Now we can define the Möbius group of S^{n-1} as follows:

$$\mathcal{M}(S^{n-1}) = \left\{\phi : S^n \to S^n \text{ such that } \hat{\pi}^{-1} \circ \phi \circ \hat{\pi} \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})\right\}, \tag{11.22}$$

the following commutative diagram visualizes the action of such Möbius transformations:

$$\begin{tikzcd} S^{n-1} \arrow[d, "\phi"'] & \hat{\mathbb{R}}^{n-1} \arrow[l, "\hat{\pi}"'] \arrow[d, dashed, "\hat{\pi}^{-1}\circ\phi\circ\hat{\pi}"] \\ S^{n-1} \arrow[r, "\hat{\pi}^{-1}"'] & \hat{\mathbb{R}}^{n-1}. \end{tikzcd}$$

Clearly $\hat{\pi}$ allows us to define the group isomorphism μ between $\mathcal{M}(\hat{\mathbb{R}}^{n-1})$ and $\mathcal{M}(S^{n-1})$ as follows:

$$\begin{array}{rccc} \mu: & \mathcal{M}(S^{n-1}) & \xrightarrow{\sim} & \mathcal{M}(\hat{\mathbb{R}}^{n-1}) \\ & \phi & \longmapsto & \mu(\phi) := \hat{\pi}^{-1} \circ \phi \circ \hat{\pi}. \end{array}$$

The definition of the Poincaré extension p' for the elements of $\mathcal{M}(S^{n-1})$ to elements of $\mathcal{M}(\mathcal{B}^n)$ is given below.

Def. 11.5.5 *Let $\phi \in \mathcal{M}(S^{n-1})$, let $\psi = \hat{\pi}^{-1} \circ \phi \circ \hat{\pi} = \mu^{-1}(\phi) \in \mathcal{M}(\hat{\mathbb{R}}^{n-1})$ and let $\tilde{\psi} \in \mathcal{M}(\mathcal{U}^n)$ be the Poincaré extension of ψ, $\tilde{\psi} = p(\psi)$. We define the Poincaré extension of ϕ as $\tilde{\phi} \equiv p'(\phi) = \eta \circ \tilde{\psi} \circ \eta^{-1} = \iota \circ p \circ \mu^{-1}(\phi) \in \mathcal{M}(\mathcal{B}^n)$.*

The following commutative diagram visualizes the action of p':

$$\begin{tikzcd} \mathcal{M}(\hat{\mathbb{R}}^{n-1}) \arrow[r, "\mu"] \arrow[d, "p"'] & \mathcal{M}(S^{n-1}) \arrow[d, dashed, "p'"] \\ \mathcal{M}(\mathcal{U}^n) \arrow[r, "\iota"'] & \mathcal{M}(\mathcal{B}^n). \end{tikzcd}$$

Two immediate consequences of this definition are the analogous versions of theorem 11.5.1 and corollary 11.5.1, that can be proven analogously.

Theorem 11.5.4 *$\phi \in \mathcal{M}(\mathcal{B}^n)$ if and only if ϕ is the Poincaré extension of an element of $\mathcal{M}(S^{n-1})$.*

Corollary 11.5.3 *The Poincaré extension p' is an isomorphism between the Möbius groups $\mathcal{M}(S^{n-1})$ and $\mathcal{M}(\mathcal{B}^n)$.*

We will now analyze the analogous version of theorem 11.5.2.

Theorem 11.5.5 *Let ϕ be a reflection or inversion w.r.t. a sphere Σ in $\hat{\mathbb{R}}^n$, then $\phi \in \mathcal{M}(\mathcal{B}^n)$ if and only if Σ is orthogonal to S^{n-1}.*

Proof. Since $\phi \in \mathcal{M}(\mathcal{B}^n)$, let us consider $\psi = \iota^{-1}(\phi) = \eta^{-1} \circ \phi \circ \eta \in \mathcal{M}(\mathcal{U}^n)$. Let us call $\Sigma' = \eta^{-1}(\Sigma)$, after a straightforward computation it is immediate to verify that ψ fixes Σ' pointwise. Moreover $\psi \neq id_{\hat{\mathbb{R}}^n}$, indeed if $\psi = id_{\hat{\mathbb{R}}^n} = \eta^{-1} \circ \phi \circ \eta$, then $\phi = id_{\hat{\mathbb{R}}^n}$, which is false. Theorem 11.4.5 allows us to conclude that ψ is the reflection or inversion w.r.t. Σ'.

Moreover, because of theorem 11.5.2, $\psi \in \mathcal{M}(\mathcal{U}^n)$ if and only if Σ' is orthogonal to $\hat{\mathbb{R}}^{n-1}$. Note that if we apply η to both Σ' and $\hat{\mathbb{R}}^{n-1}$ we obtain $\eta(\Sigma') = \Sigma$ and $\eta(\hat{\mathbb{R}}^{n-1}) = \sigma_{e_n,\sqrt{2}} \circ \rho_{e_n,0}(\hat{\mathbb{R}}^{n-1}) = \sigma_{e_n,\sqrt{2}}(\hat{\mathbb{R}}^{n-1}) = S^{n-1}$.

By corollary 11.4.3, $\eta \in \mathcal{M}(\hat{\mathbb{R}}^n)$ is conformal, thence it preserves angles, and so, in particular, it preserves orthogonality. Finally we can conclude that Σ' is orthogonal to $\hat{\mathbb{R}}^{n-1}$ if and only if Σ is orthogonal to S^{n-1}. $\square$

The following corollary is the analogous version for $\mathcal{B}^n$ of 11.5.2.

Corollary 11.5.4 *Every Möbius transformation in $\mathcal{M}(\mathcal{B}^n)$ is the composition of reflections and inversions w.r.t. spheres of $\hat{\mathbb{R}}^n$ which are orthogonal to S^{n-1}.*

We will now analyze a similar result to 11.5.3.

Theorem 11.5.6 *Let $\phi \in \mathcal{M}(\mathcal{B}^n)$, then:*

1. *if $\phi(\infty) = \infty$, then $\phi \in \mathrm{O}(n) = \mathcal{I}(\mathcal{B}^n)$;*

2. *if $\phi(\infty) \neq \infty$, let Σ be its isometric sphere and let $\phi = \psi \circ \sigma$ be its decomposition[5], with $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$ and σ the inversion w.r.t. Σ, then Σ is orthogonal to S^{n-1} and $\psi \in \mathrm{O}(n) = \mathcal{I}(\mathcal{B}^n)$.*

Proof.

1. Let us consider the case $\phi(\infty) = \infty$. By point 1. of theorem 11.4.2, $\phi \in \mathcal{S}(\hat{\mathbb{R}}^n)$ hence it can be written as $\phi(x) = b + kAx$, with $k > 0$, $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. Notice that, since $\phi \in \mathcal{M}(\mathcal{B}^n)$, the vector b should belong to $\mathcal{B}^n$, hence $|b| < 1$. Indeed if $|b| \geqslant 1$, then $\phi(0) = b \notin \mathcal{B}^n$, but $0 \in \mathcal{B}^n$ and this is contradictory with the hypothesis $\phi \in \mathcal{M}(\mathcal{B}^n)$.

   Let us suppose $b \neq 0$. Clearly $\eta^{-1} \circ \phi \circ \eta \in \mathcal{M}(\mathcal{U}^n)$. Theorem 11.5.1 and lemma 11.5.1 allow us to say that $\eta^{-1} \circ \phi \circ \eta$ is stable on $\hat{\mathbb{R}}^{n-1}$, i.e. $\eta^{-1}(\phi(\eta(\hat{\mathbb{R}}^{n-1}))) = \hat{\mathbb{R}}^{n-1}$, hence

[5] given by 2. in theorem 11.4.2.

$\phi(\eta(\hat{\mathbb{R}}^{n-1})) = \eta(\hat{\mathbb{R}}^{n-1})$, but $\eta(\hat{\mathbb{R}}^{n-1}) = S^{n-1}$, this means that $\phi(S^{n-1}) = S^{n-1}$, i.e. ϕ is stable on S^{n-1}. Since we have supposed $b \neq 0$, we can define $\tilde{b} = A^t \frac{b}{|b|}$. It is easy to verify that $|\tilde{b}| = 1$, thus $\tilde{b} \in S^{n-1}$.

Now since ϕ is stable on S^{n-1} we must have that $|\phi(\tilde{b})| = 1$. Explicitly:

$$|\phi(\tilde{b})| = \left| b + k\frac{b}{|b|} \right| = ||b| + k| = |b| + k = 1, \tag{11.23}$$

which implies that $k = 1 - |b|$, which is positive because $b \in \mathcal{B}^n$.

Clearly also $-\tilde{b} \in S^{n-1}$, i.e. $|-\tilde{b}| = 1$ and $|\phi(-\tilde{b})| = 1$. Developing the computation and using the fact that $k = 1 - |b|$ we obtain:

$$|\phi(-\tilde{b})| = \left| b - k\frac{b}{|b|} \right| = ||b| - k| = |2|b| - 1| = 1. \tag{11.24}$$

Hence $|b| = 0$ or $|b| = 1$ which is contradictory because we assumed that $b \neq 0$ and $|b| < 0$.

This means that $b = 0$, thus $\phi(x) = kAx$. Since ϕ is stable on S^{n-1}, let us consider $x \in S^{n-1}$, $|x| = 1$ and $|\phi(x)| = 1$, but $1 = |\phi(x)| = k|Ax| = k|x| = k$, hence $k = 1$ and $\phi = A \in \mathrm{O}(n) = \mathcal{I}(S^{n-1})$.

2. Let us consider the case $\phi(\infty) \neq \infty$. Let $a = \phi^{-1}(\infty) \in \mathbb{R}^n$, using the decomposition given by point 2. in theorem 11.4.2 $\phi = \psi \circ \sigma$ we have that $\phi(a) = \psi(\sigma(a)) = \infty$, so $\sigma(a) = \psi^{-1}(\infty) = \infty$, hence $\sigma(a) = \infty$. This implies that a is the center of the isometric sphere $\Sigma = S^{n-1}_{a,r}$ and $\sigma = \sigma_{a,r}$. Moreover by corollary 11.5.4, since $\phi \in \mathcal{M}(\mathcal{B}^n)$ the spheres $\Sigma = S^{n-1}_{a,r}$ and S^{n-1} are orthogonal, hence r is such that $|a|^2 = r^2 + 1$. Now, by theorem 11.5.5 we know that $\sigma \in \mathcal{M}(\mathcal{B}^n)$, moreover $\phi \in \mathcal{M}(\mathcal{B}^n)$, hence $\psi \in \mathcal{M}(\mathcal{B}^n)$, but also $\psi \in \mathcal{I}(\hat{\mathbb{R}}^n)$, so $\psi \in \mathcal{M}(\mathcal{B}^n) \cap \mathcal{I}(\hat{\mathbb{R}}^n) = \mathcal{I}(\mathcal{B}^n) = \mathrm{O}(n)$.

□

A direct consequence of the previous theorem is the following corollary.

Corollary 11.5.5 *Let $\phi \in \mathcal{M}(\mathcal{B}^n)$, then $\phi(0) = 0$ if and only if $\phi \in \mathrm{O}(n)$.*

Proof. If $\phi(\infty) = \infty$ then $\phi \in \mathrm{O}(n)$ because of point 1. in the previous theorem.

Let us consider the case of $\phi(\infty) \neq \infty$. Because of the previous theorem we have the decomposition $\phi = \psi \circ \sigma$, with $\psi \in \mathrm{O}(n)$ and σ the inversion w.r.t. the sphere $S^{n-1}_{a,r}$, with $r^2 = |a|^2 - 1$. The condition $\phi(0) = 0$ corresponds to $\phi(0) = \psi(\sigma(0)) = 0$, but, since $\psi \in \mathrm{O}(n)$, $\psi(0) = 0$, thence the previous condition is equivalent to $\sigma(0) = 0$. Now, because of property 1. in theorem 11.2.2, this means that $0 \in S^{n-1}_{a,r}$, hence $|0 - a| = |a| = r$, but $r^2 = |a|^2 - 1$, so $|a|^2 = |a|^2 - 1$ which gives a contradiction. Hence, $\phi \in \mathrm{O}(n)$ if and only if $\phi(0) = 0$. □

Chapter 12

The hyperbolic models (Antoine Guennec)

Around 300 B.C, Euclid wrote his famous 'Elements' [7], a thirteen-volume work where he presented the fundamentals of Greek geometry and number theory. In the first pages, he exposes his five postulates of planar geometry:

1. *'Let it have been postulated to draw a straight-line from any point to any point'*
2. *'And to produce a finite straight-line continuously in a straight-line'*
3. *'And to draw a circle with any center and radius'*
4. *'And that all right-angles are equal to one another'*
5. *'And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side)'.*

This last postulate is best known as the parallel postulate and it is equivalent to Playfair's axiom when combined with the first four axioms:

> *'In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point'*

For over two thousand years, mathematicians have tried to simplify Euclid's axioms of geometry, by proving the fifth axiom from the first four (known as the fifth postulate problem), but without success. However, in the 19th century, things took a surprising turn when mathematicians discovered that in fact the fifth axiom was *independent* from the first four while trying to prove the fifth postulate problem by contradiction by denying the fifth axiom. To the general astonishment of mathematicians at the time, *geometries that refute the fifth axiom (while keeping the first four), turned out to be highly consistent.*

The geometries that reject some of Euclid's postulates are fittingly designated as non-Euclidean geometries. Hyperbolic geometry is a non-Euclidean geometry where we keep the first four postulates and we refute the fifth postulate by replacing it with the following:

> *'In a plane, given a line and a point not on it, there are infinitely many lines parallel to the given line that can be drawn through the point.'*

Funnily enough, while Gauss is thought to be the one of the first mathematicians to have worked on hyperbolic geometry, he never published anything about it out of fear of the '*uproar of the Boeotians*' (1829, letter from Gauss to W. Bessel), to the extend that Gauss' visionary work on non-Euclidean geometry was only found among his papers after his death in 1855.

While the first publications on hyperbolic geometry were independently given by Nikolai Lobachevsky and János Bolyai in 1829 and 1832 respectively, it was only during the second half of the 19th century that hyperbolic geometry was fully developed by mathematicians such as Poincaré and Hilbert, with the culmination point being at the start of the 20th century with Einstein's groundbreaking use of hyperbolic geometry in his formulation of special relativity, thus showing that hyperbolic geometry was not just meant to be left in the dark cupboards of the mathematics department. More recently, hyperbolic geometry has made a come back with its use in artificial intelligence and information processing, such as in [13]or [3] which make a nice use of Poincaré's and Klein's disk embedding, respectively).

12.1 A brief overview on the four models of the hyperbolic n-space

The hyperbolic space n-space The hyperbolic space of dimension n OR The hyperbolic n-space, in contrast to S^n and $\mathbb{R}^n$, can be described in various different ways. In what follows we will by giving show the four main models that are prevalent in literature: $\mathcal{H}^n$ the hyperboloid, $\mathcal{B}^n$ the conformal ball model (also known as Poincaré disk), $\mathcal{U}^n$ the conformal upper half plane, and $\mathcal{K}^n$, the projective model (also known as Klein disk). Up to an isomorphism, for every $n \geqslant 2$, there exists a unique complete and simply connected hyperbolic manifold of dimension n. Hence, every hyperbolic models that we shall present will be isomorphic to each other. Hence all the hyperbolic models are isomorphic.

Once we have our geometric model embedded in $\mathbb{R}^n$, if we wish to refute Euclid's fifth postulate we have two choices: either straight lines are distorted (conformal model) or angles are (projective model), but not both options together, otherwise we come back to the usual Euclidean space forse questa frase va tolta perché a questo punto non è giustificata. These modifications affect the hyperbolic space in such a way that the quickest path between points is often curved compared to Euclidean geometry. Questa frase forse la toglierei perché ad esempio nel modello di Klein le geodetiche sono rette i.e. non sono distorte.

This section has the only purpose to give the reader a glimpse of the four models and a general idea of the path that we will be taking after this introduction, represented in the diagram below. Many tools will be needed to be introduced before we reach our objectives.

$$\begin{tikzcd}
 & \text{Poincaré disk} \arrow[r] & \text{Upper-half space} & \text{(Conformal model)} \\
\text{Hyperboloid} \arrow[ur] \arrow[dr] & & & \\
 & \text{Cayley-Klein model} & & \text{(Projective model)}
\end{tikzcd}$$

12.1.1 The Hyperboloid $\mathcal{H}^n$ in a nutshell

We start with what we will use as the basis of our hyperbolic n-spaces: the hyperboloid $\mathcal{H}^n$ represented in Figure 12.1. Embedded in the Lorentzian n-space $\mathbb{R}^{n,1}$, the hyperboloid is defined as upper sheet of the set of time-like vectors of Lorentz (bisogna dire quale norma) norm -1:

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}.$$

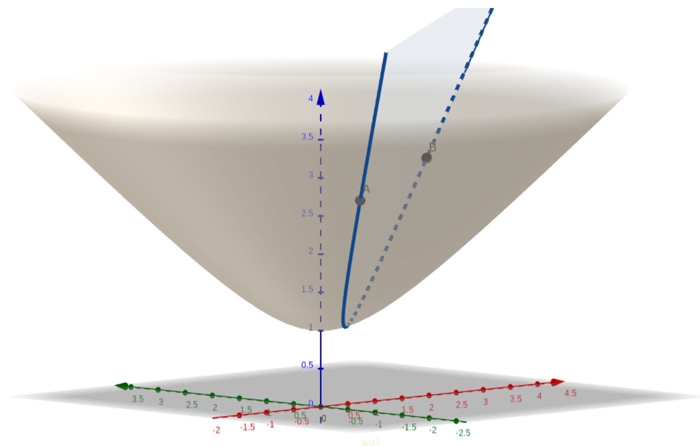


Figure 12.1: The Hyperboloid model: the line between A and B is distorted compared with the usual Euclidean straight lines. In fact, usual Euclidean lines (with the Euclidean metric) are longer than hyperbolic lines.Toglierei questa frase perché qui è detta troppo velocemente (le straight lines in che spazio sono?/le hyperbolic lines con che metrica le misuri per dire che sono più lunghe?) e il lettore a questo punto non ha abbastanza strumenti per capire questa frase.

Clearly, the map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathcal{H}^n \subset \mathbb{R}^{n+1} \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \longmapsto & \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \sqrt{1 + x_1^2 + \cdots + x_n^2} \end{pmatrix} \end{array}$$

is a bijection and so $\mathcal{H}^n$ is indeed n-dimensional space. The hyperbolic distance between two points $x, y \in \mathcal{H}^n$ is then defined using the hyperbolic cosine:

$$\cosh(d_{\mathcal{H}}(x, y)) = -x_1 y_1 - \cdots - x_n y_n + x_{n+1} y_{n+1} = -x \circ y.$$

Note that here we used the second definition given in **??** with the minus sign on the list coordinate instead of the first one.

Geodesic lines (lines that minimize the distance and are of constant speed lines of constant speed that minimize the distance) will be shown to be of the form:

$$\gamma(t) = \cosh(t)x + \sinh(t)y,$$

where t is the path's parameter and x, y is the initial condition.

Moreover the isometry group $\mathcal{I}(\mathcal{H}^n)$ will be proven to be $\mathrm{PO}(n,1)$, the positive Lorentz group. Finally, for a curve $\gamma : [a,b] \to \mathcal{H}^n$ the hyperbolic arc length along the hyperboloid is will be shown to be:

$$\|\gamma\| = \int_a^b \|\gamma'(t)\| = \int_\gamma (dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2)^{\frac{1}{2}}.$$

12.1.2 The conformal models $\mathcal{B}^n$ and $\mathcal{U}^n$ in a nutshell

What is important to remember here is the fact that *conformal* is equivalent to '*angles are maintained*'. A conformal transformation maintains the angle between two curves in the space (a rotation or translation for example rotations and translations are classic examples of conformal transformations) and a conformal model is a hyperbolic geometry model that maintains the same notion of Euclidean angles than the usual Euclidean geometry.

We will be presented with two analogous models: We are going to introduce now two analogous models: the open unit ball $\mathcal{B}^n$ (also said Poincaré disk in the 2-dimensional case $\mathcal{B}^2$) and the upper-half space $\mathcal{U}^n$.

$$\mathcal{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}, \qquad \mathcal{U}^n = \{x \in \mathbb{R}^n : x_n > 0\}.$$

The two models are very similar, one is found from the other by an homeomorphic and inversive transformation (which is also conformal!) through a Möbius transformation (which are conformal as we have seen in corollary 11.4.3) as so:

$$\eta(x) = \sigma_{e_n,\sqrt{2}}(\rho_{e_n,0}(x)) = e_n + \frac{2}{|Jx - e_n|^2}(Jx - e_n), \qquad \text{where } J = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence the two models will be shown to have isomorphic isometric groups $\mathcal{I}(\mathcal{B}^n) \simeq \mathcal{I}(\mathcal{U}^n) \simeq \mathcal{M}(\hat{E}^{n-1})$, where $\mathcal{M}(\hat{E}^k)$ is the set of Möbius transformation defined in on a k-dimension Euclidean space. From section ?? of the previous chapter we already know that $\mathcal{M}(\mathcal{U}^n) \cong \mathcal{M}(\mathcal{B}^n) \cong \mathcal{M}(\hat{\mathbb{R}}^{n-1})$, here we will show that this isomorphism preserves also the metric structure of the different models involved. The metric given to the hyperbolic conformal ball model will be inherited from the Hyperboloid model by setting the projection ζ from $\mathcal{B}^n$ to $\mathcal{H}^n$ (see figure 12.2) to be an isometry (in other words, we set $d_{\mathcal{B}}(x,y) = d_{\mathcal{H}}(\zeta(x),\zeta(y))$ and thus the metric the hyperbolic metric on $\mathcal{B}^n$ is defined as: $\mathcal{B}^n$ inherits its metric from $\mathcal{H}^n$ through the projection ζ from $\mathcal{B}^n$ to $\mathcal{H}^n$, as depicted in figure 12.2, i.e. we define the metric $d_{\mathcal{B}}$ on $\mathcal{B}^n$ in a way such that ζ is an isometry, i.e. $d_{\mathcal{B}}(x,y) = d_{\mathcal{H}}(\zeta(x),\zeta(y))$ hence:

$$\cosh(d_{\mathcal{B}}(x,y)) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}.$$

A questo punto il lettore non può capire la formula, perché non conosce la funzione ζ che non è stata ancora definita analiticamente.

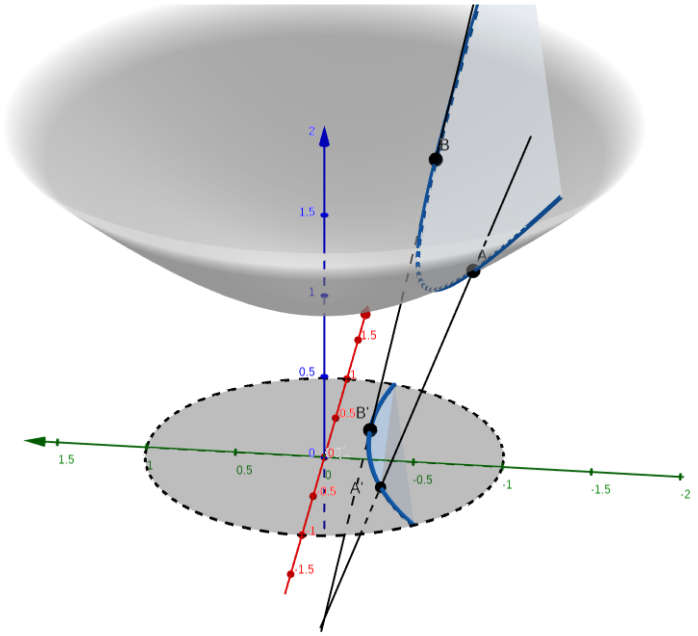


Figure 12.2: Illustration of the isometry between $\mathcal{H}^n$ and $\mathcal{B}^n$. Hyperbolic lines are transformed into arcs of Euclidean circles othogonal to S^{n-1} or diameter line of S^{n-1}. Illustration of the isometry between $\mathcal{H}^n$ and $\mathcal{B}^n$. Geodesics on $\mathcal{H}^n$ are transformed into arcs of Euclidean circles othogonal to S^{n-1} or diameters of S^{n-1}.

Moreover, hyperbolic lines geodesics in the a conformal model will be arcs of Euclidean circles and lines orthogonal to the boundary of the model (S^{n-1} in the case of $\mathcal{B}^n$ and $\mathbb{R}^{n-1} \simeq \{x \in \mathbb{R}^n : x_n = 0\}$ in the case of $\mathcal{U}^n$). See figure 12.3 for a two-dimensional depiction.

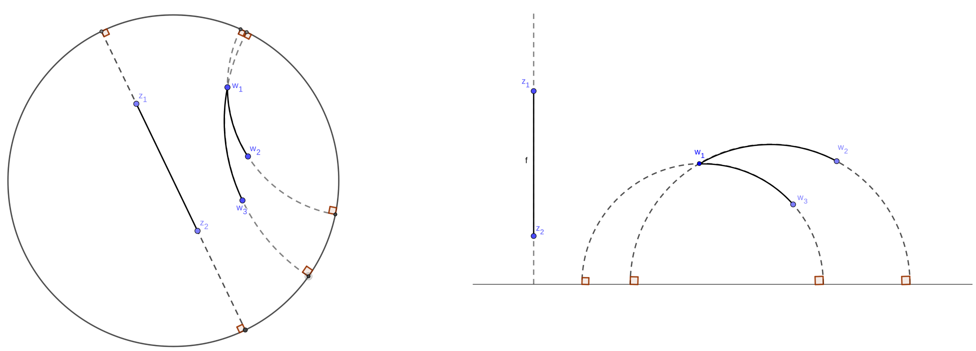


Figure 12.3: The Poincaré disc (left) is just the 2 dimensional instance of the conformal ball model. Here we see that lines that minimizes minimizing the distances are either diameters of the circle or arc of circle, orthogonal to the border S^{n-1}. In a similar way, in the two-dimensional upper-half space $\mathcal{U}^\in$, i.e. the upper-half plane (right) the lines that minimize distance are either straight vertical lines or arcs of a half circle semicircle.

12.1.3 The projective model $\mathcal{K}^n$ in a nutshell

The projective model lies between the hyperbolic model and the conformal ball model: geometrically, it is the unit ball, however, lines are not distorted when compared with the usual Euclidean lines, but angles are. While the metric on the projective model is less easy to work with, it has the advantage that its concept can be extended to any open convex sets via its cross ratio formulation (Hilbert's metric). In a 2-dimensional space, this model is often referenced as the Beltrami-Cayley-Klein model ($\mathcal{K}$ actually stands for Klein).

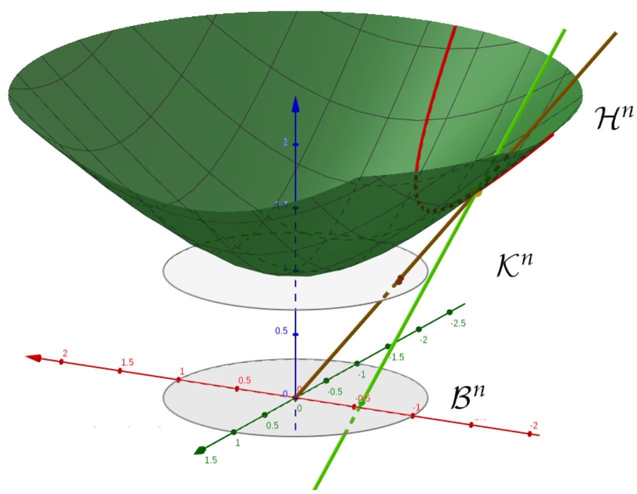


Figure 12.4: The isomorphism of $\mathcal{H}^n$ onto $\mathcal{K}^n$ versus the isomorphism of $\mathcal{H}^n$ onto $\mathcal{B}^n$

When compared with the conformal ball model, the bijective projection of $\mathcal{H}^n$ onto $\mathcal{K}^n$ (see figure 12.4) seems much more natural. In fact if we look back at theorem 9.4.1, the correspondence between the unit ball and the lines passing through zero and with a time-like orientation vector is exactly the isomorphism $\mathcal{H}^n \to \mathcal{K}^n$ given by

$$\begin{array}{ccc} \mathcal{H}^n & \longrightarrow & \mathcal{K}^n \\ \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} & \longmapsto & \frac{1}{x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \end{array}$$

Lines joining x and y in the Hyperboloid are of the form

$$L_{x,y} = \operatorname{span}(x, y) \cap \mathcal{H}^n.$$

It is easy to see that if $\tilde{x}$ and $\tilde{y}$ are the projection on $\mathcal{K}^n$ of x and y then $\operatorname{span}(\tilde{x}, \tilde{y}) = \operatorname{span}(x, y)$ and so the line $L_{x,y}$ once projected in $\mathcal{K}^n$ is

$$L_{\tilde{x},\tilde{y}} = \operatorname{span}(x, y) \cap \mathcal{K}^n \simeq \operatorname{span}(x, y) \cap \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R}^n \right\},$$

so it is a Euclidean line once extended. Hence lines in the projective model are the Euclidean lines. Just like in the conformal model the metric is derived by requiring the bijection $\mathcal{K}^n \to \mathcal{H}^n$

to be an isometry, i.e $d_{\mathcal{K}}(x, y) = d_{\mathcal{H}}(\mu(x), \mu(y))$ and so the projective model metric can be expressed using the hyperbolic cosine:

$$\cosh(d_{\mathcal{K}}(x, y)) = \frac{1 - \langle x, y\rangle}{\sqrt{(1 - |x|^2)}\sqrt{1 - |y|^2}}.$$

12.2 The hyperboloid model and the hyperbolic metric

In this section we are going to define and analyze in detail the first model of hyperbolic geometry: the hyperboloid. A quick recap about the concept of distance and angle in the Euclidean setting will help us underlying similarities and differences between spherical and hyperbolic geometry.

