Master 2 Année **2017-18**
Analytic Number Theory Université de Bordeaux

Midterm

due Nov. 16

In the sequel we use the notation $e(z) = \exp(2i\pi z)$, for $z \in \mathbb{C}$.

Exercise 1. (*Another proof of the Pólya–Vinogradov inequality*)

The goal of this exercise is to give a proof of the Pólya–Vinogradov inequality using the so-called *completion method*. We recall the setting: $q \geq 1$ is an integer and χ is a fixed *primitive* Dirichlet character modulo q. We fix an integer $N \in \{1, \dots, q-1\}$ and we want to evaluate $\sum_{1 \leq n \leq N} \chi(n)$.

1. For any integer k, let $\hat{\chi}(k) = q^{-1} \sum_{n=0}^{q-1} \chi(n) e\left(\frac{-nk}{q}\right)$. Prove that for every integer n one has
$$\chi(n) = \sum_{k=0}^{q} \hat{\chi}(k) e\left(\frac{kn}{q}\right) .$$
2. Deduce from the computation seen in class, an upper bound for $\sum_{1 \leq n \leq N} e\left(\frac{kn}{q}\right)$.
3. Recall what the absolute value of the Gaussian sum $G(a,q) = \sum_{n=1}^{q} \chi(n) e(an/q)$ is for a an integer coprime to q and conclude that
$$\left| \sum_{1 \leq n \leq N} \chi(n) \right| \ll \sqrt{q} \log q \,,$$
with an absolute implied constant.

Exercise 2. (*A simplified large sieve inequality*)

Fix integers $Q, N \geq 1$, $M \geq 0$. The goal of this exercise is to prove that the inequality
$$\sum_{1 \leq m \leq Q} \sum_{\substack{a \bmod m \\ \gcd(a,m)=1}} \left| \sum_{M < n \leq M+N} a_n e\left(\frac{an}{m}\right)) \right|^2 \leq \Delta \sum_{M < n \leq M+N} |a_n|^2 , \ \ ((a_n) \subset \mathbb{C}) \,,$$
holds for the choice $\Delta = 2\pi N + Q^2$. This is slightly weaker than what has been proved in class but in many applications it is sufficient.

1. Explain why it is enough to prove the inequality in the case $M = 0$
2. Let $f \colon [0,1] \to \infty$ be a smooth function (meaning f is indefinitely differentiable). Observe that
$$f(1/2) = \int_0^1 f(t) \mathrm{d}t + \int_0^{1/2} t f'(t) \mathrm{d}t + \int_{1/2}^1 (t-1) f'(t) \mathrm{d}t$$
and prove
$$|f(1/2| \leq \int_0^1 (|f(t)| + \frac{1}{2} |f'(t)|) \mathrm{d}t \,.$$
3. Deduce that for fixed $x \in \mathbb{R}$, $\delta > 0$ and for any function f smooth on $[x - \delta/2, x + \delta/2]$ we have:

$$|f(x)| \leq \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2}|f(t)|\mathrm{d}t + \frac{1}{2}\int_{x-\delta/2}^{x+\delta/2}|f'(t)|\mathrm{d}t\,.$$

4. Apply the above to the function $f(t) = \left(\sum_{n=1}^N a_n e(nt)\right)^2$ for the choice $x = a/q$, $m \leq Q$, $\gcd(a, m) = 1$, $\delta = Q^{-2}$.

5. Observe that the intervals $(a/q - \delta/2, a/q + \delta/2)$ are pairwise disjoint for $q \leq Q$ and a and q coprime and sum what was obtained in question 4 over q and a.

6. Conclude, using the Parseval identity and the Cauchy–Schwarz inequality.

Exercise 3. (*An L-function over a function field*)

Let $\mathbb{F}$ be a finite field of $q = p^a$ elements, where p is a prime number. We consider in this exercise the generalized Gauss sum $G(\chi, \psi) = \sum_{x\in\mathbb{F}} \chi(x)\psi(x)$, where χ (resp. ψ) is a character of $(\mathbb{F}^\times, \times)$ (resp. of $(\mathbb{F}, +)$) and its higher dimensional analogues: for $n \geq 1$, let $\mathbb{F}_n$ be a degree n extension of $\mathbb{F}$ and define

$$G_n(\chi, \psi) = \sum_{x\in\mathbb{F}_n} \chi(\mathrm{Norm}(x))\psi(\mathrm{Tr}(x))\,,$$

where Norm (resp. Tr) is the norm map (resp. the trace map) relative to $\mathbb{F}_n/\mathbb{F}$. The *zeta function* attached to the family $(G_n(\chi, \psi))_{n\geq 1}$ is defined as an element of $\mathbb{C}[[T]]$ by

$$Z(\chi, \psi) = \exp\left(\sum_{n\geq 1} \frac{G_n(\chi, \psi)}{n} T^n\right).$$

The goal of this exercise is to prove the *rationality* of $Z(\chi, \psi)$; precisely $Z(\chi, \psi) = 1 + G(\chi, \psi)T$.

We denote $R = \mathbb{F}[X]$ and $F = \mathbb{F}(X)$, which is the field of rational functions in one variable over $\mathbb{F}$.

1. For $h \in R$, define $N(h) := q^d$ where $d = \deg(h)$. If H is the subgroup of $F^\times$ consisting of quotients of monic polynomials and if G is a subgroup of H such that $h_1h_2 \in G$ implies $h_1, h_2 \in G$ then a character $\alpha\colon G \to \mathbb{C}^\times$ can be extended to a totally multiplicative function on the set of monic elements of R by setting $\alpha(h) = 0$ if $h \notin G$. Prove that the Dirichlet series defined by
$$L(s, \alpha) = \sum_{\substack{h\in R\\ h \text{ monic}}} \alpha(h)N(h)^{-s} \qquad (\mathrm{Re}(s) > 1)$$
has the following factorization $L(s, \alpha) = \prod_{P \text{ monic irred.}} (1 - \alpha(P)N(P)^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$.

2. Consider the subgroup G of H consisting of the rational functions f that are defined and non-vanishing at 0. Show that λ defined by $\lambda(h) = \chi(a_d)\psi(a_1)$ for $h = X^d - a_1X^{d-1} + \cdots + (-1)^d a_d$ can be extended to a character of G.

3. Prove that $L(s, \lambda) = 1 + G(\chi, \psi)q^{-s}$. [To do so write $L(s, \lambda) = \sum_{d\geq 0}(\sum_{\deg(h)=d} \lambda(h))q^{-ds}$, and compute the inner sum separately for $d = 0$, $d = 1$ and $d \geq 2$.]

4. Show that $\sum_{\substack{P\\ d=\deg(P)|n}} d\lambda(P)^{n/d} = G_n(\chi, \psi)$ for any $n \geq 1$.

5. Let $Z(T) := Z(\chi, \psi)$. Compute the logarithmic derivative of $Z(q^{-s})$ and show that it coincides with the logarithmic derivative of $L(s, \lambda)$.

6. Deduce that $Z(q^{-s}) = L(s, \lambda)$ and conclude.