A Few Remarks on a Theorem by J. Rauch

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ABSTRACT. In this paper, we consider semi-linear hyperbolic initial boundary value problem on multidimensional domains. We assume that the system is symmetric hyperbolic, with maximal dissipative boundary conditions, the boundary is either characteristic of constant multiplicity or noncharacteristic. In particular, we treat the case of "conservative" boundary conditions. We show that this problem can be seen as a limit when $\varepsilon \to 0^+$ of a parabolic initial boundary value problem. The parabolic operators are obtained from the hyperbolic operator by adding a viscosity $\varepsilon\mathcal{E}$, where $\mathcal{E}$ is a well chosen elliptic second order operator. We prescribe a Dirichlet boundary condition for these parabolic perturbations. This answers a question raised by J. Rauch in [13]. The elliptic operators $\mathcal{E}$ verify a "weakly dissipation" assumption. On characteristics components, strict dissipation is required. We also give a topological description of the set of the convenient symmetric viscosities for vacuum Maxwell's system with "incoming wave" condition.

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1. Introduction

We consider a symmetric hyperbolic linear operator:

$$\mathcal{H}(t,x,\partial) := A_0(t,x)\partial_t + \sum_{1\leqslant j\leqslant n} A_j(t,x)\partial_j + B(t,x).$$

The $N \times N$ matrices $(A_j)_{0\leqslant j\leqslant n}$, B are symmetric. Their coefficients are real valued, C^∞. The matrix A_0 is positive definite. We denote by Ω an open subset of $\mathbb{R}^n$ with smooth, compact boundary Γ such that Ω lies on one side of Γ. The real $T > 0$ is fixed in all the paper. We consider a linear boundary condition $M(t,x)u = 0$, where M is a $N \times N$ matrix, C^∞, on the boundary Γ. We introduce the boundary matrix: $A_{\mathfrak{n}} := \sum_{1\leqslant j\leqslant n} \mathfrak{n}_j A_j$ where $\mathfrak{n} := (\mathfrak{n}_1,\dots,\mathfrak{n}_n)$ is the interior unit normal to Γ.

Assumption 1.1. *The dimension* $d_0(t,x) := \dim\ker A_{\mathfrak{n}}(t,x)$ *does not depend of* $(t,x) \in (0,T)\times\Gamma$.

Notice that in Assumption 1.1, we only consider $x \in \Gamma$ (and not $x \in \Omega$). Thus, the boundary is either characteristic of constant multiplicity (if $d_0 = 0$) or noncharacteristic (if $d_0 \neq 0$).

Assumption 1.2. *The boundary condition is maximal dissipative, i.e., for all* $(t,x) \in (0,T)\times\Gamma$, *if* $M(t,x)u = 0$, *the quadratic form* $\langle A_{\mathfrak{n}}(t,x)\cdot,\cdot\rangle$ *is nonpositive on* $\ker M(t,x)$ *and* $\ker M(t,x)$ *is maximal for this property.*

We have chosen, for simplicity, to work with solutions which vanish in the past. Thus we consider the initial boundary value problem:

(1.1) $\mathcal{H}(t,x,\partial)u^0 = F(t,x,u^0) + f(t,x)$ when $(t,x) \in (-1,T)\times\Omega$,

(1.2) $M(t,x)u^0 = 0$ when $(t,x) \in (-1,T)\times\Gamma$,

(1.3) $u^0 = 0$ when $(t,x) \in (-1,0)\times\Omega$, where F is a C^∞ function with $F(t,x,0) = 0$ and f is a C^∞ source term with $f|_{t\leqslant 0} = 0$. According to [14], [5], [18] the problem (1.1)–(1.2)–(1.3) is locally well posed: there exists $T_0 > 0$ and a unique solution $u^0 \in C^\infty((-1,T_0)\times\Omega)$. The goal of the paper is to show that u^0 is the limit, as $\varepsilon \to 0^+$, of a well chosen viscous perturbation of the system (1.1), with homogenous Dirichlet conditions on the boundary and null initial data. In Section ***1.1***, we present some results in this sense.

When the function F is identically null, the equation (1.1) is linear. For all $T > 0$, even if we only assume that $f \in L^2$, the problem (1.1)–(1.2)–(1.3) is well posed, admitting a unique solution in $u^0 \in L^2((-1,T)\times\Omega)$ (cf. [14]). This weak framework will be considered in Section ***1.2***.

Let us recall a previous result obtained by J. Rauch in [13], which is the main motivation of this work. Following J. Rauch, we consider the L^2 linear framework, i.e., $F \equiv 0$. In [13], J. Rauch showed that when the boundary conditions M in (1.2) are strictly dissipative, which means that there is a real $c > 0$ such that, for all $(t,x) \in (0,T)\times\Omega$, for all $u \in \ker M(t,x)$,

$$\langle A_{\mathrm{n}}(t,x)u, u\rangle \leqslant -c\|(\mathrm{Id} - \Pi_0(t,x))u\|^2,$$

where $\Pi_0(t,x)$ is the orthogonal projector on $\ker A_{\mathrm{n}}(t,x)$, then there is a symmetric viscosity tensor

$$\mathcal{E}(t,x,\partial) := \sum_{1\leqslant i,j\leqslant n} \partial_i E_{i,j}(t,x)\partial_j, \tag{1.4}$$

where

(1.5) the $N\times N$ matrices $(E_{i,j})_{1\leqslant i,j\leqslant n}$ are symmetric,

(1.6) their coefficients are real and C^∞,

(1.7) $\exists\, c > 0 \mid \forall\, \zeta \in \mathbb{R}^d,\ \forall\, (t,x) \in (0,T)\times\Omega,$
$$\sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}(t,x) \geqslant c|\zeta|^2\mathrm{Id},$$

such that the solutions $(u^\varepsilon)_\varepsilon$ of

(1.8) $\mathcal{P}^\varepsilon(t,x,\partial)u^\varepsilon = F(t,x,u^\varepsilon) + f(t,x)$ when $(t,x) \in (-1,T)\times\Omega$,

(1.9) $u^\varepsilon = 0$ when $(t,x) \in (-1,T)\times\Gamma$,

(1.10) $u^\varepsilon = 0$ when $(t,x) \in (-1,0)\times\Omega$,

with $F \equiv 0$, converge in L^2 to u^0 when $\varepsilon \to 0^+$. We have denoted $(\mathcal{P}^\varepsilon)_\varepsilon$ the family of parabolic operators

$$\mathcal{P}^\varepsilon(t,x,\partial) = \mathcal{H}(t,x,\partial) - \varepsilon\mathcal{E}(t,x,\partial). \tag{1.11}$$

We point out the fact that a totally characteristic problem (i.e., when $d_0 = N$) is strictly dissipative as it is conservative.

The main ingredient in the proof of Rauch's theorem is the following result.

Theorem 1.3 ([13]). *There are symmetric matrices $E(t,x)$, uniformly positive definite, i.e.,*

$$\inf_{(t,x)\in(0,T)\times\Omega}\{\mu(t,x) \text{ eigenvalue of } E(t,x)\} > 0,$$

such that

$$\forall\ (t,x)\in(0,T)\times\Gamma,\quad \ker M(t,x) = E_{\leqslant 0}(E^{-1}(t,x)A_{\mathfrak{n}}(t,x)), \tag{1.12}$$

where $E_{\leqslant 0}(E^{-1}A_{\mathfrak{n}})$ is the sum of the eigenspaces of $E^{-1}A_{\mathfrak{n}}$, associated to nonpositive eigenvalues of $E^{-1}A_{\mathfrak{n}}$.

In fact, J. Rauch says much more than the existence of ONE viscosity such that the solutions $(u^{\varepsilon})_{\varepsilon}$ of (1.8)–(1.9)–(1.10) converge in L^2 to u^0 when $\varepsilon \to 0^+$. He states that the convenient matrices for the previous results are the one which verify the following assumption.

Assumption 1.4. *The normal matrix $E := \sum_{1\leqslant i,j\leqslant n}\mathfrak{n}_i\mathfrak{n}_j E_{i,j}$ verifies* (1.12).

For every $\varepsilon > 0$, the problem (1.8)–(1.9)–(1.10) is a classical symmetric parabolic system which admits a unique solution $u^{\varepsilon} \in L^2([-1,T];H^1(\mathbb{R}^n_+))$ (cf. [10]).

However, Rauch's theorem does not cover the case of the general dissipative boundary condition (instead of "strictly dissipative") and for example does not apply to many physical situations with conservative boundary conditions. The question of conservative boundary conditions was explicitly raised in [13]. The goal of this paper is to answer this question and extend the result of [13] to general dissipative boundary conditions.

1.1. Smooth solutions. Let us now explain our main results. We consider here the general case of a nonvanishing function F. First, it is well known (cf. [10]) that for every $\varepsilon > 0$, the semi-linear problem (1.8)–(1.9)–(1.10) is locally well-posed in H^s, $s > d/2$, if $\mathcal{E}$ is a viscosity tensor of the form (1.4) which verifies the following uniform strong ellipticity assumption. This means that for every $\varepsilon \in\ ]0,1]$, there are a real $T_{\varepsilon} > 0$ and one and only one $u^{\varepsilon} \in H^{\infty}((-1,T_{\varepsilon})\times\Omega)$ solution of (1.8)–(1.9)–(1.10).

Assumption 1.5. *There is $c > 0$ such that for all $\zeta \in \mathbb{R}^n - \{0\}$, for all $(t,x) \in (0,T)\times\Omega$, the eigenvalues $\mu(t,x)$ of*

$$\sum_{1\leqslant i,j\leqslant n}\zeta_i\zeta_j(A_0^{-1}E_{i,j})(t,x)$$

verify $\Re(\mu(t,x)) \geqslant c|\zeta|^2$.

Notice that Assumption 1.5 is more general than (1.5), (1.6) and (1.7). In Assumption 1.5, the matrices $E_{i,j}$ need not be symmetric. In order to insure some suitable estimates uniform in ε, we make the following original assumption.

Assumption 1.6 (Weakly dissipation)**.** *There exists $c > 0$ such that for all $t \in (0, T)$, for all $u \in C_0^\infty(\Omega, \mathbb{R}^N)$,*

$$(1.13) \qquad -\int_{x\in\Omega} {}^t u(x) \cdot \mathcal{E}(t, x, \partial) u(x)\, dx \geqslant -c \int_{x\in\Omega} |u(x)|^2\, dx.$$

Assumption 1.6 is a key point in the success of our method.

In order to deal with characteristic components, we need the following assumption about the normal viscosity matrix E.

Assumption 1.7 (Strong characteristic dissipation)**.** *There exists $c > 0$ such that for all $(t, x) \in (0, \infty) \times \Omega$, for all $u \in \mathbb{R}^N$,*

$$(1.14) \qquad \langle {}^t\Pi_0 E \Pi_0 u, u\rangle \geqslant c\|\Pi_0 u\|^2.$$

We can now state our first main theorem. We will discuss Assumption 1.4, 1.5, 1.6, and 1.7 in Section ***1.3***.

Theorem 1.8. (a) *Assume that $\mathcal{E}$ is a C^∞ viscosity tensor verifying Assumptions 1.4, 1.5, 1.6, and 1.7. There exist $T_1 \in]0, T_0]$ and a real $\varepsilon_0 \in]0, 1]$ such that for all $\varepsilon \in]0, \varepsilon_0]$ the problem* (1.8)–(1.9)–(1.10) *admits one and only one solution $u^\varepsilon \in H^\infty((-1, T_1)\times\Omega)$. Moreover, the solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ converge in $H^s((-1, T_1)\times\Omega)$ to u^0 when $\varepsilon \to 0^+$ for all $s \in [0, \frac{1}{2}[$.*

(b) *There exist a C^∞ viscosity tensor $\mathcal{E}$ verifying Assumptions 1.4, 1.5, 1.6, and 1.7.*

Rauch's result is optimal because the limit, when $\varepsilon \to 0$, of symmetric parabolic problems (1.8)–(1.9)–(1.10) with a symmetric positive definite viscosity tensor is a symmetric hyperbolic IBVP with strictly dissipative boundary condition. The example of linearized Euler's equations suggests that symmetric hyperbolic problems with conservative boundary condition would be the limits of partially parabolic problem. In [19], we look at parabolic problem with Dirichlet-Neumann boundary condition. In Theorem 1.8, we look at parabolic problems with Dirichlet condition and elliptic and dissipative viscosities. Theorem 1.8 shows that symmetric hyperbolic problems with conservative boundary condition are limits of viscous perturbations with Dirichlet boundary condition and unsymmetrical viscosity.

Notice that even in the noncharacteristic case, we treat a case which is not covered by paper [12].

The proof gives the following small viscosity uniform estimate

$$\|u^\varepsilon - u^0\|_{H^s((-1,T_1)\times\Omega)} = O(\varepsilon^{1/4-s/2}) \quad \forall\, s \in [0, \tfrac{1}{2}].$$

If the boundary is noncharacteristic, we have a better estimate

$$\|u^\varepsilon - u^0\|_{H^s((-1,T_1)\times\Omega)} = O(\varepsilon^{1/2-s}) \quad \forall\, s \in [0, \tfrac{1}{2}].$$

In particular, the sequence $(u^\varepsilon)_{\in]0,1]}$ is bounded in $H^{1/2}((-1,T_1)\times\Omega)$. It is also possible to prove the $L^\infty((-1,T_1)\times\Omega)$ boundedness of the sequence $(u^\varepsilon)_{\varepsilon\in]0,1]}$.

A question that naturally suggests itself is: Conversely, assume that $\mathcal{E}$ is a C^∞ viscosity tensor verifying Assumptions 1.4, 1.5, 1.6, and 1.7; are the boundary conditions automatically maximal dissipative? In other words, if Assumptions 1.1, 1.4, 1.5, 1.6, and 1.7 hold, is Assumption 1.2 satisfied? We do not know the complete answer to this question. We only give a quite partial answer with the following additional assumption.

Assumption 1.9. *There is a basis $(u_1,\dots,u_{N-\ell_+})$ of $E_{\leqslant 0}(E^{-1}A_n)$ such that for $1\leqslant i\leqslant N-\ell_+$, $A_n u_i=\lambda_i E u_i$ with $\lambda_i\leqslant 0$, and such that for $1\leqslant i\neq j\leqslant N-\ell_+$, $\langle A_n u_i,u_j\rangle=0$.*

Notice that if the matrices A_n and E are symmetric and E is positive definite, then Assumption 1.9 holds.

Proposition 1.10. *Assume that Assumptions 1.1, 1.4, 1.6, and 1.9 hold; then Assumption 1.2 is true.*

Proof. For all $u\in\ker M$, thanks to Assumptions 1.4 and 1.9, there are some real $(\alpha_i)_{1\leqslant i\leqslant N-\ell_+}$ such that $u=\sum_{i=1}^{N-\ell_+}\alpha_i u_i$; then

$$\begin{aligned}\langle A_n u,u\rangle &= \sum_{1\leqslant i,j\leqslant N-\ell_+}\alpha_i\alpha_j\langle A_n u_i,u_j\rangle\\ &= \sum_{1\leqslant i\leqslant N-\ell_+}\alpha_i^2\langle A_n u_i,u_i\rangle\\ &= \sum_{1\leqslant i\leqslant N-\ell_+}\alpha_i^2\lambda_i\langle E u_i,u_i\rangle.\end{aligned}$$

Thanks to Assumption 1.6 and Proposition 1.12, we get $\langle Eu_i,u_j\rangle\geqslant 0$ for $1\leqslant i\leqslant N-\ell_+$, and then $\langle A_n u,u\rangle\leqslant 0$. □

In [13], J. Rauch also considered boundary value problems as limits of problems in all space. In particular, he showed that condition (1.12) is relevant in this setting too. In this paper, we do not consider such problems; however, it could be interesting to do so.