12.2.1 Memories of spherical geometry

The classical way of introducing the concept of angle and spherical distance is based on the Cauchy-Schwarz inequality (lemma 10.1.1). In fact, as a direct consequence, we have that for all $x, y \in \mathbb{R}^n$, there is a number $\alpha(x, y) \in [-1, 1]$ such that

$$\langle x, y\rangle = \alpha(x, y) \|x\| \|y\|,$$

which, for non-zero vectors, satisfies the following properties: $\alpha(x, y) = 0$ if and only if x and y are orthogonal and $\alpha(x, y) = \pm 1$ if and only is x and y are linearly dependent. Being $\cos|_{[0,\pi]}$ a bijective function between $[0, \pi]$ and $[-1, 1]$, with $\cos(0) = 1$, $\cos(\pi/2) = 0$ and $\cos(\pi) = -1$, we have the identification:

$$\alpha(x, y) = \cos(\theta(x, y)),$$

where $\theta(x, y) \in [0, \pi]$ is *defined* to be the **angle** between x and y. $\theta(x, y)$ is related to the so-called spherical distance between two vectors, that we recall next.

Def. 12.2.1 *The spherical distance $d_S(x, y)$ between two vectors $x, y \in \mathbb{R}^n$ is the angle between the projections of x and y on the unit sphere S^{n-1}.*

It follows that $\theta(x, y)$ and $d_S(x, y)$ are identical when $\|x\| = \|y\| = 1$, which implies the equation

$$\cos(d_S(x, y)) = \langle x, y\rangle \iff d_S(x, y) = \arccos(\langle x, y\rangle) \tag{12.1}$$

and, since $\cos^2(d_S(x, y)) + \sin^2(d_S(x, y)) = 1$, $\sin(d_S(x, y)) = \sqrt{1 - \langle x, y\rangle^2}$, where only the positive determination of the square root makes sens here because $d_S(x, y)$ has been defined as the angle between x, y, which belongs to $[0, \pi]$, so $\sin(d_S(x, y)) \geqslant 0$.

The straight lines on the sphere through $x, y \in S^{n-1}$ is

$$\ell_{x,y} = \mathrm{span}(x, y) \cap S^{n-1},$$

and the shortest (geodesic) arc between $x, y \in S^{n-1}$ has the expression

$$\gamma(t) = \cos(t)x + \frac{\sin(t)}{\sqrt{1 - \langle x, y\rangle^2}}\Big(y - \langle x, y\rangle x\Big), \qquad t \in [0, d_S(x, y)],$$

notice that $\gamma(0) = x$ and

$$\begin{aligned}\gamma(d_S(x,y)) &= \cos(d_S(x,y))x + \frac{\sin(d_S(x,y))}{\sqrt{1-\langle x,y\rangle^2}}\Big(y - \langle x,y\rangle\, x\Big)\\ &= \langle x,y\rangle\, x + \frac{\sqrt{1-\langle x,y\rangle^2}}{\sqrt{1-\langle x,y\rangle^2}}\Big(y - \langle x,y\rangle\, x\Big)\\ &= y,\end{aligned}$$

or

$$\gamma(t) = \cos(t)x + \sin(t)y, \qquad t \in [0, d_S(x,y)],$$

if x and y are orthogonal.

We also remark that sine and cosine are the only functions verifying

$$1 = \cos(t)^2 + \sin(t)^2 = \left\| \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right\|_E^2$$

and

$$S^1 = \left\{ \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \in \mathbb{R}^2 \ : \ t \in \mathbb{R} \right\}.$$

The hyperboloid model that we will analyze now will show analogous features, the major difference being represented by the fact that the circular functions sine and cosine must be replaced by the their hyperbolic counterparts:

$$\cosh(t) = \frac{e^x + e^{-x}}{2}, \ \sinh(t) = \frac{e^x - e^{-x}}{2},$$

which, for all $t \in \mathbb{R}$, satisfy

$$\left\| \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix} \right\|_E^2 = \cosh^2(t) - \sinh^2(t) = 1$$

and

$$\left\| \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix} \right\|^2 = -\cosh^2(t) + \sinh^2(t) = -1.$$

12.2.2 The hyperboloid model and its metric

We have just seen how we can build a distance on the sphere from the scalar product and the cosine function. Here we follow exactly the same path by replacing the unit sphere with the unit hyperboloid, i.e. the one defined by $q(x) = -1$ and the cosine by the hyperbolic cosine.

Def. 12.2.2 *The hyperboloid model of hyperbolic geometry is defined as the upper connected part of the level set defined by $q(x) = -1$ in $\mathbb{R}^{1,n}$, explicitly,*

$$\mathcal{H}^n = \{x \in \mathbb{R}^{1,n} : \|x\|^2 = -1, \ x_1 > 0\}.$$

The hyperboloid model can thus be described also as **the set of all unit positive time-like vectors** in $\mathbb{R}^{1,n}$. The analysis of the hyperboloid model starts with a variation of the Cauchy-Schwarz inequality specific for positive time-like vectors.

Theorem 12.2.1 *Let $x, y \in \mathbb{R}^{1,n}$ be two positive time-like vectors. Then:*

$$x \circ y \leqslant \|x\| \, \|y\| \, ,$$

with equality if and only if x and y are linearly dependent.

Proof. Set $t := |\,\|x\|\,| > 0$ ($\|x\|^2 = -t^2$) and because $x \in \mathrm{span}(x)$ is a one-dimensional time-like vector subspace of $\mathbb{R}^{1,n}$, by theorem 10.3.4 there exists $\phi \in \mathrm{PO}(1,n)$ such that $\phi(\mathrm{span}(x)) = \mathrm{span}(e_1)$ and consequently we have $\phi(x) = te_1$. Set $z \equiv \begin{pmatrix} z_1 \\ \bar{z} \end{pmatrix} := \phi(y)$. Then,

$$\begin{aligned} \|x\|^2 \, \|y\|^2 &= \|\phi(x)\|^2 \, \|\phi(y)\|^2 = -t^2(-z_1^2 + |\bar{z}|) = t^2 z_1^2 - t^2 |\bar{z}| \\ &\leqslant t^2 z_1^2 = (te_1 \circ z)^2 = (\phi(x) \circ \phi(y))^2 = (x \circ y)^2, \end{aligned}$$

thus $\|x\|^2 \, \|y\|^2 \leqslant (x \circ y)^2$. Notice that the equality $\|x\|^2 \, \|y\|^2 = (x \circ y)^2$ holds if and only if $\bar{z} = 0$, which implies $\phi(y) \in \mathrm{span}(e_1)$ and, since the action of $\mathrm{PO}(1,n)$ on time-like vector subspaces is stable, $y \in \mathrm{span}(x)$, i.e. x and y are linearly dependent.

Finally, theorem 10.2.1 guarantees that $x \circ y < 0$, hence $(\|x\| \, \|y\|)^2 = \|x\|^2 \, \|y\|^2 \leqslant (x \circ y)^2$ is an inequality between two negative real numbers, which implies

$$x \circ y \leqslant \|x\| \, \|y\|$$

since the function $\xi \mapsto \xi^2$ is decreasing, and thus order-reversing, in $(-\infty, 0]$. □

If $x, y \in \mathcal{H}^n$, then $\|x\|^2 = \|y\|^2 = -1$, so $\|x\| = \|y\| = i$ and $\|x\|\|y\| = -1$, this leads directly to the following corollary.

Corollary 12.2.1 *Let $x, y \in \mathcal{H}^n$. Then:*

$$x \circ y \leqslant -1,$$

with equality if and only if $x = y$.

Now, at this point the fundamental observations towards the construction of the hyperbolic distance on $\mathcal{H}^n$ are that $\cosh(\alpha) \geqslant 1$ for all $\alpha \in \mathbb{R}$ and that $\cosh(-\alpha) = \cosh(\alpha)$, thus we can consider just positive entries $\alpha \geqslant 0$ and formulate the following corollary.

Corollary 12.2.2 *Let $x, y \in \mathbb{R}^{1,n}$ be two positive time-like vectors, then there exists a unique $\alpha(x,y) \geqslant 0$ such that*

$$x \circ y = \cosh(\alpha(x,y)) \, \|x\| \, \|y\| \, . \tag{12.2}$$

In particular, if $x, y \in \mathcal{H}^n$, then $\|x\|\|y\| = -1$ and so it exists only one $\alpha(x,y) \geqslant 0$ such that $\cosh(\alpha(x,y)) = -x \circ y$.

Following the lead given to us by spherical geometry, we introduce the hyperbolic distance on $\mathcal{H}^n$ as follows.

Def. 12.2.3 (Hyperbolic distance on $\mathcal{H}^n$) *The hyperbolic distance between two elements x, y of $\mathcal{H}^n$ is $d_H(x, y) = \alpha(x, y)$, where $\alpha(x, y) \geqslant 0$ is the only non-negative real number that satisfies the equation:*

$$\cosh(d_H(x, y)) = -x \circ y, \tag{12.3}$$

or, equivalently,

$$\boxed{d_H(x, y) = \operatorname{arcosh}(-(x \circ y))}\,. \tag{12.4}$$

The non-negative real number $\alpha(x, y)$ is called the ***Lorentzian time-like angle*** *between $x, y \in \mathcal{H}^n$.*

A transformation $T : \mathcal{H}^n \to \mathcal{H}^n$ is a ***hyperbolic isometry*** *on $\mathcal{H}^n$ if it verifies the following condition:*

$$d_H(T(x), T(y)) = d_H(x, y), \qquad \forall x, y \in \mathcal{H}^n. \tag{12.5}$$

The set of hyperbolic isometries on $\mathcal{H}^n$ is denoted with $\mathcal{I}(\mathcal{H}^n)$.

By (12.1), we have that the spherical distance is $d_S(x, y) = \arccos(\langle x, y\rangle)$, thus, apart from the minus sign in front of the Lorentz pseudo-scalar product, the only change that is required to pass from the spherical to the hyperbolic distance on $\mathcal{H}^n$ is to replace the inverse circular function arccos with the inverse hyperbolic function arcosh.

Clearly, d_H is positive, symmetric and $d_H(x, y) = 0$ if and only if $x = y$ by corollary 12.2.1. All that is left to prove to verify that d_H is actually a distance is the triangular inequality, which is far from being trivial.

The proof of the triangular inequality of d_H needs a result that is important by its own: the possibility to identify the isometries of $\mathcal{H}^n$ with positive Lorentz transformations. The proof of this result requires the following lemma, which is proven with a technical reasoning of vast applicability that we will encounter again in this chapter.

Lemma 12.2.1 *A generic transformation $S : \mathcal{H}^n \to \mathcal{H}^n$ that preserves the Lorentz pseudo-scalar product can be extended to a positive Lorentz transformation $\phi_S \in \mathrm{PO}(1, n)$ if and only if there exists a transformation $T : \mathcal{H}^n \to \mathcal{H}^n$ that preserves the Lorentz pseudo-scalar product and that has an arbitrary fixed point $h \in \mathcal{H}^n$, i.e. $T(h) = h$, which can be extended to a positive Lorentz transformation $\phi_T \in \mathrm{PO}(1, n)$.*

Proof. If a generic transformation $S : \mathcal{H}^n \to \mathcal{H}^n$ that preserves the Lorentz pseudo-scalar product can be extended to a positive Lorentz transformation ϕ_S, then this property is also shared by a map T of this kind that also has the additional property of having a fixed point $h \in \mathcal{H}^n$. So, the non-trivial part of the proof consists in showing that the opposite is true.

To this end, write $S(h) = x \in \mathcal{H}^n$ and recall that $\mathrm{PO}(1, n)$ is transitive, in particular, on the set of 1-dimensional time-like vector subspaces of $\mathbb{R}^{1,n}$, so it surely exists $\tilde{R} \in \mathrm{PO}(1, n)$ such that $\tilde{R}(x) = h$. Since both x and h belong to $\mathcal{H}^n$, we can consider $R := \tilde{R}\big|_{\mathcal{H}^n}$ and compute $(R \circ S)(h) = R(x) = h$, which shows that h is a fixed point for $T := R \circ S$, which surely preserves the Lorentz pseudo-scalar product since it is the composition of two functions that share this property.

Notice now that, since $\mathrm{PO}(1, n)$ is a group, it exists a transformation $\tilde{R}^{-1} \in \mathrm{PO}(1, n)$ such that the restriction $R^{-1} := \tilde{R}^{-1}\big|_{\mathcal{H}^n}$ satisfies the equation $S = R^{-1} \circ T$.

Finally, if T can be extended to a positive Lorentz transformation, i.e. if there exists $\phi_T \in \mathrm{PO}(1,n)$ such that $T = \phi_T|_{\mathcal{H}^n}$, then $S = R^{-1} \circ T = \tilde{R}^{-1}\Big|_{\mathcal{H}^n} \circ \phi_T|_{\mathcal{H}^n} = (\tilde{R}^{-1} \circ \phi_T)\Big|_{\mathcal{H}^n}$, hence we recognize the extension of S to $\mathrm{PO}(1,n)$ to be $\phi_S := \tilde{R}^{-1} \circ \phi_T$. $\square$

Theorem 12.2.2 *Every hyperbolic isometry on $\mathcal{H}^n$ can be extended to be a positive Lorentz transformation and every positive Lorentz transformation is a hyperbolic isometry on $\mathcal{H}^n$. Thus, we have the identification:*

$$\mathrm{PO}(1,n) \cong \mathcal{I}(\mathcal{H}^n).$$

Proof. If $\phi \in \mathrm{PO}(1, n-1)$, then $\phi : \mathcal{H}^n \to \mathcal{H}^n$ and, by definition (12.3), we have:

$$\cosh(d_H(x,y)) = -x \circ y = -(\phi(x) \circ \phi(y)) = \cosh(d_H(\phi(x), \phi(y))), \qquad \forall x, y \in \mathcal{H}^n,$$

but cosh is injective on $\mathbb{R}^+$, so $d_H(\phi(x), \phi(y)) = d_H(x,y)$ for all $x, y \in \mathcal{H}^n$.

Conversely, let $T : \mathcal{H}^n \to \mathcal{H}^n$ be a hyperbolic isometry, $T \equiv (T_1, \ldots, T_{n+1})$, $T_j : \mathcal{H}^n \to \mathbb{R}$ being the j-th component function of T, i.e for $j \in \{1, \ldots, n+1\}$,

$$\begin{array}{rccc} T: & \mathcal{H}^n & \longrightarrow & \mathcal{H}^n \\ & x & \longmapsto & \begin{pmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_{n+1}(x) \end{pmatrix} \equiv \begin{pmatrix} T_1(x) \\ \overline{T}(x) \end{pmatrix} \end{array}.$$

We must prove that there exists $\phi \in \mathrm{PO}(1,n)$ such that $T = \phi|_{\mathcal{H}^n}$. Since ϕ preserves the Lorentz pseudo-scalar product on $\mathbb{R}^{1,n}$, for this problem to be well-posed, we must first check if T preserves the Lorentz pseudo-scalar product on $\mathcal{H}^n$. In order to do that we use eq. (12.3) and the fact that T preserves the hyperbolic distance to write, for all $x, y \in \mathcal{H}^n$,

$$\begin{aligned} d_H(T(x), T(y)) = d_H(x,y) \quad &\iff \quad \cosh(d_H(T(x), T(y))) = \cosh(d_H(x,y)) \\ &\iff \quad T(x) \circ T(y) = x \circ y. \end{aligned}$$

Having proven that a hyperbolic isometry T preserves the Lorentz pseudo-scalar product on $\mathcal{H}^n$ has another important consequence, i.e. the possibility to invoke lemma 12.2.1: if we solve our problem w.r.t.just one hyperbolic isometry $T : \mathcal{H}^n \to \mathcal{H}^n$ with a fixed point, then we automatically solve it for all the other hyperbolic isometries of $\mathcal{H}^n$.

A particularly clever choice of such a fixed point is represented by e_1, that clearly belongs to $\mathcal{H}^n$. The reason underlying this choice can be understood by recalling that the matrix

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathrm{O}(n), \tag{12.6}$$

is a positive Lorentzian matrix thanks to corollary 10.3.3. Since a Lorentzian matrix is associated to a Lorentz transformation w.r.t. the canonical basis $(e_1, \ldots, e_n)$, the fact that the first column of Λ coincides is $(1, 0, \ldots, 0)^t$ means that e_1 is a fixed point for the transformation.

As a consequence, the only thing that remains to do in order to prove the theorem is to use the properties of T to build a suitable orthogonal matrix such that expression in (12.6) extends T from $\mathcal{H}^n$ to the whole $\mathbb{R}^{1,n}$.

We start by observing that $T(e_1) = e_1$ implies, for all $u \in \mathcal{H}^n$:

$$T(u) \circ T(e_1) = T(u) \circ e_1 = -T_1(u) + 0 + \ldots 0 = -T_1(u),$$

on the other side, since the Lorentz pseudo-scalar product is preserved by T, we have:

$$T(u) \circ T(e_1) = u \circ e_1 = -u_1 + 0 + \ldots 0 = -u_1,$$

so $T_1(u) = u_1$ for all $u \in \mathcal{H}^n$.

Consider now $x, y \in \mathcal{H}^n$ and recall that $x \circ y = -x_1y_1 + \langle \bar{x}, \bar{y} \rangle$, $\bar{x} = (x_2, \ldots, x_{n+1})^t$, $\bar{y} = (y_2, \ldots, y_{n+1})^t$, so

$$x \circ y = T(x) \circ T(y) \iff -\cancel{x_1 y_1} + \langle \bar{x}, \bar{y} \rangle = -\cancel{T(x_1)T(y_1)} + \left\langle \overline{T}(x), \overline{T}(y) \right\rangle,$$

where the first terms in each member of the second equality above cancel out because, as we have just proven, $T_1(x) = x_1$ and $T_1(y) = y_1$. So,

$$x \circ y = T(x) \circ T(y) \iff \langle \bar{x}, \bar{y} \rangle = \left\langle \overline{T}(x), \overline{T}(y) \right\rangle,$$

notice that this property is not yet enough to say that $\overline{T} = (T_2, \ldots, T_{n+1})^t$ is an orthogonal transformation on $\mathbb{R}^n$, that we could associated to the $\mathrm{O}(n)$ matrix that we are searching for, because up to now we have shown that $\overline{T}$ preserves the Euclidean scalar product only when we apply it to the vectors $\bar{x}$ and $\bar{y}$, which were obtained by extracting the last n components of $x, y \in \mathcal{H}^n$. The extension to $\mathbb{R}^n$ can be achieved by considering the following bijection:

$$\begin{array}{rccc} p: & \mathcal{H}^n & \xrightarrow{\sim} & \mathbb{R}^n \\ & u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{pmatrix} & \longmapsto & p(u) := \begin{pmatrix} u_2 \\ \vdots \\ u_{n+1} \end{pmatrix}, \end{array}$$

which allows us to build the function $\tilde{T} := \overline{T} \circ p^{-1}$, explicitly

$$\begin{array}{rccl} \tilde{T}: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & u & \longmapsto & (T_2(p^{-1}(u)), \ldots, T_{n+1}(p^{-1}(u)))^t. \end{array}$$

$\tilde{T}$ is an orthogonal transformation on $\mathbb{R}^n$, lemma 10.1.2 guarantees that $\tilde{T}$ is linear and, by denoting with $A_{\tilde{T}}$ the associated matrix w.r.t. the canonical basis of $\mathbb{R}^n$, we have that

$$\begin{pmatrix} 1 & 0 \\ 0 & A_{\tilde{T}} \end{pmatrix}$$

is the Lorentzian matrix of a transformation $\phi \in \mathrm{PO}(1, n)$ such that $\phi|_{\mathcal{H}^n} = T$. $\square$

We now start the proof of the triangular inequality for d_H: a fundamental tool for this proof is given by the so-called Lorentzian cross product, which is the hyperbolic variant of the classical cross (or vector) product $x \times y$ between two vectors x, y in $\mathbb{R}^3$.

Recall that the $x \times y$ is a vector *orthogonal to the plane that contains* x *and* y, i.e. $\langle x, x \times y \rangle = \langle y, x \times y \rangle = 0$ and defined as follows:

$$x \times y := \det \begin{pmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = (x_2y_3 - x_3y_2)e_1 - (x_1y_3 - x_3y_1)e_2 + (x_1y_2 - x_2y_1)e_3 = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Def. 12.2.4 *Let $x, y \in \mathbb{R}^{1,2}$. The* ***Lorentzian cross-product*** *is defined as*

$$x \otimes y := \eta(x \times y) = \begin{pmatrix} -x_2y_3 + x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Remark 12.2.1 *The Lorentzian cross product of x and y in $\mathbb{R}^{1,2}$ is Lorentz-orthogonal to both x and y:*

$$x \circ (x \otimes y) = x \circ \eta(x \times y) = \langle x, \eta(\eta(x \times y))\rangle = \langle x, x \times y\rangle = 0,$$

and analogously for y. Then, if x or y belong to $\mathcal{H}^2$, their Lorentzian cross product is space-like.

The proof of the following result can be obtained by direct computation.

Theorem 12.2.3 *For all $x, y, w, z \in \mathbb{R}^{1,2}$ we have:*

1. $x \otimes y = -y \otimes x$, *'antisymmetry';*
2. $(x \otimes y) \circ z = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$, *'Lorentz mixed product formula' ;*
3. $x \otimes (y \otimes z) = (x \circ y)z - (x \circ z)y$;
4. $(x \otimes y) \circ (z \otimes w) = \det \begin{pmatrix} x \circ w & x \circ z \\ y \circ w & y \circ z \end{pmatrix}$, *'Lorentz version of Lagrange identity' .*
5. $x \otimes y = -\eta(x) \times \eta(y) = \eta(y) \times \eta(x)$.

Corollary 12.2.3 *For all $x, y \in \mathbb{R}^{1,2}$ we have:*

$$\|x \otimes y\|^2 = (x \circ y)^2 - \|x\|^2 \|y\|^2 .$$

Proof. By using property 4. of theorem 12.2.3 we get:

$$\|x \otimes y\|^2 = (x \otimes y) \circ (x \otimes y) = \det \begin{pmatrix} x \circ y & x \circ x \\ y \circ y & y \circ x \end{pmatrix} = (x \circ y)^2 - \|x\|^2 \|y\|^2 .$$

□

The three statements that follow are direct consequences of the previous corollary.

Corollary 12.2.4 *If $x, y \in \mathbb{R}^{1,2}$ are space-like, then*

1. $|x \circ y| < \|x\| \, \|y\| \iff x \otimes y$ *is time-like;*
2. $|x \circ y| = \|x\| \, \|y\| \iff x \otimes y$ *is light-like;*

3. $|x \circ y| > \|x\| \, \|y\| \iff x \otimes y$ *is space-like.*

Corollary 12.2.5 *Let $x, y \in \mathbb{R}^{1,2}$ be two linearly independent, positively-oriented, time-like vectors. Then, $x \otimes y$ is space-like and*

$$\|x \otimes y\| = -\|x\| \, \|y\| \sinh(\alpha(x, y)),$$

where $\alpha(x, y)$ is the Lorentzian time-like angle between x and y.

In particular, if $x, y \in \mathcal{H}^n$, then $\|x \otimes y\| = \sinh(\alpha(x, y))$.

Proof. Thanks to corollary 12.2.3 we have:

$$\begin{aligned} \|x \otimes y\|^2 &= (x \circ y)^2 - \|x\|^2 \|y\|^2 \underset{(12.2)}{=} \|x\|^2 \|y\|^2 \cosh^2(\alpha(x, y)) - \|x\|^2 \|y\|^2 \\ &= \|x\|^2 \|y\|^2 (\cosh^2(\alpha(x, y)) - 1) = \|x\|^2 \|y\|^2 \sinh^2(\alpha(x, y)). \end{aligned}$$

From remark 12.2.1, $x \otimes y$ is Lorentz-orthogonal to x which is time-like so $x \otimes y$ must space-like by corollary 10.2.2. The space-likeness of $x \otimes y$ implies $\|x \otimes y\| > 0$, so:

$$\|x \otimes y\| = -\|x\| \, \|y\| \sinh(\alpha(x, y)).$$

□

We are now ready to prove the triangular inequality of the hyperbolic distance on $\mathcal{H}^n$. As we have said, we will have to use the properties of the Lorentz cross-product, however, since it is defined only on $\mathbb{R}^{1,2}$, it seems not appropriate to use this operation to prove a property of d_H on $\mathcal{H}^n$ for n different than 3.

In fact, as we will see in the proof below, the clever idea that will allow us to circumvent this problem consists in the very simple observation that *only three vectors are involved in the triangular inequality* so, proving the triangular inequality of d_H in the 3-dimensional vector subspace generated by those three vectors or with $\mathbb{R}^{1,2}$ will be enough to infer the same property of d_H on $\mathcal{H}^n$ thanks to the transitivity of $\mathrm{PO}(1, n)$ on time-like vector subspaces and to the isometric nature of positive Lorentz transformations.

Theorem 12.2.4 *The hyperbolic distance d_H is a metric on $\mathcal{H}^n$.*

Proof. As previously said, only the triangular inequality for d_H remains to be proven. Let $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{H}^n$ distinct and $\tilde{V} = \mathrm{span}(\tilde{x}, \tilde{y}, \tilde{z})$. Thanks to theorem 10.3.4, it exists $\phi \in \mathrm{PO}(1, n)$ such that $\phi(\tilde{V}) = \mathrm{span}(e_1, e_2, e_3) \cong \mathbb{R}^{1,2}$. We set $x = \phi(\tilde{x})$, $y = \phi(\tilde{y})$ and $z = \phi(\tilde{z})$.

As proven in theorem 12.2.2, positive Lorentz transformations preserve the hyperbolic distance, thus proving the triangular inequality for $x, y, z \in \mathbb{R}^{1,2}$ is equivalent to prove it for the vectors $\tilde{x}, \tilde{y}, \tilde{z}$. To this aim, let us use corollary 12.2.5 to write

$$\|x \otimes y\| = \sinh(d_H(x, y)) \qquad \text{and} \qquad \|y \otimes z\| = \sinh(d_H(y, z)), \tag{12.7}$$

then, by property 3. of theorem 12.2.3, we have

$$(x \otimes y) \otimes (y \otimes z) = -\underbrace{((x \otimes y) \circ y)}_{=0} z - ((x \otimes y) \circ z) y = -((x \otimes y) \circ z) y, \tag{12.8}$$

$-((x \otimes y) \circ z) \in \mathbb{R}$, thus $(x \otimes y) \otimes (y \otimes z)$ and y are linearly dependent, so $(x \otimes y) \otimes (y \otimes z)$ is either time-like or it is the zero vector. Corollary 12.2.4 implies the following inequality

$$|(x \otimes y) \circ (y \otimes z)| \leqslant \|x \otimes y\| \, \|y \otimes z\| \, . \tag{12.9}$$

Finally, we recall the formula $\cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b$ for all $a, b \in \mathbb{R}$. We have gathered all the information that we need to prove the triangular inequality for x, y, z:

$$\begin{aligned}
\cosh(d_H(x, y) + d_H(y, z)) &= \cosh(d_H(x, y)) \cosh(d_H(y, z)) + \sinh(d_H(x, y)) \sinh(d_H(y, z)) \\
&\underset{(12.3) \text{ and } (12.7)}{=} (x \circ y)(y \circ z) + \|x \otimes y\| \, \|y \otimes z\| \\
&\underset{(12.9)}{\geqslant} (x \circ y)(y \circ z) + |(x \otimes y) \circ (y \otimes z)| \\
&\geqslant (x \circ y)(y \circ z) + (x \otimes y) \circ (y \otimes z) \\
&\underset{(4. \text{ of th. } 12.2.3)}{\geqslant} \cancel{(x \circ y)(y \circ z)} + (x \circ z)(y \circ y) - \cancel{(x \circ y)(y \circ z)} \\
&= (x \circ z)\|y\|^2 = -(x \circ z) \\
&\underset{(12.3)}{=} \cosh(d_H(x, z)),
\end{aligned}$$

i.e. $\cosh(d_H(x, y) + d_H(y, z)) \geqslant \cosh(d_H(x, z))$, but cosh is a strictly increasing function on $\mathbb{R}^+$, so it preserves the order and we can write $d_H(x, y) + d_H(y, z) \geqslant d_H(x, z)$, which is the triangular inequality that we wanted to prove. □

Def. 12.2.5 *The metric space $(\mathcal{H}^n, d_H)$ is called the **hyperbolic n-space**.*

In the geometry of the sphere, the geodesic lines are given by the intersection of the sphere S^n with a 2-dimensional vector subspace of $\mathbb{R}^{n+1}$ (and thus results in circles). Once again the hyperboloid model has very similar features as those of spherical geometry.

