Questions about error terms in Sato-Tate distributions

Around Frobenius distributions and related topics 3

October 05, 2022

Equidistributed sequences

Let Ω be a closed interval in $\mathbb{R}$ with a probability measure μ. A sequence $X = \{x_n\}$ of real numbers in Ω is said to be equidistributed with respect to μ (or μ-equidistributed) if for all intervals $I \subseteq \Omega$,

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A pertinent analytic question about a μ-equidistributed sequence is about error terms in the above asymptotics.

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- We consider sequences X in a suitable family $\mathcal{F}$. Can we obtain average error terms

$$\frac{1}{|\mathcal{F}|} \sum_{X \in \mathcal{F}} D_X(I,V) \text{ and } \frac{1}{|\mathcal{F}|} \sum_{X \in \mathcal{F}} D_X(\phi,V)?$$

Does the order of the discrepancy improve upon averaging?

Fluctuations

- Can we estimate or find asymptotics for the variance

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- This leads to questions about higher moments of $D_X(I,V)$ and $D_X(\phi,V)$.
- Such questions can be approached by modelling $D_X(I,V)$ and $D_X(\phi,V)$ as sums of "independent" random variables.

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- If $R(t) = 1$, then we say that the pair correlation function of X is Poissonnian.

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- Any Hecke newform $f(z) \in \mathcal{F}_k(N)$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n,$$

where $a_f(1) = 1$ and

$$\frac{T_n(f(z))}{n^{\frac{k-1}{2}}} = a_f(n) f(z),\ n \geq 1.$$

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- By the Sato-Tate equidistribution theorem, for a non-CM eigenform $f \in \mathcal{F}_k(N)$, the above sequence is equidistributed in the interval $[0, 1]$ with respect to the measure

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That is, for any interval $I \subset [0, 1]$,

$$\lim_{x\to\infty} \frac{1}{\pi_N(x)} \#\{p \le x,\ (p, N) = 1 :\ \theta_f(p) \in I\} = \int_I \mu(t)dt,$$

where $\pi_N(x)$ is the number of primes up to x, coprime to N.

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- It is sufficient to check whether above asymptotic holds for a collection of polynomials $P_m(y)$, $m \geq 0$ where $P_m(y)$ denotes a polynomial of degree m.

- We choose $\{U_m(y)\}_{m\geq 0}$, where $U_m(y)$ is the m-th Chebyshev polynomial given by

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- Thorner (2020) proves unconditionally that

$$D_{f,I}(x) \ll \frac{\pi(x)\log(kN\log x))}{\sqrt{\log x}}.$$

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The trace formula for Hecke operators

Proposition (Estimates from the Eichler-Selberg trace formula)

Let k be a positive even integer, m be a positive integer and p be coprime to N. Then,

$$\frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} a_f(p^m) = \frac{\delta_{2|m}}{p^{m/2}} + \mathrm{O}\left(\frac{p^{mc}}{kN^{1/2-\epsilon}} \right).$$

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$$\frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} a_f(p^m) = \frac{\delta_{2|m}}{p^{m/2}} + \mathrm{O}\left(\frac{p^{mc}}{kN^{1/2-\epsilon}}\right).$$

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The trace formula for Hecke operators

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Theorem (Sarnak, Zubrilina, 2022)

$$\frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} a_f(p^m) = \frac{\delta_{2|m}}{p^{m/2}} + O\left(\frac{p^{m/8+\epsilon}}{N^{1/2-\epsilon}}\right).$$

Thus, for every $m \geq 0$,

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$$= \int_{-2}^{2} U_m(y)\widetilde{\nu}_p(y)dy,$$

where

$$\widetilde{\nu}_p(y) = \frac{p+1}{\pi\left[(p^{1/2}+p^{-1/2})^2 - y^2\right]}\sqrt{1-\frac{y^2}{4}}.$$

What can we say about $\sum_{p\leq x}\mu_p$ and

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Theorem (Sarnak, 1984)

Let p, q be distinct primes. Then, as $N + k \to \infty$ such that k is even and $p, q \nmid N$,

$$\mathbb{E}[X_p X_q] \sim \nu_p(I)\nu_q(I).$$

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Theorem (Conrey-Duke-Farmer, 1997, Nagoshi, 2006)

Let I be a fixed interval in $[0,1]$ and $X_p : \mathcal{F}_k(1) \to \{0,1\}$ is given by $X_p(f) = \chi_I(\theta_f(p))$.