1.2. Weak solutions. For linear problems, i.e., when F vanishes identically and f is in L^2, we deduce from Theorem 1.8 the following result.

Corollary 1.11. *Assume that $\mathcal{E}$ is a C^∞ viscosity tensor verifying Assumptions 1.4, 1.5, 1.6, and 1.7 such that the solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ of the problem (1.8)–(1.9)–(1.10) converge in $L^2((-1,T)\times\Omega)$ to u^0 when $\varepsilon\to 0^+$.*

Proof. By density, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0^\infty((-1,T)\times\Omega)$ converging to f in $L^2((-1,T)\times\Omega)$. We introduce, for $\varepsilon\in[0,1]$, the solution $u^{\varepsilon,n}$ of

$$\begin{aligned} &\mathcal{H}u^{\varepsilon,n} = \varepsilon\mathcal{E}u^{\varepsilon,n} + f_n && \text{when } (t,x)\in(-1,T)\times\Omega,\\ &u^{\varepsilon,n} = 0 && \text{when } (t,x)\in(-1,T)\times\Gamma,\\ &u^{\varepsilon,n} = 0 && \text{when } (-1,0)\times\Omega. \end{aligned}$$

Classic L^2 estimates yield that for all $\delta > 0$ there is n such that for all $\varepsilon\in[0,1]$, $\|u^{\varepsilon,n}-u^\varepsilon\|_{L^2((-1,T)\times\Omega)} \leqslant \delta$. On the other hand, thanks to the previous sections, there is a constant C_n such that for all $\varepsilon\in\,]0,1]$, $\|u^{\varepsilon,n}-u^{0,n}\|_{L^2((-1,T)\times\Omega)} \leqslant C_n\varepsilon^{1/4}$. Therefore, we have

$$\begin{aligned} \|u^\varepsilon-u^0\|_{L^2((-1,T)\times\Omega)} &\leqslant \|u^{\varepsilon,n}-u^\varepsilon\|_{L^2} + \|u^{\varepsilon,n}-u^{0,n}\|_{L^2} + \|u^{0,n}-u^0\|_{L^2}\\ &\leqslant 2\delta + C_n\varepsilon^{1/4}. \end{aligned}$$

For small ε, we get $\|u^\varepsilon-u^0\|_{L^2((-1,T)\times\Omega)} \leqslant 3\delta$. ◻

1.3. About the assumptions. We now comment the assumptions. First notice that the assumptions are preserved by congruence. More precisely, consider some invertible matrices $\Gamma_1(t,x)$, C^∞ with respect to (t,x). Put $\breve{u} := \Gamma_1(t,x)^{-1}u$. We look at

$$\begin{aligned} \breve{\mathcal{H}} &:= {}^t\Gamma_1\cdot\mathcal{H}\cdot\Gamma_1,\\ \breve{F} &:= {}^t\Gamma_1\cdot F\cdot\Gamma_1,\\ \breve{f} &:= {}^t\Gamma_1\cdot f\cdot\Gamma_1,\\ \breve{M} &:= M\cdot\Gamma_1. \end{aligned}$$

Then $(\breve{\mathcal{H}},\breve{F},\breve{f},\breve{M})$ satisfy the same assumptions as $(\mathcal{H},F,f,M)$. Consider a C^∞ tensor $\mathcal{E}$ of the form (1.4); then $\breve{\mathcal{E}}$ is a C^∞ tensor of the form (1.4). Moreover, $\mathcal{E}$ satisfies Assumptions 1.4, 1.5, 1.6, and 1.7 iff $\breve{\mathcal{E}}$ satisfies Assumptions 1.4, 1.5, 1.6, and 1.7.

We now detail each assumption. Let us begin with Assumption 1.4.

1.3.1. *About Assumption 1.4.* We explain why the search of a matrix E such that $\ker M = E_{\leqslant 0}(E^{-1}A)$ is so crucial in the proofs of Rauch's theorem and Theorem 1.8. The reason lies in the presence of boundary layers. The task of the boundary layer is to insure the additional boundary conditions required for the viscous perturbations. There are two kinds of boundary layers: first, the characteristic ones, which are of size $\sqrt{\varepsilon}$, and the non characteristic ones, of size ε. For semi-linear problem, as here, the boundary layers behaviour is well understood ([7], [18]). If $(y,x_n)\mapsto y+x_n\mathfrak{n}$, with $y\in\Gamma$ and x_n small parameterizes a neighborhood of $\partial\Omega$, then the boundary layers can be described by some profiles

$\mathcal{U}_b(t,y,x_n/\sqrt{\varepsilon})$ and $\mathcal{U}_c(t,y,x_n/\varepsilon)$, where $\mathcal{U}_b(t,y,\theta)$ and $\mathcal{U}_c(t,y,z)$ are C^∞ rapidly decreasing functions in θ and z. They verify the following equations:

$$A_n\partial_\theta\mathcal{U}_b = 0,$$
$$\partial_t\mathcal{U}_b + \sum_{j=1}^{n-1} A_j\partial_j\mathcal{U}_b = E\partial_\theta^2\mathcal{U}_b,$$
$$A_n\partial_z\mathcal{U}_c = E\partial_z^2\mathcal{U}_c.$$

We can see that the boundary layers are polarized, i.e., $\mathcal{U}_b \in \ker A_n$ and $\mathcal{U}_c \in E_-(E^{-1}A_n)$, where for a matrix A, $E_-(A)$ denotes the sum of the eigenspaces associated to strictly negative eigenvalues. But since the $\mathcal{U}_b$ and $\mathcal{U}_c$ task is to insure the Dirichlet condition, we need $\ker M = E_{\leqslant 0}(E^{-1}A_n)$.

1.3.2. *About Assumption 1.6.* We begin to give a consequence of Assumption 1.6 about the symbol of $\mathcal{E}$.

Proposition 1.12. *Suppose that Assumption 1.6 holds. Then for all* $(t,x) \in (0,T)\times\Omega$, *for all* $u \in \mathbb{R}^N$, *for all* $\zeta \in \mathbb{R}^n$,

$$\Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}(t,x)u, u\Big\rangle \geqslant 0. \tag{1.15}$$

Proof. We proceed in four steps. The method is inspired by the proof of Theorem 7.17 of [3].

1. Thanks to Assumption 1.6, there exists $c > 0$ such that for all $t \in (0,T)$, for all $\underline{u} \in C_0^\infty(\Omega,\mathbb{C}^N)$,

$$-\int_{x\in\Omega} \underline{u}^*(x)\cdot(\mathcal{E}+\mathcal{E}^*)(t,x,\partial)\underline{u}(x)\,dx \geqslant -c\int_{x\in\Omega} |\underline{u}(x)|^2\,dx, \tag{1.16}$$

   where we denoted by $\mathcal{E}^*$ the adjoint operator of $\mathcal{E}$:

$$\mathcal{E}^*(t,x,\partial) := \sum_{1\leqslant i,j\leqslant n} \partial_i{}^tE_{j,i}(t,x)\partial_j.$$

2. Consider $u \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^n$, $\tau > 0$, $\Phi \in C_0^\infty(\Omega,\mathbb{R})$. Set $\underline{u}(t,x) := \Phi(x)e^{i\tau\zeta x}u$ and apply Inequality (1.16). On one hand, we get

$$-\int_{x\in\Omega} {}^t\underline{u}(x)\cdot(\mathcal{E}+\mathcal{E}^*)(t,x,\partial)\underline{u}(x)\,dx = \tau^2\int_\Omega \mathcal{Q}(t,x)\cdot|\Phi(x)|^2\,dx + \cdots,$$

   where the dots indicate terms of lower order in τ, and

$$\mathcal{Q}(t,x) := \Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j(E_{i,j}+{}^tE_{j,i})(t,x)u,u\Big\rangle$$
$$= 2\Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_jE_{i,j}(t,x)u,u\Big\rangle.$$

On the other hand, we get

$$\int_{x\in\Omega} |\underline{u}(x)|^2\,dx = \int_\Omega |u|^2\,|\Phi(x)|^2\,dx.$$

3. Divide the estimate by τ^2 and let $\tau \to +\infty$; we obtain

$$\int_\Omega \mathcal{Q}(t,x)\cdot|\Phi(x)|^2\,dx \geqslant 0. \tag{1.17}$$

4. Consider $(t,x_0) \in (0,T)\times\Omega$ and choose $\Psi \in C_0^\infty(\mathbb{R})$ with $\int_\Omega |\Psi|^2 = 1$, and set $\Phi(x) := \varepsilon^{-n/2}\cdot\Psi((x-x_0)/\varepsilon)$, where $\varepsilon > 0$ is small enough so that Φ is supported in Ω. Then as $\varepsilon \to 0^+$, $|\varphi|^2$ approaches the delta-function at x_0 so Inequality (1.17) becomes $\mathcal{Q}(t,x_0) \geqslant 0$. This ends the proof of Proposition 1.12.

□

Next proposition gives some examples of viscosities which satisfy Assumption 1.6. It contains partial converses to Proposition 1.12.

Proposition 1.13. *Let $\mathcal{E}$ be an operator of the form* (1.4) *such that one of the following holds:*

(1) *there is $c > 0$ such that for all $\zeta \in \mathbb{R}^n - \{0\}$, for all $(t,x) \in (0,T)\times\Omega$, for all $u \in \mathbb{R}^N$,*

$$\Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}(t,x)u, u\Big\rangle \geqslant c|\zeta|^2\,\|u\|_2^2;$$

(2) *the coefficients of the matrices $E_{i,j}$ are constant and*

$$\forall\,\zeta \in \mathbb{R}^n \quad \forall\,u \in \mathbb{R}^N, \quad \Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}u, u\Big\rangle \geqslant 0; \tag{1.18}$$

(3) *for all $i \neq j$, $E_{i,j} = 0$ and for $1 \leqslant i \leqslant n$, for all (t,x) in $(0,T)\times\Omega$, for all $u \in \mathbb{R}^N$, $\langle E_{i,i}(t,x)u, u\rangle \geqslant 0$;*

(4) *for all (i,j), the matrix $E_{i,j}$ is diagonal with nonnegative diagonal coefficients.*

Then Assumption 1.6 is fulfilled.

Proof. Case (1) (resp. (2)) is a well-known consequence of Garding's inequality (Plancherel's theorem). Case (3) can be easily proved integrating by parts. Case (4) is a consequence of a theorem of C. Fefferman and D.H. Phong [2]. □

For all $(t,x) \in \Omega$, for all $u \in \mathbb{R}^N$, for all $\zeta \in \mathbb{R}^n$, we consider

$$H(t,x,\zeta) := \sum_{i=1}^n \zeta_i A_i(t,x), \quad E(t,x,\zeta) := \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}(t,x),$$

and $(\lambda_k(t,x,\zeta))_{1\leqslant k\leqslant n}$ the eigenvalues of $iH(t,x,\zeta)+E(t,x,\zeta)$.

Proposition 1.14.

(1) *Assume that Assumption of Proposition 1.13 holds; then there is $c>0$ such that for $1\leqslant k\leqslant n$, for all $(t,x)\in(0,T)\times\Omega$, for all $\zeta\in\mathbb{R}^n$, $\Re\lambda_k(t,x,\zeta)\geqslant c|\zeta|^2$.*

(2) *For all $(t,x)\in(0,T)\times\Omega$, for all $\zeta\in\mathbb{R}^n$, assume that*

$$\forall\, u\in\mathbb{R}^N,\quad \Big\langle \sum_{1\leqslant i,j\leqslant n} \zeta_i\zeta_j E_{i,j}(t,x)u,u\Big\rangle \geqslant 0. \tag{1.19}$$

Then for $1\leqslant k\leqslant n$, the eigenvalue $\lambda_k(t,x,\zeta)$ verifies $\Re\lambda_k(t,x,\zeta)\geqslant 0$.

Proof. We only prove (2). Minor modifications yield a proof of (1). We proceed in three steps.

1. Consider X an eigenvector associated to λ so that $(iH+E)X=\lambda X$ (we do not specify (t,x,ζ) for simplicity). Then $X^*(-iH^*+E^*)=\bar{\lambda}X^*$ so we have on one side $X^*(iH+E)X=\bar{\lambda}X^*X$ and on the other side $X^*(-iH^*+E^*)X=\bar{\lambda}X^*X$. Adding these latter two equations, we get

$$iX^*(H-H^*)X+X^*(E+E^*)X=2\Re(\lambda)X^*X. \tag{1.20}$$

2. Because H is real and symmetric, we have $H-H^*=0$ and then (1.20) becomes $X^*(E+E^*)X=2\Re(\lambda)X^*X$.
3. Thanks to the assumption, for all $(X,\zeta)\in\mathbb{C}^N\times\mathbb{R}^n$, $X^*(E+E^*)X\geqslant 0$. Because $X\neq 0$, we get $\Re(\lambda)\geqslant 0$. □

Combine Propositions 1.12 and 1.14 to see that Assumption 1.6 implies the second conclusion of Proposition 1.14. This is quite natural since such property is already required for problems without boundary cf. A. Majda and R. Pego [11], and interior estimates of G. Métivier and K. Zumbrun [12].

Note that unless it is symmetric, a viscosity can satisfy Assumption 1.5 and needs not satisfy Assumption 1.6. See for example in the case $N=2$, $n=1$, $\mathcal{E}:=\partial_1\begin{bmatrix}1&-4\\0&2\end{bmatrix}\partial_1$.

1.3.3. *About Assumption 1.7.* Assumption 1.7 is needed to solve characteristic boundary layers equations (cf. Section ***3.1.1***).

1.4. Example. Another question raised in [13] is the description of the set $\mathcal{R}$ of the matrices E which verify the condition (1.12) of Theorem 1.3. In this paper, we give an algebraic characterization of $\mathcal{R}$. In general, this characterization is not very descriptive, but it can be simpler under some conditions. An example will be given by the vacuum Maxwell's equations:

$$\partial_t E - c\cdot\operatorname{curl} B=0,\qquad \partial_t B + c\cdot\operatorname{curl} E=0,$$

with the following boundary condition, called "incoming wave" condition in [1]:

$$(E - cB \wedge n) \wedge n = 0,$$

where n is the unit outgoing normal.

2. Proof of Part (b) of Theorem 1.8

Until Section 2.3, we assume that f is C_0^∞. The first step of the proof of Theorem 1.8 is an extension of Theorem 1.3.

Theorem 2.1. *We assume that the couple* (A_{n}, M) *verifies Assumptions 1.1 and 1.2. There are invertible matrices* $\tilde{E}(t,x)$, C^∞ *such that*

(I) *for all* $(t,x) \in (0,T) \times \Gamma$, $\ker M(t,x) = E_{\leqslant 0}(\tilde{E}^{-1}(t,x)A_{\mathrm{n}}(t,x))$,
(II) *for all* $(t,x) \in (0,T) \times \Omega$, *the eigenvalues* $(\mu_i(t,x))_{1\leq i\leq s}$ *of* $(A_0^{-1}\tilde{E})(t,x)$ *are real, strictly positive and* $\inf_{(t,x)\in(0,T)\times\Omega} \min_{1\leq i\leq s} \mu_i(t,x) > 0$,
(III) *for all* $(t,x) \in (0,T) \times \Omega$, *for all* $u \in \mathbb{R}^N$, $\langle \tilde{E}(t,x)u, u\rangle \geqslant 0$.

