Lecture 3 (Sept. 15, 2025).

During the lecture we proved the following statement.

Theorem 1. *Let X be compact and let μ be a Radon measure on X. Let (x_n) be a sequence of elements of X. Then (x_n) is μ-equidist. in X if and only if for all $E \in \mathcal{B}(X)$ such that $\mu(\overline{E} \setminus \mathring{E}) = 0$, one has*

$$\frac{1}{n}|\{i \leq n \colon x_i \in E\}| \to \mu(E) \qquad (n \to \infty)\,.$$

To clarify the argument for the converse implication given during the lecture we repeat it here.

First the assumption means that $n^{-1}\sum_{i=1}^n \mathbb{1}_E(x_i)$ tends to $\int_X \mathbb{1}_E \mathrm{d}\mu$ for every Borel subset E of X of μ-negligible boundary. By linearity this extends to any function $g = \sum_{i=1}^k \alpha_i \mathbb{1}_{E_i}$ where the E_i are pairwise disjoint Borel subsets of X of μ-negligible boundary and the α_i are real numbers.

Let $f \colon X \to \mathbf{R}$ be continuous and let $\varepsilon > 0$. We exploit the following lemma.

Lemma 1. *There exists pairwise disjoint open subsets $X_1, \ldots, X_k$ of X, all having μ-negligible boundary, such that the complement N_ε of $\cup_{1 \leq i \leq k} X_i$ is closed and μ-negligible and*

$$\forall i \in \{1, \ldots, k\}\,, \qquad \Big|\sup_{x \in X_i} f(x) - \inf_{x \in X_i} f(x)\Big| < \varepsilon\,.$$

Proof of the Lemma. First note that for any h outside of a Lebesgue negligible subset Z of $\mathbf{R}$, one has $\mu(f^{-1}\{h\}) = 0$ (proof left as an exercise). Next compactness of X and continuity of f allow us to cover $f(X)$ by a finite number of intervals $[h_{i-1}, h_i]$ $(1 \leq i \leq k)$ satisfying:

$$\forall i \geq 1,\ h_{i-1} \notin Z,\ h_i \notin Z,\ h_{i-1} < h_i \text{ and } |h_i - h_{i-1}| < \varepsilon\,.$$

(Here we use the fact that Z contains no open interval of $\mathbf{R}$ to make sure the h_i can be chosen outside of Z.) For each i, set $X_i = f^{-1}(]h_{i-1}, h_i[)$. The X_i are pairwise disjoint open subsets of X and the complement N_ε of the union of the X_i is $\cup_{1 \leq i \leq k} f^{-1}\{h_i\}$ which is μ-negligible (by the fact that $h_i \notin Z$, for all i). The fact that f has fluctuations of magnitude at most ε on each X_i comes from the definition of the sequence (h_i) and finally note that

$$\overline{X_i} \setminus \mathring{X}_i \subset f^{-1}([h_{i-1}, h_i]) \setminus X_i = f^{-1}\{h_{i-1}, h_i\}$$

which is μ-negligible since $h_i \notin Z$ for all i. □

For each $i \in \{1, \ldots, k\}$, choose $z_i \in X_i$ and consider $g_\epsilon = \sum_{i=1}^k f(z_i)\mathbb{1}_{X_i}$. By the Lemma we have

$$\sup_{x \in X \setminus N_\varepsilon} |f(x) - g_\varepsilon(x)| < \varepsilon\,. \tag{1}$$

The lemma, combined with the discussion preceding it, shows moreover that under our assumption:

$$\frac{1}{n}\sum_{i=1}^n g_\varepsilon(x_i) \to \int_X g_\varepsilon \mathrm{d}\mu \qquad (n \to \infty)\,. \tag{2}$$

Finally we compute:

$$\left|\frac{1}{n}\sum_{j=1}^n f(x_j) - \int_X f\mathrm{d}\mu\right| \leq \left|\int_X g_\varepsilon \mathrm{d}\mu - \int_X f\mathrm{d}\mu\right| + \left|\frac{1}{n}\sum_{j=1}^n g_\varepsilon(x_j) - \int_X g_\varepsilon \mathrm{d}\mu\right| + \frac{1}{n}\sum_{j=1}^n \Big|f(x_j) - g_\varepsilon(x_j)\Big|\,.$$

On the right-hand side, the first summand is upper bounded by ε (combine (1) with the fact that $\mu(N_\varepsilon) = 0$). The second summand is upper bounded by ε for big enough n by (2). As for the third summand, we invoke (1) once more to see that it is upper bounded by

$$\varepsilon + \frac{1}{n} \sum_{\substack{i=1 \\ x_i \in N_\varepsilon}}^{n} |f(x_i)| \leq \varepsilon + \|f\|_\infty \Big(\frac{1}{n} |\{i \leq n \colon x_i \in N_\varepsilon\}| \Big)$$

and we conclude by applying our assumption to N_ε (closed and μ-negligible):

$$\frac{1}{n} |\{i \leq n \colon x_i \in N_\varepsilon\}| \to \mu(N_\varepsilon) = 0 \qquad (n \to \infty)\,.$$