Def. 12.2.6 *A **hyperbolic line** in $\mathcal{H}^n$ is the intersection of $\mathcal{H}^n$ with a 2-dimensional time-like vector subspace of $\mathbb{R}^{1,n}$.*

Since a 2-dimensional time-like vector subspace of $\mathbb{R}^{1,n}$ must pass through the origin, its intersection with $\mathcal{H}^n$ will always be a hyperbola, so, in turn, **a hyperbolic line in $\mathcal{H}^n$ is just a hyperbola**.

Lemma 12.2.2 *Two distinct elements x, y of the hyperboloid $\mathcal{H}^n$ are linearly independent and so they span a 2-dimensional time-like vector subspace of $\mathbb{R}^{1,n}$.*

Proof. By absurd, let $x, y \in \mathcal{H}^n$, $x \neq y$, be linearly dependent, then $y = \lambda x$, $\lambda \in \mathbb{R} \backslash \{1\}$, then $\|x\|^2 = -1 = \|y\|^2 = \|\lambda x\|^2 = \lambda^2 \|x\|^2$, $\lambda = -1$. However, λ cannot be -1 because otherwise y would not belong to $\mathcal{H}^n$ anymore since its first coordinate would be negative. □

Remark 12.2.2 Given two distinct $x, y \in \mathcal{H}^n$, we have

$$\ell_{x,y} := \mathcal{H}^n \cap \operatorname{span}(x, y)$$

that is the unique hyperbolic line of $\mathcal{H}^n$ that contains both x and y.

We will show that these hyperbolic lines are the 'straight lines' of the hyperbolic metric, i.e. the curves that minimize the hyperbolic distance between two points.

Def. 12.2.7 *Three points x, y and z of $\mathcal{H}^n$ are said to be hyperbolically collinear if there is a hyperbolic line ℓ passing through x, y and z.*

Lemma 12.2.3 *If $x, y, z \in \mathcal{H}^n$ are such that*

$$d_H(x,z) = d_H(x,y) + d_H(y,z),$$

then x, y and z are hyperbolically collinear.

Proof. As shown in the proof of the triangular inequality for d_H, it is possible to consider the Lorentzian cross product of vectors belonging to $\mathcal{H}^n$ by associating them to vectors belonging to $\mathbb{R}^{1,2}$, in what follows this assumption will be implicitly assumed.

Let $x, y, z \in \mathcal{H}^n$ verify the equality $d_H(x,z) = d_H(x,y) + d_H(y,z)$ and apply cosh to both members, then, using the already quoted property $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$ for all $a, b \in \mathbb{R}$, we get:

$$\begin{aligned}
\cosh(d_H(x,z)) &= \cosh(d_H(x,y) + d_H(y,z)) \\
&= \cosh(d_H(x,y)) \cosh(d_H(y,z)) + \sinh(d_H(x,y)) \sinh(d_H(y,z)) \\
&= (-x \circ y)(-y \circ z) + \|x \otimes y\| \, \|y \otimes z\| \\
&= (x \circ y)(y \circ z) + \|x \otimes y\| \, \|y \otimes z\| \,,
\end{aligned}$$

but $\cosh(d_H(x,z)) = -x \circ z$, so

$$-x \circ z - (x \circ y)(y \circ z) = \|x \otimes y\| \, \|y \otimes z\| \,.$$

We can interpret the left-hand side of the previous equality as the following determinant:

$$\det \begin{pmatrix} x \circ z & x \circ y \\ y \circ z & y \circ y \end{pmatrix} = (x \circ z) \|y\|^2 - (x \circ y)(y \circ z) = -x \circ z - (x \circ y)(y \circ z),$$

but, thanks to property 4. of theorem 12.2.3, we have

$$\det \begin{pmatrix} x \circ z & x \circ y \\ y \circ z & y \circ y \end{pmatrix} = (x \otimes y) \circ (y \otimes z),$$

which implies

$$(x \otimes y) \circ (y \otimes z) = \|x \otimes y\| \, \|y \otimes z\| \,.$$

Thanks to remark 12.2.1, $x \otimes y$ and $y \otimes z$ are space-like vectors, thus their norm is positive, so $(x \otimes y) \circ (y \otimes z) = |(x \otimes y) \circ (y \otimes z)|$ and we can write:

$$|(x \otimes y) \circ (y \otimes z)| = \|x \otimes y\| \, \|y \otimes z\| \,.$$

Property 2. of corollary 12.2.4 implies that $(x \otimes y) \otimes (y \otimes z)$ is light-like, moreover, thanks to eq. (12.8),

$$(x \otimes y) \otimes (y \otimes z) = -((x \otimes y) \circ z) y,$$

but $-((x \otimes y) \circ z) \in \mathbb{R}$ and y is time-like, hence $(x \otimes y) \otimes (y \otimes z)$ is a light-like vector collinear with a time-like vector, which is possible if and only if $(x \otimes y) \otimes (y \otimes z) = 0$, i.e. $((x \otimes y) \circ z) y = 0$, but $y \in \mathcal{H}^n$, so the only possibility that remains is that $(x \otimes y) \circ z = 0$.

Finally, by property 4. of theorem 12.2.3 we have:

$$\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = ((x \otimes y) \circ z) = 0$$

and so x, y, z are linearly dependent, so each vector belong to $\mathcal{H}^n$ and to the span of the other two vectors, thus, by definition, x, y, z are hyperbolically collinear. $\square$

In order to prove that hyperbolic lines minimize the hyperbolic distance, we start with the definition and analysis of hyperbolic geodesic arcs.

Def. 12.2.8 (Geodesic arc) *A geodesic arc in a generic metric space (X, d) is a distance preserving function $\gamma : [a, b] \subseteq \mathbb{R} \to X$, with $a < b$.*

Explicitly, this means that $\forall t, s \in [a, b]$, $s \leqslant t$, we have: $d(\gamma(s), \gamma(t)) = d(s, t)$, but $d(s, t) = t - s$, so the request for a geodesic arc can be explicitly restated as follows:

$$d(\gamma(s), \gamma(t)) = t - s, \qquad \forall t, s \in [a, b],\ s \leqslant t.$$

Def. 12.2.9 (Hyperbolic geodesic arc) *A geodesic arc in the metric space $(\mathcal{H}^n, d_H)$ is called a hyperbolic geodesic arc.*

Theorem 12.2.5 *Let $\gamma : [a, b] \to \mathcal{H}^n$ be a curve. The following statements are equivalent.*

1. *The curve γ is a hyperbolic geodesic arc.*
2. *There exist Lorentz-orthonormal vectors $x, y \in \mathbb{R}^{1,n}$ such that*

$$\gamma(t) = \cosh(t - a)x + \sinh(t - a)y. \tag{12.10}$$

3. *The curve satisfies the differential equation $\gamma'' - \gamma = 0$.*

Proof.

$\boxed{1 \implies 2}$: we assume γ to be a geodesic arc on $(\mathcal{H}^n, d_H)$. Then for all $t \in [a, b]$, we have

$$\begin{aligned} d_H(\gamma(a), \gamma(b)) &= b - a = t - a + b - t \\ &= d_H(\gamma(a), \gamma(t)) + d_H(\gamma(t), \gamma(b)), \end{aligned} \tag{12.11}$$

which, by lemma 12.2.3, shows that $\gamma(t), \gamma(a)$ and $\gamma(b)$ are hyperbolically collinear for all $t \in [a, b]$, i.e $\gamma(t) \in \operatorname{span}(\gamma(a), \gamma(b))$, and so

$$\gamma([a, b]) \subset \ell_{\gamma(a), \gamma(b)}.$$

Since the image of γ belongs to $\mathcal{H}^n$, $\operatorname{span}((\gamma(a), \gamma(b))$ is a time-like vector subspace of $\mathbb{R}^{1,n}$, so, thanks to the transitivity of $\mathrm{PO}(1, n)$ on the set of time-like vector subspaces of $\mathbb{R}^{1,n}$, there exists $\phi \in \mathrm{PO}(1, n)$ such that $\phi(\operatorname{span}(\gamma(a), \gamma(b))) = \operatorname{span}(e_1, e_2) \cong \mathbb{R}^{1,1}$ and $\phi(\gamma(a)) = e_1$.

For all $t \in [a,b]$, let $z_t := \phi(\gamma(t))$. To obtain eq. (12.10), the decomposition of z_t on the $\mathbb{R}^{1,1}$ orthonormal basis $(e_1, \epsilon e_2)$, where $\epsilon = \pm 1$, will prove very helpful. In fact, if we write

$$z_t = \langle z_t, e_1 \rangle e_1 + \langle z_t, e_2 \rangle e_2 \tag{12.12}$$

and we apply ϕ^{-1} to both members we obtain

$$\phi^{-1}(z_t) = \langle z_t, e_1 \rangle \phi^{-1}(e_1) + \langle z_t, e_2 \rangle \phi^{-1}(e_2),$$

having used the linearity of Lorentz transformations. By definition, the last equation can be re-written as follows:

$$\gamma(t) = \langle z_t, e_1 \rangle \gamma(a) + \langle z_t, e_2 \rangle \phi^{-1}(e_2),$$

notice now that $x := \gamma(a) \in \mathcal{H}^n$ is a time-like vector and $y := \phi^{-1}(e_2)$ is a space-like vector because e_2 is space-like Lorentz transformations do not modify the likeness of vectors. Thus, x and y are Lorentz-orthogonal, plus, since ϕ^{-1} preserves the Lorentz norm, and $\|x\| = \left\|\phi^{-1}(e_1)\right\| = \|e_1\| = 1$ and $\|y\| = \left\|\phi^{-1}(e_2)\right\| = \|e_2\| = 1$, hence x and y are Lorentz-orthonormal vectors.

It follows that the only thing that remains to do is to prove that $\langle z_t, e_1 \rangle = \cosh(t-a)$ and $\langle z_t, e_2 \rangle = \sinh(t-a)$. Let us start with the first coefficient:

$$\begin{aligned}
\langle z_t, e_1 \rangle &= -z_t \circ e_1 = -\phi(\gamma(t)) \circ \phi(a) \\
&\underset{(\phi \in \mathrm{PO}(1,n))}{=} -\gamma(t) \circ \gamma(a) = \cosh(d_H(\gamma(t), \gamma(a))) \\
&= \cosh(t-a).
\end{aligned}$$

Now, regarding the second coefficient, we remark that z_t belongs to $\mathcal{H}^n$, so, using the decomposition in eq. (12.12), we must have

$$-\langle z_t, e_1 \rangle^2 + \langle z_t, e_2 \rangle^2 = -1 \iff -\cosh^2(t-a) + \langle z_t, e_2 \rangle^2 = -1,$$

which implies that $\langle z_t, e_2 \rangle = \pm \sinh(t-a)$. By choosing the positive determination, we get precisely formula (12.10).

$\boxed{2 \implies 1}$: we assume $\gamma : [a,b] \to \mathcal{H}^n$ is such that there is $x, y \in \mathbb{R}^{1,n}$, Lorentz-orthonormal, that verify $\gamma(t) = \cosh(t-a)x + \sinh(t-a)y \ \forall t \in [a,b]$. This means that $\gamma(t) \in \mathrm{span}(x,y)$, and so (x,y) is a Lorentz-orthonormal basis for this vector subspace. We recall that, by definition of Lorentz-orthonormal basis, x is time-like and y is space-like.

Now, given $s, t \in [a,b]$, $s \leqslant t$, we have

$$\begin{aligned}
\cosh(d_H(\gamma(s), \gamma(t))) &= -\gamma(s) \circ \gamma(t) \\
&= -(\cosh(s-a)x + \sinh(s-a)y) \circ (\cosh(t-a)x + \sinh(t-a)y) \\
&\quad \text{(by Lorentz-orthogonality of } x \text{ and } y) \\
&= -(\cosh(t-a)\cosh(s-a) \underbrace{\|x\|^2}_{=-1} + \sinh(t-a)\sinh(s-a) \underbrace{\|y\|^2}_{=1}) \\
&= \cosh(t-a)\cosh(s-a) - \sinh(t-a)\sinh(s-a) \\
&= \cosh((t-a) - (s-a)) \\
&= \cosh(t-s),
\end{aligned}$$

thus, since $\cosh(\xi)$ is injective for $\xi \geqslant 0$, $d_H(\gamma(s), \gamma(t)) = t - s$ and so γ is a geodesic arc.

$\boxed{2 \implies 3}$: if $\gamma(t) = \cosh(t-a)x + \sinh(t-a)y \ \forall t \in [a,b]$, then

$$\left.\begin{array}{r}\cosh''(t-a) = \cosh(t-a) \\ \sinh''(t-a) = \sinh(t-a)\end{array}\right\} \implies \gamma''(t) - \gamma(t) = 0, \quad \forall t \in [a,b].$$

$\boxed{3 \implies 2}$: suppose $\gamma''(t) - \gamma(t) = 0 \ \forall t \in [a,b]$. From ODE calculus, we know that the general solution of the previous differential equation is:

$$\gamma(t) = \cosh(t-a)\gamma(a) + \sinh(t-a)\gamma'(a). \tag{12.13}$$

Thus, proving 2. comes down to proving that $\gamma(a)$ and $\gamma'(a)$ are Lorentz-orthonormal.

To this aim, we notice that, since $\gamma(t) \in \mathcal{H}^n$ for all $t \in [a,b]$, $\gamma(t) \circ \gamma(t) = -1$ for all $t \in [a,b]$, so $\gamma \circ \gamma : [a,b] \to \mathbb{R}$ is the constant function $t \mapsto -1$, thus $(\gamma \circ \gamma)'(t) = 0$. On the other side, by applying the Leibniz rule on the Lorentz pseudo-scalar product we get

$$(\gamma \circ \gamma)'(t) = \gamma'(t) \circ \gamma(t) + \gamma(t) \circ \gamma'(t) = 2\,\gamma(t) \circ \gamma'(t) \qquad \forall t \in [a,b],$$

where, in the last step, we have used the symmetry of $\circ$. By mixing these results we find $\gamma(t) \circ \gamma'(t) = 0$ for all $t \in [a,b]$, hence, in particular, $\gamma(a)$ and $\gamma'(a)$ are Lorentz-orthogonal. Moreover, using (12.13), for all $t \in [a,b]$ we have,

$$\begin{aligned}\|\gamma(t)\|^2 &= -1 = \gamma(t) \circ \gamma(t) \\ &= (\cosh(t-a)\gamma(a) + \sinh(t-a)\gamma'(a)) \circ (\cosh(t-a)\gamma(a) + \sinh(t-a)\gamma'(a)) \\ &= \cosh^2(t-a)\underbrace{\|\gamma(a)\|^2}_{=-1} + \sinh^2(t-a)\left\|\gamma'(a)\right\|^2 = -\cosh^2(t-a) + \sinh^2(t-a)\left\|\gamma'(a)\right\|^2,\end{aligned}$$

where the terms proportional to $\gamma(a) \circ \gamma'(a)$ have not been written because of the Lorentz-orthogonality between $\gamma(a)$ and $\gamma'(a)$. We conclude that

$$-\cosh^2(t-a) + \sinh^2(t-a)\left\|\gamma'(a)\right\|^2 = -1 \qquad \forall t \in [a,b],$$

which implies $\|\gamma'(a)\|^2 = 1$, i.e. $\gamma(a)$ and $\gamma'(a)$ are Lorentz-orthonormal. □

Remark 12.2.3 In the theory of dynamical systems, the differential equation satisfied by a hyperbolic geodesic arc, i.e. $\gamma'' - \gamma = 0$, is that of the harmonic repulsor, whose phase portrait is known to be given by hyperbolae. Instead, the differential equation satisfied by a spherical geodesic arc, i.e. $\gamma'' + \gamma = 0$, is that of the harmonic oscillator, whose phase portrait is represented by circles.

When we extend the arc parameterization interval $[a,b]$ to the whole $\mathbb{R}$, we say that a geodesic arc γ is a **geodesic line**.

Corollary 12.2.6 *A function $\gamma : \mathbb{R} \to \mathcal{H}^n$ is a hyperbolic geodesic line if and only if there are $x, y \in \mathbb{R}^{1,n}$ Lorentz-orthonormal such that*

$$\begin{aligned}\gamma(t) &= \cosh(t)x + \sinh(t)y \\ &= \cosh(t)\gamma(0) + \sinh(t)\gamma'(0).\end{aligned}$$

Corollary 12.2.7 *The hyperbolic geodesic lines of $\mathcal{H}^n$ are its hyperbolic lines.*

Proof. Let $x, y \in \mathcal{H}^n$, $x \neq y$, and $\ell_{x,y} = \mathcal{H}^n \cap \operatorname{span}(x, y)$ be a hyperbolic line passing trough x and y, which defines, geometrically, a hyperbola connecting the points on the hyperboloid $\mathcal{H}^n$ identified by the vectors x and y. Thanks to the transitivity of $\mathrm{PO}(1, n)$ on $\mathcal{V}_m^T$, the set of all time-like m-dimensional vector subspaces of $\mathbb{R}^{1,n}$, $m \leqslant n$, there is a $\phi \in \mathrm{PO}(1, n)$ such that

$$\phi(\operatorname{span}(x, y)) = \operatorname{span}(e_1, e_2) \simeq \mathbb{R}^{1,1}.$$

Then, if we apply ϕ to $\ell_{x,y}$ we transform the hyperbola connecting x and y on $\mathcal{H}^n$ to a rectangular hyperbola on $\mathbb{R}^{1,1}$ relative to the canonical basis (e_1, e_2). We use $\mathcal{H}^1$ to denote this object, which is well-known to be parameterized by the hyperbolic functions as follows:

$$\phi(\ell_{x,y}) = \mathcal{H}^1 = \left\{\gamma(t) = \cosh(t)e_1 + \sinh(t)e_2,\ t \in \mathbb{R}\right\}$$

and so, thanks to the linearity of ϕ, we get

$$\ell_{x,y} = \left\{\cosh(t)\phi^{-1}(e_1) + \sinh(t)\phi^{-1}(e_2),\ t \in \mathbb{R}\right\}.$$

Since ϕ preserves the likeness, the orientation and the norm of vectors, we have that $\ell_{x,y}$ is written as in formula (12.10), thus it is a hyperbolic geodesic line. □

Def. 12.2.10 *A metric space X is* ***geodesically complete*** *if each geodesic arc $\gamma : [a, b] \to X$ extends to a unique geodesic line $\lambda : \mathbb{R} \to X$.*

The previous results show us that each hyperbolic geodesic arc extends uniquely to a hyperbolic geodesic line, i.e. it can be seen as a piece of an infinite hyperbola, thus $\mathcal{H}^n$ **is geodesically complete**.

The final result that we discuss is the equivalence between the hyperbolic topology on $\mathcal{H}^n$ generated by d_H and the Euclidean topology on $\mathcal{H}^n$ inherited by $\mathbb{R}^{n+1}$ with the Euclidean distance d_E. In the proof of this result we will use the Taylor-MacLaurin series expansion for cosh:

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{k=0}^{m} \frac{x^{2k}}{(2k)!} + \mathrm{O}(x^{2m+1}) \tag{12.14}$$

Theorem 12.2.6 *The metric topology on $\mathcal{H}^n$ given by d_H is equivalent to d_E.*

Proof. For all $x \in \mathcal{H}^n$ and $r > 0$, let us define the open neighborhoods of radius r around x w.r.t. the Euclidean and the hyperbolic distance, respectively, as follows:

$$B_E(x, r) := \{y \in \mathcal{H}^n \ : \ d_E(x, y) < r\}, \quad B_H(x, r) := \{y \in \mathcal{H}^n \ : \ d_H(x, y) < r\}.$$

If we prove that $B_E(x, r) \subseteq B_H(x, r)$ and that $B_H(x, r) \subseteq B_E(x, r)$ for all $x \in \mathcal{H}^n$ and $r > 0$, then the theorem will be proven.

$\boxed{B_E(x, r) \subseteq B_H(x, r)}$: consider $x, y \in \mathcal{H}^n$ distinct, then, since $|x-y|^2 = (x-y)_1^2 + \cdots + (x-y)_n^2$ and $\|x - y\|^2 = -(x - y)_1^2 + \cdots + (x - y)_n^2$, we have $d_E(x, y)^2 = |x - y|^2 > \|x - y\|^2$, moreover,

$\|x-y\|^2 = (x-y)\circ(x-y) = \|x\|^2 - 2x\circ y + \|y\|^2 = -2x\circ y - 2 = 2\cosh(d_H(x,y)) - 2 = 2(\cosh(d_H(x,y)) - 1)$, i.e.

$$\|x-y\|^2 = 2(\cosh(d_H(x,y)) - 1), \tag{12.15}$$

so

$$d_E(x,y)^2 > 2(\cosh(d_H(x,y)) - 1) \underset{(12.14)}{>} 2\left(1 + \frac{d_H(x,y)^2}{2} - 1\right) = d_H(x,y)^2, \tag{12.16}$$

so, by positivity, $d_E(x,y) > d_H(x,y)$. Let now $y \in B_E(x,r)$, then $d_H(x,y) < d_E(x,y) < r$, so $y \in B_H(x,r)$ too, thus $B_E(x,r) \subseteq B_H(x,r)$ for all $x \in \mathcal{H}^n$ and all $r > 0$.

$\boxed{B_H(x,r) \subseteq B_E(x,r)}$: we start by noticing that, thanks to corollary 12.2.6, once we fix an arbitrary $x \in \mathcal{H}^n$, all the hyperbolic lines passing through x are parameterized by a unit space-like vector y Lorentz-orthogonal to x, i.e.

$$L_x := \{\ell_{x,z} = \operatorname{span}(x,z) \cap \mathcal{H}^n,\ z \in \mathcal{H}^n \backslash \{x\}\} \cong \{y \in \operatorname{span}(x)^L,\ \|y\|^2 = 1\} =: S_x^L.$$

Again corollary 12.2.6 tells us that the hyperbolic geodesic line associated to $y \in S_x^L$ is

$$\gamma_y(t) = \cosh(t)x + \sinh(t)y \qquad t \in \mathbb{R}.$$

γ_y is clearly continuous in the Euclidean topology on the whole $\mathbb{R}$, in particular, the continuity in $t = 0$ can be explicitly written as follows:

$$\forall \varepsilon > 0\ \exists \delta_y(\varepsilon) > 0\ :\ |t| < \delta_y(\varepsilon) \implies d_E(\gamma_y(0), \gamma_y(t)) < \varepsilon, \tag{12.17}$$

having interpreted the images of γ_y as points in $(\mathbb{R}^{n+1}, d_E)$. The key observation that let us introduce the hyperbolic distance in our reasoning is that, by definition of hyperbolic geodesic,

$$d_H(x, \gamma_y(t)) = d_H(\gamma_y(0), \gamma_y(t)) = |t - 0| = |t| \qquad t \in \mathbb{R},$$

so that expression (12.17) can be replaced by

$$\forall \varepsilon > 0\ \exists \delta_y(\varepsilon) > 0\ :\ d_H(x, \gamma_y(t)) < \delta_y(\varepsilon) \implies d_E(\gamma_y(0), \gamma_y(t)) < \varepsilon$$

or, equivalently,

$$d_H(x,z) < \delta_y(\varepsilon) \implies d_E(x,z) < \varepsilon \qquad z \in \gamma_y(\mathbb{R}). \tag{12.18}$$

Moreover, by the transitivity of $\mathrm{PO}(1,n)$ on time-like vector subspaces, there exists $\phi \in \mathrm{PO}(1,n)$ such that $\phi(x) = e_1$ and so we have

$$\phi(\operatorname{span}(x)^L) = \phi(\operatorname{span}(e_1)^L) = \operatorname{span}(e_2, \dots, e_{n+1}) \cong \mathbb{R}^n.$$

Now recall that S_x^L is the set of space-like vectors y belonging to $\operatorname{span}(x)^L$ such that $\|y\|^2 = 1$, but the Lorentz norm of a space-like vector is positive, so also $\|y\| = 1$. Since ϕ preserves the Lorentz norm, $\phi(S_x^L)$ is the set of vectors belonging to $\operatorname{span}(e_2, \dots, e_{n+1}) \cong \mathbb{R}^n$ with unit Lorentz norm, however, the Lorentz and the Euclidean norms coincide on $\operatorname{span}(e_2, \dots, e_{n+1})$, so $\phi(S_x^L) = \{y \in \mathbb{R}^n\ :\ |y| = 1\} \equiv S^{n-1}$, which is compact. By the continuity of ϕ^{-1}, we get that also $S_x^L = \phi^{-1}(S^{n-1})$ is compact. The compactness of S_x^L allows us to set

$$\delta(\varepsilon) := \inf_{y \in S_x^L} \{\delta_y(\varepsilon)\} > 0$$

which allows us to get rid of the dependence on y in the implication (12.18) and to write

$$d_H(x,z) < \delta(\varepsilon) \implies d_E(x,z) < \varepsilon \qquad \forall \varepsilon > 0,$$

i.e. for all radius $\varepsilon > 0$, $z \in \gamma_y(\mathbb{R})$, it exists a radius $\delta(\varepsilon) > 0$ such that $B_H(x, \delta(\varepsilon)) \subseteq B_E(x, \varepsilon)$ which concludes the proof. □

12.2.3 The hyperbolic arc length

Let $\gamma : [a,b] \to \mathcal{H}^n$ be a curve. In this section we shall discuss how the metric given by the hyperbolic distance on $\mathcal{H}^n$ can be extended to compute the arc length. For that, we need to recall that a partition $P = \{t_0, \dots, t_m\}$ of $[a,b]$ is an ordered finite set such that

$$a = t_0 < t_1 < \cdots < t_m = b.$$

We set a partial ordering on partitions

$$Q \leqslant P \iff P \subseteq Q$$

and set

$$|P| = \inf_{k \in \{1,\dots,m\}} |t_k - t_{k-1}|,$$

$|P|$ is the finest partition interval. Notice that $|P| \to 0$ can be interpreted as 'P converges to $[a,b]$' and that if $Q \leqslant P$, then $|Q| \leqslant |P|$.