Proof. As J. Rauch in his proof of Theorem 1.3 (cf. [13]), we proceed in two steps. First we look at a special case and then we reduce the general case to this one. Next lemma sums up results of the first step. We consider $\ell_+, \ell_- \in \mathbb{N}$ such that $\ell_+ + \ell_- \leqslant N$. We introduce the matrix

$$\Delta := \begin{bmatrix} \mathrm{Id}_{\ell_+} & 0 & 0 \\ 0 & \mathrm{Id}_{\ell_-} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and we consider some $N \times N$ matrices $M_\Delta(t,x)$ which depend of $(t,x) \in (0,T) \times \Gamma$ in a C^∞ way.

Lemma 2.2. *Assume that the couple* (Δ, M_Δ) *verifies Assumptions 1.1 and 1.2 with* Δ *instead of* A_{n} *and* M_Δ *instead of* M. *Then there are invertible matrices* $\tilde{E}_\Delta(t,x)$, C^∞ *such that*

(i) *for all* $(t,x) \in (0,T) \times \Gamma$, $\ker M_\Delta(t,x) = E_{\leqslant 0}(\tilde{E}_\Delta^{-1}(t,x)\Delta)$,
(ii) *for all* $(t,x) \in (0,T) \times \Omega$, *the eigenvalues* $(\mu_i(t,x))_{1\leq i\leq s}$ *of* $\tilde{E}_\Delta(t,x)$ *are real, strictly positive and* $\inf_{(t,x)\in(0,T)\times\Omega} \min_{1\leq i\leq s} \mu_i(t,x) > 0$,
(iii) *for all* $(t,x) \in (0,T) \times \Omega$, *for all* $u \in \mathbb{R}^N$, $\langle \tilde{E}_\Delta(t,x)u, u\rangle \geqslant 0$.

The next subsection is devoted to the proof of Lemma 2.2.

2.1. Proof of Lemma 2.2. As consequences of Assumptions 1.1 and 1.2, for all $(t,x) \in (0,T) \times \Gamma$, $\ker \Delta \subset \ker M_\Delta(t,x)$ and $\dim \ker M_\Delta(t,x) = N - \ell_+$. We denote by $\ell_0 := N - (\ell_+ + \ell_-)$ the dimension of $\ker \Delta$. We begin with two particular cases:

(1) If $\ell_- = 0$, then $\dim \ker M_\Delta = \ell_0$ so $\ker M_\Delta = \ker \Delta = E_{\leqslant 0}(\Delta)$.
(2) If $\ell_+ = 0$, then $\ker M_\Delta = \mathbb{R}^N = E_{\leqslant 0}(\Delta)$.

Thus in cases (1) and (2), (i)–(ii)–(iii) are satisfied with $\tilde{E}_\Delta = \mathrm{Id}_N$. Look now at other cases.

(3) If $\ell_+, \ell_- \neq 0$, the analysis is subtler.

We write

$$u := \begin{bmatrix} u_+ \\ u_- \\ u_0 \end{bmatrix},$$

where u_+, u_- and u_0 are column vectors of $\mathbb{R}^{\ell_+}$, $\mathbb{R}^{\ell_-}$ and $\mathbb{R}^{\ell_0}$. We denote by $\|\cdot\|$ the Euclidean norms on $\mathbb{R}^{\ell_+}$, $\mathbb{R}^{\ell_-}$ and the norm on the space of $\ell_+ \times \ell_-$ matrices induced by the latter two.

2.1.1. *Definition of* $\tilde{E}_\Delta$. We are going to exhibit some convenient matrices $\tilde{E}_\Delta(t,x)$. Their definition is quite complicated. We begin with the following lemma.

Sublemma 2.3. *There are some* C^∞, $\ell_+ \times \ell_-$ *matrices* $S(t,x)$ *such that for all* $(t,x) \in (0,T) \times \Gamma$,

$$\ker M_\Delta = \{u \in \mathbb{R}^N \mid u_+ = Su_-\} \quad \text{and} \quad \|S\| \leqslant 1.$$

Proof. For simplicity, we do not take care of the variables (t,x). We proceed in three steps.

1. We have, for all $u \in \mathbb{R}^n$, the following property:

$$u_+ \neq 0 \quad \text{and} \quad M_\Delta u = 0 \;\Rightarrow\; u_- \neq 0. \tag{2.1}$$

   Indeed, if there is $u \in \mathbb{R}^n$ such that $u_+ \neq 0$, $M_\Delta u = 0$ and $u_- = 0$, then $\langle \Delta u, u\rangle = \|u_+\|^2 > 0$! This is impossible because the couple (Δ, M_Δ) verifies Assumption 1.2.

2. Thus, the function $\Phi : \ker M_\Delta \to \mathbb{R}^{\ell_-} \times \mathbb{R}^{\ell_0}$, defined by $u \mapsto (u_-, u_0)$, is injective. As $\dim \ker M_\Delta = \ell_- + \ell_0$, Φ is a homeomorphism. As a consequence, if $\ell_0 \neq 0$, there are an $\ell_+ \times \ell_-$ matrix S and an $\ell_+ \times \ell_0$ matrix T, such that for all $u \in \mathbb{R}^N$

$$M_\Delta u = 0 \iff u_+ = Su_- + Tu_0.$$

   Using (2.1) one more time, we get $T = 0$. If $\ell_0 = 0$, we simply see that there still exists an $\ell_+ \times \ell_-$ matrix S such that for all $u \in \mathbb{R}^N$

$$M_\Delta u = 0 \iff u_+ = Su_-.$$

3. There is $u_- \in \mathbb{R}^{\ell_-}$ such that $\|u_-\| = 1$ and $\|Su_-\| = \|S\|$. Put $u_+ = Su_-$. Then $\langle \Delta u, u\rangle = \|u_+\|^2 - \|u_-\|^2 = \|S\|^2 - 1$. As such a vector u is in $\ker M_\Delta$, because of Assumption 1.2, $\langle \Delta u, u\rangle \leqslant 0$. Hence $\|S\| \leqslant 1$. ◻

Remark 2.4. The conservative case corresponds to equality $\|S\| = 1$, and the strictly dissipative one to $\|S\| < 1$.

Using extension and the previous sublemma, there are some $\ell_+ \times \ell_-$ matrices $S(t,x)$, C^∞ with respect to $(t,x) \in (0,T)\times\Omega$ such that for all $(t,x) \in (0,T)\times\Gamma$, $\ker M_\Delta = \{u \in \mathbb{R}^N \mid u_+ = Su_-\}$ and for all $(t,x) \in (0,T) \times \Omega$, $\|S\| \leqslant 1$. Choose $\rho > \sup_{(t,x)\in(0,T)\times\Omega} \|S(t,x)\|^2$, and introduce for all $(t,x) \in (0,T)\times\Omega$, the matrices

$$\underline{\tilde{E}}_\Delta := \begin{bmatrix} \mathrm{Id}_{\ell_+} + \rho^{-1}SS^* & 2S & 0 \\ 2\rho^{-1}S^* & \mathrm{Id}_{\ell_-} + \rho^{-1}S^*S & 0 \\ 0 & 0 & \mathrm{Id}_{\ell_0} \end{bmatrix}.$$

Sublemma 2.5. *For all $(t,x) \in (0,T) \times \Omega$, the matrix $\underline{\tilde{E}}_\Delta$ is invertible.*

Proof. We fix $(t,x) \in (0,T) \times \Omega$ in all the proof. We proceed in three steps.

1. We introduce the matrices

$$\tilde{S} := \rho^{-1/2}S, \quad F := \begin{bmatrix} \mathrm{Id}_{\ell_+} & \tilde{S} & 0 \\ \tilde{S}^* & \mathrm{Id}_{\ell_-} & 0 \\ 0 & 0 & \mathrm{Id}_{\ell_0} \end{bmatrix}.$$

We use a Schur formula and get $\det(F) = \det(\mathrm{Id}_{\ell_-} - \tilde{S}\tilde{S}^*)$. As $\|\tilde{S}\| < 1$, the matrix $\mathrm{Id}_{\ell_-} - \tilde{S}\tilde{S}^*$ is symmetric positive definite. So $\det(F) \neq 0$ and F is invertible.

2. We introduce the matrix

$$\underline{E}_\Delta := F^2 = \begin{bmatrix} \mathrm{Id}_{\ell_+} + \tilde{S}\tilde{S}^* & 2\tilde{S} & 0 \\ 2\tilde{S}^* & \mathrm{Id}_{\ell_-} + \tilde{S}^*\tilde{S} & 0 \\ 0 & 0 & \mathrm{Id}_{\ell_0} \end{bmatrix}.$$

As the matrix F is invertible, the matrix $\underline{E}_\Delta$ is also invertible.

3. We introduce the matrix

$$O := \begin{bmatrix} \mathrm{Id}_{\ell_+} & 0 & 0 \\ 0 & \sqrt{\rho}\,\mathrm{Id}_{\ell_-} & 0 \\ 0 & 0 & \mathrm{Id}_{\ell_0} \end{bmatrix}$$

and we notice that

$$\underline{\tilde{E}}_\Delta = O^{-1}\underline{E}_\Delta O. \tag{2.2}$$

As a consequence, the matrix $\underline{\tilde{E}}_\Delta$ is invertible. ◻

As a consequence of Lemma 2.5, we can define $\tilde{E}_\Delta := \underline{\tilde{E}}_\Delta^{-1}$. We are going to show that $\tilde{E}_\Delta$ satisfies the conclusions of Lemma 2.2. We proceed in three steps.

2.1.2. *Proof of Lemma 2.2* (i). We begin to show that $\tilde{E}_\Delta$ satisfies (i). We start with a lemma. We define $E_\Delta = \underline{E}_\Delta^{-1}$.

Sublemma 2.6. *For all* $(t,x) \in (0,T)\times\Omega$,

$$\ker M_\Delta(t,x)O^{-1} = E_{\leqslant 0}(E_\Delta^{-1}(t,x)\Delta(t,x)). \tag{2.3}$$

Proof. As in the proof of Lemma 2.5, we fix $(t,x)\in(0,T)\times\Omega$. We proceed in two steps.

1. A calculus yields

$$F\Delta F = \begin{bmatrix} \mathrm{Id}_{\ell_+} - \tilde{S}\tilde{S}^* & 0 & 0 \\ 0 & -\mathrm{Id}_{\ell_-} + \tilde{S}^*\tilde{S} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $v \in \mathbb{R}^N$. As for u, we denote

$$v := \begin{bmatrix} v_+ \\ v_- \\ v_0 \end{bmatrix},$$

where v_+, v_- and v_0 are column vectors of $\mathbb{R}^{\ell_+}$, $\mathbb{R}^{\ell_-}$ and $\mathbb{R}^{\ell_0}$. Because $\mathrm{Id}_{\ell_-} - \tilde{S}\tilde{S}^*$ is symmetric positive definite, we get for all $v\in\mathbb{R}^N$

$$v \in E_{\leqslant 0}(F\Delta F) \iff v_+ = 0.$$

2. We remark that $E_{\leqslant 0}(E_\Delta^{-1}\Delta) = F\cdot E_{\leqslant 0}(F\Delta F)$. Moreover, if $v\in\mathbb{R}^N$ verifies $v_+ = 0$, then

$$Fv = \begin{bmatrix} \tilde{S}^* v_- \\ v_- \\ v_0 \end{bmatrix}.$$

As a consequence, we have for all $u\in\mathbb{R}^N$

$$u \in E_{\leqslant 0}(E_\Delta^{-1}\Delta) \iff u_+ = \tilde{S}^* u_-.$$

Look back at the definition of $\tilde{S}$ to see that for all $u\in\mathbb{R}^N$

$$u \in E_{\leqslant 0}(E_\Delta^{-1}\Delta) \iff u_+ = \rho^{-1/2} S u_-,$$

and then we get (2.3). ∎

With (2.2), we get $\tilde{E}_\Delta^{-1}\Delta = O^{-1}E_\Delta^{-1}O\Delta$. Because Δ and O are diagonal, they commute. Thus we get $\tilde{E}_\Delta^{-1}\Delta = O^{-1}E_\Delta^{-1}\Delta O$. Therefore

$$E_{\leqslant 0}(\tilde{E}_\Delta^{-1}A_{\mathfrak{n}}) = O^{-1}E_{\leqslant 0}(E_\Delta^{-1}A_{\mathfrak{n}}).$$

Using Lemma 2.6, we get $E_{\leqslant 0}(\tilde{E}_\Delta^{-1}A_{\mathfrak{n}}) = O^{-1}\ker MO^{-1} = \ker M$. This proves (i).

2.1.3. *Proof of Lemma 2.2* (ii). We now show that $\tilde{E}_\Delta$ satisfies (ii).
As $\sup_{(0,T)\times\Omega}\|S\| < 1$, we obtain

$$\inf_{(t,x)\in(0,T)\times\Omega}\{\mu(t,x) \text{ eigenvalue of } F(t,x)\} > 0,$$

so

$$\inf_{(t,x)\in\Omega}\{\mu(t,x) \text{ eigenvalue of } E_\Delta(t,x)\} > 0. \tag{2.4}$$

The eigenvalues of $\tilde{E}_\Delta$ are real positive definite because $\tilde{E}_\Delta$ is conjugated to a symmetric positive definite matrix, namely E. Moreover, the passage matrix O does not depend of (t,x). (2.4) yields

$$\inf_{(t,x)\in(0,T)\times\Omega}\{\mu(t,x) \text{ eigenvalue of } \tilde{E}_\Delta(t,x)\} > 0.$$

2.1.4. *Proof of Lemma 2.2* (iii). It remains to show that $\tilde{E}_\Delta$ satisfies (iii). It is a trivial consequence of the following lemma.

Lemma 2.7. *Choose* $u \neq 0$ *and* $\gamma := \langle \underline{E}_\Delta u, u\rangle$. *Assume* $\rho \geqslant 1$. *Then* $\gamma \geqslant 0$.

Proof. A calculation yields

$$\begin{aligned}\gamma = \|u_+\|^2 + \|u_-\|^2 + 2\langle Su_-, u_+\rangle \\ + \rho^{-1}(\|S^*u_+\|^2 + \|Su_-\|^2 + 2\langle Su_-, u_+\rangle) + \|u_0\|^2.\end{aligned}$$

Assume for example that $\|u_-\|^2 - \|Su_-\|^2 \geqslant \|u_+\|^2 - \|S^*u_+\|^2$. As $\rho \geqslant 1$, then $\beta := \|u_-\|^2 - \|Su_-\|^2 - \rho^{-1}(\|u_+\|^2 - \|S^*u_+\|^2)$ is non-negative. As a consequence, $\gamma = (1+\rho^{-1})\|u_+ + Su_-\|^2 + \beta + \|u_0\|^2$ is also non-negative. If $\|u_-\|^2 - \|Su_-\|^2 \leqslant \|u_+\|^2 - \|S^*u_+\|^2$, we proceed in a similar way. ◻

This ends the proof of Lemma 2.2. ◻

Remark 2.8. Look at the special case $\ell_+ = \ell_- = 1$, $\ell_0 = 0$, $S = 1$; then $\gamma = (1+\rho^{-1})(u_+ + u_-)^2$ vanishes when $u_+ + u_- = 0$.

