THE SCHUR–ZASSENHAUS THEOREM

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When N is a normal subgroup of G, can we reconstruct G from N and G/N? In general, no. For instance, the groups $\mathbf{Z}/(p^2)$ and $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ (for prime p) are nonisomorphic, but each has a cyclic subgroup of order p and the quotient by it also has order p. As another example, the nonisomorphic groups $\mathbf{Z}/(2p)$ and D_p (for odd prime p) have a normal subgroup that is cyclic of order p, whose quotient is cyclic of order 2.

If we impose the condition that N and G/N have relatively prime order, then something nice can be said: G is a semidirect product of N and G/N. This is the Schur-Zassenhaus theorem, which we will discuss below. It doesn't uniquely determine G, as there could be several non-isomorphic semi-direct products of the abstract groups N and G/N, but each one is a group with normal subgroup N and quotient by it isomorphic to G/N. For instance, if $N \cong \mathbf{Z}/(p)$ for odd prime p and $G/N \cong \mathbf{Z}/(2)$ then G must be a semi-direct product $\mathbf{Z}/(p) \rtimes \mathbf{Z}/(2)$. The only two semidirect products are the direct product (which is isomorphic to $\mathbf{Z}/(2p)$) and the nontrivial semidirect product (which is isomorphic to D_p).

Theorem 1 (Schur-Zassenhaus)**.** *Let G be a finite group and write $\#G = ab$ where $(a, b) = 1$. If G has a normal subgroup of order a then it has a subgroup of order b.*

Let's see why this theorem tells us G is a semidirect product. Letting N be the normal subgroup of order a and H be a subgroup of order b, $N \cap H$ is trivial since $(a, b) = 1$, so $G = NH \cong N \rtimes H$ is a semidirect product with N as the normal factor.

We will present two proofs of this theorem. Both proofs will be incomplete at the end. Each proof will reduce to the case when N is abelian, at which point the machinery of group cohomology can be applied. While group cohomology provides a general tool to describe the groups having a particular normal subgroup with a particular quotient group (up to isomorphism), it requires the normal subgroup be abelian, and we are making no such assumption. So the proof of the Schur-Zassenhaus theorem amounts to a reduction process to the case when N is abelian.

The first proof of the theorem will use the following lemma.

Lemma 2. *If $N \lhd G$ and $P \in \mathrm{Syl}_p(N)$ then $G = N \cdot \mathrm{N}_G(P)$. In particular, if $P \lhd N$ then $P \lhd G$.*

Proof. Pick $g \in G$. Since $P \subset N$ and $N \lhd G$, $gPg^{-1} \subset N$. Then by Sylow II for the group N, there is an $n \in N$ such that $gPg^{-1} = nPn^{-1}$, so $n^{-1}gPg^{-1}n = P$. That means $n^{-1}g \in \mathrm{N}_G(P)$, so $g \in n\,\mathrm{N}_G(P)$. Thus $G = N \cdot \mathrm{N}_G(P)$.

If $P \lhd N$ then $N \subset \mathrm{N}_G(P)$, so $N \cdot \mathrm{N}_G(P) = \mathrm{N}_G(P)$. Thus $G = \mathrm{N}_G(P)$, so $P \lhd G$. □

Here is the first proof of the Schur–Zassenhaus theorem.

Proof. Assume the theorem is false and let G be a counterexample of minimal order. So any group with order less than $\#G$ satisfies the theorem. Easily $a > 1$ and $b > 1$.

Let $N \lhd G$ with $\#N = a$. We aim to get a contradiction.

Step 1: Show N is a minimal normal subgroup of G: there are no normal subgroups of G lying strictly between $\{e\}$ and N.

Suppose $N' \lhd G$ with $\{e\} \subset N' \subset N$ and $N' \neq \{e\}$ or N. We look at the group G/N' with order $< \#G$. Since $N/N' \lhd G/N'$ and $\#(G/N') = \#(N/N')b$ with the two factors being relatively prime, by minimality of G there is a subgroup of G/N' with order b. It has the form K/N', so $\#K = \#N'b < ab$. Since $\#N'$ and b are relatively prime, by minimality of G there is a subgroup of order b in K and hence in G. This is a contradiction, so N' doesn't exist.