Def. 12.2.11 *Let $\gamma : [a,b] \to \mathcal{H}^n$ be a curve and let $P = \{t_0, \dots, t_m\}$ be a partition of $[a,b]$. We define the* ***hyperbolic P-inscribed length*** *of γ as:*

$$L_H(\gamma, P) = \sum_{i=1}^{m} d_H(\gamma(t_i), \gamma(t_{i-1})).$$

Moreover, the curve γ is ***rectifiable*** *if there is a real number $L(\gamma)$ such that for all $\varepsilon > 0$, there is a partition P_ε of $[a,b]$ such that for any partition Q verifying $Q \leqslant P_\varepsilon$, then*

$$|L(\gamma) - L_H(\gamma, Q)| < \varepsilon.$$

Lemma 12.2.4 *If γ is rectifiable, then for any partition P of $[a,b]$,*

$$L_H(\gamma, P) \leqslant L(\gamma).$$

Proof. Let P be a partition of $[a,b]$. First we note that if Q is a partition of $[a,b]$ such that $Q \leqslant P$, then

$$L(\gamma, P) \leqslant L(\gamma, Q) \tag{12.19}$$

by the triangular inequality of d_H. Since γ is rectifiable, for any ε there is a P_ε such that for all $Q \leqslant P_\varepsilon$,

$$|L(\gamma) - L_H(\gamma, Q)| < \varepsilon.$$

Moreover, $Q_\epsilon := P_\varepsilon \cup P$ is such that $Q_\varepsilon \leqslant P_\varepsilon$ and $Q_\varepsilon \leqslant P$. Finally, for all $\varepsilon > 0$ we have

$$\begin{aligned} L_H(\gamma, P) - L(\gamma) &\underset{(12.19)}{\leqslant} L_H(\gamma, Q_\varepsilon) - L(\gamma) \\ &\leqslant \lim_{\varepsilon \to 0} L_H(\gamma, Q_\varepsilon) - L(\gamma) = 0, \end{aligned}$$

hence

$$L_H(\gamma, P) \leqslant L(\gamma).$$

□

Def. 12.2.12 *Let $x, y \in \mathcal{H}^n$. We define the Lorentzian distance as*

$$d_L(x, y) = \|x - y\|.$$

Lemma 12.2.5 *The Lorentzian distance d_L verifies the following properties.*

1. *$d_L(x, y) \geqslant 0$ with equality if and only if $x = y$*
2. *$d_L(x, y) = d_L(y, x)$*

Proof. Let $x, y \in \mathcal{H}^n$. Then,

$$\begin{aligned} \|x - y\|^2 &= \|x\|^2 - 2(x \circ y) + \|y\|^2 \\ &\underset{(12.2.1)}{\geqslant} -2 - 2 \underbrace{\|x\| \, \|y\|}_{=-1} = 0, \end{aligned}$$

with equality if and only if they are linearly dependent, which implies $x = y$ since x, y belong to $\mathcal{H}^n$. □

Remark 12.2.4 *The Lorentzian distance is not a metric since it does not verify the triangular inequality. In fact, if we take $x, y, z \in \mathcal{H}^n$ hyperbolically collinear and such that y is between x and z, then it can be proven that*

$$d_L(x, z) > d_L(x, y) + d_L(y, z).$$

While the Lorentzian distance is not a metric, it will be useful because it can approximate the hyperbolic metric locally. To show this, we use formula (12.15) to write

$$\|x - y\|^2 = 2(\cosh(d_H(x, y)) - 1) \underset{y \to x}{\sim} 2\left(1 - \frac{d_H(x, y)^2}{2} - 1\right)^2 = d_H(x, y)^2$$

and so, by positivity,

$$d_L(x, y) \underset{y \to x}{\sim} d_H(x, y).$$

Def. 12.2.13 *Let $\gamma : [a, b] \to \mathcal{H}^n$ be a curve and let $P = \{t_0, \dots, t_m\}$ be a partition of $[a, b]$. We define the* ***Lorentzian P-inscribed length*** *of γ as:*

$$L_L(\gamma, P) = \sum_{i=1}^{m} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Moreover, the curve γ is ***Lorentz-rectifiable*** *if there is a real number $\mathcal{L}(\gamma)$ such that for all $\varepsilon > 0$, there is a partition P_ε of $[a, b]$ such that for any partition Q verifying $Q \leqslant P_\varepsilon$, then*

$$|\mathcal{L}(\gamma) - L_L(\gamma, Q)| < \varepsilon.$$

Since we do not have the triangular inequality for the Lorentzian distance, lemma 12.2.4 does not hold in the case of Lorentz-rectifiable curves.

Lemma 12.2.6 *If $\mathcal{L}(\gamma)$ exists, then it is unique.*

Proof. Assume γ is Lorentz-rectifiable with $\mathcal{L}_1(\gamma)$ and $\mathcal{L}_2(\gamma)$. If $\mathcal{L}_1(\gamma) \neq \mathcal{L}_1(\gamma)$, then there exists a $\varepsilon > 0$ such that $|\mathcal{L}_1(\gamma) - \mathcal{L}_2(\gamma)| > \varepsilon$. Let P, Q be partitions of $[a, b]$ such that

- $|\mathcal{L}_1(\gamma) - L_L(\gamma, P')| < \frac{\varepsilon}{2}$
- $|\mathcal{L}_2(\gamma) - L_L(\gamma, Q')| < \frac{\varepsilon}{2}$,

for all partitions P', Q' of $[a, b]$ verifying $P' \leqslant P$ and $Q' \leqslant Q$. The partition $R := P \cup Q$ is such that $R \leqslant P$ and $R \leqslant Q$, and we come to the following contradiction:

$$|\mathcal{L}_1(\gamma) - \mathcal{L}_2(\gamma)| \leqslant |\mathcal{L}_1(\gamma) - \mathcal{L}(\gamma, R)| + |\mathcal{L}_2(\gamma) - \mathcal{L}(\gamma, R)| < \varepsilon.$$

□

Def. 12.2.14 *Let $\gamma : [a, b] \to \mathcal{H}^n$ be a curve. We define the hyperbolic arc length as*

$$|\gamma|_H := \begin{cases} L(\gamma) & \text{if } \gamma \text{ is rectifiable} \\ \infty & \text{otherwise} \end{cases},$$

similarly, we define the Lorentzian length of γ as

$$\|\gamma\| = \begin{cases} \mathcal{L}(\gamma) & \text{if } \gamma \text{ is Lorentz-rectifiable} \\ \infty & \text{otherwise} \end{cases}.$$

Theorem 12.2.7 *Let $\gamma : [a, b] \to \mathcal{H}^n$ be a curve. Then γ is rectifiable in $\mathcal{H}^n$ if and only if γ Lorentz-rectifiable. Furthermore, the hyperbolic length is the same as the Lorentz length of γ, i.e.*

$$|\gamma|_H = \|\gamma\|.$$

Proof. We need to collect some preliminary results. Let $\eta > 0$, using the Taylor-MacLaurin series of cosh (12.14) we have

$$\begin{aligned}
\eta^2 \leqslant 2(\cosh(\eta) - 1) &= 2\left(1 + \frac{\eta^2}{2} + \frac{\eta^4}{24} + \sum_{k=3}^{m} \frac{\eta^{2k}}{(2k)!} + \mathrm{O}(x^{2m+1}) - 1\right) \\
&= 2\left(\frac{\eta^2}{2} + \frac{\eta^4}{24}\left(\sum_{k=0}^{m} 4!\frac{\eta^{2k}}{(2k+4)!} + \mathrm{O}(x^{2m+1})\right)\right) \\
&\leqslant 2\left(\frac{\eta^2}{2} + \frac{\eta^4}{24}\left(\sum_{k=0}^{m} \frac{\eta^{2k}}{(2k)!} + \mathrm{O}(x^{2m+1})\right)\right) \\
&= \eta^2 + \frac{\eta^4}{12}\cosh(\eta).
\end{aligned}$$

Consequently, if $\cosh(\eta) \leqslant 12$,

$$2(\cosh(\eta) - 1) \leqslant \eta^2(1 + \eta^2). \tag{12.20}$$

If we replace η in eq. (12.20) with $d_H(x,y)$ and we suppose that $d_H(x,y) \leqslant \operatorname{arcosh}(12)$, then, since $d_L(x,y)^2 = 2(\cosh(d_H(x,y)) - 1)$ by eq. (12.15), we get

$$d_L(x,y)^2 \leqslant d_H(x,y)^2(1 + d_H(x,y)^2) \iff d_L(x,y) \leqslant d_H(x,y)\sqrt{1 + d_H(x,y)^2}.$$

On the other side, eq. (12.16) implies that $d_H(x,y)^2 \leqslant 2(\cosh(d_H(x,y)) - 1) = d_L(x,y)^2$, hence $d_H(x,y) \leqslant d_L(x,y)$ for all $x, y \in \mathcal{H}^n$ not necessarily distinct.

So, for all $x, y \in \mathcal{H}^n$ such that $d_H(x,y) \leqslant \operatorname{arcosh}(12)$, it holds that

$$d_H(x,y) \leqslant d_L(x,y) \leqslant d_H(x,y)\sqrt{1 + (d_H(x,y))^2}. \tag{12.21}$$

We can now start with the proof of the equivalence.

$\boxed{\Longrightarrow}$: we start by assuming that γ is rectifiable. Let $\varepsilon > 0$ and P a partition of $[a,b]$ such that for all $Q \leqslant P$ we have by lemma 12.2.4

$$|\gamma|_H - L_H(\gamma, P) < \epsilon.$$

Let $\delta > 0$ and set

$$\mu(\gamma,\delta) = \sup_{a \leqslant s < t \leqslant b} \big\{ d_H(\gamma(s), \gamma(t)) : |t - s| < \delta \big\}.$$

Note that since $[a,b]$ is compact and γ is continuous, γ is uniformly continuous and so $\mu(\gamma,\delta) \underset{\delta \to 0}{\to} 0$. Let $\delta > 0$ such that $\cosh(\mu(\gamma,\delta)) \leqslant 12$ and

$$|\gamma|_H \sqrt{1 + \mu(\gamma,\delta)^2} < |\gamma|_H + \varepsilon,$$

and P' a partition of $[a,b]$ such that $P' \leqslant P$ and $|P| < \delta$. Then for all partitions $Q = \{t_0, \ldots, t_m\}$ of $[a,b]$ such that $Q \leqslant P'$ we have:

$$|\gamma|_H - \epsilon \leqslant L_H(\gamma, Q) \leqslant L_L(\gamma, Q)$$

on one side, and

$$\begin{aligned}
L_L(\gamma, Q) &= \sum_{i=1}^{m} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
&\leqslant \sum_{i=1}^{m} d_H(\gamma(t_i), \gamma(t_{i-1}))\sqrt{1 + d_H^2(\gamma(t_i), \gamma(t_{i-1}))} \\
&\leqslant L_H(\gamma, Q)\sqrt{1 + \mu(\gamma,\delta)^2} \\
&\leqslant |\gamma|_H + \varepsilon.
\end{aligned}$$

Hence, by combining both inequalities,

$$\big||\gamma|_H - L_L(\gamma, Q)\big| < \varepsilon, \qquad \forall Q \leqslant P'$$

and by the unicity of the Lorentzian arc length (lemma 12.2.6), γ is Lorentz-rectifiable and $|\gamma|_H = \|\gamma\|$.

$\boxed{\Longleftarrow}$: we now suppose γ is Lorentz-rectifiable. Let $\varepsilon > 0$ and P be a partition of $[a,b]$ such that if $Q \leqslant P$, $\big|\|\gamma\| - L_L(\gamma, Q)\big| < \varepsilon$. Then,

$$L_H(\gamma, Q) - \|\gamma\| \leqslant L_L(\gamma, Q) - \|\gamma\| \leqslant \varepsilon$$

for all $Q \leqslant P$, and so γ is rectifiable. $\square$

Before proving the theorem regarding the metric of the arc length, we first make the following remark: for any $\mathcal{C}^1$-curve $\gamma : [a,b] \to \mathcal{H}^n$, we have that $\gamma(t)$ is Lorentz-orthogonal to $\gamma'(t)$ for all $t \in [a,b]$, i.e.

$$\gamma(t) \circ \gamma'(t) = 0,$$

in fact, by differentiating the Lorentz pseudo-scalar product and using the Leibniz property together with the symmetry of $\circ$ we get:

$$\begin{aligned}(\gamma(t) \circ \gamma(t))'(t) &= 2(\gamma(t) \circ \gamma'(t)) \\ &= (t \mapsto \|\gamma(t)\|^2 \equiv -1)' = 0.\end{aligned}$$

Theorem 12.2.8 *Let $\gamma : [a,b] \to \mathcal{H}^n$ a $\mathcal{C}^1$-curve. Then γ is rectifiable and the hyperbolic length of γ is given by*

$$\|\gamma\|_H = \int_a^b \left\|\gamma'\right\| dt.$$

Proof. Let $F : [a,b]^{n+1} \to \mathbb{R}$ defined by

$$F(t_1, \dots, t_{n+1}) = \big(-\gamma_1'(t_1)^2 + \cdots + \gamma_{n+1}'(t_{n+1})^2 \big)^{\frac{1}{2}}.$$

Since γ is $\mathcal{C}^1$ and $\gamma'(t)$ is space-like for all t because it is Lorentz-orthogonal to $\gamma(t) \in \mathcal{H}^n$, F is continuous on $[a,b]^{n+1}$ which is compact, thus F is uniformly continuous. The set

$$\big\{ |F(t) - F(s)|,\ t, s \in [a,b]^{n+1} \big\}$$

is bounded since F is continuous on a compact set.

For any fixed $\delta > 0$ we define

$$\mu(F,\delta) = \sup_{t,s \in [a,b]^{n+1}} \big\{ |F(t) - F(s)|,\ |t_i - s_i| \leqslant \delta,\ i \in \{1, \dots, n+1\} \big\}.$$

As in the previous proof, F is uniformly continuous so $\mu(F,\delta) \underset{\delta \to 0}{\to} 0$ and if we set $P = \{t_0, \dots, t_m\}$ such that $|P| \leqslant \delta$, we have by the mean-value theorem $\exists s_{ij} \in [t_{j-1}, t_j]$

$$|\gamma_i(t_j) - \gamma_i(t_{j-1})| = \gamma_i'(sij)(t_j - t_{j-1})$$

and if we set $s_j = (s_{1,j}, \dots, s_{n+1,j})$, then

$$\begin{aligned}\|\gamma_i(t_j) - \gamma_i(t_{j-1})\| &= \big(-[\gamma_1(t_j) - \gamma_1(t_{j-1})]^2 + [\gamma_2(t_j) - \gamma_2(t_{j-1})]^2 + \cdots + [\gamma_{n+1}(t_j) - \gamma_{n+1}(t_{j-1})]^2 \big)^{\frac{1}{2}} \\ &= \big(-\gamma_1'(s_{1,j})^2 + \gamma_1'(s_{2,j})^2 + \cdots + \gamma_{n+1}'(s_{n+1,j})^2 \big)^{\frac{1}{2}} (t_j - t_{j-1}) \\ &= F(s_j)(t_j - t_{j-1}).\end{aligned}$$

Additionally, we set

$$S(\gamma, P) = \sum_{j=1}^{m} \left\|\gamma'(t_j)\right\| (t_j - t_{j-1})$$

and we remind

$$L_L(\gamma, P) = \sum_{j=1}^{m} \|\gamma(t_j) - \gamma(t_{j-1})\| .$$

As such, we have

$$\begin{aligned}
\left|S(\gamma, P) - L_L(\gamma, P)\right| &= \left|\sum_{j=1}^{m} \left\|\gamma'(t_j)\right\| (t_j - t_{j-1}) - F(s_j)(t_j - tj - 1)\right| \\
&\leqslant \mu(F, \delta)(b-a) \qquad\qquad (*_2)
\end{aligned}$$

and furthermore,

$$\begin{aligned}
\left|\int_a^b \left\|\gamma'(t)\right\| dt - S(\gamma, P)\right| &= \left|\sum_{i=1}^{m} \Big(\int_{t_{j-1}}^{t_j} \left\|\gamma'(t)\right\| - \left\|\gamma'(t_j)\right\| (t_j - t_{j-1}) dt \Big)\right| \\
&\leqslant \sum_{i=1}^{m} \int_a^b \underbrace{\left| \left\|\gamma'(t)\right\| - \left\|\gamma'(t_j)\right\| (t_j - t_{j-1})\right|}_{\leqslant \mu(F,\delta)} dt \\
&\leqslant \mu(F, \delta)(b-a) \qquad\qquad (*_2)
\end{aligned}$$

Finally, by combining $(*_1)$and $(*_2)$ we obtain

$$\begin{aligned}
\left|\int_a^b \left\|\gamma'(t)\right\| dt - L_L(\gamma, P)\right| &\leqslant \left|\int_a^b \left\|\gamma'(t)\right\| dt - S(\gamma, P)\right| + \left|S(\gamma, P) - L_L(\gamma, P)\right| \\
&\leqslant 2\mu(F, \delta)
\end{aligned}$$

and since $\mu(F, \delta) \underset{\delta \to 0}{\to} 0$ and $|P| \underset{\delta \to 0}{\to} 0$

$$\|\gamma\| = \int_a^b \left\|\gamma'(t)\right\| dt = \lim_{|P| \to 0} L_L(\gamma, P)$$

□

Def. 12.2.15 *Let $\gamma : [a, b] \to \mathcal{H}^n$ curve. If $d_x = (dx_1, , \ldots, dx_{n+1})$, then*

$$\|dx\| = (-dx_1^2 + dx_2^2 + \cdots + dx_{n+1}^2)^{\frac{1}{2}}$$

and

$$\int_\gamma \|dx\| := \|\gamma\| .$$

Additionally if γ is a $\mathcal{C}^1-$curve,

$$\|\gamma\| = \int_\gamma \|dx\| = \int_a^b \left\|\gamma'(t)\right\| dt.$$

The differential $\|dx\|$ is called the ***element of hyperbolic arc length*** *of $\mathcal{H}^n$.*

12.2.4 The hyperboloid as a Riemannian manifold

In this short subsection we will prove that the hyperboloid $\mathcal{H}^n$ can be considered a Riemannian manifold. To this aim, we recall that the Lorentz pseudo-scalar product is a bilinear, symmetric non-degenerate form. Hence, if we set

$$f: \begin{array}{ccc} \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (x \circ x), \end{array}$$

then differential of f is

$$df_x(y) = 2(x \circ y).$$

In fact,

$$f(x+y) = (x+y) \circ (x+y) = f(x) + \underbrace{2(x \circ y)}_{\text{linear}} + \underbrace{(y \circ y)}_{\text{quadratic}},$$

since the differential represents the unique linear approximation of f, the result follows.

Theorem 12.2.9 *The hyperboloid $\mathcal{F}^n = f^{-1}(\{-1\})$ is a Riemannian n-manifold.*

Proof. The differential $df_x(y) = 2(x \circ y)$ is surjective and -1 is not a critical value of f, so, by the level set theorem 1.2.1, the hyperboloid is a differential manifold of dimension n. Moreover, thanks to eq. (2.34), for every $x \in \mathcal{F}^n$, the tangent space $T_x\mathcal{F}^n$ is given by

$$T_x\mathcal{F}^n = \ker df_x = \operatorname{span}(x)^L.$$

Now, since x is time-like, $\operatorname{span}(x)^L$ is a n-dimensional space-like vector subspace of $\mathbb{R}^{n+1}$ and so, for all $y \in T_x\mathcal{F}^n$ we have $y \circ y > 0$. Hence, the Lorentz pseudo-scalar product is positive-definite on the tangent spaces $T_x\mathcal{F}^n$ and so $\mathcal{F}^n$ is a Riemannian manifold. □

Finally, $\mathcal{H}^n$ can be defined as the biggest subset of F^n that contains e_1 and that is simply connected and so $\mathcal{H}^n$ is a complete, simply connected Riemannian manifold of dimension n with metric tensor $g_x(u, v) = u \circ v$.

12.3 The conformal model $\mathcal{B}^n$

The conformal model comes hand in hand with the previous section on Möbius transformation. The conformal model of hyperbolic geometry lies in the open unit ball $\mathcal{B}^n$ or the the upper-half space $\mathcal{U}^n$ and is a model that maintains up to a certain extent the notion of Euclidean angles, hence it's name. Additionally, to explain the naming further, the set of conformal transformations that is stable on $\mathcal{B}^n$ (or $\mathcal{U}^n$ respectively) is the isometry group of the model and by Liouville's theorem of conformal transformations, is the set of Möbius transformations stable on $\mathcal{B}^n$ ($\mathcal{U}^n$ respectively).
We begin by redefining the Lorentzian scalar product in $\mathbb{R}^{n+1}$:

$$x \circ y = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}$$

and we identify the open unit ball with it's injection in $\mathbb{R}^{n+1}$, with the notation $\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$:

$$\begin{aligned} \mathcal{B}^n &= \left\{x \in R^n \ \ |x| < 1\right\} \\ &= \left\{x \in \mathbb{R}^{n+1} \ \ |\bar{x}| < 1\right\} \end{aligned}$$

. We wish to transfer the hyperbolic metric of the hyperboloid model $\mathcal{H}^n$ onto the the open unit ball $\mathcal{B}^n$. This can be done via a **stereographic projection** ζ by projecting $x \in \mathcal{B}^n$ away from $-e_{n+1}$ until it meets $\mathcal{H}^n$. Explicitly, for $x \in \mathcal{B}^n$, ζ is of the form

$$\zeta(x) = x + s(x + e_{n+1})$$

such that $\|\zeta(x)\|^2 = -1$. By developing the computations, we obtain $s = \frac{1+\|x\|^2}{1-\|x\|^2}$. Note that in the case of $x \in \mathcal{B}^n$, the Lorentzian norm and the Euclidean norm coincide. Hence,

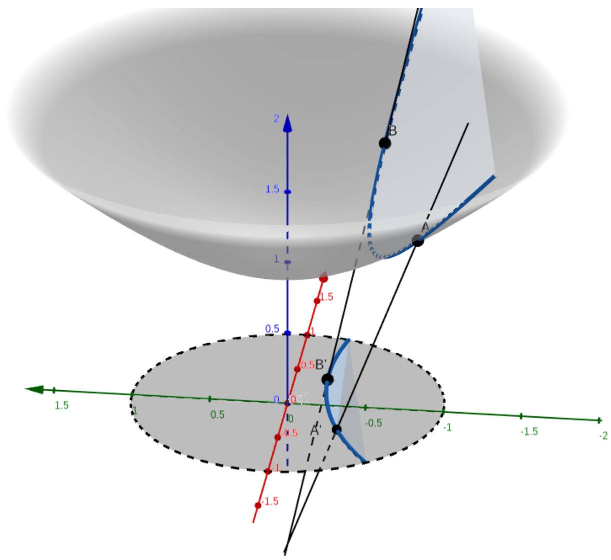


Figure 12.5: Illustration of the isometry between $\mathcal{H}^n$ and $\mathcal{B}^n$.

$$\begin{aligned}
\zeta(x) &= x + \frac{1+|x|^2}{1-|x|^2}(x + e_{n+1}) \\
&= \left(\frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}\right)
\end{aligned}$$

Lemma 12.3.1 *$\zeta : \mathcal{B}^n \to \mathcal{H}^n$ is bijective with inverse*

$$\zeta^{-1}(y) = \left(\frac{y_1}{1+y_{n+1}}, \ldots, \frac{y_n}{1+y_{n+1}}\right)$$

Proof. Let $y \in \mathcal{H}^n$. Since $x \in \mathcal{B}^n, \zeta(x) \in \mathcal{H}^n$ and $-e_{n+1}$ are aligned and belong to the same Euclidean line, we have similarly $\zeta^{-1}(y), y$ and $-e_{n+1}$ that must also be aligned. Hence, ζ^{-1} must be of the form

$$\zeta^{-1}(y) = y + s(-e_{n+1} - y)$$

such that $\zeta^{-1}(y) \circ e_{n+1} = 0$.

$$\zeta^{-1}(y) = 0 \iff (y(1-s) - se_{n+1}) \circ e_{n+1} = 0 \iff -y_{n+1}(1-s) + s = 0 \quad (12.22)$$

$$\iff s(1 + y_{n+1}) = 1 \iff s = \frac{1}{1+y_{n+1}} \quad (12.23)$$

Hence,

$$\zeta^{-1}(y) = \big(\frac{y_1}{1+y_{n+1}}, \dots, \frac{y_n}{1+y_{n+1}}\big).$$

Injectivity : Suppose $\zeta(x) = \zeta(y)$. Then,

$$\frac{1+|x|^2}{1-|x|^2} = \frac{1+|y|^2}{1-|y|^2} \iff |x| = |y|$$

and

$$\frac{2x_i}{1-|x|^2} = \frac{2y_i}{1-|y|^2} \overset{|x|=|y|}{\Longrightarrow} x_i = y_i$$

for all $i \in \{1, \dots, n\}$. Therefore, $x = y$
Surjectivity : Let $y \in \mathcal{H}^n$. First, we have

- $\left\|\zeta^{-1}(y)\right\|^2 = \frac{|\bar{y}|^2}{(1+y_{n+1})} = \frac{y_{n+1}-1}{(1+y_{n+1})^2} = \frac{y_{n+1}-1}{1+y_{n+1}}$
- $1 - |\zeta^{-1}(y)|^2 = \frac{2}{1+y_{n+1}}$

. Thus, by combining the two computations,

$$\begin{aligned} \zeta(\zeta^{-1}(y)) &= \big(2\frac{y_1}{1+y_{n+1}}\frac{1}{1-|\zeta^{-1}(y)|^2}, \dots, 2\frac{y_n}{1+y_{n+1}}\frac{1}{1-|\zeta^{-1}(y)|^2}, \frac{1+|\zeta^{-1}(y)|^2}{1-|\zeta^{-1}(y)|^2}\big) \\ &= \big(y_1, \dots, y_n, y_{n+1}\big) \end{aligned}$$

□

12.3.1 The hyperbolic metric on the unit ball

Through the bijection ζ between $\mathcal{H}^n$ and $\mathcal{B}^n$, we wish to extend the hyperbolic metric onto $\mathcal{B}^n$. To do so, we force ζ to be an isometry.

Def. 12.3.1 *We define the hyperbolic metric on $\mathcal{B}^n$, also called **Poincaré metric**, as follows: for $x, y \in \mathcal{B}^n$,*

$$d_B(x,y) = d_H(\zeta(x), \zeta(y))$$

The metric space $(\mathcal{B}^n, d_B)$ is called the conformal ball model.

Once again, the hyperbolic cosine will give us an elegant formulation of the metric.

Theorem 12.3.1 *The Poincaré metric d_B is given by*

$$\cosh(d_B(x,y)) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

Proof.

$$\begin{aligned}
\cosh(d_H(\zeta(x),\zeta(y))) &= -\zeta(x)\circ\zeta(y) \\
&= -\sum_{i=1}^{n}\frac{4x_iy_i}{(1-|x|^2)(1-|y|^2)}+\frac{(1+|x|^2)(1+|y|^2)}{(1-|x|^2)(1-|y|^2)} \\
&= \frac{-4\langle x,y\rangle+1+|x|^2+|y|^2+|x|^2|y|^2}{(1-|x|^2)(1-|y|^2)} \\
&= \frac{(1-|x|^2)(1-|y|^2)+2|x|^2+2|y|^2-4\langle x,y\rangle}{(1-|x|^2)(1-|y|^2)} \\
&= 1+\frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}
\end{aligned}$$

□

To interpret this metric, we can think of a 1 meter ruler with an infinite amount of graduations and where graduations become smaller and smaller. In the figure 12.6 shown below, we can see such a representation. The space is more and more compacted as we reach the border of the disc.

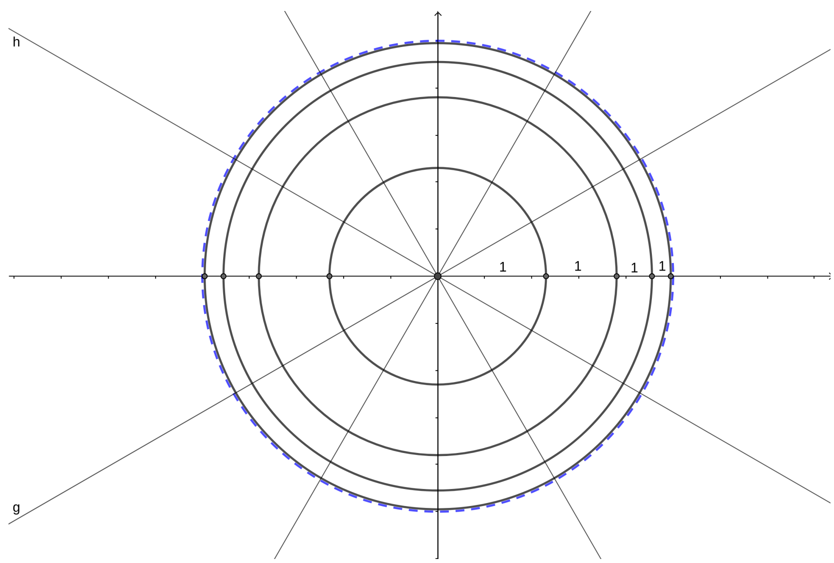


Figure 12.6: The Poincaré disc: between each radius, we go 1 unit of distance. we see that at edges, the space is very heavily compacted

Theorem 12.3.2 *The element of hyperbolic arc length of the conformal model of the unit ball is given by*

$$\|dx\|_B = \frac{2|dx|}{1+|x|^2}$$

Proof. Let $x \in \mathcal{B}^n$ and $y = \zeta(x)$. Then, we have

$$y_i = \frac{2x_i}{1-|x|^2} \qquad \text{and} \qquad y_{n+1} = \frac{1+|x|^2}{1-|x|^2}$$

Then, for $h \in \mathcal{B}^n$ arbitrarily close to zero and $i \in \{1, .., n\}$ we have the following computation

$$\begin{aligned}
\zeta_i(x+h) &= \frac{2(x_i+h_i)}{1-|x+h|^2} = \frac{2(x_i+h_i)}{(1-|x|^2)(1-2\left\langle \frac{x}{1-|x|^2}, h\right\rangle + \frac{|h|^2}{1-|x|^2})} \\
&\stackrel{1}{=} \frac{2(x_i+h_i)}{1-|x|^2}\Big(1+\frac{2\langle x,h\rangle}{1-|x|^2} + \mathrm{O}(|h|^2)\Big) \\
&= \frac{2x_i}{1-|x|^2} + \frac{2h_i}{1-|x|^2} + \frac{4x_i\langle x,h\rangle}{(1-|x|^2)^2} + \mathrm{O}(|h|^2)
\end{aligned}$$

and from this we are able to deduce

$$dy_i = \frac{2dx_i}{1-|x|^2} + \frac{4x_i\langle x, dx\rangle}{(1-|x|^2)^2}.$$

Similarly, for $h \in \mathcal{B}^n$ arbitrarily close to zero we have

$$\begin{aligned}
\zeta_{n+1}(x+h) &= \frac{1+|x+h|^2}{1-|x+h|^2} =^1 \frac{1+|x|^2+2\langle x,h\rangle+|h|^2}{1-|x|^2}\Big(1+\frac{2\langle x,h\rangle}{1-|x|^2}+\mathrm{O}(|h|^2)\Big) \\
&= \frac{1+|x|^2}{1-|x|^2} + \frac{4\langle x,h\rangle}{(1-|x|^2)^2} + \mathrm{O}(|h|^2)
\end{aligned}$$

and so

$$dy_{n+1} = \frac{4\langle x, dx\rangle}{(1-|x|^2)^2}.$$

From this we are able to obtain

- $dy_i^2 = \frac{4}{(1-|x|^2)^2}\big(dx_i^2 + \frac{4x_i dx_i\langle x,dx\rangle}{1-|x|^2} + \frac{4x_i^2\langle x,dx\rangle^2}{(1-|x|^2)^2}\big)$
- $\sum\limits_{i=1}^{n} dy_i^2 = \frac{4}{(1-|x|^2)^2}\big(|dx|^2 + \frac{4\langle x,y\rangle^2}{(1-|x|^2)^2}\big)$
- $dy_{n+1}^2 = \frac{16\langle x,dx\rangle^2}{(1-|x|^2)^4}$

, which when combined

$$\begin{aligned}
\|dx\|_B = \quad \|dy\| &= \sqrt{\sum_{i=1}^{n} dy_i^2 - dy_{n+1}^2} \\
&= \sqrt{\tfrac{4|dx|^2}{(1-|x|^2)^2}} = \frac{2|dx|}{1-|x|^2}
\end{aligned}$$

□

As announced in the preface of this section, Möbius transformation plays the major role in the conformal model. Here we have a first taste: Möbius transformation act isometrically on the conformal ball model.