2.2. The general case. We now consider the general case. We consider a couple (A_n, M) which verifies Assumptions 1.1 and 1.2. We proceed in two steps. First we simultaneously reduce the matrices A_n and A_0 in order to use Lemma 2.2. Then we exhibit a matrix $\tilde{E}$ satisfying (I), (II) and (III).

1. Because, for all $(t,x) \in (0,T) \times \Omega$, the matrices A_n and A_0 are symmetric and A_0 is positive definite, there are some invertible matrices $P(t,x)$ such that $A_0 = {}^tPP$, $A_n = {}^tPDP$, where the matrix D is of the form

$$D := \begin{bmatrix} D_+ & 0 & 0 \\ 0 & D_- & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   where D_+ (respectively D_-) is a $\ell_+ \times \ell_+$ (resp. $\ell_- \times \ell_-$) diagonal matrix with strictly positive (resp. negative) coefficients. Moreover, as A_n and A_0, the matrices $P(t,x)$ are smooth with respect to (t,x). We introduce the matrices

$$Q := \begin{bmatrix} D_+^{-1/2} & 0 & 0 \\ 0 & (-D_-)^{-1/2} & 0 \\ 0 & 0 & \mathrm{Id}_{d_0} \end{bmatrix}, \quad M_\Delta := M(QP)^{-1}.$$

   We get $A_n = {}^t(QP)\Delta(QP)$ and see easily that the couple (Δ, M_Δ) verifies Assumptions 1.1 and 1.2. Thanks to Lemma 2.2, there are some invertible matrices $\tilde{E}_\Delta(t,x)$ which verify (i), (ii) and (iii).

2. We introduce, for all $(t,x) \in (0,T) \times \Omega$, the matrix $\tilde{E} := {}^t(QP)\tilde{E}_\Delta(QP)$. The equality $\tilde{E}^{-1}A_n = (QP)^{-1}\tilde{E}_\Delta^{-1}\Delta(QP)$ yields

$$E_{\leqslant 0}(\tilde{E}^{-1}A_n) = (QP)^{-1} \cdot E_{\leqslant 0}(\tilde{E}_\Delta^{-1}\Delta).$$

   Using (iii), we get

$$E_{\leqslant 0}(\tilde{E}^{-1}A_n) = (QP)^{-1} \ker M_\Delta = \ker M.$$

   This proves the point (I) of Theorem 2.1.
   Because Q and O are diagonal, they commute. Thus we get

$$A_0^{-1}\tilde{E} = P^{-1}{}^tQO^{-1}E_\Delta OQP = (OP)^{-1} \cdot ({}^tQE_\Delta Q) \cdot (OP).$$

   Hence $A_0^{-1}\tilde{E}$ is conjugated to ${}^tQE_\Delta Q$, which is symmetric positive definite. So (II) holds.
   It is easy to check (III). This ends the proof of Theorem 2.1. □

Remark 2.9. Notice that the previous proof implies Theorem 1.3 of J. Rauch. When the boundary conditions are strictly dissipative, we have

$$\sup_{(0,T)\times\Omega} \|S\| < 1,$$

and we can take $\rho = 1$. The matrix $\tilde{E}$ is thereby symmetric. Indeed the proof consists in reducing the problem with the substitution of S by $\tilde{S}$. This strategy is inspired by Exercise 14.5 of [16]. Theorem 2.1 gives also another case for which the conclusion of the Proposition 15.2.6 of [16] is true.

We go on with the end of the proof of part (b) of Theorem 1.8. We consider the viscosity tensor $\mathcal{E} := \sum_{i=1}^{n} \partial_i \tilde{E} \partial_i$. The normal matrix of $\mathcal{E}$ is $\tilde{E}$. Thanks to (I) the tensor $\mathcal{E}$ satisfies Assumption 1.4. As for all $\zeta \in \mathbb{R}^n - \{0\}$,

$$\sum_{1 \leqslant i,j \leqslant n} \zeta_i \zeta_j (A_0^{-1} E_{i,j}) = |\zeta|^2 A_0^{-1} \tilde{E}$$

and thanks to (II), Assumption 1.5 is satisfied. Using (III) and integrating by parts yield Assumption 1.6. We conclude the proof of Theorem 1.8 showing that the tensor $\mathcal{E}$ satisfies Assumption 1.7. We proceed in two steps.

1. We denote by $\Pi_{0,\Delta}$ the matrix $\Pi_{0,\Delta} := {}^t(QP)\Pi_0(QP)$. Because

$$A_{\mathfrak{n}} = {}^t(QP)\Delta(QP),$$

   $\Pi_{0,\Delta}$ is the orthogonal projector onto $\ker \Delta$. As $\tilde{E} := {}^t(QP)\tilde{E}_\Delta(QP)$ and QP is invertible, it is sufficient to prove that there is $c > 0$ such that for all $(t,x) \in (0,\infty) \times \Omega$, for all $u \in \mathbb{R}^N$,

$$\langle {}^t\Pi_{0,\Delta} \tilde{E}_\Delta \Pi_{0,\Delta} u, u \rangle \geqslant c \|\Pi_{0,\Delta} u\|^2. \tag{2.5}$$

2. For all $u \in \mathbb{R}^N$, $\Pi_{0,\Delta} u = (0,0,u_0)$ is fixed by the matrix $\underline{E}_\Delta$ and so by $\tilde{E}_\Delta$ too. As a consequence, take $c := 1$, and inequality (2.5) holds. Thus Assumption 1.7 is satisfied and part (b) of Theorem 1.8 is proved.

3. Proof of Part (a) of Theorem 1.8

To simplify, we will assume in this section that $\Gamma := \{x_n = 0\}$ and $\Omega := \{x_n > 0\}$. The general case can be reduced to this particular one. Notice that $A_{\mathfrak{n}} = A_n$.

We follow [7], [18] to construct approximate solutions as boundary layer expansions. Let us recall briefly the method. Consider the spaces

$$\mathcal{N}_\theta := H^\infty((-1,T) \times \Omega, S(\mathbb{R}^+\theta)), \quad \mathcal{N}_z := H^\infty((-1,T) \times \Omega, S(\mathbb{R}_z^+)),$$

where S is the Schwartz space of C^∞ rapidly decreasing functions, and the profile space

$$\mathcal{P}(T) := \Big\{ \mathcal{U}(t,x,z,\theta) = \mathcal{U}_a(t,x) + \mathcal{U}_b(t,x,\theta) + \mathcal{U}_c(t,x,z), \\ \text{where } \mathcal{U}_a \in H^\infty((-1,T) \times \Omega),\ \mathcal{U}_b \in \mathcal{N}_\theta \text{ and } \mathcal{U}_c \in \mathcal{N}_z \Big\}.$$

The function $\mathcal{U}_a$ is the regular part or the interior part of the profile, $\mathcal{U}_b$ is a characteristic boundary layer, and $\mathcal{U}_c$ is a noncharacteristic boundary layer. We define the set

$$\beta(T) := \Big\{ (u^\varepsilon)_{\varepsilon\in]0,1]} \in (L^2((-1,T)\times\Omega))^{]0,1]} \mid \forall \ell \in \mathbb{N}\ \exists \varepsilon' \in]0,1] \\ \text{such that } \sup_{\varepsilon\in]0,\varepsilon']} \|(\varepsilon\partial_{t,x})^\ell u^\varepsilon\|_{L^2((0,T)\times\Omega)} < \infty \Big\}.$$

The profile space $\mathcal{P}(T)$ was introduced in [18]. Such tools are also used in [17] and [19]. They are some improvements of tools of [7]. Note particularly that we do not use the profiles $\mathcal{U}_d(t,x,z,\theta)$ of [7]. We substitute the Proposition 5.1 of [18] for the algebra property of the profile space of [7]. If $\mathcal{U} \in \mathcal{P}(T)$, then the family $(u^\varepsilon)_{\varepsilon\in]0,1]}$ defined by $u^\varepsilon(t,x) := \mathcal{U}(t,x,x_n/\varepsilon,x_n/\sqrt{\varepsilon})$ is in $\beta(T)$. Next theorem shows that we can describe the solutions u^ε of the perturbated problems as boundary layer expansions at all orders.

Theorem 3.1. *There are a real $T_1 \in]0,T_0]$ and a real $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon \in]0,\varepsilon_0]$ (1.8)–(1.9)–(1.10) admits one and only one solution $u^\varepsilon \in H^\infty((-1,T_1)\times\Omega)$. Moreover, there are some profiles $(\mathcal{U}^j)_{j\geqslant 0}$ in $\mathcal{P}(T_1)$ such that for all $\varepsilon \in]0,1]$, for all $k \in \mathbb{N}$, there is $R^\varepsilon \in \beta(T_1)$ such that*

$$u^\varepsilon(t,x) = \sum_{j=0}^{k} \sqrt{\varepsilon}^j\,\mathcal{U}^j\left(t,x,\frac{x_n}{\varepsilon},\frac{x_n}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}^{k+1} R^\varepsilon(t,x).$$

We proceed in two steps.

3.1. Approximate solutions. Our goal in Section ***3.1*** is to prove the following result.

Theorem 3.2. *There is a real $T_1 \in]0,T_0]$ such that for all $k \in \mathbb{N}$, there are $(\mathcal{U}^j)_{0\leqslant j\leqslant k}$ in $\mathcal{P}(T_1)$ such that the family $(a^\varepsilon)_{\varepsilon\in]0,1]}$ defined by*

$$a^\varepsilon(t,x) := \sum_{j=0}^{k} \sqrt{\varepsilon}^j\,\mathcal{U}^j\left(t,x,\frac{x_n}{\varepsilon},\frac{x_n}{\sqrt{\varepsilon}}\right) \tag{3.1}$$

verifies

$$\begin{aligned} \mathcal{P}^\varepsilon a^\varepsilon &= F(t,x,a^\varepsilon) + f^\varepsilon + \varepsilon^M g^\varepsilon && \text{when } (t,x) \in (-1,T_1)\times\Omega,\\ a^\varepsilon &= 0 && \text{when } (t,x) \in (-1,T_1)\times\Gamma,\\ a^\varepsilon &= 0 && \text{when } (t,x) \in (-1,0)\times\Omega, \end{aligned}$$

with $(g^\varepsilon)_{\varepsilon\in]0,1]} \in \beta(T_1)$.

Proof. The general process is the same as in [7], [18], [17], [19], but since the assumptions on the system are not the same, we need to check that the construction is still valid. We only give a sketch of proof. Plugging the expansion (3.1) instead of u^ε in (1.8)–(1.9)–(1.10) yields a sequence of profile problems as in [7], [18], [17], [19].

Because Assumption 1.1 only concerns the dimension $d_0(t,x)$ of $\ker A_n(t,x)$ *for* $x \in \Gamma$, we write, for all $(t,x) \in \Omega_T$, $A_n(t,x) = A_n(t,y,0) + x_n A_n^\flat(t,x)$. This strategy was used in [7], [14], [18]. For characteristic boundary layer, the underlying idea is the following. Plugging

$$u^\varepsilon(t,x) := \mathcal{U}_b(t,x,x_n/\sqrt{\varepsilon})$$

in $A_n(t,x)\partial_n u^\varepsilon$ yields

$$A_n(t,x)\partial_n u^\varepsilon = \frac{1}{\sqrt{\varepsilon}} A_n(t,y,0)(\partial_\theta \mathcal{U}_b)\left(t,x,\frac{x_n}{\sqrt{\varepsilon}}\right) + A_n(t,x)\partial_n \mathcal{U}_b\left(t,x,\frac{x_n}{\sqrt{\varepsilon}}\right) + A_n^\flat[(\theta\partial_\theta)\mathcal{U}_b(t,x,\theta)]\big|_{\theta=x_n/\sqrt{\varepsilon}}. \tag{3.2}$$

If $\mathcal{U}_b$ in $\mathcal{N}_\theta$, then the function $(\theta\partial_\theta)\mathcal{U}_b(t,x,\theta)$ still is in $\mathcal{N}_\theta$. Therefore, (3.2) provides an expansion of $A_n(t,x)\partial_n u^\varepsilon$ with

$$\frac{1}{\sqrt{\varepsilon}} A_n(t,y,0)(\partial_\theta \mathcal{U}_b)\left(t,x,\frac{x_n}{\sqrt{\varepsilon}}\right)$$

as leading term. It could be possible to use Assumption 1.1 about $A_n(t,y,0)$.

We follow the same strategy with $E_{nn}(t,x)$ writing

$$E_{n,n}(t,x) = E_{n,n}(t,y,0) + x_n E_{n,n}^\flat(t,x).$$

We will denote $E_{n,n}(t,y,0)$ by $\mathring{E}_{nn}(t,y)$.

The zero order profile $\mathcal{U}_b^0$ verifies $A_n(t,y,0)\partial_\theta \mathcal{U}_b^0 = 0$ so $(\mathrm{Id} - \Pi_0)\mathcal{U}_b^0 = 0$. For the next orders, i.e., for $j \geqslant 1$, the profiles $\mathcal{U}_b^j$ are not polarized anymore. Then, we have to treat the terms $\Pi_0 \mathcal{U}_b^j$ and $(\mathrm{Id} - \Pi_0)\mathcal{U}_b^j$ separately. First, we get for $(\mathrm{Id} - \Pi_0)\mathcal{U}_b^j$ an equation of the form

$$A_n(t,y,0)\partial_\theta(\mathrm{Id} - \Pi_0)\mathcal{U}_b^j + (\mathrm{Id} - \Pi_0)K = 0, \tag{3.3}$$

where K only depends of the profiles $(\mathcal{U}_b^\ell)_{\ell<j}$. Because ∂_θ is an isomorphism of $\mathcal{N}_\theta$, (3.3) has one and only one solution in $\mathcal{N}_\theta$. For $\Pi_0 \mathcal{U}_b^j$, we get some hyperbolic-parabolic problem. We detail it in Section ***3.1.1***. The main difference with [18] lies in the lack of symmetry of these problems. Thanks to Assumption 1.7, these problems are well-posed. Only $\Pi_0 \mathcal{U}_b^0$ satisfies a nonlinear problem.

The profiles $(\mathcal{U}_c^j)_{j\in\mathbb{N}}$ verify some linear second order ODE with respect to z, with (t,x) as parameters (cf. Section 3.1.2). The equation for $\mathcal{U}_c^0$ is homogeneous whereas for $j \geq 1$ the equation for $\mathcal{U}_c^j$ involves a source term which only depends on lower orders profiles $(\mathcal{U}_c^\ell)_{\ell<j}$. The profile $\mathcal{U}_c^0$ verifies the polarization condition $(\mathrm{Id}-\Pi_-)\mathcal{U}_c^0 = 0$. For the higher order profiles, the part $(\mathrm{Id}-\Pi_-)\mathcal{U}_c^j$ is imposed by the source term. We only have a degree of freedom on the part $\Pi_-\mathcal{U}_c^j$. For it, we prescribe a Dirichlet condition in $z = 0$.