Step 2: Show N is an abelian p-group.

Let P be a nontrivial Sylow subgroup of N, so by Theorem 2 we have $G = N\,\mathrm{N}_G(P)$. Then $G/N \cong \mathrm{N}_G(P)/(N \cap \mathrm{N}_G(P))$ and the order of $\mathrm{N}_G(P)$ is $\#(N \cap \mathrm{N}_G(P))b$ with $\#(N \cap \mathrm{N}_G(P))$ a factor of a (hence relatively prime to b). Since $N \cap \mathrm{N}_G(P)$ is a normal subgroup of $\mathrm{N}_G(P)$, if $N_G(P)$ is a proper subgroup of G then by minimality of G there is a subgroup of order b in $\mathrm{N}_G(P)$, and hence in G. This isn't possible, so $N_G(P) = G$, which means $P \lhd G$. Therefore, by the Sylow theorems, P is a normal subgroup of N, so $P = N$ by Step 1. Then $Z(P)$ is a nontrivial normal subgroup of P, so $Z(P) = P$ by Step 1 again, which means N is an abelian p-group.

Step 3: Show $N \cong (\mathbf{Z}/(p))^k$.

Considering the structure of finite abelian p-groups, this step is equivalent to showing $N^p = \{x^p : x \in N\}$ is trivial. Assume N^p is nontrivial. It is preserved as a set by all group automorphisms of N, so in particular $gN^pg^{-1} = N^p$ for any $g \in G$. Thus $N^p \lhd G$, so $N/N^p \lhd G/N^p$. Since N/N^p is a p-group while the index $[G/N^p : N/N^p] = [G : N]$ is relatively prime to p, by induction G/N^p has a subgroup of order $[G : N]$. The subgroup is H/N^p for some $H \subset G$, so $[H : N^p] = [G : N]$ is not divisible by p. Since $N^p \lhd H$, N^p is a p-group with index prime to p in H, and $\#H < \#G$, by induction again there is a subgroup K of H with order $[H : N^p] = [G : N]$. This K is also in G, so G has a subgroup of order $[G : N]$. This is a contradiction, so N^p is trivial.

Step 4: Get a final contradiction.

Let G act on N by conjugation. Since $N \cong (\mathbf{Z}/(p))^k$, automorphisms of N can be interpreted as elements of $\mathrm{GL}_k(\mathbf{Z}/(p))$. Therefore the conjugation action of G on N is a group homomorphism $G \to \mathrm{GL}_k(\mathbf{Z}/(p))$. Since N is abelian, it acts trivially on itself, so our action descends to a homomorphism $G/N \to \mathrm{GL}_k(\mathbf{Z}/(p))$. At this point the reader is referred to the literature for the rest of the proof. Two possible approaches are representation theory [2, p. 146] or group cohomology (the vanishing of $\mathrm{H}^2(G/N, N)$; a cohomological neophyte can find this done without any reference to cohomology in [3, pp. 246–247], but it is not very illuminating). $\square$

Here is a second proof. Again we will reduce to the case of an abelian normal subgroup.

Proof. Let $N \lhd G$ with $\#N$ and $[G : N]$ relatively prime. We want to prove G has a subgroup of order $[G : N]$. Of course we can assume N is a nontrivial proper subgroup of G.

We induct on $\#G$. Assume $\#G > 1$ and the theorem is verified for subgroups with smaller order. Let p be a prime factor of $\#N$ and P be a p-Sylow subgroup of N, so P is nontrivial. Because $[G : N]$ is prime to $\#N$, p does not divide $[G : N]$ so P is also a p-Sylow subgroup of G. Since $P \subset N$ and $N \lhd G$, all G-conjugates of P are in N. Therefore all the