[1] we use the approximation $\frac{1}{1-X} = 1 + X + \mathrm{O}(X^2)$

Lemma 12.3.2 *If ϕ is a Möbius transformation stable on $\mathcal{B}^n$ and $x, y \in \mathcal{B}^n$, then*

$$\frac{|\phi(x)-\phi(y)|^2}{(1-|\phi(x)|^2)(1-|\phi(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

Proof. If $\phi(0) = 0$, then $\phi \in \mathrm{O}(n)$ and so ϕ is an isometry and the result is automatically given to us. Suppose $\phi(0) \neq 0$. We then have the decomposition $\phi = \psi\sigma$ with ψ a Euclidean isometry and σ a inversion of a sphere $S(a, r)$ of $\hat{\mathbb{R}}^n$, orthogonal to S^{n-1}. Hence, to prove this lemma, all that is needed is to prove the result for σ. First, we recall that since $S(a, r)$ is orthogonal to S^{n-1}, $r^2 = |a|^2 - 1$ and since σ is a inversion,

$$|\sigma(x)-\sigma(y)|^2 \overset{11.2.2}{=} \frac{r^4|x-y|^2}{|x-a||y-a|}.$$

Furthermore,

$$\begin{aligned}
\phi(x) &= a + \frac{r^2}{|x-a|^2}(x-a) \\
\implies |\phi(x)|^2 &= |a|^2 + \frac{2r^2\langle a, x\rangle - 2r^2 + r^4}{|x-a|^2} \\
\implies |\phi(x)|^2 - 1 &= \frac{r^2\big(|x-a|^2 + 2\langle a, x-a\rangle + |a|^2 - 1\big)}{|x-a|^2} \\
&= \frac{r^2(|x|^2-1)}{|x-a|^2}
\end{aligned}$$

Hence, we come to the conclusion

$$\begin{aligned}
\frac{|\phi(x)-\phi(y)|^2}{(1-|\phi(x)|^2)(1-|\phi(y)|^2)} &= \frac{r^4|x-y|^2}{|x-a|^2|y-a|^2}\frac{|x-a|^2|y-a|^2}{r^4(1-|x|^2)(1-|y|^2)} \\
&= \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}
\end{aligned}$$

. ∎

As a direct of this lemma, we obtain our first result step to showing that the isometry group of $\mathcal{B}^n$ is it's Möbius group.

Theorem 12.3.3 *If $\phi \in \mathcal{M}(\mathcal{B}^n)$, then ϕ acts as an isometry on $\mathcal{B}^n$:*

$$d_B(x, y) = d_B(\phi(x), \phi(y)),$$

for all $x, y \in \mathcal{B}^n$

Corollary 12.3.1 *For all $x \in \mathcal{B}^n$ we have*

$$d_B(0, x) = \log\left(\frac{1+|x|}{1-|x|}\right)$$

Proof. Let $x \in \mathcal{B}$. Then

$$\cosh(d_B(0, x)) = 1 + \frac{2|x|^2}{1-|x|^2} = \frac{1+|x|^2}{1-|x|^2}$$

and by recalling that $\operatorname{arccosh}(y) = \log(y + \sqrt{y^2 - 1})$, we come to the computation

$$\begin{aligned} d_B(0, x) &= \log\left(\frac{1+|x|^2}{1-|x|^2}\sqrt{\frac{(1+|x|^2)^2}{(1-|x|^2)^2} - 1}\right) \\ &= \log\left(\frac{1+|x|^2}{1-|x|^2} + \frac{2|x|}{1-|x|^2}\right) \\ &= \log\left(\frac{1+|x|}{1-|x|}\right) \end{aligned}$$

□

12.3.2 The isometry group of $\mathcal{B}^n$

We know at least that the isometry of $\mathcal{B}$ is as big as it's Möbius group. In fact, it cannot be bigger. Just like the Lorentz group for the hyperboloid model, we will need transitivity of the Möbius group on B^n in order to advance further.

Lemma 12.3.3 *The action of $\mathcal{M}(\mathcal{B}^n)$ on $\mathcal{B}^n$ is transitive.*

Proof. Let $a \in \mathcal{B}^n$, $a \neq 0$ and set $\sigma_a = \sigma(\frac{a}{|a|^2}, r)$ such that $r^2 = \frac{1}{|a|^2} - 1$. Then $\sigma \in \mathcal{M}(\mathcal{B}^n)$ since σ is a inversion of a sphere orthogonal to S^{n-1} and $\sigma_a(0) = a$. □

Theorem 12.3.4 *Every Möbius transformation of $\mathcal{B}^n$ restricts to an isometry on $\mathcal{B}^n$ and every isometry of $\mathcal{B}^n$ extends to a Möbius transformation*

Proof. As seen just above, every Möbius transformation stable on B^n is an isometry, so all that's left to prove is that every isometry of $\mathcal{B}^n$ is a restriction of a Möbius transformation. Let ϕ a isometric transformation on $\mathcal{B}^n$. We start by setting

$$\psi = \begin{cases} \sigma\phi & \text{if } \phi(0) \neq 0 \\ \phi & \text{if } \phi(0) = 0 \end{cases}$$

where σ is the inversion such that $\sigma(\phi(0)) = 0$. Then, $\psi(0) = 0$ and ψ is an isometry of $\mathcal{B}$. We notice that for $x, y \in \mathcal{B}$, we have

$$\begin{aligned} d_B(\psi(x), 0) = d_B(x, 0) \quad &\Longleftrightarrow \quad \frac{|\psi(x)|^2}{1 - |\psi(x)|^2} = \frac{|x|^2}{1-|x|^2} \\ &\Longleftrightarrow \quad |\psi(x)| = |x| \end{aligned}$$

and in the same way $|\psi(y)| = |y|$. From this we deduce that ψ is a also Euclidean isometry on $\mathcal{B}^n$:

$$\begin{aligned} d_B(\psi(x), \psi(y)) = d_B(x, y) \quad &\Longleftrightarrow \quad \frac{|\psi(x) - \psi(y)|^2}{(1 - |\psi(x)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \\ &\Longleftrightarrow \quad |\psi(x) - \psi(y)| = |x - y| \end{aligned}$$

. Hence, ψ preserves Euclidean distances and it can extended to $\overline{\mathcal{B}^n}$ by setting

$$\bar{\psi}(x) = 2\psi(\frac{x}{2}).$$

Because $\bar{\psi}$ preserves Euclidean distances and so the Euclidean inner product, $\psi(e_1), \ldots, \psi(e_n)$ is a orthonormal basis and so for $x \in \overline{B^n}$, $\bar{\psi}(x) = \sum_{i=1}^{n} c_i \bar{\psi}(e_i) = \sum_{i=1}^{n} x_i e_i$ with $\sum_{i=1}^{n} c_i^2 \leqslant 1$ and

$$\begin{aligned} \langle \bar{\psi}(x), \bar{\psi}(e_i) \rangle &= \langle x, e_i \rangle = x_i \\ &= c_i \end{aligned}$$

. From this we deduce that $\bar{\psi}$ is linear and $\bar{\psi} = \psi$. Hence, ψ is the restriction of an orthogononal transformation and so ϕ extends to a Möbius transformation.
Unicity : Suppose ϕ_1 and ϕ_2 are two Möbius transformations that both extend ϕ, ie $\phi = \phi_1\big|_{\mathcal{B}^n} = \phi_2\big|_{\mathcal{B}^n}$. Then for any sphere Σ in B^n, $\phi_2^{-1}\ \phi_1(\Sigma) = \Sigma$ and so by theorem 11.4.5, $\phi_1 = \phi_2$. $\square$

Corollary 12.3.2 *$\mathcal{I}(\mathcal{B}^n)$ and $\mathcal{M}(\mathcal{B}^n)$ are isomorphic.*

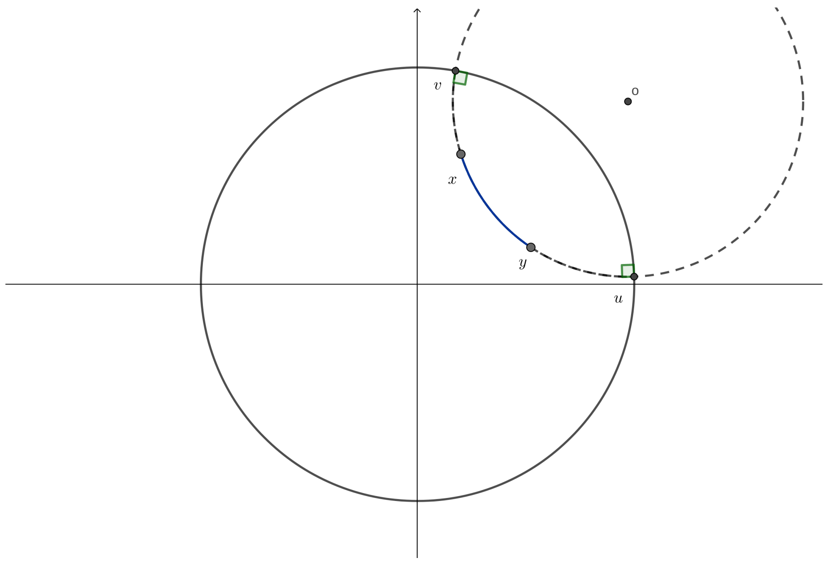
The conformal model is well known as having two ways to compute it's metric: one direct method that we have seen before, and another method by using the cross ratio. Weirdly enough, in literature we often see either one or the other and not both at the same time. Even less often, it's rarely shown exactly how both are related. A possible explanation of this, is that relating both metrics requires the use of many results from Möbius transformations:

1. The image of a sphere of $\hat{\mathbb{R}}^n$ of a Môbius transformation is also a sphere.
2. Möbius transformations preserve angles.
3. Möbius transformations preserves the cross-product.

Theorem 12.3.5 *Let $x, y \in \mathcal{B}^n$. Then,*

$$d_B(x, y) = \log([x, y, u, v]) = \log\big(\frac{|x-u||y-v|}{|x-v||y-u|}\big)$$

where u, v are the two points of intersection between a circle or line orthogonal to S^{n-1} containing x and y and S^{n-1} as in the figure below



Proof. Let $x, y \in \mathcal{B}^n$ distinct. We first suppose that $y = 0$. In this case, all that is needed to be done is to rewrite corollary 12.3.1

$$\begin{aligned} d_B(x,0) &= \log\big(\frac{1+|x|}{1-|x|}\big) \\ &= \log\big(\frac{|x-(-\frac{x}{|x|})||0-\frac{x}{|x|}|}{|x-\frac{x}{|x|}||0-(-\frac{x}{|x|})|}\big) \\ &= \log\big([x, 0, -\frac{x}{|x|}, \frac{x}{|x|}]\big) \end{aligned}$$

. Hence, $x, y, u = -\frac{x}{|x|}$ and $x = \frac{x}{|x|}$ all belong to the line $L_x = \{tx : t \in \mathbb{R}\}$ which is a orthogonal line to S^{n-1}. If $y \neq 0$, we set $\tilde{y} = \frac{y}{|y|^2}$ and $\sigma = \sigma(\tilde{y}, \sqrt{|\tilde{y}|^2 - 1})$. As such, $.\sigma(y) = 0$ and so

$$\begin{aligned} d_B(x,y) &= d_B(\sigma(x), \sigma(y)) = d_B(\sigma(x), 0) \\ &= [\sigma(x), \sigma(y), \tilde{u}, \tilde{v}] \end{aligned}$$

where $\tilde{u}$ and $\tilde{v}$ are the two points of intersection of the line $L_{\sigma(x)} = \{t\sigma(x) : t \in \mathbb{R}\}$ and S^{n-1}. We set $u = \sigma^{-1}(\tilde{u})$ and $v = \sigma^{-1}(\tilde{v})$. Since, $L_{\sigma(x)}$ is a Euclidean line, it become a circle $C_{x,y}$ under the transformation σ by the preservation of spheres of $\hat{\mathbb{R}}^n$ by the Möbius group and the fact that $\sigma(V(x,y)) = V(x,y)$. Furthermore, since $\sigma(x), \sigma(y), \sigma(u), \sigma(v)$ all belong to $L_{\sigma(x)}$, x, y, u, v all lie on $C_{x,y}$. Because $\sigma \in \mathcal{M}(\mathcal{B}^n)$, σ is stable on S^{n-1}, so u and v both belong to S^{n-1}. Finally, σ is conformal so $L_{\sigma(x)}$ is orthogonal to S^{n-1} implies $C_{x,y}$ is also orthogonal to S^{n-1}. ☐

Remark 12.3.1 *In the case that x and y do not lie on the same lines passing through 0, the circle C orthogonal to S^{n-1} and passing through x and y. A simple construction follows: if a is the center of the circle and r it's radius, then the orthogonality of the circle C with S^{n-1} forces a to be on the vector subspace $V(x,y)$ and lies on a line $L \subset V(x,y)$ passing through $\frac{x+y}{2}$ and orthogonal in $V(x,y)$ to the line $L_{x,y} = \{x + t(x-y) : t \in \mathbb{R}\}$. Moreover, the condition $r^2 = |a|^2 - 1$ forces a to be the unique point on the line L that verify this condition.*

A shorter proof but more profound proof, would be to remark that $x' = \frac{x}{|x|^2}$ and $y' = \frac{y}{|y|^2}$ must also lie this circle and since the circle can be defined from only three points, we are done.

From, this we get the intuition of the two geodesics of the conformal ball model: lines and circles orthogonal to $\mathcal{B}^n$. Naturally, the isometry group $\mathcal{I}(\mathcal{B}^n) = \mathcal{M}(\mathcal{B})$ should transfer any geodesic of $(\mathcal{B}^n, d_B)$ to other geodesics. In fact, we can generalise this to **m-spheres** or **m-planes** orthogonal to S^{n-1} defined as

- a $m - plane$ orthogonal to S^{n-1} is a vector subspace of $\mathbb{R}^n$ of dimension m
- a $m - sphere$ is the intersection between a sphere $S(a,r)$ and a vector subspace of $\mathbb{R}^n$ of dimension $m+1$

Theorem 12.3.6 *$\mathcal{M}(\mathcal{B}^n)$ is transitive on the set of combined set of m-spheres and m-planes orthogonal to S^{n-1}, with $m \in \{1, \ldots, n-1\}$.*

Proof. For the stability of the action of $\mathcal{M}(\mathcal{B}^n)$, the conformality of the Möbius transformations combined with the fact that Möbius transformations is stable on the set of spheres of S^{n-1} suffices. Let $m \in \{1, \ldots, n-1\}$ and set $V = \text{span}\{e_1, \ldots, e_m\}$. For any m-plane $\tilde{V}$ orthogonal to S^{n-1}, we have a rotation $\phi \in \mathrm{O}(n) \subset \mathcal{M}(\mathcal{B}^n)$ such that $\phi(V) = \tilde{V}$ (by transitivity of $\mathrm{O}(n)$ on vector subspaces of $\mathbb{R}^n$, a consequence of the Gramm-Schmidt decomposition). Furthermore, we can use this same argument to reduce the case of two m-spheres $\Sigma = S(a,r) \cap V_m$ and $\Sigma' = S(b,s) \cap V'_m$ orthogonal to S^{n-1}, to the case where they belong to the same vector subspace of dimension $m+1$ ($V_m = V'_m$), with a and b to be on the same line, ie $b = ka$ with $|k| > \frac{1}{|a|}$. If we set $a' = \frac{|a|-r}{|a|}a$ and $b' = \frac{|b|-s}{|b|}b$ then a' and b' are both in $\mathcal{B}^n$ and are points of Σ and Σ' respectively lying on the line $L_a = \{\lambda a : \lambda \in \mathbb{R}\}$. Then, transferring a' to b' via

$$\sigma = \sigma(\frac{b'}{|b'|^2}, r'_b)\sigma(\frac{a'}{|a'|^2}, r'_a) \in \mathcal{M}(\mathcal{B}^n)$$

with $r'^2_a = \frac{1}{|a'|^2} - 1$ and $r'_b = \frac{1}{|b'|^2} - 1$, allows us to transfer Σ to Σ'. All that's left is to show we can transfer Σ to a m-plane V (any will do). We do so by reusing $\sigma_a = \sigma(\frac{a'}{|a'|^2}, r'_a)$ which transfers $S(a,r)$ to a hyperplane P since we have $a' \in S(a,r)$ and $\sigma(a') = 0$, but 0 cannot belong to a Euclidean sphere orthogonal to S^{n-1}, so $\sigma(\Sigma) = P(a,0)$ a hyperplane with normal vector a. This is assured by the comformality of the transformation, if α is C^1-curve on $S(a,r)$ such that $\alpha(0) = a'$ and β a curve defined by $\beta(t) = \big(\frac{|a|-r}{|a|} + t\big)a$, then $\alpha'(0)$ and $\beta'(0)$ are orthogonal and so are their image by σ_a. Hence, if $\Sigma = S(a,r) \cap V$, with V a $m+1$ vector subspace, we have

$$\sigma_a(\Sigma) = V' = \langle a \rangle^{\perp} \cap V$$

a m-plane orthogonal to S^{n-1}. □

Lemma 12.3.4 *Let $x, y \in \mathcal{B}^n$ be two linearly dependent distinct points. Then, $z \in \mathcal{B}$ verifies*

$$d_B(x,y) = d_B(x,z) + d_B(z,y)$$

if and only if z lies between x and y (ie $z = tx + (1-t)y$).

Proof. We start with the assumption that $z = 0$ is between x and y. With such a condition, we have $\frac{x}{|x|} = -\frac{y}{|y|}$ and

$$\begin{aligned}
d_B(x,y) &= \log\left([x, y, -\frac{x}{|x|}, \frac{x}{|x|}]\right) = \log\left([x, y, \frac{y}{|y|}, -\frac{y}{|y|}]\right) \\
&= \log\left(\frac{|x + \frac{x}{|x|}||y + \frac{y}{|y|}|}{|x - \frac{x}{|x|}||y - \frac{y}{|y|}|}\right) \\
&= \log\left(\frac{|x + \frac{x}{|x|}|}{|x - \frac{x}{|x|}|}\right) + \log\left(\frac{|y + \frac{y}{|y|}|}{|y - \frac{y}{|y|}|}\right) \\
&= d_B(x,0) + d_B(0,y)
\end{aligned}$$

If we now consider $x, y \in \mathcal{B}$ are any linearly dependent points and z a point between x and y, then we can send z to 0 through the inversion σ_z (as in lemma 12.3.3) and because σ_z leaves the line $L_{x,y}$ invariant on $\mathcal{B}^n$ and maintains the image of z between the images of x and y,

$$\begin{aligned}
d_B(x,y) &= d_B(\sigma_z(x), \sigma_z(y)) = d_B(\sigma_z(x), 0) + d_B(0, \sigma_z(y)) \\
&= d_B(x,z) + d_B(z,y)
\end{aligned}$$

Conversely, suppose

$$d_B(x,y) = d_B(x,z) + d_B(z,y),$$

with z not linearly dependent with x and y.

$$\vdots$$

(*to be continued, proof in progress ...*)

□

Corollary 12.3.3 *The geodesics arcs of conformal ball model are the 1-planes and 1-spheres orthogonal to S^{n-1}.*

We could have shown the geodesic of the conformal model directly from the Hyperboloid model, by how the geodesics are transferred. However, it is useful to note that even if we chose to present the conformal model as a "descendant" of the hyperboloid model, it is very much a model of hyperbolic geometry that holds by itself. In fact quite often, the conformal ball model is presented as the Poincaré disc (the 2-dimensional version) by it's own and completely separated from the hyperboloid model.

12.3.3 The upper-half plane $\mathcal{U}^n$

Once the conformal model of the open unit ball $\mathcal{B}^n$ is done, the conformal model of the upper-half plane $\mathcal{U}^n$ is a walk in a park. Through the conformality of the Möbius transformation $\eta = \pi\rho$ (the bijection $\mathcal{U}^n \to \mathcal{B}^n$), with $\pi = \sigma(e_n, \sqrt{2})$ and $\rho = p(e_n, 0)$, the upper-half plane easily inherits all of the properties of the conformal model of the open ball $\mathcal{B}^n$. Naturally, we define the hyperbolic metric on $\mathcal{U}^n$ by defining η as an hyperbolic isometry between $\mathcal{B}^n$ and $\mathcal{U}^n$

Def. 12.3.2 *The hyperbolic metric on $\mathcal{U}^n$ is given by*

$$d_U(x, y) = d_B(\eta(x), \eta(y))$$

Similarly to the case of $\mathcal{B}^n$, we have two 'clean' version of the hyperbolic metric on $\mathcal{U}^n$.

Theorem 12.3.7 *Let $x, y \in \mathcal{U}^n$ distinct. Then,*

$$\cosh(d_U(x, y)) = 1 + \frac{|x - y|^2}{2x_n y_n}$$

Proof.

$$\begin{aligned}
\cosh(d_U(x, y)) &= \cosh(d_B(\eta(x), \eta(y))) \\
&= 1 + \frac{2|\eta(x) - \eta(y)|}{(1 - |\eta(x)|^2)(1 - \eta(y)^2)} \\
&= 1 + 2\frac{4|\eta(x) - \eta(y)|^2}{|\eta(x) - e_n|^2 |\eta(y) - e_n|^2} \times \frac{|\eta(x) - e_n|^2}{4x_n} \frac{|\eta(y) - e_n|^2}{4y_n} \\
&= 1 + \frac{|x - y|^2}{2x_n y_n}
\end{aligned}$$

□

Since the transformation between $\mathcal{B}^n$ and $\mathcal{U}^n$ is a Möbius transformation, thus conformal, we have the exactly the same results than on $\mathcal{B}^n$:

Theorem 12.3.8 *We have the following:*

- *the Isometry group of $\mathcal{U}^n$ is isomorphic to it's Möbius group $\mathcal{M}(\mathcal{U}^n)$*
- *The geodesics of $\mathcal{U}^n$ are the lines and circles orthogonal to E^{n-1}*
- *If x and y are two distinct points of U^n, then*

$$d_U(x, y) = \log([x, y, u, v])$$

where u and v are the points of intersection between the geodesic γ_{xy} that passes through x and y and E^{n-1}

Theorem 12.3.9 *The element of hyperbolic arc length on the upper-half plane is given by:*

$$\|dx\|_U = \frac{|dx|}{x_n}$$

Proof. Let $x \in \mathcal{U}^n$ and $y = \eta(x)$. Then, from

$$y = e_n + \frac{2}{|\rho(x) - e_n|^2}(\rho(x) - e_n),$$

we have for $i < n$:

$$y_i = \frac{2x_i}{|x + e_n|^2} \qquad \text{and} \qquad y_n = 1 - \frac{2(x_n + 1)}{|x + e_n|^2} \ .$$

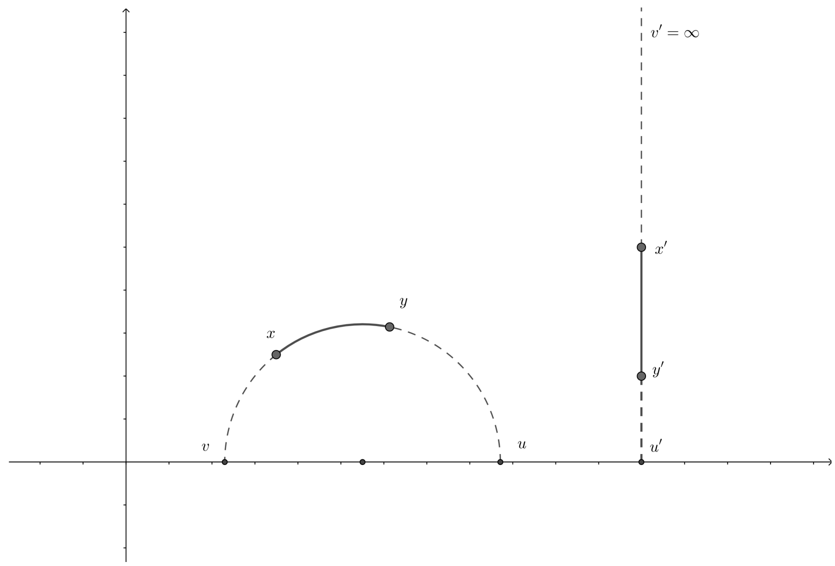


Figure 12.7: The two type of lines of the upper-half plane: half-circles and Euclidean lines orthogonal to E^{n-1}

Then for h arbitrarily close to 0 and $i \in \{1, \ldots, n-1\}$ we find

$$\begin{aligned}
\eta_i(x+h) &= \frac{2(x_i+h_i)}{|x+h+e_n|^2} \quad = \quad \frac{2(x_i+h_i)}{|x+e_n|^2} \\
&= \frac{2(x_i+h_i)}{|x+e_n|^2 + 2\langle h, x+e_n\rangle + |h|^2} \\
&\overset{2}{=} \frac{2(x_i+h_i)}{|x+e_n|^2}\left(1 - 2\frac{\langle h, x+e_n\rangle}{|x+e_n|^2} + \mathrm{O}(|h|^2)\right) \\
&= \eta_i(x) + \frac{2h_i}{|x+e_n|^2} - \frac{4x_i\langle h, x+e_n\rangle}{|x+e_n|^4} + \mathrm{O}(|h|^2)
\end{aligned}$$

$$\begin{aligned}
\eta_n(x+h) &= 1 - \frac{2(x_n+h_n+1)}{|x+h+e_n|^2} \\
&= 1 - \frac{2(x_n+1)+2h_n}{|x+e_n|^2 + 2\langle h, x+e_n\rangle + |h|^2} \\
&\overset{4}{=} 1 - \frac{(2(x_n+1)+2h_n)}{|x+e_n|^2}\left(1 - \frac{2\langle h, x+e_n\rangle}{|x+e_n|^2} + \mathrm{O}(|h|^2)\right) \\
&= \eta_n(x) + \frac{2h_n}{|x+e_n|^2} + \frac{4(x_n+1)\langle h, x+e_n\rangle}{|x+e_n|^4} + \mathrm{O}(|h|^2)\ .
\end{aligned}$$

Hence we have,

$$dy_i = \frac{2dx_i}{|x+e_n|^2} - \frac{4x_i\langle dx, x+e_n\rangle}{|x+e_n|^4} \qquad \text{and} \qquad dy_n = -\frac{2dx_n}{|x+e_n|^2} + \frac{4(x_n+1)\langle dx, x+e_n\rangle}{|x+e_n|^4}$$

[2] we use the approximation $\frac{1}{1+x} = 1 - x + \mathrm{O}(x^2)$

which brings us to the following computation

$$\begin{aligned}
|dy|^2 &= dy_n^2 + \sum_{i=1}^{n} dy_i^2 \\
&= \left(-\frac{2dx_n}{|x+e_n|^2} + \frac{4(x_n+1)\langle dx, x+e_n\rangle}{|x+e_n|^4} \right)^2 + \sum_{i=1}^{n-1} \left(\frac{2dx_i}{|x+e_n|^2} - \frac{4x_i\langle dx, x+e_n\rangle}{|x+e_n|^4} \right)^2 \\
&= \frac{4}{|x+e_n|^4} \left(|dx|^2 - \frac{4\langle dx, x+e_n\rangle^2}{|x+e_n|^2} + \frac{4|x+e_n|^2\langle dx, x+e_n\rangle^2}{|x+e_n|^4} \right) \\
&= \frac{4|dx|^2}{|x+e_n|^4} \\
\implies |dy| &= \frac{2|dx|}{|x+e_n|^2} .
\end{aligned}$$

Furthermore, by reusing **??** we have

$$1 - |y|^2 = \frac{4x_n}{|x+e_n|^2}$$

which allows us to conclude with

$$\begin{aligned}
\|dx\|_U &= \|dy\|_B = \frac{2|dy|}{1-|y|^2} \\
&= \frac{4|dx|}{|x+e_n|^2} \frac{|x+e_n|^2}{4x_n} \\
&= \frac{|dx|}{x_n}
\end{aligned}$$

□

On a final note: we can observe that the upper half plane when compared to the other models is the 'furthest' away in the sense that it is the most different in it's geometry. This allows us to obtain a very different point of view (very useful in some occasions) and it's arc length metric is the most practical of the four hyperbolic model.