3.1.1. *Characteristic boundary layer.* Let us deal with the characteristic boundary layer. We introduce the matrix $K := {}^t\Pi_0 A_n^\flat \Pi_0$ and $\mathbb{H} := {}^t\Pi_0 \mathcal{H} \Pi_0$ which is a symmetric hyperbolic operator on the space of the functions W which verify $(\mathrm{Id}-\Pi_0)W = 0$. The boundary $\{x_n = 0\}$ is totally characteristic for this operator. Let us introduce the operator $\Xi := \mathbb{H} -{}^t\, \Pi_0 \mathring{E}_{n,n}\partial_\theta^2$ and define the hyperbolic-parabolic linear problem

$$
\begin{cases}
(\mathrm{Id}-\Pi_0)W = 0 & \text{when } (t,x,\theta) \in (-1,T)\times\Omega\times\mathbb{R}_\theta^+,\\
\Xi W = \Pi_0 f(t,x,p,W) & \text{when } (t,x,\theta) \in (-1,T)\times\Omega\times\mathbb{R}_\theta^+,\\
W|_{\theta=0} = b & \text{when } (t,x,\theta) \in (-1,T)\times\Omega,\\
W = 0 & \text{when } (t,x,\theta) \in (-1,0)\times\Omega\times\mathbb{R}_\theta^+,
\end{cases}
\tag{3.4}
$$

where f is C^∞, p in $\mathcal{N}_\theta$ and b is in $H^\infty((0,T)\times\Omega)$, verify $(\mathrm{Id}-\Pi_0)b = 0$ and $b|_{t\leqslant 0} = 0$.

Theorem 3.3. *There is one and only one W in $\mathcal{N}_\theta$, solution of* (3.4).

This result is a variant of Theorem 2.1 of [7]. In [7] the matrix $\mathring{E}_{n,n}$ involved in the operator Ξ is symmetric positive definite. That is not the case here anymore. As a consequence we cannot simply apply Theorem 2.1 of [7]. However, the proof of Theorem 2.1 of [7] extends easily as a proof of Theorem 3.3. In particular, the two first steps are the same. The first one is a reduction to homogeneous boundary condition and the second one is a reduction into a $d_0 \times d_0$ system taking into account the polarization condition $(\mathrm{Id}-\Pi_0)W = 0$. Thanks to Assumption 1.7 these problems are well-posed, since an integration by parts gives exactly the same energy estimates as in [7] (estimate (2.1.7) of [7]). This is a key point in the proof, and here Assumption 1.7 plays an important role.

3.1.2. *Noncharacteristic boundary layer.* The noncharacteristic boundary layer profile $\mathcal{U}_c^0$ has to verify

$$\partial_{zz}\mathcal{U}_c^0 = E_{nn}^{-1}A_n\partial_z\mathcal{U}_c^0 \text{ when } (t,x)\in(-1,T)\times\Omega\times\mathbb{R}_z^+, \tag{3.5}$$

$$\mathcal{U}_c^0|_{z=0} = -\Pi_- u^0|_{x_n=0} \text{ when } (t,x)\in(-1,T)\times\Omega. \tag{3.6}$$

At the higher orders, i.e., for $j \geqslant 1$ the noncharacteristic boundary layer profile $\mathcal{U}_c^j$ has to verify

(3.7) $\partial_{zz}\mathcal{U}_c^j - E_{nn}^{-1}A_n\partial_z\mathcal{U}_c^j = q^j$ when $(t,x) \in (-1,T)\times\Omega\times\mathbb{R}_z^+$,

(3.8) $\Pi_-\mathcal{U}_c^j|_{z=0} = -\Pi_-(u^0|_{x_n=0} + \mathcal{U}_b^j|_{\theta=0})$ when $(t,x) \in (-1,T)\times\Omega$,

where q^j depends only on $(\mathcal{U}_c^j)_{\ell\leqslant j-1}$. This problem has one and only one solution in $\mathcal{N}_z$. □

3.2. Convergence. In this section, we state a convergence theorem. We will denote $Z_0 := \partial_t$, $Z_i := \partial_i$ for $1 \le i \le n-1$ and $Z_n := h(x_n)\partial_n$, where h is a C^∞ and bounded function on $\mathbb{R}_+$ such that $h(x_n) = x_n$ when $0 \le x_n \le 1$. The family $(Z_i)_{0\le i\le n}$ generates the algebra of C^∞ tangent vector fields to Γ. When $\alpha := (\alpha_0,\dots,\alpha_n) \in \mathbb{N}^{n+1}$, we denote by Z^α the derivative $Z^\alpha := Z_0^{\alpha_0}\cdots Z_n^{\alpha_n}$.

Theorem 3.4. *Let $T>0$, $m > (1+n)/2$ and $M' \geqslant 2m$. Let $(a^\varepsilon)_{\varepsilon\in]0,1]}$ be a family of approximate regular solutions of the problem* (1.8)–(1.9)–(1.10)*, i.e.,*

$$\begin{aligned}
&\mathcal{P}^\varepsilon a^\varepsilon = F(t,x,a^\varepsilon) + f + \varepsilon^{M'}g^\varepsilon && \text{when } (t,x)\in(-1,T)\times\Omega,\\
&a^\varepsilon = 0 && \text{when } (t,x)\in(-1,T)\times\Gamma,\\
&a^\varepsilon = 0 && \text{when } (t,x)\in(-1,T)\times\Omega,
\end{aligned}$$

with $(\varepsilon^m g^\varepsilon)_{\varepsilon\in]0,1]}$ bounded in $H^m((0,T)\times\Omega)$. Assume that the family $(a^\varepsilon)_{\varepsilon\in]0,1]}$ verifies the following estimates:

$$\forall\,\alpha\in\mathbb{N}^{n+1},\ \forall\,k\in\mathbb{N}\qquad \sup_{\varepsilon\in]0,1]}\|Z^\alpha(\varepsilon\partial_n)^k a^\varepsilon\|_{L^\infty((-1,T)\times\Omega)} < \infty. \tag{3.9}$$

Then there is a real $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon\in]0,\varepsilon_0]$, (1.8)–(1.9)–(1.10) *has one exact solution and $(\varepsilon^{-m}(u^\varepsilon - a^\varepsilon))_{\varepsilon\in]0,1]}$ is bounded in $H^m((-1,T)\times\Omega)$.*

Combine Theorem 3.2 and Theorem 3.4 to prove Theorem 3.1.

4. Proof of Theorem 3.4

4.1. Setting of the problem and iterative schemes.

4.1.1. *Setting of the problem.* There is a smooth $N\times N$ matrix G such that for all (t,x,u,v)

$$F(t,x,u+v) = F(t,x,u) + G(t,x,u,v)\cdot v.$$

We define $w^\varepsilon := \varepsilon^{-M}(u^\varepsilon - a^\varepsilon)$, where $M>0$ will be chosen later, and obtain the equivalent problem:

$$\begin{cases}
\mathcal{P}^\varepsilon w^\varepsilon = G(t,x,a^\varepsilon,\varepsilon^M w^\varepsilon)\cdot w^\varepsilon + \varepsilon^{M'-M}g^\varepsilon & \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{when } (t,x)\in(-1,T)\times\Omega,\\
w^\varepsilon = 0 \qquad\qquad\qquad\qquad\qquad \text{when } (t,x)\in(-1,T)\times\Gamma,\\
w^\varepsilon = 0 \qquad\qquad\qquad\qquad\qquad \text{when } (t,x)\in(-1,T)\times\Omega.
\end{cases} \tag{4.1}$$

The classical theory of parabolic IBVP gives the existence for all $\varepsilon > 0$ of a strictly nonnegative time $T_\varepsilon > 0$ and of a regular solution w^ε. We have to obtain ε-uniform estimates. We introduce a new function unknown. We denote by Π the constant matrix

$$\Pi := \begin{bmatrix} I_{N-d_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

There are invertible C^∞ $N \times N$ matrices $\Gamma_1(t,x)$ and $\Gamma_2(t,x)$, such that for all $(t,y) \in (0,T) \times \mathbb{R}^{n-1}$, $(\Gamma_2 A_n \Gamma_1)(t,y,0) = \Pi$. For each $\varepsilon \in]0,1]$, we denote by $\tilde{w}^\varepsilon := \Gamma_1^{-1} w^\varepsilon$ the new unknown. Then for each $\varepsilon \in]0,1]$, the function w^ε verifies Problem (4.1) if and only if $\tilde{w}^\varepsilon$ verifies

$$\begin{cases} \tilde{\mathcal{P}}^\varepsilon \tilde{w}^\varepsilon = \tilde{G}(t,x,a^\varepsilon,\varepsilon^M \tilde{w}^\varepsilon) \cdot \tilde{w}^\varepsilon + \varepsilon^{M'-M} \tilde{g}^\varepsilon & \\ & \text{when } (t,x) \in (-1,T) \times \Omega, \\ \tilde{w}^\varepsilon = 0 & \text{when } (t,x) \in (-1,T) \times \Gamma, \\ \tilde{w}^\varepsilon = 0 & \text{when } (t,x) \in (-1,T) \times \Omega, \end{cases} \tag{4.2}$$

where $\tilde{\mathcal{P}}^\varepsilon := \Gamma_2 \mathcal{P}^\varepsilon \Gamma_1$, $\tilde{G}(t,x,a,\tilde{w}) := \Gamma_2 G(t,x,a,\Gamma_1 \tilde{w})$ and $\tilde{g}^\varepsilon := \Gamma_2 g^\varepsilon$. The operator $\tilde{\mathcal{P}}^\varepsilon$ is of the form:

$$\tilde{\mathcal{P}}^\varepsilon = \tilde{\mathcal{H}}(t,x,\partial) - \varepsilon \tilde{\mathcal{E}}(t,x,\partial), \tag{4.3}$$

where

$$\tilde{\mathcal{H}}(t,x,\partial) := \tilde{A}_0(t,x)\partial_t + \sum_{1 \leqslant j \leqslant n} \tilde{A}_j(t,x) Z_j + \Pi \partial_n + \tilde{B}(t,x), \tag{4.4}$$

$$\tilde{\mathcal{E}}(t,x,\partial) := \sum_{1 \leqslant i,j \leqslant n} \partial_i \tilde{E}_{i,j}(t,x) \partial_j + \sum_{1 \leqslant i \leqslant n} \tilde{E}_i(t,x) \partial_i. \tag{4.5}$$

Thanks to this change of unknown function, the commutator of the normal matrix of $\tilde{\mathcal{H}}$ with conormal derivatives will be simplified.

4.1.2. *Iterative schemes.* We consider the family of iterative schemes denoted $(\tilde{w}^{\varepsilon,\nu})_{\nu \in \mathbb{N}, \varepsilon \in]0,1]}$, defined by $\tilde{w}^{\varepsilon,0} = 0$ and

$$\begin{aligned} &\tilde{\mathcal{P}}^\varepsilon \tilde{w}^{\varepsilon,\nu+1} = \tilde{B}^{\varepsilon,\nu} \tilde{w}^{\varepsilon,\nu+1} + \varepsilon^{M'-M} \tilde{g}^\varepsilon && \text{when } (t,x) \in (-1,T) \times \Omega, \\ &\tilde{w}^{\varepsilon,\nu+1} = 0 && \text{when } (t,x) \in (-1,T) \times \Gamma, \\ &\tilde{w}^{\varepsilon,\nu+1} = 0 && \text{when } (t,x) \in (-1,0) \times \Omega, \end{aligned}$$

where $\tilde{B}^{\varepsilon,\nu}(t,x) := \tilde{G}(t,x,a^\varepsilon,\varepsilon^M \tilde{w}^{\varepsilon,\nu})$. In order to obtain estimates for the $(\tilde{w}^{\varepsilon,\nu})_{\nu \in \mathbb{N}, \varepsilon \in]0,1]}$, we first look at the family of linear problems:

$$\tilde{\mathcal{P}}^\varepsilon w^\varepsilon = \mathfrak{B}^\varepsilon w^\varepsilon + \mathfrak{g}^\varepsilon \text{ when } (t,x) \in (-1,T) \times \Omega, \tag{4.6}$$

(4.7) $\quad w^\varepsilon = 0$ when $(t,x) \in (-1,T) \times \Gamma$,

(4.8) $\quad w^\varepsilon = 0$ when $(t,x) \in (-1,0) \times \Omega$,

where $(\mathfrak{B}^\varepsilon)_{\varepsilon \in]0,1]}$ is a family of $N \times N$ matrices defined by $\mathfrak{B}^\varepsilon := \tilde{G}(t,x,a^\varepsilon,\varepsilon^M \mathfrak{w}^\varepsilon)$. The family of functions $\mathfrak{w}^\varepsilon$ is given.

4.1.3. *Toolbox.* We denote by $\langle \cdot, \cdot \rangle$ the scalar product of $L^2(\mathbb{R}^n_+)$, and by $\| \cdot \|_2$ the associated norm. We introduce, for $\mu > 0$, $\lambda \geqslant 1$ and $m \in \mathbb{N}$, the weighted norms

$$|w\|_{0,\lambda} := \|e^{-\lambda t} w\|_{L^2((0,T)\times\Omega)},$$
$$|w\|_{m,\mu,\lambda} := \sum_{0 \leqslant k \leqslant m} \mu^{m-k} |Z^k w\|_{0,\lambda},$$
$$\|w\|_{m,\mu,\lambda} := \sum_{0 \leqslant k \leqslant m} \mu^{m-k} |\partial^k w\|_{0,\lambda}.$$

We denote by $\langle \cdot, \cdot \rangle_{0,\lambda}$ the scalar product of $L^2((0,T) \times \Omega)$ endowed with the measure $e^{-2\lambda t}\, dt\, dx$. We use the following Moser's type inequality.

Lemma 4.1. *Let m be an integer. There is $c_m > 0$ such that for all functions w_1 and $w_2 \in H^m((0,T) \times \Omega)$, for all $\mu \geqslant \lambda \geqslant \lambda_m$,*

$$\mu^{m-|\alpha|} |Z^{\alpha_1} w_1 . Z^{\alpha_2} w_2\|_{0,\lambda} \leqslant c_m (|w_1\|_{m,\mu,\lambda} \|w_2\|_\infty + \|w_1\|_\infty \cdot |w_2\|_{m,\mu,\lambda}),$$
$$\mu^{m-|\alpha|} \|\partial^{\alpha_1} w_1 . \partial^{\alpha_2} w_2\|_{0,\lambda} \leqslant c_m (\|w_1\|_{m,\mu,\lambda} \|w_2\|_\infty + \|w_1\|_\infty \cdot \|w_2\|_{m,\mu,\lambda}),$$

if α_1, $\alpha_2 \in \mathbb{N}^{1+n}$, $|\alpha_1| + |\alpha_2| \leqslant m$.