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Above results can be generalized to a higher (fixed) level N.

Theorem (Prabhu-S, 2019)

Let N be a positive integer. If $(p, N) = 1$, define $X_p : \mathcal{F}_k(N) \to \{0, 1\}$ as $X_p(f) = \chi_I(\theta_f(p))$. Let $k = k(x)$ such that

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as $x \to \infty$. For any integer $r \geq 0$,

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$$V_{G,L} = \int_0^1 G_L(t)^2 \mu(t)dt - \left(\int_0^1 G_L(t)\mu(t)dt \right)^2.$$

Theorem (Contd.)

- *For any integer $r \geq 0$,*

$$\frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} \left(\frac{\sum_{p \leq x} X_p(f) - \pi(x) \int_0^1 G_L(t)\mu(t)dt}{\sqrt{\pi(x) V_{G,L}}} \right)^r$$

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Theorem (Baier-Prabhu-S, 2020)

For $\lambda, \omega > 0$, suppose the Fourier transform $\widehat{g}$ satisfies $\widehat{g}(t) \ll e^{-\lambda |t|^{\omega}}$, as $|t| \to \infty$. Then, the conclusions of previous theorem hold if $k = k(x) \geq 2$ satisfies $\frac{(\log k)}{(\log x)^{1+1/\omega}} \to \infty$ as $x \to \infty$.

Motivation and proofs

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- In 2006, Nagoshi proved the following: If $\log k / \log x \to \infty$ as $x \to \infty$, then

$$\lim_{x \to \infty} \frac{1}{s_k(1)} \sum_{f \in \mathcal{F}_k(1)} \left(\frac{\sum_{p \le x} a_f(p)}{\sqrt{\pi(x)}} \right)^r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt.$$

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- We approximate

$$\sum_{m=0}^{L} S^-(m) a_f(p^{2m}) \le \chi_I(\theta_f(p)) \le \sum_{m=0}^{L} S^+(m) a_f(p^{2m}),$$

and reduce our questions to the evaluation of moments

$$\frac{1}{s_k(N)} \sum_{f\in\mathcal{F}_k(N)} \left(\sum_{p\le x} \sum_{m=1}^{L} S^{\pm}(m) a_f(p^{2m}) \right)^r.$$

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- It is natural to consider

$$\frac{1}{s_k(N)} \sum_{f \in \mathcal{F}_k(N)} \left(\sum_{p \leq x} \sum_{m=1}^{\infty} G(m) a_f(p^{2m}) \right)^r ,$$

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  and seek conditions on $G(m)$ which ensure the convergence of these moments.
- This motivates us to choose the kind of test functions which ensure appropriate decay of $G(m)$. The contribution from $a_f(p^{2m})$ is measured by trace formulas.

Straightened Hecke angles

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- In the 1990s, Katz and Sarnak asked questions that compares the spacings between straightened Hecke angles to Poissonian spacing spacings.
- One way to address these questions is via the pair correlation function, which looks at the spacings between unordered elements of a uniformly distributed sequence.

Pair correlation of straightened Hecke angles

Question (Katz, Sarnak)

For any $s > 0$, define the interval

$$I_x = \left[\frac{-s}{\pi_N(x)}, \frac{s}{\pi_N(x)}\right].$$

the pair correlation function of the sequence $\{H(\theta_f(p)) : p \text{ prime}, (p, N) = 1\}$ is defined as:

$$R(x, s)(f) :=$$

$$\frac{1}{\pi_N(x)} \# \left\{ \begin{array}{c} (p, q) : p \neq q \leq x, (p, N) = (q, N) = 1, \\ H(\theta_f(p)) - H(\theta_f(q)) \in I_x + \mathbb{Z} \end{array} \right\}.$$

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For any $s > 0$, does $\lim_{x \to \infty} R(x, s)(f)$ exist and is it equal to $2s$?