p-Sylow subgroups of G are in N, hence by counting p-Sylows in G and in N we get

$$[G : \mathrm{N}_G(P)] = [N : \mathrm{N}_G(P) \cap N].$$

Writing these indices as ratios and rearranging terms,

$$[G : N] = [\mathrm{N}_G(P) : \mathrm{N}_G(P) \cap N]. \tag{1}$$

Case 1: P is not normal in G. Then $\mathrm{N}_G(P)$ is a proper subgroup of G. The group $\mathrm{N}_G(P) \cap N$ is normal in $\mathrm{N}_G(P)$ since $N \lhd G$, the order of $\mathrm{N}_G(P) \cap N$ divides $\#N$, and the index of $\mathrm{N}_G(P) \cap N$ in $\mathrm{N}_G(P)$ is $[G : N]$ by (1), so $\mathrm{N}_G(P)$ and its normal subgroup $\mathrm{N}_G(P) \cap N$ satisfy the hypotheses of the theorem. Since $\# \mathrm{N}_G(P) < \#G$, by induction $\mathrm{N}_G(P)$ has a subgroup of order $[\mathrm{N}_G(P) : \mathrm{N}_G(P) \cap N] = [G : N]$. This is a subgroup of G too, so we're done.

Case 2: $P \lhd G$. Then $P \lhd N$ and $N/P \lhd G/P$ with $\#(N/P)$ dividing $\#N$ and $[G/P : N/P] = [G : N]$. This order and index are relatively prime, and $\#(G/P) < \#G$, so by induction the theorem holds for G/P and its subgroup N/P: there is a subgroup in G/P of order $[G/P : N/P] = [G : N]$. Write the subgroup as H/P, so H is a subgroup of G and

$$[H : P] = \#(H/P) = [G : N] \tag{2}$$

is not divisible by p. (If $P = N$ then $H = G$.)

Since P is a nontrivial p-group, its center $Z := Z(P)$ is nontrivial. Also $Z \lhd H$ (the center of a normal subgroup is also a normal subgroup), so $P/Z \lhd H/Z$. The group P/Z is a p-group (possibly trivial, if P is abelian) while $[H/Z : P/Z] = [H : P] = [G : N]$ is prime to p, so (since $\#(H/Z) < \#H \leq \#G$) by induction H/Z contains a subgroup K/Z of order $[H : P]$. (If P is abelian then $K = H$.)

Now we have $Z \lhd K$ with Z a p-group and

$$[K : Z] = \#(K/Z) = [H : P] = [G : N]$$

being prime to p, so K and its normal subgroup Z satisfy the hypotheses of the theorem. Now *if* $\#K < \#G$ then we can apply induction to conclude K has a subgroup of order $[K : Z] = [G : N]$, and this is also a subgroup of G, so we're done. What if $K = G$? Since $K \subset H \subset G$, if $K = G$ then $H = G$ so $[G : P] = [G : N]$ by (2). Therefore $N = P$ since $P \subset N$, so N is a normal Sylow subgroup of G.

If N is a normal p-Sylow in G and it is not abelian, we can use induction yet again to finish the proof. Run through the argument two paragraphs up (with $P = N$, $H = G$, and $Z = Z(P) = Z(N)$ the center of N). We get a subgroup K/Z of G/Z with order $[G : N]$. Now $\#K = \#Z[G : N]$. If $Z \neq N$ (*i.e.*, N is non-abelian) then $\#Z < \#N$ so $\#K < \#N[G : N] = \#G$ and we are done as before.

What if N is normal in G and N is abelian? In this case we can, as in the previous proof, consider $N^p = \{x \in N : x^p = 1\}$. This is a normal subgroup of N and in fact it is normal in G too. Running through the previous paragraph with N^p in place of Z we are done by another induction unless $N^p = N$, which means all the elements of N have order p. So we are left to contemplate the same case as at the end of the first proof: N is a normal p-Sylow subgroup of G and is isomorphic to $(\mathbf{Z}/(p))^k$ for some k. The end of the proof is now the same as in the first proof: use either representation theory or group cohomology. $\square$

Remark 3. The Schur–Zassenhaus theorem actually has an important second part, which we omitted: any two subgroups of order b in G are conjugate to each other. See [3, p. 248] for the proof of that.

Let's put the Schur–Zassenhaus theorem to work. We ask, out of idle curiosity, whether $p|\#G$ implies $p|\#\operatorname{Aut}(G)$. The answer, of course, is no: try $G = \mathbf{Z}/(p)$. As we now show, this counterexample essentially explains all the others.