Corollary 4.2. *Let m be an integer. Let $(w^\varepsilon)_\varepsilon$ be a family of functions in $H^m((0,T) \times \Omega)$, There is $c_m > 0$ such that for all $\varepsilon \in]0,1]$,*

$$\mu^{m-|\alpha|} |[\mathfrak{B}^\varepsilon, Z^\alpha] w^\varepsilon\|_{0,\lambda} \leqslant c_m (|w^\varepsilon\|_{m,\mu,\lambda} + \varepsilon^M \|w^\varepsilon\|_\infty \cdot |\mathfrak{w}^\varepsilon\|_{m,\mu,\lambda}),$$
$$\mu^{m-|\alpha|} \|[\mathfrak{B}^\varepsilon, \partial^\alpha] w^\varepsilon\|_{0,\lambda} \leqslant c_m (\|w^\varepsilon\|_{m,\mu,\lambda} + \varepsilon^M \|w^\varepsilon\|_\infty \cdot \|\mathfrak{w}^\varepsilon\|_{m,\mu,\lambda}).$$

Proof. We only prove the first inequality. The second one can be proved in the same way. We introduce a notation: if $\alpha \in \mathbb{N}$, $\alpha \neq 0$ and φ is a vector-valued function whose components are denoted φ_i, we denote by $Z^{\langle \alpha \rangle} \varphi$ the collection of the terms of the form $Z^{\alpha_1} \varphi_{i_1} \cdots Z^{\alpha_r} \varphi_{i_r}$, where $1 \leqslant r \leqslant \alpha$, $\alpha_1 + \cdots + \alpha_r = \alpha$ and $\alpha_i \geqslant 1$ and if $\alpha = 0$, $Z^{\langle \alpha \rangle} \varphi = 0$. The commutator is of the form

$$\Psi(t,x,a^\varepsilon,\varepsilon^M \mathfrak{w}^\varepsilon) \cdot Z^{\langle \alpha_1 \rangle}(\mathfrak{a}^\varepsilon) \cdot Z^{\langle \alpha_2 \rangle}(\varepsilon^M \mathfrak{w}^\varepsilon) \cdot Z^{\alpha_3} w^\varepsilon, \tag{4.9}$$

with $\alpha_1 + \alpha_2 + \alpha_3 = |\alpha|$. Thanks to estimates (3.9), we get

$$\mu^{m-|\alpha|} |(4.9)\|_{0,\lambda} \leqslant C \mu^{m-|\alpha|} |(Z^{\langle \alpha_2 \rangle}(\varepsilon^M \mathfrak{w}^\varepsilon) \cdot Z^{\alpha_3} w^\varepsilon\|_{0,\lambda}.$$

Using Lemma 4.1,

$$\mu^{m-|\alpha|}|(4.9)\|_{0,\lambda} \leqslant C\big(|w^\varepsilon\|_{m,\lambda} + \varepsilon^M \|w^\varepsilon\|_\infty \cdot |\mathfrak{w}^\varepsilon\|_{m,\lambda}\big). \qquad \square$$

4.2. Linear estimate. Next proposition sums up estimates for the family of linear problems (4.6)–(4.7)–(4.8).

Proposition 4.3. *Let $R > 0$. For all $m \in \mathbb{N}$, there are λ_m, $C_m \geqslant 1$, such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \|\mathfrak{w}^\varepsilon\|_\infty \leqslant R$, then for all $\lambda \geqslant \lambda_m$,*

$$\varepsilon^m \|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} \leqslant \frac{C_m}{\lambda}\big(\|\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda} + (\varepsilon^M \|w^\varepsilon\|_\infty) \cdot \varepsilon^m \|\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}\big).$$

Notice that there is a loss of a factor ε^m from the source term $\mathfrak{g}^\varepsilon$ to the solution w^ε. More precisely, each derivative of w^ε needs a factor ε to be bounded by the correspondent derivative of $\mathfrak{g}^\varepsilon$. This explains that we use two different weights: λ/ε and λ in the norms which appear.

Proof. We reason by induction. To initiate the induction ($m = 0$), we start with L^2 estimates.

4.2.1. *L^2 linear estimate.* A key point is to use two forms of Equation (4.6) and the two kinds of estimates associated to these forms. The first step mainly uses the symmetry of the operator $\mathcal{H}$ and Assumption 1.6. The second step uses Assumption 1.5. Thanks to Assumption 1.5, there exists some matrix $P_0(t,x,\zeta)$, smooth in its arguments, for $\zeta \neq 0$, homogeneous of degree 0 in ζ, positive definite (i.e., $P_0 \geqslant cI > 0$), such that $-((P_0 A_0^{-1}\mathcal{E})(t,x,\zeta) + ((A_0^{-1}\mathcal{E})^* P_0)(t,x,\zeta))$ is also positive definite, i.e.,

$$-((P_0 A_0^{-1}\mathcal{E})(t,x,\zeta) + ((A_0^{-1}\mathcal{E})^* P_0)(t,x,\zeta)) \geqslant c|\zeta|^2 I > 0. \tag{4.10}$$

To simplify the notations, we will write P_0 instead of $P_0(t,x)$. We choose $K > 0$ such that $Q := \frac{1}{2}(P_0 + P_0^*) + K\Lambda^{-1}$ is positive definite on $L^2((0,T)\times\Omega)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{0,\lambda,T}$. We have denoted by Λ the operator associated to the symbol $\langle\zeta\rangle := (1+\zeta^2)^{1/2}$. We denote by S the $N \times N$ matrix $S := {}^t\Gamma_1\Gamma_2^{-1}$.

Proposition 4.4. *There is $\lambda_0 \geqslant 1$, such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \|\mathfrak{w}^\varepsilon\|_\infty \leqslant R$, then for all $\lambda \geqslant \lambda_0$,*

$$\begin{aligned}\lambda|w^\varepsilon\|_{0,\lambda}^2 + \varepsilon^2|\partial_x w^\varepsilon\|_{0,\lambda}^2 & \\ \leqslant \lambda_0(|\langle w^\varepsilon, S\mathfrak{g}^\varepsilon\rangle_{0,\lambda}| &+ \varepsilon|\langle A_0^{-1}\Gamma_2^{-1}\mathfrak{g}^\varepsilon, Q\Gamma_1 w^\varepsilon\rangle_{0,\lambda}|).\end{aligned} \tag{4.11}$$

Proof. We proceed in two steps. We begin with a first L^2 estimate of w^ε.

Lemma 4.5. *There is $\lambda_0 \geqslant 1$, such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \|\mathfrak{w}^\varepsilon\|_\infty \leqslant R$, then for all $\lambda \geqslant \lambda_0$,*

$$\lambda | w^\varepsilon \|^2_{0,\lambda} \leqslant \lambda_0 | \langle w^\varepsilon, S\mathfrak{g}^\varepsilon \rangle_{0,\lambda} |. \tag{4.12}$$

Proof. For all $t \in (0,T)$, we multiply Equation (4.6) on the left by the matrix S. We introduce the following symmetric operators

$$\check{\mathcal{H}} := {}^t\Gamma_1 \tilde{\mathcal{H}} \Gamma_1, \quad \check{\mathcal{E}} := {}^t\Gamma_1 \tilde{\mathcal{E}} \Gamma_1, \quad \check{\mathcal{P}}^\varepsilon := \check{\mathcal{H}}(t,x,\partial) - \varepsilon \check{\mathcal{E}}(t,x,\partial).$$

The operators $\check{\mathcal{H}}$ and $\check{\mathcal{E}}$ are of the form

$$\check{\mathcal{H}}(t,x,\partial) = \check{A}_0(t,x)\partial_t + \sum_{1\leqslant j\leqslant n} \check{A}_j(t,x)\partial_j + \check{B}(t,x),$$
$$\check{\mathcal{E}}(t,x,\partial) = \sum_{1\leqslant i,j\leqslant n} \partial_i \check{E}_{i,j}(t,x)\partial_j + \sum_{1\leqslant i\leqslant n} \check{E}_i(t,x)\partial_i.$$

We introduce the matrices $\check{\mathfrak{B}}^\varepsilon := S\mathfrak{B}^\varepsilon$ and the functions $\check{\mathfrak{g}}^\varepsilon := S\mathfrak{g}^\varepsilon$. Thus we get the following equation:

$$\check{\mathcal{P}}^\varepsilon w^\varepsilon = \check{\mathfrak{B}}^\varepsilon w^\varepsilon + \check{\mathfrak{g}}^\varepsilon, \quad \text{when } (t,x) \in (-1,T)\times\Omega.$$

Then we do a scalar product with w^ε, then a space integration. We get

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathcal{H}} w^\varepsilon - \varepsilon \int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathcal{E}} w^\varepsilon = \int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathfrak{B}}^\varepsilon w^\varepsilon + \int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathfrak{g}}_\varepsilon.$$

We are going to estimate the three first terms of previous equality.

1. Let us begin to deal with the term $\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon . \check{\mathcal{H}} w^\varepsilon$. Thanks to the symmetry and the homogenous Dirichlet conditions (4.7), we get

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{A}_0 \partial_t w^\varepsilon = \frac{1}{2}\partial_t\Big(\int_{x\in\mathbb{R}^n_+} |\check{A}_0 w^\varepsilon|^2\Big) - \frac{1}{2}\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot (\partial_t \check{A}_0)\cdot w^\varepsilon,$$
$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{A}_i \cdot \partial_i w^\varepsilon = -\frac{1}{2}\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot (\partial_i \check{A}_i)\cdot w^\varepsilon, \quad \text{for } 1 \leqslant i \leqslant n.$$

Thus, we get

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathcal{H}} w^\varepsilon = \frac{1}{2}\partial_t\Big(\int_{x\in\mathbb{R}^n_+} |\check{A}_0 w^\varepsilon|^2\Big) - \frac{1}{2}\int_{x\in\mathbb{R}^n_+} \sum_{0=1}^{n} {}^t w^\varepsilon \cdot (\partial_i \check{A}_i)\cdot w^\varepsilon.$$

2. There is an increasing function $C(\cdot)$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathfrak{B}}^\varepsilon w^\varepsilon \leqslant C(R) \int_{x\in\mathbb{R}^n_+} |w^\varepsilon|^2.$$

3. Assumption 1.6 yields $-\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon \cdot \check{\mathcal{E}} w^\varepsilon \geqslant -\int_{x\in\mathbb{R}^n_+} |w^\varepsilon|^2$. Thus we have, for all $t \in (0,T)$,

$$\partial_t \|w^\varepsilon\|_2^2(t) \leqslant C(\|w^\varepsilon\|_2^2(t) + |\langle \check{\mathfrak{g}}^\varepsilon, w^\varepsilon\rangle|^2(t)).$$

Then, multiply by $e^{-2\lambda t}$, integrate for $t \in (0,T)$, and use a Gronwall lemma which yield estimate (4.12). □

We now use Assumption 1.5 and its consequence (4.10) to get an estimate of $\partial_x w^\varepsilon$.

Lemma 4.6. *There is $\lambda_0 \geqslant 1$, such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \|\mathfrak{w}^\varepsilon\|_\infty \leqslant R$, then for all $\lambda \geqslant \lambda_0$,*

$$\varepsilon^2 |\partial_x w^\varepsilon\|_{0,\lambda}^2 \leqslant \lambda_0(\varepsilon|\langle A_0^{-1}\Gamma_2^{-1}\mathfrak{g}^\varepsilon, Q\Gamma_1 w^\varepsilon\rangle_{0,\lambda}| + |\langle w^\varepsilon, S\mathfrak{g}^\varepsilon\rangle_{0,\lambda}|). \tag{4.13}$$

Proof. We introduce $\underline{Q} := {}^t\Gamma_1 Q \Gamma_1$.

$$\frac{d}{dt}\langle w^\varepsilon, \underline{Q} w^\varepsilon\rangle = 2\langle \partial_t w^\varepsilon, \underline{Q} w^\varepsilon\rangle + \langle w^\varepsilon, (\partial_t \underline{Q}) w^\varepsilon\rangle.$$

First we notice that $\langle w^\varepsilon, (\partial_t \underline{Q}) w^\varepsilon\rangle = \langle w^\varepsilon, (\partial_t \underline{P}_0) w^\varepsilon\rangle$. Thus $|\langle w^\varepsilon, (\partial_t \underline{Q}) w^\varepsilon\rangle| \leqslant C\|w^\varepsilon\|_2$. Then we define the operator $\hat{\mathcal{P}}^\varepsilon := \tilde{A}_0^{-1}\tilde{\mathcal{P}}^\varepsilon = \hat{\mathcal{H}} - \varepsilon\hat{\mathcal{E}}$, where

$$\hat{\mathcal{H}}(t,x,\partial) := \partial_t + \sum_{1\leqslant j\leqslant n} \hat{A}_j(t,x)\partial_j + \hat{B}(t,x),$$
$$\hat{\mathcal{E}}(t,x,\partial) := \sum_{1\leqslant i,j\leqslant n} \partial_i \tilde{E}_{i,j}(t,x)\partial_j + \sum_{1\leqslant i\leqslant n} \tilde{E}_i(t,x)\partial_i.$$

We introduce the matrices $\hat{\mathfrak{B}}^\varepsilon := \tilde{A}_0^{-1}\tilde{\mathfrak{B}}^\varepsilon$ and the functions $\hat{\mathfrak{g}}^\varepsilon := \tilde{A}_0^{-1}\tilde{\mathfrak{g}}^\varepsilon$. Multiply Equation (4.6) by $\tilde{A}_0^{-1}$ to find, for $(t,x) \in (0,T)\times\mathbb{R}^n$,

$$\hat{\mathcal{P}}^\varepsilon w^\varepsilon = \hat{\mathfrak{B}}^\varepsilon w^\varepsilon + \hat{\mathfrak{g}}^\varepsilon$$
$$\partial_t w^\varepsilon = -\sum_{1\leqslant j\leqslant n} \hat{A}_j \partial_j w^\varepsilon - \hat{B} w^\varepsilon - \hat{\mathfrak{B}}^\varepsilon w^\varepsilon + \varepsilon\hat{\mathcal{E}}(t,x,\partial)w^\varepsilon + \hat{\mathfrak{g}}^\varepsilon.$$

Then take the scalar product with $\underline{Q}w^\varepsilon$:

$$\langle \partial_t w^\varepsilon, \underline{Q}w^\varepsilon\rangle := -\sum_{1\leqslant j\leqslant n}\langle \hat{A}_j\partial_j w^\varepsilon, \underline{Q}w^\varepsilon\rangle - \langle(\hat{B}+\hat{\mathfrak{B}}^\varepsilon)w^\varepsilon, \underline{Q}w^\varepsilon\rangle$$
$$+\,\varepsilon\langle \hat{\mathcal{E}}(t,x,\partial)w^\varepsilon, \underline{Q}w^\varepsilon\rangle + \langle \hat{\mathfrak{g}}^\varepsilon, \underline{Q}w^\varepsilon\rangle.$$

We are going to estimate each term of the right member of the previous equality.