12.4 The Projective model $\mathcal{K}^n$

The projective model $\mathcal{K}^n$ as it's name implies, is the embedding of the hyperboloid model $\mathcal{H}^n$ in the projective space $\mathbb{RP}^n$. Quite often, this model is presented in it's $2-$dimensional version as the Klein disc (hence $\mathcal{K}$ for Klein). In fact we have actually seen this model previously in 9.4, without naming it as such. We start by reminding the projective group and space:

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \diagup_{\mathbb{R}^\times} \quad \text{and} \quad PGL(n+1,\mathbb{R}) = GL(n+1) \diagup_{\mathbb{R}^\times} .$$

Moreover, we will use in what follows the notation for $x \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}^\times$

$$x \cdot \mathbb{R}^\times = [x] = [\lambda x].$$

The embedding of $\mathbb{R}^n$ in $\mathbb{RP}^n$ is given by

$$\mathbb{R}^n \simeq \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} : x \in \mathbb{R}^n \right\} \subset \mathbb{RP}^n.$$

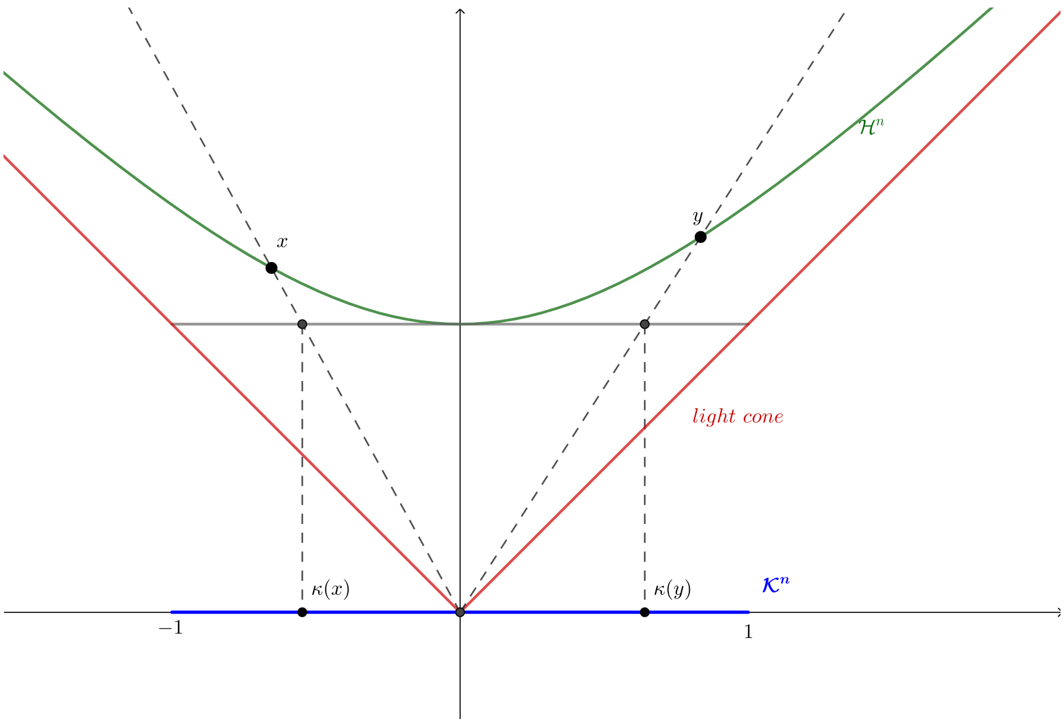


Figure 12.8: The gnomonic projection between $\mathcal{H}^n$ and $\mathcal{K}^n$

While the conformal model stands by itself in it's own way it is less the case of the projective model $\mathcal{K}^n$, as it's geometry descends directly from the hyperboloid model.

$$\begin{array}{ccccc} \mathcal{H}^n & \longrightarrow & \mathbb{RP}^n & \longrightarrow & \mathcal{K}^n \subset \mathbb{R}^n \\ x & \longmapsto & [x] = \left[\frac{x}{x_{n+1}}\right] & \longmapsto & (\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}) \end{array}.$$

Additionally, by the fact that every Euclidean line passing through 0 in the cone of time-like vectors contains a unique element of $\mathcal{H}^n$, our previous work in 9.4 and more specifically theorem 9.4.1 allows us to deduce that this mapping is a bijection and the projective model is given by

$$\mathcal{K}^n = \{x \in \mathbb{R}^n : |x| < 1\}.$$

Notice here that while the projective model of hyperbolic geometry is the same set than the conformal ball model, their geometry as we shall see is very different: the conformal ball model maintains the notion of Euclidean angles but has curved space whereas the projective space will maintain Euclidean lines. Explicitly, if we define κ as the mapping $\mathcal{H}^n \to \mathcal{K}^n$, then for $x \in \mathcal{H}^n$ and $y \in \kappa^n$ we have

$$\kappa(x) = (\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}})$$

and

$$\kappa^{-1}(y) = \frac{y + e_{n+1}}{|\,\|y + e_{n+1}\|\,|} \quad .$$

12.4.1 The hyperbolic metric on the projective model and it's group of isometry

Consistently with what we have done with the conformal model, we define the hyperbolic metric on $\mathcal{K}^n$ by setting κ, the bijection between $\mathcal{H}^n$ and $\mathcal{K}^n$, as an hyperbolic isometry between both spaces.

Def. 12.4.1 *The hyperbolic metric on $\mathcal{K}^n$ for is defined as*

$$d_K(x, y) = d_H(\kappa^{-1}(x), \kappa^{-1}(y))$$

for all $x, y \in \mathcal{K}^n$.

Similarly to before, we have a simple hyperbolic cosine version of the metric:

Theorem 12.4.1 *For all $x, y \in= \mathcal{K}^n$, we have*

$$\cosh(d_K(x, y)) = \frac{1 - \langle x, y\rangle}{\sqrt{1 - |x|^2}\sqrt{1 - |y|^2}}$$

Proof.

$$\begin{aligned} \cosh(d_K(x, y)) &= \cosh(d_H(\kappa^{-1}(x), \kappa^{-1}(y))) \\ &= -\big(\frac{x + e_{n+1}}{|\,\|x + e_{n+1}\|\,|}\big) \circ \big(\frac{y + e_{n+1}}{|\,\|y + e_{n+1}\|\,|}\big) \\ &= \frac{1 - \langle x, y\rangle}{\sqrt{1 - |x|^2}\sqrt{1 - |y|^2}} \end{aligned}$$

□

As one can see the metric isn't as nice as the other metrics of the other models: the Euclidean inner product $\langle x, y\rangle$ the metric not only depends on how far you are from the point (in a Euclidean way), but the direction also distorts the space. This is the reason why projective model is often said to be the model that doesn't keep Euclidean angles. In fact, the hyperbolic arc length isn't any more welcoming...

Theorem 12.4.2 *The element of hyperbolic arc length on the projective model $\mathcal{K}^n$ is given by*

$$\|dx\|_K = \frac{\sqrt{(1 - |x|^2)|dx|^2 + \langle x, dx\rangle^2}}{1 - |x|^2}$$

We can also obtain the metric by using the cross-ratio in a similar way to the conformal ball model.

Corollary 12.4.1 *For all $x \in \mathcal{K}^n$ we have*

$$d_K(0, x) = \frac{1}{2}\log\big(\frac{1 + |x|}{1 - |x|}\big) = \frac{1}{2}\log([x, 0, -\frac{x}{|x|}, \frac{x}{|x|}])$$

Proof.

$$\begin{aligned} d_K(0,x) &= \operatorname{arcosh}(\frac{1}{\sqrt{1-|x|^2}}) = \log\left(\frac{1}{\sqrt{1-|x|^2}} + \sqrt{(\frac{1}{\sqrt{1-|x|^2}})^2 - 1}\right) \\ &= \log\big(\frac{1+|x|}{\sqrt{1-|x|^2}}\big) = \frac{1}{2}\log\big(\frac{1+|x|}{1-|x|}\big) \\ &= \frac{1}{2}\log([x, 0, -\frac{x}{|x|}, \frac{x}{|x|}]) \end{aligned}$$

□

The thing to notice here is that $-\frac{x}{|x|}$ and $\frac{x}{|x|}$ are the two points of intersection between the Euclidean line that passes through 0 and x and S^{n-1}. In fact, as we shall see later on, we can extend this formulation of the metric for any points $x, y \in \mathcal{K}^n$.

Def. 12.4.2 *The action of a projective transformation $\phi \in PGL(n+1, \mathbb{R})$ on $\mathbb{R}^n$ is defined by*

$$\phi : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ x = \begin{bmatrix} x \\ 1 \end{bmatrix} & \longmapsto & \left[\phi\begin{pmatrix} x \\ 1 \end{pmatrix}\right] = \begin{bmatrix} y \\ y_{n+1} \end{bmatrix} = \frac{y}{y_{n+1}} \end{array} .$$

Note that projective transformation are not always well defined on all of $\mathbb{R}^n$ since $y_{n+1} = \left\langle \phi\begin{pmatrix} x \\ 1 \end{pmatrix}, e_{n+1} \right\rangle$ can be zero. We now search for the set of projective transformation that leave $\mathcal{K}^n$ invariant.

Lemma 12.4.1 *Let $\phi \in GL(n+1, \mathbb{R})$. Then ϕ leaves the light cone $\{z \in \mathbb{R}^{n+1} : \|z\| \leqslant 1\}$ invariant if and only if there is a scalar $\lambda > 0$ such that $\lambda\phi$ is a Lorentz transformation.*

Proof. Suppose we have $\phi \in GL(n+1, \mathbb{R})$ such that it leaves the light cone invariant. By continuity of ϕ, ϕ also leaves the inside of the light cone (ie the set time like vectors) invariant and by the same argument it also leaves the set of light-like vectors invariant. Hence, $\phi(e_{n+1})$ is time like. Furthermore, by the transitivity of $\mathrm{O}(n,1)$ on the $1-$dimensional time-like vector subspace, there is a Lorentz transformation $A\ in \mathrm{O}(n,1)$ such that

$$A\phi(e_{n+1}) = \lambda e_{n+1},$$

with $\lambda > 0$. All that's left to show is that $\lambda^{-1}A\phi \in \mathrm{O}(n+1) \cap \mathrm{PO}(n,1)$ (see 10.3.3). Let $x \in R^{n+1}$ be linearly independent to e_{n+1} and $B_x \in \mathrm{O}(n+1) \cap \mathrm{O}(n,1)$ such that $\tilde{\phi} = \lambda^{-1}B_x A\phi$ leaves $V(x, e_{n+1})$ invariant and fixes e_{n+1}. Consequently, we may assume $n = 1$ and because $\tilde{\phi}$ leaves e_{n+1} unchanged, it is of the form

$$\tilde{\phi} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

Since $\tilde{\phi}$ is stable on the set of light-like vectors, we have

$$\begin{aligned} \left\|\tilde{\phi}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\|^2 = 0 = \left\|\tilde{\phi}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|^2 &\iff a^2 - (b-1)^2 = a^2 - (b+1)^2 \\ &\iff b = 0 \\ &\implies a = \pm 1. \end{aligned}$$

Hence, $\tilde{\phi} \in \mathrm{O}(n+1) \cap \mathrm{O}(n,1)$ which implies $\lambda^{-1} A\phi \in \mathrm{O}(n+1) \cap \mathrm{O}(n)$ and that $\lambda^{-1}\phi$ is a Lorentz transformation. □

Lemma 12.4.2 *A projective transformation $[\phi] \in PGL(n+1, \mathbb{R})$ leaves $\mathcal{K}^n$ invariant if and only if any element of it's class (ie $\lambda\phi$ with $|\lambda| > 0$) leaves the light cone invariant.*

Proof. This proof is a direct consequence of 9.4.1 □

Hence by combining both lemmas we come to the conclusion that every projective transformation that leaves $\mathcal{K}^n$ invariant is the class of a unique positive Lorentz transformation.

Theorem 12.4.3 *Every isometry of $\mathcal{K}^n$ extends to a unique projective transformation that leaves $\mathcal{K}^n$ invariant and every projective transformation that leaves $\mathcal{K}^n$ invariant can be restricted to an isometry.*

Proof. The isometries of $\mathcal{H}^n$ are it's Lorentz transformation and via the isometry $\kappa : \mathcal{H}^n \to \mathcal{K}$ correspond to the isometries of $\mathcal{K}^n$. Hence, by applying both lemmas we obtain the theorem. □

Corollary 12.4.2 $\mathcal{I}(\mathcal{K}^n) = PO(n,1) \diagup_{\mathbb{R}^\times}$

Corollary 12.4.3 *A isometry of $\mathcal{K}^n$ fixes 0 if and only if it is the restriction of a orthogonal transformation of $\mathbb{R}^n$ on $\mathcal{K}^n$.*

While the projective model does not maintain the Euclidean notion of angles, it offers a big advantage compared to other models: the hyperbolic lines of the projective model are exactly the Euclidean lines. This makes this model very useful for convexity arguments. It's worth mentioning that we cannot have a hyperbolic model that retains both the Euclidean lines and the conform with the Euclidean angles since otherwise we return back to the Euclidean model of geometry: in hyperbolic geometry we cannot have the cake and eat it !

Theorem 12.4.4 *The hyperbolic lines of $\mathcal{K}^n$ are the Euclidean lines restricted to $\mathcal{K}^n$.*

As announced previously, we can give a version of the hyperbolic metric on $\mathcal{K}^n$ using the cross-ratio. In the literature, this is known as the Cayley-Klein metric. This formulation of the hyperbolic metric can also be extended to bounded convex sets, in which case it is called the Hilbert metric.

Theorem 12.4.5 *The hyperbolic metric on $\mathcal{K}^n$ is given by:*

$$d_K(x,y) = \frac{1}{2}\log\big([x,y,u,v]\big),$$

for all $x, y \in \mathcal{K}^n$ distinct and u, v the two points of intersection of the Eulcidean line ℓ_x, y and S^{n-1} such that $|x-u| > |y-u|$ and $|y-v| > |x-v|$.

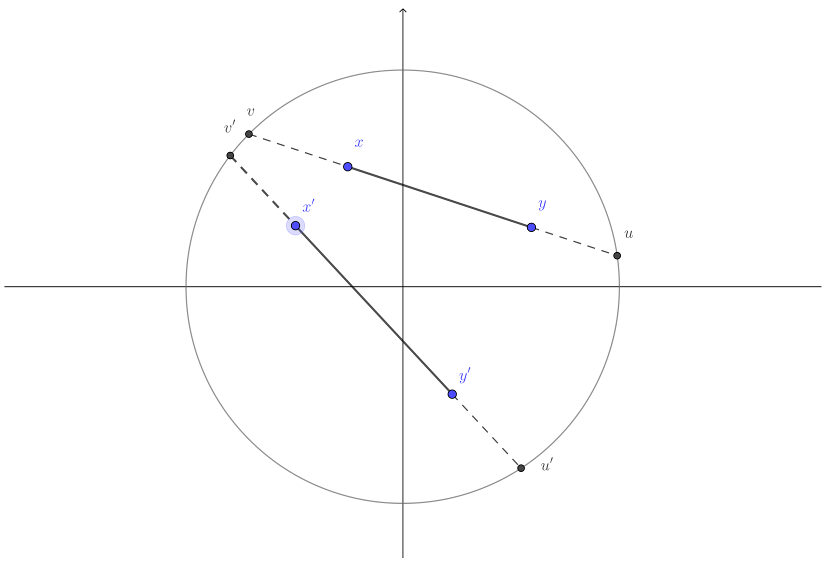


Figure 12.9: Example of the cross-ratio points in the case of Klein disc (2−dimensional case)

12.4.2 Birkhoff's version of Hilbert's metric on convex sets

In this section we will study a generalisation of the Cayley-Klein metric to any convex set: the Hilbert metric. To do so, we will introduce a second metric Birkhoff metric that will allow us to extend the easily prove that the Hilbert metric is well defined on any convex set. This section is mainly based on [11].

Def. 12.4.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set, non-empty. We define the Hilbert metric on Ω as*

$$\delta(x, y) = \begin{cases} \log\left([x, y, u, v]\right) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in \Omega$ and with $u, v \in \partial\Omega$ defined as the points of intersection of the Euclidean line passing through x and y and the border of Ω such that $|x - u| > |y - u|$ and $|y - v| > |x - v|$.

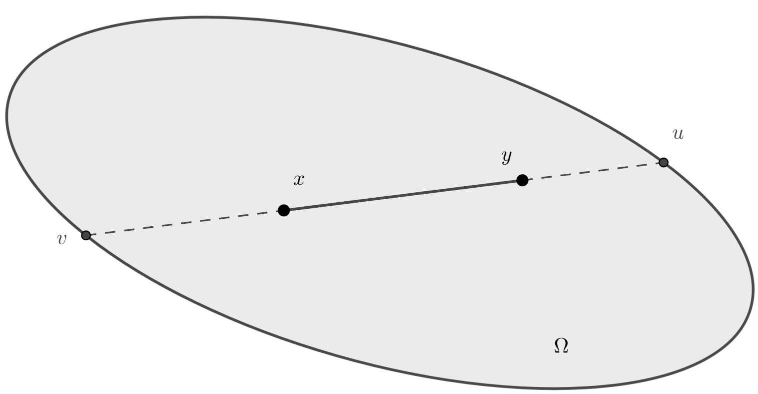


Figure 12.10: Example of the Hilbert metric on a 2−dimensional convex set

To show that the Hilbert metric is well defined we will be using a path different from how the Hilbert metric is usually introduced by introducing Birkhoff's version of the Hilbert metric on cones.

Def. 12.4.4 *Let V be a vector space and $\mathcal{C} \subset V$ a subset. We say that $\mathcal{C}$ is a cone if $\mathcal{C}$ verifies*

1. *$\mathcal{C}$ is convex: $\forall x, y \in \mathcal{C}$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in \mathcal{C}$*
2. *$\lambda\mathcal{C} \subseteq \mathcal{C}$, for all $\lambda > 0$.*
3. *$\bar{\mathcal{C}} \cap (-\bar{\mathcal{C}}) = \{0\}$*

Mathematical tradition dictates us to make the following remark: by combining 1. and 2. of the definition of a cone we obtain that a cone is stable under addition. In what follows, we shall consider V to be a vector space and $\mathcal{C} \subset V$ a cone. In order to define Birkhoff's metric, we introduce a partial ordering on cone.

Def. 12.4.5 *We define the partial ordering on the $\mathcal{C}$ for $x, y \in \mathcal{C}$ as*

$$x \leqslant_c y \iff y - x \in \mathcal{C}.$$

*Furthermore, we say that y **dominates** x if there exists $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha y \leqslant_c x \leqslant_c \beta y$$

and the equivalence relationship given by this partial ordering as

$$x \sim_c y \iff y \textit{ \textbf{dominates} } x \textit{ and } x \textit{ \textbf{dominates} } y.$$

In the case that y dominates x, we note the following quantities:

$$\begin{aligned} M\big(\frac{x}{y}\big) &= \inf\{\beta \in \mathbb{R} : x \leqslant_c \beta y\} \\ m\big(\frac{x}{y}\big) &= \sup\{\alpha \in \mathbb{R} : \alpha_c \leqslant_c x\} \end{aligned}$$

Lemma 12.4.3 *If $x, y \in \mathcal{C}\backslash\{0\}$, then $x \sim_c y$ if and only if there is $0 < \alpha \leqslant \beta$ such that*

$$\alpha y \leqslant_c x \leqslant \beta y.$$

Moreover, if $x \sim_c y$ we have

$$m\big(\frac{x}{y}\big) = \sup\{\alpha > 0 : y \leqslant \frac{1}{\alpha} x\} = M\big(\frac{y}{x}\big)^{-1}$$

Def. 12.4.6 *We define the Birkhoff metric on the cone $\mathcal{C}$ as*

$$d(x,y) = \begin{cases} \log\left(\frac{M(\frac{x}{y})}{m(\frac{x}{y})}\right) & \textit{if } x \sim_c y \textit{ and } y, x \neq 0 \\ 0 & \textit{if } x = y = 0 \\ \infty & \textit{otherwise} \end{cases}$$

Theorem 12.4.6 *Let $x, y, z \in \mathcal{C}\backslash\{0\}$ such that $x \sim_c y \sim_c z$. Then,* *1.* $d(x,y) \geqslant 0$

2. $d(x,y) = d(y,x)$

3. $d(x,z) \leqslant d(x,y) + d(y,z)$

4. $d(x,y) = d(\lambda x, \mu y)$ *for all* $\lambda, \mu > 0$

Moreover, if V *is a Banach space, then* $d(x,y) = 0$ *if and only if* $x = \lambda y$ *for some* $\lambda \geqslant 0$.

Proof. Let $x, y, z \in \mathcal{C}\backslash 0$ such that $x \sim_c y \sim_c z$.

1. We take note that if $0 < \alpha < m(\frac{x}{y})$ and $0 < M(\frac{x}{y}) < \beta$, we have $\alpha y \leqslant_c x \leqslant_c \beta y$ and $y \leqslant_c \frac{\beta}{\alpha} y$ since

$$\begin{aligned} \frac{\beta}{\alpha} y - y \in \mathcal{C} &\iff \beta y - \alpha y \in \mathcal{C} \\ &\iff (\underbrace{\beta y - x}_{\in \mathcal{C}}) + (\underbrace{x - \alpha y) \in \mathcal{C}}_{(\in \mathcal{C})}) \in \mathcal{C}. \end{aligned}$$

Hence, $(\frac{\beta}{\alpha} - 1) y \in \mathcal{C}$ and consequently $\frac{\beta}{\alpha} - 1 \leqslant 0$ since $\bar{\mathcal{C}} \cap (-\bar{\mathcal{C}}) = \{0\}$. To conclude, if we note $(\alpha_n)_{n \geqslant 0}$ and $(\beta_n)_{n \geqslant 0}$ two real sequences such that $0 < \alpha_n < m(\frac{x}{y})$, $0 < M(\frac{x}{y}) < \beta_n$,

$$\lim_{n \to \infty} \alpha_n = m(\frac{x}{y}) \quad \text{and} \quad \lim_{n \to \infty} \beta_n = M(\frac{x}{y}),$$

then we obtain through the limit

$$\frac{M(\frac{x}{y})}{m(\frac{x}{y})} = \lim_{n \to \infty} \frac{\alpha_n}{\beta_n} \geqslant 1.$$

2. To prove the second point, we simply need to use $m(\frac{x}{y}) = M(\frac{y}{x})^{-1}$:

$$\begin{aligned} d(x,y) &= \log\big(\frac{M(\frac{x}{y})}{m(\frac{x}{y})}\big) = \log\big(M(\frac{x}{y}) M(\frac{y}{x})\big) \\ &= \log\big(\frac{M(\frac{y}{x})}{m(\frac{y}{x})}\big) = d(y,x)\ . \end{aligned}$$

3. Let α,β as before, $0 < \lambda < m\big(\frac{y}{z}\big)$ and $0 < M\big(\frac{y}{z}\big) < \mu$. Then, we have $\alpha y \leqslant_c x$ and $\lambda z \leqslant_c y$, which when combined gives us $\alpha\lambda z \leqslant x$, thus

$$0 < \alpha\lambda < m\big(\frac{x}{y}\big).$$

Similarly by combining $x \leqslant_c \beta y$ and $y \leqslant_c \mu z$, we obtain $x \leqslant_c \beta\mu z$ and so

$$M\big(\frac{x}{z}\big) \leqslant \beta\mu$$

and by pushing α,β,λ and μ to their respective *sup* or *inf* limits, we obtain

$$m\big(\frac{y}{z}\big) m\big(\frac{x}{y}\big) \leqslant m\big(\frac{x}{z}\big) \quad \text{and} \quad M\big(\frac{x}{z}\big) \leqslant M\big(\frac{x}{y}\big) M\big(\frac{y}{z}\big).$$

This allows us to directly conclude:

$$\begin{aligned} d(x,z) &= \log\left(\frac{M(\frac{x}{z})}{m(\frac{x}{z})}\right) \\ &\leqslant \log\left(\frac{M(\frac{x}{y})}{m(\frac{x}{y})}\frac{M(\frac{y}{z})}{m(\frac{y}{z})}\right) \\ &\leqslant d(x,y)+d(y,z)\ . \end{aligned}$$

4. Let $\lambda, \mu > 0$. Then,

$$M\big(\frac{\lambda x}{\mu y}\big) = \frac{\lambda}{\mu}M\big(\frac{x}{y}\big) \quad \text{and} \quad m\big(\frac{\lambda x}{\mu y}\big) = \frac{\lambda}{\mu}m\big(\frac{x}{y}\big),$$

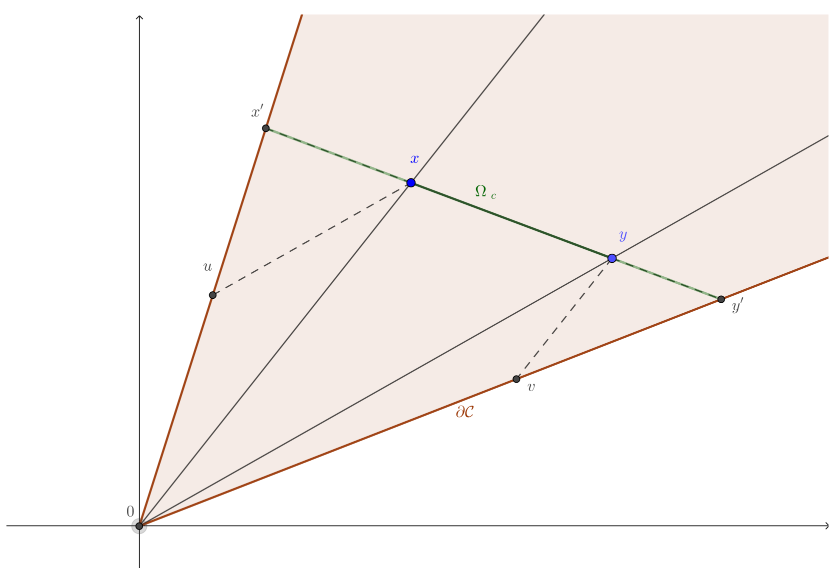
and so

$$d(x,y) = d(\lambda x, \mu y).$$

$\square$

Theorem 12.4.7 *Let $\mathcal{C} \subset V$ a closed cone in a $(n+1)-$dimensional vector space with a non-empty interior (ie $\overset{o}{\mathcal{C}} \neq \{\emptyset\}$) and $H \subset V$ be a n-dimensional affine hyperplane such that $\Omega_c = H \cap \overset{o}{\mathcal{C}}$ is a open, bounded and convex set. Then, the restriction of the Birkhoff metric d on Ω_c coincides with the Hilbert metric δ.*

Proof. Let $x, y \in \Omega_c$ distinct, $\alpha = m\big(\frac{x}{y}\big) = M\big(\frac{y}{x}\big)^{-1}$ and $\beta = M\big(\frac{x}{y}\big)$. We remark that because $\mathcal{C}$ is closed, we have αy and $x \leqslant \beta y$. We set $u = x - \alpha y \in \partial\mathcal{C}$, $v = y - \frac{1}{\beta}x \in \partial\mathcal{C}$, $\ell_{x,y}$ the Euclidean line passing through x and y and $x', y' \in \partial\Omega_c$ the two points of intersection between $\ell_{x,y}$ and $\partial\Omega_c$ such that $|y - x'| > |x - x'|$ and $|x - y'| > |y - y'|$.