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Question (Katz, Sarnak)

For any $s > 0$, define the interval

$$I_x = \left[\frac{-s}{\pi_N(x)}, \frac{s}{\pi_N(x)}\right].$$

the pair correlation function of the sequence $\{H(\theta_f(p)) : p \text{ prime}, (p, N) = 1\}$ is defined as:

$$R(x, s)(f) :=$$

$$\frac{1}{\pi_N(x)} \# \left\{ \begin{array}{c} (p, q) : \ p \neq q \leq x, \ (p, N) = (q, N) = 1, \\ H(\theta_f(p)) - H(\theta_f(q)) \in I_x + \mathbb{Z} \end{array} \right\}.$$

For any $s > 0$, does $\lim_{x \to \infty} R(x, s)(f)$ exist and is it equal to $2s$? If the answer is yes, we say that the sequence $\{H(\theta_f(p))\}$ has Poissonnian pair correlation.

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- We define

$$\widetilde{R}_\delta(x, s)(f) :=$$

$$\frac{1}{\pi_N(x)\mu(I_\delta)} \# \left\{ \begin{array}{c} (p, q) : \ p \neq q \leq x, \ (p, N) = (q, N) = 1, \\ \theta_f(p), \theta_f(q)) \in I_\delta, \\ H(\theta_f(p)) - H(\theta_f(q)) \in I_x \end{array} \right\}.$$

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- By the result of Thorner, we get the following: Let $F(x)$ be a monotonically increasing function with $\lim_{x\to\infty} F(x) = \infty$. If $\delta(x) \to 0$ is chosen such that

$$\mu(I_\delta) \geq \frac{\log(kN\log x)F(x)}{\sqrt{\log x}},$$

then $\#\{p \leq x : (p,N)=1,\, \theta_f(p) \in I_\delta\} \sim \pi_N(x)\mu(I_\delta)$.

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- For $0 < \psi < 1$, henceforth, we denote $A := 2\sin^2 \pi\psi$. Let us consider intervals
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- The advantage of localizing our intervals around ψ is that the Sato-Tate density $2\sin^2 \pi t \sim A$ is essentially constant in short intervals around ψ and the straightening of the Hecke angles amounts to rescaling them.

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- The advantage of localizing our intervals around ψ is that the Sato-Tate density $2\sin^2 \pi t \sim A$ is essentially constant in short intervals around ψ and the straightening of the Hecke angles amounts to rescaling them.
- Thus,
$$\mathcal{L}_f := \#\left\{p \leq x : \, (p, N) = 1, \, \theta_f(p) \in \mathcal{I}_L\right\}$$
$$\sim \pi_N(x) \int_{\psi - \frac{1}{AL}}^{\psi + \frac{1}{AL}} 2\sin^2 \pi t \, dt \sim A \frac{2}{AL} \pi_N(x) = \frac{2\pi_N(x)}{L}.$$

- If $\theta_f(p)$, $\theta_f(p) \in \mathcal{I}_L$, then, as $x \to \infty$,

$$H(\theta_f(p)) - H(\theta_f(q)) = \int_{\theta_f(q)}^{\theta_f(p)} 2\sin^2 \pi t \, dt \sim A(\theta_f(p) - \theta_f(q)).$$

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- Let

$$\widetilde{I_x} = \left[\frac{-s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right].$$

- As $x \to \infty$,

$$\widetilde{R}_{1/AL}(x,s)(f)$$

$$\sim \frac{1}{\mathcal{L}_f} \sum_{p \neq q \leq x} \chi_{\mathcal{I}_L}(\theta_f(p)) \chi_{\mathcal{I}_L}(\theta_f(q)) \chi_{\widetilde{I_x}}(\theta_f(p) - \theta_f(q)).$$