1. We get

$$\begin{aligned}\langle \hat{\mathcal{E}}w^\varepsilon, \underline{Q}w^\varepsilon\rangle &= \langle \tilde{A}_0^{-1}\tilde{\mathcal{E}}w^\varepsilon, \underline{Q}w^\varepsilon\rangle\\ &= \langle \Gamma_1^{-1}A_0^{-1}\mathcal{E}\Gamma_1 w^\varepsilon, {}^t\Gamma_1 Q\Gamma_1 w^\varepsilon\rangle\\ &= \langle A_0^{-1}\mathcal{E}\Gamma_1 w^\varepsilon, \underline{Q}\Gamma_1 w^\varepsilon\rangle.\end{aligned}$$

Thanks to property (4.10), there are $c>0$ and $C\in\mathbb{R}$ such that

$$-\langle \hat{\mathcal{E}}(t,x,\partial)w^\varepsilon, Qw^\varepsilon\rangle \geqslant -c\|\partial_x w^\varepsilon\|_2^2 + C\|w^\varepsilon\|_2^2.$$

2. Because of the lack of symmetry, we could deal with the hyperbolic part as we have done in Lemma 4.5, integrating by parts. Instead, for $1\leqslant i\leqslant n$, we use Cauchy-Schwarz inequality to obtain the singular estimate:

$$|\langle \hat{A}_i\partial_i w^\varepsilon, \underline{Q}w^\varepsilon\rangle| \leqslant C\|w^\varepsilon\|_2\,\|\partial_i w^\varepsilon\|_2 \leqslant \frac{1}{\varepsilon}C\|w^\varepsilon\|_2^2 + \frac{c\varepsilon}{2}\|\partial_i w^\varepsilon\|_2^2.$$

3. There is an increasing function $C(\cdot)$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

$$|\langle(\hat{B}+\hat{\mathfrak{B}}^\varepsilon)w^\varepsilon, \underline{Q}w^\varepsilon\rangle| \leqslant C(R)\|w^\varepsilon\|_2^2.$$

4. A straightforward calculation yields $\langle \hat{\mathfrak{g}}^\varepsilon, \underline{Q}w^\varepsilon\rangle = \langle A_0^{-1}\Gamma_2^{-1}\mathfrak{g}^\varepsilon, Q\Gamma_1 w^\varepsilon\rangle$. Thus, we get

$$\frac{d}{dt}\langle w^\varepsilon, \underline{Q}w^\varepsilon\rangle + \frac{c\varepsilon}{2}\|\partial_x w^\varepsilon\|_2^2 \leqslant C\left(|\langle A_0^{-1}\Gamma_2^{-1}\mathfrak{g}^\varepsilon, Q\Gamma_1 w^\varepsilon\rangle| + \frac{1}{\varepsilon}\|w^\varepsilon\|_2^2\right).$$

Multiply by $e^{-2\lambda t}$, integrate for $t\in(0,T)$, and use a Gronwall lemma to obtain

$$\lambda\langle w^\varepsilon, \underline{Q}w^\varepsilon\rangle_{0,\lambda} + \frac{c\varepsilon}{2}\|\partial_x w^\varepsilon\|_{0,\lambda}^2 \leqslant C\left(|\langle A_0^{-1}\Gamma_2^{-1}\mathfrak{g}^\varepsilon, Q\Gamma_1 w^\varepsilon\rangle_{0,\lambda}| + \frac{1}{\varepsilon}\|w^\varepsilon\|_{0,\lambda}^2\right).$$

Use the definite positiveness of $\underline{Q}$ and Lemma 4.5 to find estimate (4.13). □

Remark 4.7. In this proof, we only use the fact that $\underline{Q}$ is nonnegative on $L^2((0,T)\times\Omega)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{0,\lambda}$.

Mix estimates (4.12) and (4.13) to find estimate (4.11) and prove Proposition 4.4. ◻

4.2.2. Conormal estimates. To get higher order estimates, we proceed in two steps. First, we get conormal estimates.

Proposition 4.8. *Let $R > 0$. For all $m \in \mathbb{N}$, there is $\lambda_m \geqslant 1$, such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \|\mathfrak{w}^\varepsilon\|_\infty \leqslant R$, then for all $\lambda \geqslant \lambda_m$,*

$$\begin{aligned}(4.14)\quad & \lambda \varepsilon^m |w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} + \varepsilon^{m+1} |\partial_x w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} \\ & \leqslant \frac{\lambda_m}{\lambda} (||\mathfrak{g}^\varepsilon||_{m,\lambda,\lambda} + (\varepsilon^M ||w^\varepsilon||_\infty) \cdot \varepsilon^m ||\mathfrak{w}^\varepsilon||_{m,\lambda/\varepsilon,\lambda}).\end{aligned}$$

Proof. Consider an inductive reasoning. Using Proposition 4.4 and Cauchy-Schwarz inequality shows (4.14) for $m = 0$. Consider $m \geqslant 1$ and $\alpha \in \mathbb{N}^{1+n}$ such that $|\alpha| \leqslant m$. Apply the derivative Z^α to the equation (4.6). We get an equation of the form

$$\begin{aligned} &\tilde{\mathcal{P}}^\varepsilon Z^\alpha w^\varepsilon = \mathfrak{B}^\varepsilon Z^\alpha w^\varepsilon + \tilde{\mathfrak{g}}_\varepsilon && \text{when } (t,x) \in (0,T) \times \Omega, \\ &w^\varepsilon = 0 && \text{when } (t,x) \in (0,T) \times \Gamma, \\ &w^\varepsilon = 0 && \text{when } (t,x) \in (-1,T) \times \Omega, \end{aligned}$$

where

$$\tilde{\mathfrak{g}}_\varepsilon := Z^\alpha \mathfrak{g}_\varepsilon + [\mathfrak{B}^\varepsilon, Z^\alpha] w^\varepsilon + [\tilde{\mathcal{P}}^\varepsilon, Z^\alpha] w^\varepsilon.$$

We apply inequality (4.11) with $Z^\alpha w^\varepsilon$ and $\tilde{\mathfrak{g}}^\varepsilon$ instead of w^ε and $\mathfrak{g}^\varepsilon$ respectively, and multiply by $(\lambda/\varepsilon)^{2(m-\alpha)}$:

$$\begin{aligned}(4.15)\quad & \lambda \left(\left(\frac{\lambda}{\varepsilon}\right)^{m-\alpha} |Z^\alpha w^\varepsilon\|_{0,\lambda} \right)^2 + \varepsilon^2 \left(\left(\frac{\lambda}{\varepsilon}\right)^{m-\alpha} |\partial_x Z^\alpha w^\varepsilon\|_{0,\lambda} \right)^2 \\ & \leqslant \lambda_0 \left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)} \Bigg(|\langle Z^\alpha w^\varepsilon, S \tilde{\mathfrak{g}}^\varepsilon \rangle_{0,\lambda}| \\ & \qquad + \varepsilon \left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)} |\langle A_0^{-1} \Gamma_2^{-1} \mathfrak{g}^\varepsilon, Q \Gamma_1 w^\varepsilon \rangle_{0,\lambda}| \Bigg).\end{aligned}$$

We only detail how to deal with the first term of the right member of (4.15). To bound the second one, we can proceed in the same way. We are going to bound each of the following terms

$$(4.16)\quad \varepsilon^{2m} (\lambda/\varepsilon)^{2(m-\alpha)} |\langle Z^\alpha w^\varepsilon, S Z^\alpha \mathfrak{g}^\varepsilon \rangle_{0,\lambda}|,$$

$$(4.17)\quad \varepsilon^{2m} (\lambda/\varepsilon)^{2(m-\alpha)} |\langle Z^\alpha w^\varepsilon, S[\tilde{A}_i Z_i, Z^\alpha] w^\varepsilon \rangle_{0,\lambda}|, \text{ for } 0 \leqslant i \leqslant n,$$

(4.18) $\varepsilon^{2m}(\lambda/\varepsilon)^{2(m-\alpha)}|\langle Z^\alpha w^\varepsilon, S[\Pi\partial_n, Z^\alpha]w^\varepsilon\rangle_{0,\lambda}|,$

(4.19) $\varepsilon^{2m}(\lambda/\varepsilon)^{2(m-\alpha)}|\langle Z^\alpha w^\varepsilon, S[\mathfrak{B}^\varepsilon, Z^\alpha]w^\varepsilon\langle_{0,\lambda}|,$

(4.20) $\varepsilon^{2m+1}(\lambda/\varepsilon)^{2(m-\alpha)}|\langle Z^\alpha w^\varepsilon, S[\partial_i\tilde{E}_{i,j}\partial_j, Z^\alpha]w^\varepsilon\rangle_{0,\lambda}|$, for $0 \leqslant i,\ j \leqslant n$,

(4.21) $\varepsilon^{2m+1}(\lambda/\varepsilon)^{2(m-\alpha)}|\langle Z^\alpha w^\varepsilon, S[\tilde{E}_i\partial_i, Z^\alpha]w^\varepsilon\rangle_{0,\lambda}|$, for $0 \leqslant i \leqslant n$

by

$$\lambda_m\Big(\varepsilon^m|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}\cdot|\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda} + \varepsilon^{2m}\{|w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} + \varepsilon^M\|w^\varepsilon\|_\infty\cdot|\mathfrak{w}^\varepsilon\|_{m,\lambda,\varepsilon,\lambda}\}\cdot|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} + \delta\varepsilon^{2m+2}|\partial_x w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} + C\varepsilon^{2m}|\partial_x w^\varepsilon\|^2_{m-1,\lambda/\varepsilon,\lambda}\Big).$$

1. Using Cauchy-Schwarz inequality, we get

$$(4.16) \le \varepsilon^m|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}\cdot|\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda}. \tag{4.22}$$

2. We continue with estimates of the terms (4.17). Using Cauchy-Schwarz inequality, we get

$$\left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)}|\langle[\tilde{A}_iZ_i, Z^\alpha]w^\varepsilon, Z^\alpha w^\varepsilon\rangle\|_{0,\lambda} \leqslant \left(\frac{\lambda}{\varepsilon}\right)^{m-\alpha}|[\tilde{A}_iZ_i, Z^\alpha]w^\varepsilon\|_{0,\lambda}\cdot|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}.$$

   The commutator $[\tilde{A}_iZ_i, Z^\alpha]w^\varepsilon$ is of the form $(Z^{\alpha_1}\tilde{A}_i)\cdot(Z^{\alpha_2}Z_iw^\varepsilon)$, with $|\alpha_1|+|\alpha_2| \leqslant |\alpha|$, $|\alpha_2| \leqslant |\alpha|-1$. Therefore

$$\left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)}|\langle[\tilde{A}_iZ_i, Z^\alpha]w^\varepsilon, Z^\alpha w^\varepsilon\rangle_{0,\lambda/\varepsilon,\lambda}| \leqslant C|w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda}.$$

3. We postpone the estimate of (4.18) for a while and look now at the estimate of (4.19). Thanks to Corollary 4.2 and Cauchy-Schwarz inequality, we get

$$\left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)}|\langle[\mathfrak{B}^\varepsilon, Z^\alpha]w^\varepsilon, Z^\alpha w^\varepsilon\rangle_{0,\lambda}| \le (|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} + \varepsilon^M||w^\varepsilon\|_\infty\cdot|\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda})|\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}.$$

4. We now deal with the commutators of the form $[\partial_i\tilde{E}_{i,j}\partial_j, Z^\alpha]w^\varepsilon$. The most difficult term is $[\partial_n\tilde{E}_{n,n}\partial_n, Z^\alpha]w^\varepsilon$, which involves two normal derivatives. We

only treat this particular term. The commutator $[\partial_n \tilde{E}_{n,n}\partial_n, Z^\alpha]w^\varepsilon$ is of the form $\partial_n Z^{\alpha_1}\tilde{E}_{n,n}\partial_n Z^{\alpha_1}w^\varepsilon$, with $|\alpha_1|+|\alpha_2| \leqslant |\alpha|$, $|\alpha_2| \leqslant |\alpha|-1$. Integrating by parts, we get

$$|\langle[\partial_n \tilde{E}_{n,n}\partial_n, Z^\alpha]w^\varepsilon, Z^\alpha w^\varepsilon\rangle_{0,\lambda}| = |\langle Z^{\alpha_1}\tilde{E}_{n,n}\partial_n Z^{\alpha_1}w^\varepsilon, \partial_n {}^t S Z^\alpha w^\varepsilon\rangle_{0,\lambda}|.$$

Using Cauchy-Schwarz inequality, we get

$$\left(\frac{\lambda}{\varepsilon}\right)^{2(m-\alpha)} |\langle[\partial_n \tilde{E}_{n,n}\partial_n, Z^\alpha]w^\varepsilon, Z^\alpha w^\varepsilon\rangle_{0,\lambda}| \leqslant C|\partial_n w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} \cdot |\partial_n w^\varepsilon\|_{m-1,\lambda/\varepsilon,\lambda}.$$

Multiplying by ε^{2m+1}, we get

$$(4.20) \leqslant C(\varepsilon^{m+1}|\partial_n w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}) \cdot (\varepsilon^m|\partial_n w^\varepsilon\|_{m-1,\lambda/\varepsilon,\lambda}),$$

and then

$$(4.20) \leqslant \delta\varepsilon^{2m+2}|\partial_n w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} + C\varepsilon^{2m}|\partial_n w^\varepsilon\|^2_{m-1,\lambda/\varepsilon,\lambda},$$

where δ is small enough to allow the first term to be absorbed by viscosity. The second one is bounded by iteration assumption.

5. We do not detail the estimate of (4.21), which is simpler than the previous one.
6. It remains the estimate of (4.18). Because the matrix Π is constant, the commutator $[\Pi\partial_n, Z^\alpha]w^\varepsilon$ is of the form $a(x_n)Z^\alpha\Pi\partial_n$, where a is a scalar function. We suppress $\Pi\partial_n w^\varepsilon$ from the equation. Then we go back to previous cases.

Thus we get the following estimate

$$\lambda\varepsilon^{2m}|w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} + \varepsilon^{2m+2}|\partial_x w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} \leqslant \lambda_m\Big(\varepsilon^m|w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda} \cdot |\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda} + \varepsilon^{2m}\{|w^\varepsilon\|^2_{m,\lambda/\varepsilon,\lambda} + \varepsilon^M\|w^\varepsilon\|_\infty \cdot |\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}\} \cdot |w^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}\Big),$$

and then estimate (4.14). This ends the proof of Proposition 4.8. □

4.2.3. *Normal estimates.* We complete the proof of Proposition 4.3 getting higher order normal derivatives estimates. We fix $m \in \mathbb{N}^*$. Consider the inductive reasoning based on the following assumption:

$\Pi(j)$: there are some λ_j, $C_j \geqslant 1$ such that for all $\lambda \geqslant \lambda_j$,

$$\varepsilon^j\|w^\varepsilon\|_{j,\lambda/\varepsilon,\lambda} \leqslant \frac{C_j}{\lambda}(\|\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda} + (\varepsilon^M\|w^\varepsilon\|_\infty) \cdot \varepsilon^m\|\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}).$$

L^2 estimates of Proposition 4.4 insure $\Pi(0)$. Assume that $1 \leqslant j \leqslant m$ and that $\Pi(\ell)$ is true for $0 \leqslant \ell \leqslant j-1$. We are going to prove $\Pi(j)$. To do that, we do a second iterative scheme. We consider the following inductive assumption:

$$\Pi(j,k): \text{ there are some } \lambda_{j,k}, C_{j,k} \geqslant 1 \text{ such that for all } \lambda \geqslant \lambda_{j,k},$$
$$\varepsilon^j |\partial_n^k w^\varepsilon\|_{j-k,\lambda/\varepsilon,\lambda} \leqslant \frac{\lambda_{j,k}}{\lambda}(\|\mathfrak{g}^\varepsilon\|_{m,\lambda,\lambda} + (\varepsilon^M \|w^\varepsilon\|_\infty) \cdot \varepsilon^m \|\mathfrak{w}^\varepsilon\|_{m,\lambda/\varepsilon,\lambda}).$$

First notice that $\|\cdot\|_{m,\mu,\lambda} = \sum_{0\leqslant k\leqslant m} \|\partial_n^k \cdot\|_{m-k,\mu,\lambda}$. For $k=0$ and $k=1$, the conormal estimates (4.14) show that $\Pi(k)$ holds. Assume now that $1 \leqslant k \leqslant j-1$ and that $\Pi(j,p)$ is true for $1 \leqslant p \leqslant k$. We are going to prove $\Pi(j,k+1)$. As $k+1 \geqslant 2$, we can write $k+1 = k'+2$. We want to estimate $\varepsilon^j \|\partial_n^{k+1} w^\varepsilon\|_{j-k-1,\lambda/\varepsilon,\lambda} = \varepsilon^{j-1}\|\partial_n^{k'}(\varepsilon\partial_n^2 w^\varepsilon)\|_{j-k-1,\lambda/\varepsilon,\lambda}$. We get from Equation (4.6): $\varepsilon\partial_n^2 w^\varepsilon = \sum_{1\leqslant i\leqslant 5} T_i$, where

$$\begin{aligned}
T_1 &:= -\varepsilon \tilde{E}_{n,n}^{-1} \cdot (\partial_n \tilde{E}_{n,n})\partial_n w^\varepsilon,\\
T_2 &:= -\varepsilon \tilde{E}_{n,n}^{-1} \cdot \tilde{\mathcal{E}}_T w^\varepsilon,\\
T_3 &:= -\tilde{E}_{n,n}^{-1} \cdot \tilde{\mathcal{H}} w^\varepsilon,\\
T_4 &:= \tilde{E}_{n,n}^{-1} \cdot \mathfrak{B}^\varepsilon,\\
T_5 &:= \tilde{E}_{n,n}^{-1} \cdot \mathfrak{g}^\varepsilon.
\end{aligned}$$

We have denoted by $\tilde{\mathcal{E}}_T$ the operator:

$$\tilde{\mathcal{E}}_T := \sum_{\substack{1\leqslant i,j\leqslant n\\ (i,j)\neq(n,n)}} \partial_i \tilde{E}_{i,j}\partial_j.$$

However, we can control $\varepsilon^{j-1}|\partial_n^{k'} T_i||_{j-k-1,\lambda/\varepsilon,\lambda}$, for $1 \leqslant i \leqslant 5$.