Since x' and y' do not lie between x and y, we have $\lambda, \mu > 1$ such that

$$x' = y + \lambda(x-y) \quad \text{and} \quad y' = x + \mu(y-x).$$

Now let $\phi \in V^*$ be a linear functional such that[3]

$$H = \{z \in V : \phi(z) = 1\}$$

and we remark that

$$\begin{aligned} y + \lambda(x-y) = x' = \frac{u}{\phi(u)} = \frac{x - \alpha y}{1-\alpha} \quad &\Longrightarrow \quad \alpha = \frac{\lambda - 1}{\lambda} \\ x + \mu(y-x) = y' = \frac{v}{\phi(v)} = \frac{y - \beta^{-1} x}{1 - \beta^{-1}} \quad &\Longrightarrow \quad \beta = \frac{\mu}{\mu - 1} \end{aligned}$$

which leads us to

$$\frac{|y-x'|}{|x-x'|} = \frac{\lambda}{1-\lambda} = \frac{1}{\alpha} = M\big(\frac{y}{x}\big) = m\big(\frac{x}{y}^{-1}\big)$$

and

$$\frac{|x-y'|}{|y-y'|} = \frac{\mu}{1-\mu} = \beta = M\big(\frac{x}{y}\big).$$

Finally we may conclude:

$$d(x,y) = \log\left(\frac{M(\frac{x}{y})}{m(\frac{x}{y})}\right) = \log\Big(\frac{|y-x'|}{|x-x'|}\frac{|x-y'|}{|y-y'|}\Big) = \delta(x,y)$$

$\square$

Corollary 12.4.4 *The Hilbert metric δ is well defined on any open bounded convex set.*

Proof. If $\Omega \subset \mathbb{R}^n$ is a convex, bounded and open set, then we embed it in $\mathbb{R}^{n+1}$ as $\Omega' = \{\begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \Omega\}$, set $\mathcal{C} = \{\lambda x : x \in \bar{\Omega}', \lambda \geqslant 0\}$ and $H = \{\begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R}^{n+1}\}$. $\square$

On a last word, this corollary is in fact the weak version to a much stronger result: we can extend the Hilbert metric to any open, convex, possibly unbounded, subset of an infinite-dimensional Banach space !

[3] If a is a normal vector of H and $t \in \mathbb{R}$ such that $ta \in H$, then we have

$$H = \{x \in V : \langle x, a\rangle = t\} = \{x \in V : \underbrace{\Big\langle \frac{a}{t} \Big\rangle, x}_{=\phi(x)} = 1\}$$

Part III:

Applications

The only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colors.

B. Riemann, 1854

Chapter 13

The standard formulation of special relativity (Valérie Garcin, Nicoletta Prencipe and Edoardo Provenzi)

The concept of an absolute space has been abandoned since Galilean relativity, in which 'space-time' is interpreted as the metric space $(\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3, dt^2 \otimes d\ell^2)$, where dt and $d\ell$ are the Euclidean metric on $\mathbb{R}$ and $\mathbb{R}^3$, respectively. Special relativity is known to be an extension of Galilean relativity in which, along with the motion of objects with mass, also the peculiar behavior of electromagnetic signals propagation is taken into account. As we will recap soon, considering also this kind of signals impose to give up the concept of an absolute time and to build a 'spacetime' where both space and time are relative to an observer and not absolute.

Formally, Galilean relativity is based on the following two postulates:

1. the space is homogeneous and isotropic and the time is homogeneous[1];

2. laws of physics[2] have the same form in all inertial (i.e. not accelerated) reference frames, i.e. no inertial reference frame is privileged.

In special relativity Einstein added the following, fundamental, postulate:

3. the speed of light in vacuum has a constant value c when measured in all inertial reference frames.

These postulates constitute the minimal set of axioms able to determine in a unique way the metric of spacetime and the analytic form of the coordinate changes from one inertial frame to another.

We will start with the metric issue. Using a standard nomenclature, we call *event* e a point in $\mathbb{R}^4$ written in coordinates as[3] $x^\mu = (ct, \mathbf{x})$, where t and $\mathbf{x} = (x^i)$, $i = 1, 2, 3$, are, respectively, the time instant and the spatial position of the event as measured by an inertial observer with respect to her/his inertial reference frame $\mathscr{R}$. Let us consider, in particular, the following two events: the first, $e_1 = (ct_1, x_1^i)$, consists in a light signal emanating at the time t_1 from the spatial position (x_1^i); the second, $e_2 = (ct_2, x_2^i)$, consists in the same light signal arriving at the time t_2 in the spatial position (x_2^i). Since the signal propagates with constant

[1]In this context, isotropy means invariance under rotations, while homogeneity means invariance under multiplication by a real constant.

[2]In Galileian relativity, the laws of physics refer only to mechanics, while in Einstein's special theory of relativity one considers also electromagnetism.

[3]Using ct instead of t is customary in special relativity: physically, this amounts at replacing the time t with the corresponding space ct traveled by a ray of light during t.

speed c, the distance that is traveled is $c(t_2 - t_1)$, however, since we have endowed $\mathbb{R}^3$ with the Euclidean metric, this same distance equals $\left(\sum_{i=1}^{3}(x_2^i - x_1^i)^2\right)^{1/2}$, so the coordinates of the events e_1 and e_2 in the fixed inertial frame $\mathscr{R}$ are related by the equation:

$$c^2(t_2 - t_1)^2 - \sum_{i=1}^{3}(x_2^i - x_1^i)^2 = 0 \iff c^2(t_2 - t_1)^2 - \|\mathbf{x}_2 - \mathbf{x}_1\|^2 = 0, \tag{13.1}$$

$\|x_2 - x_1\|^2$ being the Euclidean distance in $\mathbb{R}^3$ between x_1 and x_2. Of course, eq. (13.1) remains valid for all spacetime differences, also infinitesimal ones, thus we can write the differential version of eq. (13.1) as $c^2dt^2 - \|d\mathbf{x}\|^2 = 0$. In special relativity, the quantity

$$ds^2 = c^2dt^2 - \|d\mathbf{x}\|^2, \tag{13.2}$$

is called *spacetime interval.* From eq. (13.1) it follows that the spacetime interval between two events connected by a signal traveling at the speed of light is null. Since the speed of light is an upper limit for velocity, this amounts at promoting it as a reference and at normalizing to 0 the spacetime distance between any two events, no matter how far in space or time, connected by a light-speed signal.

Let us underline a key invariance property of ds^2 that will be used to single out the analytical form of the coordinate change between inertial observers. Postulates 1 and 3 imply that the spacetime interval ds^2 between two events described in the inertial reference frame $\mathscr{R}$ and the spacetime interval ds'^2 between the same couple of events described in any other inertial reference frame $\mathscr{R}'$ is exactly the same: $ds'^2 = ds^2$, see e.g. [**?**], page 7 or [**?**], page 117, for a rigorous proof.

If we write a generic event $e \in \mathbb{R}^4$ as a column vector $(x^0 = ct, x^1, x^2, x^3)^t = (x^\mu)^t$ and the infinitesimal difference between any two events as $dx = (dx^\mu)^t$, then the spacetime interval can be written as the (non positive-definite) quadratic form $ds^2 = (dx^\mu)^t\eta_{\mu\nu}(dx^\nu)$, where $\eta = (\eta_{\mu\nu})$ is the matrix $\mathrm{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$. The metric space $\mathcal{M} = (\mathbb{R}^4, \eta)$ is called *Minkowski spacetime* and η is the matrix associated to the Minkowski metric tensor such that $\eta = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$. The associated pseudo-norm, i.e. $\|u\|^2_{\mathcal{M}} = (u^0)^2 - [(u^1)^2 + (u^2)^2 + (u^3)^2]$ is called *Minkowski norm* of $u \in \mathcal{M}$.

Noticeable subsets of $\mathcal{M}$ are the lightcone and the world-lines. The *lightcone* is the subset of $\mathcal{M}$ given by $\mathcal{L} = \{(ct, x, y, z) \in \mathbb{R}^4 \ : \ ds^2 = 0 \iff c^2t^2 - x^2 - y^2 - z^2 = 0\}$. The volume surrounded by $\mathcal{L}$ together with $\mathcal{L}$ itself will be denoted with $\mathring{L}$. A *world-line* in $\mathcal{M}$ is any connected set of events between an initial and a final one. World-lines of inertial motions are easily seen to be segments of straight lines in $\mathcal{M}$.

We have the following categorization of events in terms of the spacetime interval:

- $ds^2 = 0$, the events e_1, e_2 are connected by a signal traveling at the speed of light, they belong to the lightcone $\mathcal{L}$;

- $ds^2 > 0$, or $\|d\mathbf{x}\|^2 < c^2dt^2$, i.e. the spatial separation between the events e_1, e_2 is less that the distance traveled by a light ray, which implies that they are connected by a world-line with speed inferior than c, they lie in the interior of the lightcone, the so-called *time-like* zone of the Minkowski space. It is also called *causality region*, because changes in the event e_1 cause changes in the event e_2. It is clear that these events are contained in $\mathring{L}$;

- $ds^2 < 0$, or $\|d\mathbf{x}\|^2 < c^2 dt^2$, i.e. the spatial separation between the events e_1, e_2 is greater that the distance traveled by a light ray, i.e. the events e_1, e_2 cannot be physically connected, they lie outside the lightcone, the so-called *space-like* zone of the Minkowski space, also said *non-causal region*.

We are now ready to discuss the problem to relate the coordinates of two inertial frames. First of all, it is simple to deduce from postulate 1 that the coordinate transformation $\omega : \mathbb{R}^4 \to \mathbb{R}^4$, $x^\mu \mapsto x'^\mu = \omega(x^\mu)$ from $\mathscr{R}$ to $\mathscr{R}'$ of an event must be *linear* (under the reasonable hypothesis to be differentiable).

In fact, by postulate 1, there are no special instants and positions in $\mathbb{R}^4$, so, the Euclidean distance between two events remains the same when these are translated by a fixed vector $b \in \mathbb{R}^4$. This is true independently on the coordinate system used to write the events in two arbitrary inertial reference frames $\mathscr{R}$ and $\mathscr{R}'$. Let $x = x^\mu$ and $y = y^\mu$ be the coordinates of the two events in $\mathscr{R}$ and $\omega^\mu(x)$ and $\omega^\mu(y)$ the coordinates of the same events in $\mathscr{R}'$. Since $(x^\mu + b^\mu) - (y^\mu + b^\mu) = x^\mu - y^\mu$, we must have $\omega^\mu(x + b) - \omega^\mu(y + b) = \omega^\mu(x) - \omega^\mu(y)$. If we derive the two sides of the last equation with respect to x^ν, $\nu = 0, 1, 2, 3$, we obtain $\frac{\partial \omega^\mu}{\partial x^\nu}(x + b) = \frac{\partial \omega^\mu}{\partial x^\nu}(x)$, for all $b \in \mathbb{R}^4$, since y does not depend on x.

Thanks to the fact that b is arbitrary, $x + b$ represents any vector in $\mathbb{R}^4$, so the function $\frac{\partial \omega^\mu}{\partial x^\nu}$ is constant, which implies that $\frac{\partial \omega^\mu}{\partial x^\nu}(x) = \Lambda^\mu_{\ \nu} \in \mathbb{R}$ for all $x \in \mathbb{R}^4$, $\mu, \nu = 0, 1, 2, 3$, i.e.

$$x'^\mu = \omega^\mu(x) = \Lambda^\mu_{\ \nu} x^\nu + a^\mu. \tag{13.3}$$

The fact that the coordinate transformation between inertial reference frames must be linear does not imply that *any* linear function ω^μ implements such a transformation. In fact, the invariance of the spacetime interval imposes strong constraints on the matrix $\Lambda = (\Lambda^\mu_{\ \nu})$. To see this, let us write the difference vector dx^μ in the inertial reference frame $\mathscr{R}'$ by using eq. (13.3): $dx'^\mu = y'^\mu - x'^\mu = \Lambda^\mu_{\ \nu} y^\nu + a^\mu - (\Lambda^\mu_{\ \nu} x^\nu + a^\mu) = \Lambda^\mu_{\ \nu} dx^\nu$. Thus, on one side,

$$ds'^2 = \eta_{\mu\nu} dx'^\mu dy'^\nu = \eta_{\mu\nu} \Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta} dx^\alpha dy^\beta, \tag{13.4}$$

and, on the other side,

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dy^\beta, \tag{13.5}$$

so, the equality $ds'^2 = ds^2$ implies the following constraint on Λ:

$$\eta_{\mu\nu} \Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta} = \eta_{\alpha\beta} \iff \Lambda^t \eta \Lambda = \eta. \tag{13.6}$$

The set of all these matrices forms a group, called the *Lorentz group* and denoted by the symbol $O(1,3) = \{\Lambda \in GL(4, \mathbb{R}) \ : \ \Lambda^t \eta \Lambda = \eta\}$, or $\mathscr{L}$. Of course, every matrix $\Lambda \in O(1,3)$ is invertible, in fact, by computing the determinant of both sides of $\Lambda^t \eta \Lambda = \eta$ and using Binet's theorem we get $\det(\Lambda) = \pm 1$.

What we have shown so far is that postulate 1 and the constancy of the speed of light in inertial reference frames imply that the coordinates used to describe the same even in two generic inertial reference frames are related by a nonhomogeneous linear transformation of the type $x' = \Lambda x + a$, $\Lambda \in O(1,3)$, $a \in \mathbb{R}^4$.

The set of these transformations forms the so-called the *Poincaré group* defined by $\mathscr{P} = \{(\Lambda, a) \ : \ \Lambda \in O(1,3), \ a \in \mathbb{R}^4\}$, endowed with composition law given by $(\Lambda_1, a_1) \cdot (\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2)$.

The Poincaré group contains the Lorentz group and the group of translations as subgroups: $O(1,3)$ is isomorphic to the subgroup of $\mathscr{P}$ given by the elements $(\Lambda, 0)$, while the elements of $\mathscr{P}$ of the form $(1, a)$ are translations by the constant vector $a \in \mathbb{R}^4$.

The coordinate transformations $x' = \Lambda x + a$ are called *Poincaré transformations*, and those corresponding to $a = 0$, i.e. $x' = \Lambda x$ are called *Lorentz transformations*.

Since we have not put any further restrictions on these maps, we have managed to show that the coordinate transformation between two inertial reference frames coincide with the Poincaré transformation. The translation part of these maps is trivial, in the following section we will analyze the structure of the Lorentz group in order to better understand the geometrical action of the Lorentz transformations on the inertial reference frames.

Part IV:

Appendices

Appendix A

Einstein's convention (Edoardo Provenzi)

In differential geometry, we often deal with expressions with many indices and sums. To simplify the notation, it is common to use **Einstein's convention** and implicitly assume a sum over repeated indices *above and below* in an algebraic expression, the sum being of course performed over the range of index variability, e.g. if $i = 1, \ldots, n$, then

$$a^i b_i := \sum_{i=1}^{n} a_i b_i.$$

This notation is consistent as long as we agree to write the indices below for the basis vectors of $\mathbb{R}^n$ and above for the components w.r.t. them. The convention for the dual space $(\mathbb{R}^n)^*$ is inverted. Coherently with that, the **canonical basis of** $\mathbb{R}^n$ will be denoted with $(e_i)_{i=1}^n$, while its **dual basis** will be written as $(\varepsilon^j)_{j=1}^n$, $\varepsilon^j \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \equiv (\mathbb{R}^n)^*$, the two bases are linked via the pairing:

$$\varepsilon^j(e_i) = \delta_i^j, \qquad i, j = 1, \ldots, n.$$

Given the vector $x = x^i e_i \in \mathbb{R}^n$, $x^i \in \mathbb{R}$, for all $i = 1, \ldots, n$, the action of the linear functional ε^j on x is:

$$\varepsilon^j(x) = \varepsilon^j(x^i e_i) = x^i \varepsilon^j(e_i) = x^i \delta_i^j = x^j,$$

i.e. ε^j **simply extracts the j-th component of the vector $x \in \mathbb{R}^n$ w.r.t. the canonical basis** $(e_i)_{i=1}^n$.

Vectors in $\mathbb{R}^n$, or any other vector space V, will always be considered as column vectors, while their duals, belonging to $(\mathbb{R}^n)^*$, or V^*, will be considered as row vectors.

It is very important to make explicit the use of the Einstein convention when we deal with matrices associated with *linear maps between vector spaces* and with *bilinear forms on a vector space*. Let $f : V \to W$ be a linear function between the vector spaces V and W of dimension n and m, respectively, then, if we denote the matrix associated to f with $A = (a_j^i)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, i.e. we write the **row index above** and the **column index below**, the Einstein convention can be coherently applied to compute the product of A with a *column vector* $v = (v^1, \ldots, v^n)^t$ of V, in fact:

$$Av = \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^m & \cdots & a_n^m \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \begin{pmatrix} a_1^1 v^1 + \cdots + a_n^1 v^n \\ \vdots \\ a_1^m v^1 + \cdots + a_n^m v^n \end{pmatrix} = (a_j^i v^j),$$

which is a column vector with m rows belonging to W. Notice that, if the matrix is square, then the **trace** of A is $\mathrm{Tr}(A) = a_i^i$.

Now we examine the case of a bilinear form. Let $g : V \times V \to \mathbb{R}$ be an $\mathbb{R}$-bilinear form over the vector space V of dimension n, then, by fixing a basis $(u_1, \ldots, u_n)$ of V we can associate to g the matrix $G = (g_{ij})$, where the matrix elements are defined by the formula:

$$g(u_i, u_j) := g_{ij},$$

so that $G = (g_{ij})_{i,j=1,\ldots,n}$. Notice that now the matrix elements of G are written with two indices below, this is the only way of being coherent with Einstein's notation, in fact, if $v, w \in V$, $v = (v^i)$ and $w = (w^j)$ where v^i and w^j, $i, j = 1, \ldots, n$, are the components of v and w w.r.t. the basis (u_i) of V, then

$$g(v, w) = g_{ij} v^i w^j := \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} v_i w_j,$$

will be a real scalar, as correctly expected.

Appendix B

Recap of ordinary calculus in $\mathbb{R}^n$

(Edoardo Provenzi)

We collect here some basic results and definitions of ordinary calculus in $\mathbb{R}^n$. We assume that the reader is already familiar with this topic, the aim of this appendix is just to recap the most important concepts of standard calculus.

In particular, we stress some concepts, as e.g. the spaces between which partial derivatives act or the role of the dual basis of $\mathbb{R}^n$ or that of the evaluation map, that are sometimes hidden when presenting ordinary calculus but that are essential for the development of differential calculus on manifolds.

It is convenient to fix the notation that will be used, unless otherwise specified, in this appendix:

- $x_0 \in \Omega \subseteq \mathbb{R}^n$, Ω open set
- $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$
- $L(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$
- $U(0)$ is an open neighborhood of the null vector $0 \in \mathbb{R}^n$
- $U(x_0)$ is the open neighborhood of x_0 obtained by translation of $U(0)$ by the vector x_0:
$$U(x_0) = \{x_0 + h,\ h \in U(0)\}$$
- a **curve**, or **path**, in $\mathbb{R}^d$, $d \geqslant 1$, is a continuous function $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^d$, where I is an open real interval.
- Modulo a translation and a rescaling, it is always possible to consider I to be $(-\varepsilon, \varepsilon)$ for a suitable $\varepsilon > 0$.
- We say that γ **passes through** $x_0 \in \mathbb{R}^d$ if $\gamma(0) = x_0$.

The main idea behind differential calculus in $\mathbb{R}^n$ is the concept of **local linearization**, which leads directly to the definition of derivative. For functions of only one real variable there is only one derivative, but for functions of more than one real variable two different (and not equivalent) derivatives can be considered: the total and the directional derivative along a vector. We start recalling the definition of the total derivative, which has been formalized as follows by Fréchet.

Def. B.0.1 *f is said to be Fréchet-differentiable (or simply differentiable) in $x_0 \in \Omega$ if there exist:*

- *an open neighborhood $U(x_0) \subseteq \Omega$*
- *a linear operator*[1] *$Df(x_0) \in L(\mathbb{R}^n, \mathbb{R}^m)$, that, in general, depends on x_0*
- *a rest function $\rho_{x_0} : U(0) \subseteq \mathbb{R}^n \to \mathbb{R}^m$,*

such that:

1. $f(x_0 + h) = f(x_0) + Df(x_0)h + \rho_{x_0}(h), \qquad \forall h \in U(0)$

2. $\rho_{x_0}(0) = 0$

3. $\frac{\|\rho_{x_0}(h)\|}{\|h\|} \xrightarrow[\|h\| \to 0]{} 0.$

f is differentiable on Ω if it is differentiable in every point of Ω.

This definition is the precise formalization of the intuitive statement that it is possible to approximate the action of f on nearby points $x = x_0 + h$ around x_0 by a linear function and that the error in doing this tends to zero faster than the distance between x and x_0, i.e. $\|h\| = \|x - x_0\|$.

Def. B.0.2 *$Df(x_0)$ is called the* ***total derivative****, the* ***Fréchet derivative****, or simply the derivative of f in x_0.*

Condition **1.** and **3.** imply an important equation, to find its expression let us rewrite **1.** as $f(x_0 + h) - f(x_0) - Df(x_0)h = \rho_{x_0}(h)$, so that $\|f(x_0 + h) - f(x_0) - Df(x_0)h\| = \|\rho_{x_0}(h)\|$ thus, dividing by $\|h\|$ and taking the limit $\|h\| \to 0$, thanks to **3.** we obtain:

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)h\|}{\|h\|} = 0. \tag{B.1}$$

Theorem B.0.1 (Uniqueness of the total derivative) *If $Df(x_0)$ exists, then it is unique.*

Proof. We must proof that if $D_1 f(x_0)$ and $D_2 f(x_0)$ are two total derivatives of f in x_0, then they must agree as linear operators belonging to $L(\mathbb{R}^n, \mathbb{R}^m)$.

To this aim, observe that (B.1) implies, in particular, that the numerator tends to 0 as $\|h\| \to 0$, i.e. $\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0$ such that $\|h\| < \delta_\varepsilon$ implies both

$$\|f(x_0 + h) - f(x_0) - D_1 f(x_0)h\| < \frac{\varepsilon}{2}\|h\| \text{ and } \|f(x_0 + h) - f(x_0) - D_2 f(x_0)h\| < \frac{\varepsilon}{2}\|h\|, \tag{B.2}$$

having used the arbitrariness of ε. Now, thanks to the triangular inequality, we have:

$$\begin{aligned} \|D_1 f(x_0)h - D_2 f(x_0)h\| &= \|f(x_0 + h) - f(x_0) - D_2 f(x_0)h - (f(x_0 + h) - f(x_0) - D_1 f(x_0)h)\| \\ &< \|f(x_0 + h) - f(x_0) - D_2 f(x_0)h\| + \|f(x_0 + h) - f(x_0) - D_1 f(x_0)h\| \\ &\underset{(B.2)}{<} \varepsilon\|h\|. \end{aligned}$$

[1] Sometimes $Df(x_0)$ is written as $f'(x_0)$.

Noticing that $D_1 f(x_0)h - D_2 f(x_0)h = D_1 f(x_0) - D_2 f(x_0))h$, we can write

$$\frac{\|(D_1 f(x_0) - D_2 f(x_0))h\|}{\|h\|} < \varepsilon \qquad \forall \|h\| < \delta_\varepsilon.$$

When $h = 0$, it is clear that $D_1 f(x_0)h = D_2 f(x_0)h = 0$ because the total derivative is linear, so, let us consider $h \neq 0$, then, from the previous expression we get:

$$\|D_1 f(x_0) - D_2 f(x_0)\| := \sup_{h \neq 0} \frac{\|(D_1 f(x_0) - D_2 f(x_0))h\|}{\|h\|} < \varepsilon \qquad \forall \|h\| < \delta_\varepsilon,$$

which implies that $D_1 f(x_0) = D_2 f(x_0)$. □

Because of the uniqueness of the total derivative, many authors say that $Df(x_0)$ **provides the best linear approximation of f in a neighborhood of x_0.**

Remark: if we replace $\mathbb{R}^n$ and $\mathbb{R}^m$ by any two finite-dimensional normed spaces, then the definitions and results above remain valid.

The reason why $Df(x_0)$ is called the total derivative is that it contains, as special cases, all the derivatives of f in x_0 along any possible directions, as we are going to formalize.

Def. B.0.3 *The* ***straight line passing through*** x_0 ***and directed as the vector*** $v \in \mathbb{R}^n$ *is the curve in* $\mathbb{R}^n$ *defined as follows:*

$$\begin{array}{rccl} r_{x_0,v}: & \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & r_{x_0,v}(t) = x_0 + tv. \end{array}$$

In order to define the concept of directional derivative, we just need to observe that the composed function $f \circ r_{x_0,v} : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ passing through $f(x_0) = f(r_{x_0,v}(0))$.

Def. B.0.4 *Given* $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* $x_0 \in \Omega$, *if the following limit (in* $\mathbb{R}^m$*) exists*[2]

$$D_v f(x_0) = \lim_{t \to 0} \frac{(f \circ r_{x_0,v})(t) - f(x_0)}{t} = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \equiv (f \circ r_{x_0,v})^{\boldsymbol{\cdot}}(0), \tag{B.3}$$

then we call it the ***directional derivative of the function in*** x_0 ***along the vector*** v.

We say that f *is* ***Gateaux differentiable*** *in* x_0 *if the directional derivatives of* f *in* x_0 *exists for every direction* v. f *is Gateaux differentiable on* Ω *if it is Gateaux differentiable in every point of* Ω.

As a particular vector v we can choose e_i, the i-th element of the canonical basis of $\mathbb{R}^n$, in this case the directional derivative of f in x_0 is called **partial derivative** of f in x_0 and denoted with

$$\frac{\partial f}{\partial x^i}(x_0) := D_{e_i} f(x_0).$$

Each function

$$\begin{array}{rccl} f: & \Omega \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ & x & \longmapsto & f(x) = (y_1, \ldots, y_m), \end{array}$$

[2]Notice that $D_v f(x_0)$ *is a vector* in $\mathbb{R}^m$ and *not* a linear operator.

is uniquely associated to an ordered collection of m real-valued functions, the so-called **component functions** $f^1, \dots, f^m : \Omega \to \mathbb{R}$, defined as follows:

$$\forall x \in \Omega, \quad f(x) = (y_1, \dots, y_m) =: (f^1(x), \dots, f^m(x)).$$

The partial derivatives $\frac{\partial f^j}{\partial x^i}(x_0)$, $i = 1, \dots, n$, $j = 1, \dots, m$, of the component functions can be organized in a $m \times n$ matrix with real entries called **Jacobian matrix** of f in x_0 and denoted with $Jf(x_0) \in M(m \times n, \mathbb{R})$:

$$(Jf(x_0))^j_i = \frac{\partial f^j}{\partial x^i}(x_0) \iff Jf(x_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x_0) & \dots & \frac{\partial f^1}{\partial x^n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x_0) & \dots & \frac{\partial f^m}{\partial x^n}(x_0) \end{pmatrix}. \tag{B.4}$$

Let us now prove that the vector $D_v f(x_0) \in \mathbb{R}^m$ can be recovered by $Df(x_0) \in L(\mathbb{R}^n, \mathbb{R}^m)$ simply by applying this linear operator to the vector v, it is in this sense that the total derivative contains all the information on the directional derivatives.

The easiest and more profound way to prove this relationship is by first examining the special case $n = 1$, i.e. curves $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$.

In standard differential calculus we prove that a function of one real variable, as γ, is differentiable in $x_0 \in (-\varepsilon, \varepsilon)$ if and only if the limit

$$\dot\gamma(x_0) := \lim_{t \to 0} \frac{\gamma(x_0 + t) - \gamma(x_0)}{t} \equiv \frac{d\gamma}{dt}(x_0) \in \mathbb{R}^m$$

exists and it is finite. In this case, $\dot\gamma(x_0)$ is called *the value of the derivative* of γ in x_0.

The fact that, in this special case, the existence of the total derivative of γ in x_0, i.e. the linear operator $D\gamma(x_0) \in L(\mathbb{R}, \mathbb{R}^m)$, is equivalent to the existence of its derivative $\dot\gamma(x_0)$ in x_0 should not be surprising if we think about the canonical identification of the vector space $L(\mathbb{R}, \mathbb{R}^m)$ with $\mathbb{R}^m$ via the linear isomorphism given by

$$\begin{array}{ccc} L(\mathbb{R}, \mathbb{R}^m) & \xrightarrow{\sim} & \mathbb{R}^m \\ T & \longmapsto & T1, \end{array} \tag{B.5}$$

i.e. the application of any linear operator $T \in L(\mathbb{R}, \mathbb{R}^m)$ to the only element of the canonical basis of $\mathbb{R}$, i.e. 1.