Corollary 4. *Fix a prime p. For a finite group G with order divisible by p, the following are equivalent:*

(1) *$\#\operatorname{Aut}(G)$ is not divisible by p,*
(2) *$G \cong \mathbf{Z}/(p) \times H$ where $\#H$ and $\#\operatorname{Aut}(H)$ are not divisible by p.*

In particular, if $p^2|\#G$ then $p|\#\operatorname{Aut}(G)$.

Proof. Assume (1) holds and let P be a p-Sylow subgroup of G. We expect to show $G \cong P \times H$ and $P \cong \mathbf{Z}/(p)$.

For any $x \in P$ there is the automorphism $\gamma_x \in \operatorname{Aut}(G)$ which is conjugation by x. Since x has p-power order, so does γ_x (recall $\gamma_x^n = \gamma_{x^n}$ for all n). By hypothesis $\#\operatorname{Aut}(G)$ is not divisible by p, so the only element of p-power order in $\operatorname{Aut}(G)$ is the identity. Thus $\gamma_x = \mathrm{id}_G$ for all $x \in P$, which means $P \subset Z(G)$. In particular, $P \lhd G$ by Sylow II and P is abelian. Therefore the Schur-Zassenhaus theorem tells us $G \cong PH$ for some subgroup H with order not divisible by p. Since $P \subset Z(G)$, $G \cong P \times H$. Because the groups P and H have relatively prime order and commute in G, $\operatorname{Aut}(G) \cong \operatorname{Aut}(P) \times \operatorname{Aut}(H)$ in the natural way. Therefore p doesn't divide $\#\operatorname{Aut}(P)$ or $\#\operatorname{Aut}(H)$.

Which finite abelian p-groups P have $\#\operatorname{Aut}(P)$ not divisible by p? Write P as a direct product of cyclic groups, say

$$P = \mathbf{Z}/(p^{r_1}) \times \cdots \times \mathbf{Z}/(p^{r_k}).$$

Since $\operatorname{Aut}(\mathbf{Z}/(p^r)) \cong (\mathbf{Z}/(p^r))^\times$ has order $p^{r-1}(p-1)$, we see that if some $r_i > 1$ then that $\mathbf{Z}/(p^{r_i})$ has an automorphism of order p, so P does as well (act by the chosen automorphism on the i-th factor and fix elements in the other factors). Thus, if $\#\operatorname{Aut}(P)$ is not divisible by p we must have $r_i = 1$ for all i, so $P \cong (\mathbf{Z}/(p))^k$ is a direct sum of copies of $\mathbf{Z}/(p)$. That means $\operatorname{Aut}(P) \cong \mathrm{GL}_k(\mathbf{Z}/(p))$, whose order is divisible by $p^{k(k-1)/2}$, and thus is divisible by p unless $k = 1$. So we must have $P \cong \mathbf{Z}/(p)$, which concludes the proof that (1) implies (2).

To show (2) implies (1), $\operatorname{Aut}(\mathbf{Z}/(p) \times H) \cong \operatorname{Aut}(\mathbf{Z}/(p)) \times \operatorname{Aut}(H) \cong (\mathbf{Z}/(p))^\times \times \operatorname{Aut}(H)$, and this has order not divisible by p since $\#\operatorname{Aut}(H)$ is not divisible by p. $\square$

Example 5. If $\#G$ is even and $\#\operatorname{Aut}(G)$ is odd then $G \cong \mathbf{Z}/(2) \times H$ where H is a group of odd order with $\operatorname{Aut}(H)$ of odd order too. The smallest such nontrivial H has order $729 = 3^6$ with automorphism group of order $19683 = 3^9$.

When $p|\#\operatorname{Aut}(G)$, one way to search for elements of order p in $\operatorname{Aut}(G)$ is by looking for an inner automorphism: if $g \in G$ has order p and g is not in the center of G then conjugation by G is an (inner) automorphism of G with order p. Since inner automorphisms are a cheap construction, we ask: when are there non-inner automorphisms of order p, assuming that we know $p|\#\operatorname{Aut}(G)$ (and $p|\#G$)? For p-groups there is a complete answer. When G is a finite abelian p-group, it has an automorphism of order p as long as $G \not\cong \mathbf{Z}/(p)$, and that automorphism is not inner since G is abelian. When G is a finite non-abelian p-group, Gatschütz [1] showed that there is an automorphism of order p which is not inner, using cohomology.

References

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