Smooth analogue of pair correlation function

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- Let g, ρ be real valued, even functions $\in C^\infty(\mathbb{R})$ in the Schwartz class with Fourier transforms supported in the interval $[-1, 1]$.
- Define, for $L = L(x) \geq 1$, the functions

$$\rho_L(\theta) := \sum_{n \in \mathbb{Z}} \rho(L(\theta + n)),\ G_x(\theta) := \sum_{n \in \mathbb{Z}} g\left(\pi_N(x)(\theta + n)\right).$$

- The smoothened pair correlation function is

$$R_2(g, \rho)(f) :=$$

$$\frac{L}{2\pi_N(x)} \sum_{\substack{p,q \leq x \\ (p,N)=(q,N)=1 \\ p \neq q}} \rho_L(\theta_f(p) - \psi)\rho_L(\theta_f(q) - \psi)G_x(\theta_f(p) - \theta_f(q)).$$

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- Poissonnian pair correlation: $R_2(g, \rho)(f) \sim A^2 \widehat{g}(0) \rho * \rho(0)$ as $x \to \infty$.

Expected value of the pair correlation function

Theorem (Balasubramanyam, S, 2019)

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$$+ \mathrm{O}\left(\frac{1}{L}\right) + \mathrm{O}\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + \mathrm{O}\left(\frac{L \pi_N(x) x^{\pi_N(x) c} \sqrt{N}}{|\mathcal{F}_k(N)|}\right).$$

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2. *If we choose L such that $L = \mathrm{o}\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$, and consider families $\mathcal{F}_k(N)$ such that $\frac{\log\left(k\sqrt{N}/4^{\nu(N)}\right)}{x} \to \infty$, then,*
$$\frac{1}{|\mathcal{F}_k(N)|} \sum_{f \in \mathcal{F}_k(N)} R_2(g, \rho)(f) \sim A^2 \widehat{g}(0) \rho * \rho(0) \text{ as } x \to \infty.$$

Second moment and variance

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Let N, k, ρ_L, G_x, R_2 be as defined above.

$$\frac{1}{|\mathcal{F}_k(N)|}\sum_{f\in\mathcal{F}_k(N)}\left(R_2(g,\rho)(f)-A^2\widehat{g}(0)\rho*\rho(0)\right)^2$$

$$\ll \frac{1}{L}+\frac{L(\log\log x)^2}{\pi_N(x)}+\frac{L^2(\log\log x)^3}{\pi_N(x)^2}+\frac{L^2(\log\log x)^4}{\pi_N(x)^3}$$

$$+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{L^3(\log\log x)^4}{\pi_N(x)^4}$$

$$+\frac{L^4\pi_N(x)^2x^{(8L+8\pi_N(x))c'}4^{\nu(N)}}{k\sqrt{N}},$$

where $c' > 0$ is an absolute constant.

Theorem (Contd.)

If we choose $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$, *and consider families* $\mathcal{F}_k(N)$ *with levels* $N = N(x)$ *and even weights* $k = k(x)$ *such that*

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Question: Are the above growth conditions sufficient to give us

$$\frac{1}{|\mathcal{F}_k(N)|} \sum_{f\in\mathcal{F}_k(N)} (R_2(g,\rho)(f))^r \sim (A^2\widehat{g}(0)\rho*\rho(0))^r?$$

Terms of the pair correlation function

$$R_2(g,\rho)(f) :=$$

$$\frac{1}{\mathcal{L}_f} \sum_{p \neq q \leq x} \rho_L(\theta_f(p) - \psi)\rho_L(\theta_f(q) - \psi) g_x\left(\theta_f(p) - \theta_f(q)\right)$$

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$$= \frac{1}{\mathcal{L}_f L^2 \pi_N(x)}\sum_{p\neq q\leq x}\left[\sum_{l\geq 0} H(l)a_f(p^{2l})\right]\left[\sum_{l'\geq 0} H(l')a_f(q^{2l'})\right]$$

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$$\times\left[4G(0)+\sum_{n\geq 1}2G(n)(a_f(p^{2n})-a_f(p^{2n-2}))(a_f(q^{2n})-a_f(q^{2n-2}))\right],$$

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where

$$H(l)=\widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)-\widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi)$$

and $G(n)=\widehat{g}\left(n/\pi_N(x)\right)$.

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