Let us use again the special element 1 of $\mathbb{R}$ to define the directional derivative of γ in x_0 and examine its relationship with the total derivative. 1 identifies the only possible direction in $\mathbb{R}$, so the straight line in $\mathbb{R}$ passing through $x_0 \in \mathbb{R}$ and directed as the vector $1 \in \mathbb{R}$ is:

$$\begin{array}{cccc} r_{x_0,1} : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & r_{x_0,1}(t) = x_0 + t. \end{array}$$

The curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ admits a directional derivative in $x_0 \in (-\varepsilon, \varepsilon)$ towards the only possible direction defined by $1 \in \mathbb{R}$ if it exists and it is finite the vector of $\mathbb{R}^m$ defined by the limit:

$$D_1\gamma(x_0) = \lim_{t \to 0} \frac{(\gamma \circ r_{x_0,1})(t) - \gamma(x_0)}{t} = \lim_{t \to 0} \frac{\gamma(x_0 + t) - \gamma(x_0)}{t} \equiv \dot\gamma(x_0),$$

i.e.

$$D_1\gamma(x_0) - \dot\gamma(x_0) = 0 \iff \lim_{t \to 0} \frac{\gamma(x_0 + t) - \gamma(x_0) - \dot\gamma(x_0)t}{t} = 0, \tag{B.6}$$

but the limit above is nothing but the 1-dimensional version of eq. (B.1). In eq. (B.1) the role of $\dot{\gamma}(x_0)$ is played by the total derivative $Df(x_0)$ which is an operator belonging to $L(\mathbb{R}^n, \mathbb{R}^m)$, this apparent mismatch can be corrected thanks to the canonical isomorphism (B.5), which allows us identifying $\dot{\gamma}(x_0)$ with the linear operator $D\gamma(x_0)1$.

So, eq. (B.6) becomes

$$D_1\gamma(x_0) - \dot{\gamma}(x_0) = 0 \iff \dot{\gamma}(x_0) = D\gamma(x_0)1,$$

and we come to the conclusion that the directional derivative of γ in x_0 along the direction $1 \in \mathbb{R}$ exists if and only if the total derivative $D\gamma(x_0)$ exists and, moreover, $D_1\gamma(x_0)$ is nothing but the application of the total derivative $D\gamma(x_0)$ to 1, as represented by the suggestive equation:

$$\boxed{\dot{\gamma}(x_0) = D_1\gamma(x_0) = D\gamma(x_0)1}\ , \tag{B.7}$$

in which 1 plays two different roles: in the expression $D_1\gamma(x_0)$ it must be interpreted as a vector defining the only possible direction of derivation in $\mathbb{R}$, while in the expression $D\gamma(x_0)1$ it must be though as the only canonical basis element of the vector space $\mathbb{R}$.

The extension of this result to a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is almost straightforward if we build, analogously to what we have done before, the curve $f \circ r_{x_0,v}$ in $\mathbb{R}^m$ passing through $f(x_0)$ by composing f with the straight line $r_{x_0,v}(t) = x_0 + tv$, $x_0 \in \Omega$, $v \in \mathbb{R}^n$.

Supposing that the curve $f \circ r_{x_0,v}$ is differentiable in 0, we have[3]:

$$\begin{aligned}
(f \circ r_{x_0,v})^{\cdot}(0) &\underset{(B.7)}{=} D(f \circ r_{x_0,v})(0)1 \\
&\underset{\text{(chain rule)}}{=} Df(r_{x_0,v}(0))Dr_{x_0,v}(0)1 \\
&\underset{(B.7)}{=} Df(x_0)\dot{r}_{x_0,v}(0) \\
&= Df(x_0)\frac{d(x_0 + tv)}{dt}(0) \\
&= Df(x_0)v,
\end{aligned}$$

but $(f \circ r_{x_0,v})^{\cdot}(0)$ is precisely $D_v f(x_0)$ thanks to eq. (B.3), so we have proven that, if f is Fréchet differentiable in x_0, then f is also Gateaux differentiable in x_0 and the directional derivative can be simply obtained by applying the total derivative to the vector v:

$$\boxed{D_v f(x_0) = Df(x_0)v}\ . \tag{B.8}$$

Counter-examples show that the reverse is not true: even if a function has directional derivatives in every direction in a point, it can be not differentiable. Thus, the Fréchet differentiability of a function of multiple real variables is stronger than the Gateaux derivability, whereas for one variable the two concepts collapse due to the canonical isomorphism $L(\mathbb{R}, \mathbb{R}^m) \simeq \mathbb{R}^m$.

Remark: this way of proving the relationship between directional and total derivative for functions of several real variables is neither the easiest, nor the standard one. However, we chose to present it because this way of reasoning is the closest to the one used in differential geometry, as the reader can appreciate starting from chapter 2.

[3] we omit the composition sign between linear operators, as conventional.

There is a last special case to consider, that of a scalar function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. In this case, the linear operator $Df(x_0)$ is an element of $L(\mathbb{R}^n, \mathbb{R}) \equiv (\mathbb{R}^n)^*$, the dual space of $\mathbb{R}^n$, i.e. $Df(x_0)$ is a linear functional on $\mathbb{R}^n$. $\mathbb{R}^n$ and its dual are canonically isomorphic via the correspondence

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\sim} & (\mathbb{R}^n)^* \\ (e_1, \dots, e_n) & \longmapsto & (\varepsilon^1, \dots, \varepsilon^n), \end{array}$$

where $\varepsilon^j(e_i) = \delta_i^j$, $i, j = 1, \dots, n$, is the **dual canonical basis of** $\mathbb{R}^n$. This isomorphism allows us to identify $Df(x_0) \in (\mathbb{R}^n)^*$ with a vector of $\mathbb{R}^n$ called the **gradient of** f **in** x_0 and denoted with $\nabla f(x_0)$.

The representation of $\nabla f(x_0)$ in components, with respect to the canonical basis of $\mathbb{R}^n$, is given by the column vector:

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x^1}(x_0), \dots, \frac{\partial f}{\partial x^n}(x_0) \right)^t,$$

i.e. the Jacobian matrix in x_0 of a real-valued function of n real variables collapses to a vector whose components are the partial derivatives of f calculated in x_0.

The directional derivative of f along a vector $v \in \mathbb{R}^n$ can be obtained via the general formula (B.8). In this case, since the Jacobian matrix is simply a row, its action on v reduces to the scalar product of $\nabla f(x_0)$ with v:

$$D_v f(x_0) = \langle \nabla f(x_0), v \rangle, \tag{B.9}$$

which can be also seen as a particular instance of the finite-dimensional version of the Riesz isomorphism theorem: the action of the linear functional $Df(x_0) \in (\mathbb{R}^n)^*$ of $v \in \mathbb{R}^n$ is the scalar product of the vector of $\mathbb{R}^n$ uniquely associated to $Df(x_0)$, i.e. $\nabla f(x_0)$, and v.

Thanks to the linearity of the limit, $D_v f$ is linear w.r.t f, but we can say more: if we express the vector v as the linear combination $k_1 v_1 + k_2 v_2$, $v_1, v_2 \in \mathbb{R}^n$, $k_1, k_2 \in \mathbb{R} \setminus \{0\}$, then, by the bilinearity of the real scalar product, we get:

$$D_v f(x_0) = \langle \nabla f(x_0), k_1 v_1 + k_2 v_2 \rangle = k_1 \langle \nabla f(x_0), v_1 \rangle + k_2 \langle \nabla f(x_0), v_2 \rangle = k_1 D_{v_1} f(x_0) + k_2 D_{v_2} f(x_0),$$

i.e.

$$\boxed{D_{k_1 v_1 + k_2 v_2} f(x_0) = k_1 D_{v_1} f(x_0) + k_2 D_{v_2} f(x_0)}\,, \tag{B.10}$$

so the directional derivative $D_v f$ is linear w.r.t both f and v. This property is crucial in chapter 7.

B.0.1 Noticeable examples of gradients and total derivatives

We show here how to compute the gradients and total derivatives of particularly important functions.

Directional derivatives of the squared Euclidean norm and of the Euclidean scalar product

In the proofs that will follow we will often use the equality

$$\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b \rangle,$$

which holds for all $a, b \in \mathbb{R}^n$.

Theorem B.0.2 *Let $x, a \in \mathbb{R}^n$, $f(x) = \|x\|^2$ and $g_a(x) = \|x-a\|^2$, then $\forall x \in \mathbb{R}^n$ it holds that:*

- $\nabla f(x) = 2x$
- $\nabla g_a(x) = 2(x-a)$.

This theorem has a clear interpretation: the computation of the gradient of the square Euclidean norm and of its translations is formally identical to that of the first derivative of the square function in $\mathbb{R}$ and its translations.

Proof. By direct computation:

$$\begin{aligned}
D_v f(x) &= \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon v) - f(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\|x+\varepsilon v\|^2 - \|x\|^2}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\|x\|^2 + \|\varepsilon v\|^2 + 2\langle x, \varepsilon v\rangle - \|x\|^2}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\varepsilon^2 \|v\|^2 + 2\varepsilon\langle x, v\rangle}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \left(\varepsilon \|v\|^2 + 2\langle x, v\rangle\right) = 2\langle x, v\rangle.
\end{aligned}$$

By (B.9), $D_v f(x) = \langle \nabla f(x), v\rangle = 2\langle x, v\rangle$, i.e. $\langle \nabla f(x), v\rangle = \langle 2x, v\rangle$, or $\langle \nabla f(x) - 2x, v\rangle = 0$ for all directions v, but this is possible if and only if $\nabla f(x) - 2x = 0$, i.e. $\nabla f(x) = 2x$.

Analogously,

$$\begin{aligned}
D_v g_a(x) &= \lim_{\varepsilon \to 0} \frac{\|x+\varepsilon v - a\|^2 - \|x-a\|^2}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\|(x-a)+\varepsilon v\|^2 - \|x-a\|^2}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\|x-a\|^2 + \|\varepsilon v\|^2 + 2\langle x-a, \varepsilon v\rangle - \|x-a\|^2}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\varepsilon^2 \|v\|^2 + \varepsilon\langle 2(x-a), v\rangle}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \left(\varepsilon \|v\|^2 + \langle 2(x-a), v\rangle\right) = \langle 2(x-a), v\rangle.
\end{aligned}$$

The same argument used above leads to the equation $\langle \nabla g_a(x) - 2(x-a), v\rangle = 0$ for all directions v, hence $\nabla g_a(x) = 2(x-a)$. □

Theorem B.0.3 *Let $x, a \in \mathbb{R}^n$, $f_a(x) = \langle a, x\rangle$, then $\nabla f_a(x) = a$.*

Interpretation: the computation of the gradient of the Euclidean scalar product function between two vectors in $\mathbb{R}^n$ is formally identical to that of the first derivative of the function in $\mathbb{R}$ given by the product between a a scalar coefficient and a real variable.

Proof. By direct computation:

$$
\begin{aligned}
D_v f_a(x) &= \lim_{\varepsilon \to 0} \frac{\langle a, x + \varepsilon v \rangle - \langle a, x \rangle}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\langle a, x \rangle + \varepsilon \langle a, v \rangle - \langle a, x \rangle}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\varepsilon \langle a, v \rangle}{\varepsilon} \\
&= \langle a, v \rangle.
\end{aligned}
$$

So $\langle \nabla f(x) - a, v \rangle = 0$ for all directions v, i.e. $\nabla f(x) = a$. □

Corollary B.0.1 *Let $x, a \in \mathbb{R}^n$, $f_a : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_a(x) = \langle a, x \rangle a$, then*

$$
Jf_a(x) = \begin{pmatrix} a_1 a_1 & a_2 a_1 & \dots & a_n a_1 \\ a_1 a_2 & a_2 a_2 & \dots & a_n a_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_1 a_n & a_2 a_n & \dots & a_n a_n \end{pmatrix} \equiv (a_i a_j)_{1 \leqslant i,j \leqslant n}. \tag{B.11}
$$

Proof. It is enough to consider the component functions $(f_a)_j(x) = \langle a, x \rangle a_j$, $j = 1, \dots, n$ and then apply to the previous theorem, obtaining $\nabla (f_a)_j(x) = a a_j$. Since the rows of $Jf_a(x)$ are $\nabla (f_a)_j(x)$, we get the result. □

Theorem B.0.4 *Let $x, a \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = \frac{1}{\|x\|^2}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $g(x) = \frac{x}{\|x\|^2}$, then $\forall x \in \mathbb{R}^n$ it holds that:*

- $\nabla f(x) = -\frac{2x}{\|x\|^4}$
- $J_g(x) = \frac{1}{\|x\|^2} \left(I_n - 2 \frac{(x_i x_j)_{1 \leqslant i,j \leqslant n}}{\|x\|^2} \right)$,

where $(x_i x_j)_{1 \leqslant i,j \leqslant n}$ is the matrix given by

$$
(x_i x_j)_{1 \leqslant i,j \leqslant n} = \begin{pmatrix} x_1 x_1 & x_2 x_1 & \dots & x_n x_1 \\ x_1 x_2 & x_2 x_2 & \dots & x_n x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1 x_n & x_2 x_n & \dots & x_n x_n \end{pmatrix}.
$$

So, even when $\|x\|^2$ appears at the denominator of a fraction we can compute the gradient or the Jacobian matrix by considering $\|x\|^2$ as a real variable and using the derivation rules.

Proof. By direct computation:

$$
\begin{aligned}
D_v f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\frac{1}{\|x + \varepsilon v\|^2} - \frac{1}{\|x\|^2} \right] &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\frac{1}{\|x\|^2 + 2\varepsilon \langle x, v \rangle + \varepsilon^2 \|v\|^2} - \frac{1}{\|x\|^2} \right] \\
= \frac{1}{\|x\|^2} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\frac{1}{1 + \varepsilon \left\langle \frac{2x}{\|x\|^2}, v \right\rangle + \varepsilon^2 \frac{\|v\|^2}{\|x\|^2}} - 1 \right],
\end{aligned}
$$

recalling the Taylor expansion $\frac{1}{1+\xi} \underset{\xi\to 0}{=} 1-\xi+\mathcal{O}(\xi^2)$ we have that

$$\frac{1}{1+\varepsilon\left\langle \frac{2x}{\|x\|^2},v\right\rangle+\varepsilon^2\frac{\|v\|^2}{\|x\|^2}}-1 \underset{\varepsilon\to 0}{\sim} \cancel{1}-\varepsilon\left\langle \frac{2x}{\|x\|^2},v\right\rangle-\varepsilon^2\frac{\|v\|^2}{\|x\|^2}-\cancel{1}=-\varepsilon\left\langle \frac{2x}{\|x\|^2},v\right\rangle-\varepsilon^2\frac{\|v\|^2}{\|x\|^2},$$

so that

$$D_v f(x)=\frac{1}{\|x\|^2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[-\varepsilon\left\langle \frac{2x}{\|x\|^2},v\right\rangle-\varepsilon^2\frac{\|v\|^2}{\|x\|^2}\right]=\left\langle -\frac{2x}{\|x\|^4},v\right\rangle$$

By the same argument used in the proof of the previous theorems we get $\nabla f(x)=-\frac{2x}{\|x\|^4}$.

The formula for $J_g(x)$ follows immediately from that of $\nabla f(x)$ and the Leibnitz property of the directional derivative applied to the component functions $g_j(x)=x_j\frac{1}{\|x\|^2}$, $j=1,\dots,n$, of $g(x)$. □

Theorem B.0.5 *Let $x\in\mathbb{R}^n$, $b\in\mathbb{R}^m$, $A\in M(m\times n,\mathbb{R})$ and $f_{A,b}(x)=\frac{1}{2}\|Ax-b\|^2$, then $\nabla f_{A,b}(x)=A^t(Ax-b)$.*

Proof. Let us compute $f_{A,b}(x+\varepsilon v)$:

$$\begin{aligned} f_{A,b}(x+\varepsilon v) &= \frac{1}{2}\|A(x+\varepsilon v)-b\|^2=\frac{1}{2}\|(Ax-b)+\varepsilon Av\|^2 \\ &= \frac{1}{2}\left(\|Ax-b\|^2+\varepsilon^2\|Av\|^2+2\varepsilon\langle Ax-b,Av\rangle\right). \end{aligned}$$

Then:

$$\begin{aligned} D_v f_{A,b}(x) &= \lim_{\varepsilon\to 0}\frac{\|Ax-b\|^2+\varepsilon^2\|Av\|^2+2\varepsilon\langle Ax-b,Av\rangle-\|Ax-b\|^2}{2\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\varepsilon^2\|Av\|^2+2\varepsilon\langle Ax-b,Av\rangle}{2\varepsilon} \\ &= \lim_{\varepsilon\to 0}\left(\frac{\varepsilon\|Av\|^2}{2}+\langle Ax-b,Av\rangle\right)=\langle Ax-b,Av\rangle \\ &= \langle A^t(Ax-b),v\rangle. \end{aligned}$$

So $\langle\nabla f_{A,b}(x)-A^t(Ax-b),v\rangle=0$ for all directions u, i.e. $\nabla f_{A,b}(x)=A^t(Ax-b)$. □

The total derivative of the determinant

Finally, we show how to compute the total derivative of the determinant in some special cases. First of all we notice that $\det: M(n,\mathbb{R})\to\mathbb{R}$, so for all $M\in M(n,\mathbb{R})$, $D\det(M)\in L(M(n,\mathbb{R}),\mathbb{R})\cong(M(n,\mathbb{R}))^*\cong(\mathbb{R}^{n^2})^*$, i.e. $D\det(M)$ is a linear functional on $M(n,\mathbb{R})$, so, when it is applied to a matrix of $M(n,\mathbb{R})$, it gives back a real number.

First of all, let us compute $D\det(I_n)$, I_n being the identity matrix $n\times n$. For all $h\in\mathbb{R}$, $h\to 0$ and for all matrix $M\in M(n,\mathbb{R})$, I_n+hA is a infinitesimal perturbation of I_n, thus, by definition of total derivative it holds that:

$$\det(I_n+hM)=\det(I_n)+D\det(I_n)hM+\rho_{I_n}(hM), \tag{B.12}$$

where $\rho_{I_n} : M(n, \mathbb{R}) \to \mathbb{R}$ is such that $\frac{\rho_{I_n}(hM)}{h} \underset{h\to 0}{\to} 0$, i.e. $\rho_{I_n}(hM) = o(h)$. In order to make eq. (B.12) explicit, we recall that the coefficient of the higher order term of the characteristic polynomial $p(t) = \det(M - tI_n)$, $t \in \mathbb{R}$, of a generic matrix $M \in M(n, \mathbb{R})$ is $(-1)^n$. Thus, thanks to the fundamental theorem of algebra we can write:

$$\det(M - tI_n) = (-1)^n \prod_{i=1}^{n} (t - \lambda_i), \qquad \lambda_i \in \mathbb{C},$$

where λ_i are the complex eigenvalues of M. If we multiply both members of the previous equation by $(-1)^n$ we get

$$(-1)^n \det(M - tI_n) = \prod_{i=1}^{n} (t - \lambda_i),$$

which, taking into account the property $\det(cM) = c^n \det(M)$ for all $c \in \mathbb{R}$ and for all $n \times n$ matrix M, can be re-written as:

$$\det(-M + tI_n) = \prod_{i=1}^{n} (t - \lambda_i).$$

Let us now operate the change of variable defined by $t = -\frac{1}{h}$, $h \neq 0$. As a consequence, the previous equation can be written in terms of h as follows:

$$\det(-M - \frac{1}{h} I_n) = \prod_{i=1}^{n} (-\frac{1}{h} - \lambda_i) = \prod_{i=1}^{n} -\frac{1}{h}(1 + h\lambda_i) = \left(-\frac{1}{h}\right)^n \prod_{i=1}^{n} (1 + h\lambda_i),$$

i.e., since $\left(-\frac{1}{h}\right)^n = \frac{(-1)^n}{h^n}$,

$$(-1)^n h^n \det(-M - \frac{1}{h} I_n) = \prod_{i=1}^{n} (1 + h\lambda_i),$$

using again the property $\det(cM) = c^n \det(M)$ we get:

$$\det(I_n + hM) = \prod_{i=1}^{n} (1 + h\lambda_i).$$

By direct computation, we can expand the right-hand side of the previous equation as follows:

$$\prod_{i=1}^{n} (1 + h\lambda_i) = 1 + h \sum_{i=1}^{n} \lambda_i + o(h) = \det(I_n) + h\mathrm{Tr}(M) + o(h),$$

thus

$$\det(I_n + hM) = \det(I_n) + h\mathrm{Tr}(M) + o(h), \tag{B.13}$$

which, by comparison with eq. (B.12), gives

$$\boxed{D \det(I_n) M = \mathrm{Tr}(M)}, \qquad \forall M \in M(n, \mathbb{R}),$$

i.e. $D\det(I_n)$ is the linear functional that, when applied to any $M \in M(n,\mathbb{R})$, gives back its trace, i.e.

$$\boxed{D\det(I_n) = \mathrm{Tr}}\,.$$

Thanks to this result, we can compute $D\det(A)$, for a generic $A \in GL(n,\mathbb{R})$. Let $M \in M(n,\mathbb{R})$, then, since A is invertible, we can write, for all $h \in \mathbb{R}$, $h \to 0$:

$$\begin{aligned}
\det(A+hM) &= \det(A(I_n + A^{-1}hM)) \\
&= \det(A)\det(I_n + A^{-1}hM) \\
&\quad \text{by using } (B.13) \text{ we get:} \\
&= \det(A)(1 + h\mathrm{Tr}(A^{-1}M) + o(h)) \\
&= \det(A) + h\mathrm{Tr}(\det(A)A^{-1}M) + o(h),
\end{aligned}$$

thus, by comparison with (B.12), we find:

$$\boxed{D\det(A)M = \det(A)\mathrm{Tr}(A^{-1}M)}\,, \quad \forall A \in GL(n,\mathbb{R}),\ M \in M(n,\mathbb{R}),$$

and so, in particular,

$$\boxed{D\det(A)M = \mathrm{Tr}(A^{-1}M)}\,, \quad \forall A \in SL(n,\mathbb{R}),\ M \in M(n,\mathbb{R}).$$

Since $\mathrm{Tr}(A^{-1}A) = \mathrm{Tr}(I_n) = n$, if we apply the total derivative to the matrix A itself we get:

$$D\det(A)A = n\det(A), \qquad \forall A \in GL(n,\mathbb{R}),$$

and

$$D\det(A)A = n, \qquad \forall A \in SL(n,\mathbb{R}).$$

B.1 The classes of functions $\mathscr{C}^1, \ldots, \mathscr{C}^k, \ldots, \mathscr{C}^\infty$

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ belongs to the class $\mathscr{C}^0(\Omega)$ if is it continuous in every point of Ω.

The notion of continuous differentiability is more complicated than in the case of functions of only one variable. Let us start with the continuous differentiability.

Def. B.1.1 ($\mathscr{C}^1$-differentiability) *A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to belong to the class $\mathscr{C}^1(\Omega)$ if it is Fréchet differentiable for any point $x_0 \in \Omega$ and if the Fréchet derivative function, i.e. the map that associates to each point of Ω the Fréchet derivative of f in it:*

$$\begin{array}{rccl} Df: & \Omega \subseteq \mathbb{R}^n & \longrightarrow & L(\mathbb{R}^n,\mathbb{R}^m) \cong \mathbb{R}^{nm} \\ & x_0 & \longmapsto & Df(x_0), \end{array}$$

is continuous.

Suppose $f \in \mathscr{C}^1(\Omega)$, then it is possible to introduce the following continuous linear functional on $\mathscr{C}^1(\Omega)$ that has a great importance in differential geometry:

$$\boxed{\left.\frac{\partial}{\partial x^i}\right|_{x_0} := ev_{x_0} \circ \frac{\partial}{\partial x^i}} \in \mathscr{C}^1(\Omega)^*, \tag{B.14}$$

where ev_{x_0} is the evaluation map in x_0, so:

$$\begin{array}{rccl} \frac{\partial}{\partial x^i}\Big|_{x_0}: & \mathscr{C}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \frac{\partial}{\partial x^i}\Big|_{x_0}(f) = \left(ev_{x_0} \circ \frac{\partial}{\partial x^i}\right)(f) = \frac{\partial f}{\partial x^i}(x_0), \end{array}$$

so, *first* we compute the partial derivative of f w.r.t x^i and *then* we evaluate the resulting function in x_0.

Suppose that $f \in \mathscr{C}^1(\Omega)$, then we can ask ourselves if the Fréchet derivative function is Fréchet differentiable in a point $x_0 \in \Omega$. If this is the case, then we say that f is two-times Fréchet differentiable in x_0 and we denote its second Fréchet derivative in x_0 as $D^2 f(x_0)$.

Of course $D^2 f(x_0)$ will still be a linear operator, but this time it will belong to the vector space $L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$ because the domain of Df is still $\Omega \subseteq \mathbb{R}^n$, but its range is the vector space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e. $L(\mathbb{R}^n, \mathbb{R}^m)$.

A useful result of linear algebra allows us to naturally identify $L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)) \cong \mathbb{R}^{n^2 m}$ with the vector space $\mathrm{Bil}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)$ of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$:

$$\begin{array}{rccl} \phi: & L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)) & \xrightarrow{\sim} & \mathrm{Bil}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m) \\ & T & \longleftrightarrow & \phi_T, \quad \phi_T(x, y) := (Tx)y, \quad \forall x, y \in \mathbb{R}^n, \end{array}$$

perfectly well-defined: $T \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$, so T acts linearly on $x \in \mathbb{R}^n$ to get $Tx \in L(\mathbb{R}^n, \mathbb{R}^m)$, which acts linerly on y to get $(Tx)y \in \mathbb{R}^m$. The naturalness of the isomorphism comes from the fact that no other structure than the very nature of the elements of the spaces involved in the definition is used.

These considerations justify the following definition.

Def. B.1.2 ($\mathscr{C}^2$-differentiability) *If $D^2 f(x_0)$ exists for every $x_0 \in \Omega$, then*

$$\begin{array}{rccl} D^2 f: & \Omega \subseteq \mathbb{R}^n & \longrightarrow & \mathrm{Bil}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m) \\ & x_0 & \longmapsto & D^2 f(x_0), \end{array}$$

is called second total derivative function of f.

f is said to belong to the class $\mathscr{C}^2(\Omega)$ if the function $D^2 f$ exists and it is continuous in every point of Ω.

Theorem B.1.1 (Schwarz's theorem) *If $f \in \mathscr{C}^2(\Omega)$, then $D^2 f(x_0) \in \mathrm{Bil}_S(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)$ for all $x_0 \in \Omega$, where Bil_S stays for symmetric bilinear functions.*

Of course, we can iterate the procedure and consider $D^k f(x_0)$, the k-th total derivative of f in x_0, which will be an element of the multilinear maps from k copies of $\mathbb{R}^n$ to $\mathbb{R}^m$:

$$D^k f(x_0) \in \mathrm{Mul}^k(\mathbb{R}^n \times \ldots \times \mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{n^k m},$$

i.e. $D^k f$ transforms linearly each variable of the Cartesian product $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k times), taken separately, to an element of $\mathbb{R}^m$.

Def. B.1.3 ($\mathscr{C}^k$-differentiability) *If $D^k f(x_0)$ exists for every $x_0 \in \Omega$, then*

$$\begin{array}{rccl} D^k f: & \Omega \subseteq \mathbb{R}^n & \longrightarrow & \mathrm{Mul}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^m) \\ & x_0 & \longmapsto & D^k f(x_0), \end{array}$$

is called the k-th total derivative function of f.

f is said to belong to the class $\mathscr{C}^k(\Omega)$ if the function $D^k f$ is continuous on Ω.

Schwarz's theorem implies that, if $f \in \mathscr{C}^k(\Omega)$, then $D^k f(x_0) \in \mathrm{Mul}^k_S(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^m)$ for all $x_0 \in \Omega$, where Mul^k_S stays for *symmetric* multilinear functions.

Def. B.1.4 ($\mathscr{C}^\infty$-differentiability or smoothness) *f is said to belong to the class $\mathscr{C}^\infty(\Omega)$, or simply to be smooth on Ω, if $D^k f$ exists and it is continuous on Ω for all $k \in \mathbb{N}$.*

The continuous linear functional $\frac{\partial}{\partial x^i}\big|_{x_0} := ev_{x_0} \circ \frac{\partial}{\partial x^i} \in \mathscr{C}^\infty(\Omega)^*$ plays a crucial role in differential geometry.

Appendix C

Recap of projective geometry (Nicoletta Prencipe and Edoardo Provenzi)

Index

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