1. Estimates of T_1 and T_5 are straightforward:

$$\varepsilon^{j-1}|\partial_n^{k'} T_1\|_{j-k-1,\lambda/\varepsilon,\lambda} \leqslant C\varepsilon^j \sum_{p=0}^{k'} |\partial_n^p w^\varepsilon\|_{j-k-1,\lambda/\varepsilon,\lambda},$$
$$\varepsilon^{j-1}|\partial_n^{k'} T_5\|_{j-k-1,\lambda\varepsilon/,\lambda} \leqslant C|\mathfrak{g}^\varepsilon\|_{j,\lambda/\varepsilon,\lambda}.$$

2. We follow with estimate of T_2:

$$\varepsilon^{j-1}|\partial_n^{k'} T_2\|_{j-k-1,\lambda/\varepsilon,\lambda} \leqslant \begin{cases} C\varepsilon^j \displaystyle\sum_{p=0}^{k} |\partial_n^p w^\varepsilon\|_{j-k,\lambda/\varepsilon,\lambda} & \text{if } n \in \{i,j\},\\ C\varepsilon^j \displaystyle\sum_{p=0}^{k'} |\partial_n^p w^\varepsilon\|_{j-k+1,\lambda/\varepsilon,\lambda} & \text{if } n \notin \{i,j\}.\end{cases}$$

3. To estimate $\varepsilon^{j-1}|\partial_n^{k'} T_3||_{j-k-1,\lambda/\varepsilon,\lambda}$, we have to distinguish two kinds of terms:

$$\varepsilon^{j-1}|\partial_n^{k'} \tilde{E}_{n,n}^{-1} \cdot \tilde{A}_i Z_i w^\varepsilon\|_{j-k-1,\lambda/\varepsilon,\lambda} \quad \text{for } 1 \leqslant i \leqslant n, \tag{4.23}$$

$$\varepsilon^{j-1}|\partial_n^{k'} \tilde{E}_{n,n}^{-1} \cdot \Pi\partial_n w^\varepsilon\|_{j-k-1,\lambda/\varepsilon,\lambda}. \tag{4.24}$$

We get

$$(4.23) \leqslant C\varepsilon^{j-1}|\partial_n^{k'} w^\varepsilon\|_{j-1-k',\lambda/\varepsilon,\lambda}\,,$$
$$(4.24) \leqslant C\varepsilon^{j-1}|\partial_n^{k} w^\varepsilon\|_{j-k-1,\lambda/\varepsilon,\lambda}\,.$$

4. We now look at T_4:

$$|\partial_n^{k'} T_4\|_{j-k-1,\lambda/\varepsilon,\lambda} \leqslant \frac{\varepsilon}{\lambda}|\partial_n^{k-1} T_4\|_{j-(k-1)-1,\lambda/\varepsilon,\lambda}.$$

We use Lemma 4.2 to get

$$\varepsilon^{j-1}|\partial_n^{k'} T_4\|_{j-k-1,\lambda/\varepsilon,\lambda} \leqslant \frac{\varepsilon^j}{\lambda}(\|w^\varepsilon\|_{j-1,\lambda/\varepsilon,\lambda} + \varepsilon^M\|w^\varepsilon\|_\infty\,\|\mathfrak{w}^\varepsilon\|_{j-1,\lambda/\varepsilon,\lambda}).$$

We end the proof using the inductive assumptions. ◻

4.3. Boundedness of the iterative scheme.

Before studying the iterative scheme, we recall a Sobolev imbedding lemma adapted from [6].

Lemma 4.9 (Sobolev imbedding)**.** *Let $m > (n+1)/2$. There is a function C such that for all functions $w \in H^m$, for all $\mu > \lambda \geqslant 1$,*

$$\|w\|_{L^\infty(\Omega\times(0,T))} \leqslant \mu^{-1/4}C(\lambda)\|w\|_{m,\mu,\lambda}.$$

Take $M = m$ and consider

$$R > 0,\ \lambda \geqslant \max(\lambda_m, 2C_m),\ h := \sup_{0<\varepsilon\leqslant 1}(\varepsilon^m\|g^\varepsilon\|_{m,\lambda,\lambda}),$$

and $\varepsilon_0 > 0$ small enough such that $(\varepsilon_0/\lambda)^{1/4}C(\lambda)h \leqslant \min(R,1)$.

Proposition 4.10. *The family $(w^{\varepsilon,\nu})_{\nu\in\mathbb{N}}$ satisfy the (ε,ν)-uniform estimates:*

$$\left.\begin{aligned}\varepsilon^M\|\tilde{w}^{\varepsilon,\nu}\|_{L^\infty(\Omega\times(0,T)} &\leqslant R,\\ \varepsilon^m\|\tilde{w}^{\varepsilon,\nu}\|_{m,\lambda/\varepsilon,\lambda} &\leqslant h,\end{aligned}\right\}\quad \forall\, \nu\in\mathbb{N},\ \forall\,\varepsilon\in\,]0,\varepsilon_0]\,. \tag{4.25}$$

Proof. We reason by induction on $\nu \in \mathbb{N}$. As $\tilde{w}^{\varepsilon,0} = 0$, it is true for $\nu = 0$. Assume that $\|\tilde{w}^{\varepsilon,\nu}\|_{L^\infty(\Omega\times(0,T)} \leqslant R$ and $\varepsilon^m \|\tilde{w}^{\varepsilon,\nu}\|_{m,\lambda/\varepsilon,\lambda} \leqslant h$. We proceed in four steps.

1. As $\|\tilde{w}^{\varepsilon,\nu}\|_{L^\infty(\Omega\times(0,T)} \leqslant R$, we can apply Proposition 4.3 and get

$$\begin{aligned}(4.26)\quad & \varepsilon^m \|\tilde{w}^{\varepsilon,\nu+1}\|_{m,\lambda/\varepsilon,\lambda} \\ & \leqslant \frac{C_m}{\lambda}\left(\varepsilon^{M'-M}\|g^\varepsilon\|_{m,\lambda,\lambda} + (\varepsilon^M \|\tilde{w}^{\varepsilon,\nu+1}\|_\infty)\cdot \varepsilon^m \|\tilde{w}^{\varepsilon,\nu}\|_{m,\lambda/\varepsilon,\lambda}\right).\end{aligned}$$

2. As $M' - M \geqslant m$, we get $\varepsilon^{M'-M}\|g^\varepsilon\|_{m,\lambda,\lambda} \leqslant h$. Thanks to Lemma 4.9, we get $\|\tilde{w}^{\varepsilon,\nu+1}\|_{L^\infty(\Omega\times(0,T)} \leqslant (\lambda/\varepsilon)^{-1/4} C(\lambda)\|\tilde{w}^{\varepsilon,\nu+1}\|_{m,\lambda/\varepsilon,\lambda}$. Combine these estimates with (4.26) and the inductive assumption to find

$$\varepsilon^m \|\tilde{w}^{\varepsilon,\nu+1}\|_{m,\lambda/\varepsilon,\lambda} \leqslant \frac{C_m}{\lambda}\left(h + \left(\frac{\lambda}{\varepsilon}\right)^{-1/4} C(\lambda) h \varepsilon^m \|\tilde{w}^{\varepsilon,\nu+1}\|_{m,\lambda/\varepsilon,\lambda}\right).$$

3. As $(\lambda/\varepsilon)^{-1/4}C(\lambda)h \leqslant 1$ and $\lambda \geqslant 2C_m$, we get $\varepsilon^m \|\tilde{w}^{\varepsilon,\nu+1}\|_{m,\lambda/\varepsilon,\lambda} \leqslant h$.
4. Thanks to Lemma 4.9 and because $(\lambda/\varepsilon)^{-1/4}C(\lambda)h \leqslant R$, we get

$$\varepsilon^m \|\tilde{w}^{\varepsilon,\nu+1}\|_{L^\infty(\Omega\times(0,T)} \leqslant R.$$ ◻

Classic methods yield the end of the proof of Theorem 3.4. We do not detail them.

5. A Characterization of $\mathcal{R}$

First notice that we can assume, without loss of generality, that

$$A_n := \begin{bmatrix} \mathrm{Id}_{\ell_-} & 0 & 0 \\ 0 & -\mathrm{Id}_{\ell_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To prove it, let us adopt for convenience a more acute notation: $\mathcal{R}_{A_n,M}$. If A_n is symmetric, there is an invertible matrix O such that $A_n = {}^t O \Delta O$, with

$$\Delta := \begin{bmatrix} \mathrm{Id}_{\ell_-} & 0 & 0 \\ 0 & -\mathrm{Id}_{\ell_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we get $\mathcal{R}_{A_n,M} := {}^t O \mathcal{R}_{\Delta, MO^{-1}} O$ and the boundary condition $MO^{-1}u = 0$ is strictly dissipative for Δ.

We split $v \in \mathbb{R}^N$ in

$$v = \begin{bmatrix} v_+ \\ v_- \\ v_0 \end{bmatrix}.$$

To shorten, we will write $v = (v_+, v_-, v_0)$. We note ℓ_0 the dimension of $\ker A_{\mathrm{n}}$, so the size of the vector v_0 is also ℓ_0. Then there is an $\ell_+ \times \ell_-$ matrix S, with a norm induced by the euclidian norms less than 1, such that $\ker M = \{v \in \mathbb{R}^N \mid v_+ = Sv_-\}$. We look at E^{-1} with

$$E^{-1} := \begin{bmatrix} E_1 & E_4 & E_5 \\ {}^tE_4 & E_2 & E_6 \\ {}^tE_5 & {}^tE_6 & E_3 \end{bmatrix}.$$

Proposition 5.1. *The following assertions are equivalent:*

(1) *The matrix E is in the Rauch's set $\mathcal{R}$.*
(2) *The space $E_-({}^tE_4S - E_2)$, sum of the eigenspaces of ${}^tE_4S - E_2$ associated to strictly nonpositive eigenvalues, is equal to $\mathbb{R}^{\ell_-}$ and $E_1Sv_- = \lambda Sv_- + E_4v_-$ for all eigencouples (v_-, λ) associated to ${}^tE_4S - E_2$ with $\lambda < 0$.*

Proof. First we assume (1) and prove (2). We proceed in five steps.

1. We denote by $E^\sharp$ the $(\ell_+ + \ell_-) \times (\ell_+ + \ell_-)$ matrix

$$E^\sharp := \begin{bmatrix} E_1 & -E_4 \\ {}^tE_4 & -E_2 \end{bmatrix}.$$

As

$$E^\sharp = \begin{bmatrix} E_1 & E_4 \\ {}^tE_4 & E_2 \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{\ell_-} & 0 \\ 0 & -\mathrm{Id}_{\ell_-} \end{bmatrix},$$

the matrix $E^\sharp$ is $\mathbb{R}$-diagonalizable and has ℓ_- strictly negative eigenvalues $\lambda^1 \leqslant \cdots \leqslant \lambda^{\ell_-} < 0$. We denote by (v_+^1, v_-^1), $\dots$, $(v_+^{\ell_-}, v_-^{\ell_-})$ some linearly independent eigenvectors of $E^\sharp$ associated to $\lambda^1 \leqslant \cdots \leqslant \lambda^{\ell_-}$. For $1 \leqslant j \leqslant \ell_-$, we denote respectively by v_0^j and v^j the vectors

$$v_0^j := ({}^tE_5v_+^j - {}^tE_6v_-^j)/\lambda^j \quad \text{and} \quad v^j := (v_+^j, v_-^j, v_0^j).$$

Notice that

$$E^{-1}A_{\mathrm{n}} = \begin{bmatrix} E_1 & -E_4 & 0 \\ {}^tE_4 & -E2 & 0 \\ {}^tE_5 & -{}^tE_6 & 0 \end{bmatrix}.$$

Thus for $1 \leqslant j \leqslant \ell_+$, v^j is an eigenvector of $E^{-1}A_{\mathrm{n}}$ associated to λ^j.

2. Thanks to (1), for $1 \leqslant j \leqslant \ell_-$, v^j is in $\ker M$. So $v_+^j = Sv_-^j$.

3. From the equations $E^{-1}A_n v^j = \lambda^j v^j$, we get ${}^tE_4 v_+^j - E_2 v_-^j = \lambda^j v_-^j$ and then $({}^tE_4 S - E_2)v_-^j = \lambda^j v_-^j$. As the eigenvalues λ^j are strictly negative, the vectors $v_-^1, \dots, v_-^{\ell_-}$ are in $E_-({}^tE_4 S - E_2)$.
4. The vectors $v_-^1, \dots, v_-^{\ell_-}$ are linearly independent because the vectors (v_+^1, v_-^1), $\dots$, $(v_+^{\ell_-}, v_-^{\ell_-})$ are linearly independent and $v_+^j = Sv_-^j$ for $1 \leqslant j \leqslant \ell_+$. Thus $E_-({}^tE_4 S - E_2) = \mathbb{R}^{\ell_-}$.
5. Moreover, from the equations $E^{-1}A_n v^j = \lambda^j v^j$, we get $E_1 S v_-^j = \lambda^j S v_-^j + E_4 v_-^j$. This ends the proof of (2).

We now go on with the converse and assume (2). We proceed in three steps.

1. Thanks to (2), $E_-({}^tE_4 S - E_2) = \mathbb{R}^{\ell_-}$ and there are some linearly independent eigenvectors $v_-^1, \dots, v_-^{\ell_-}$ of $E_-({}^tE_4 S - E_2)$ associated to $\lambda^1 \leqslant \cdots \leqslant \lambda^{\ell_-} < 0$. We defined for $1 \leqslant j \leqslant \ell_-$, $v_+^j := Sv_-^j$, $v_0^j := ({}^tE_5 v_+^j - {}^tE_6 v_-^j)/\lambda^j$ and $v^j := (v_+^j, v_-^j, v_0^j)$. Thus, for $1 \leqslant j \leqslant \ell_-$, as, thanks to (2), $E_1 S v_-^j = \lambda^j S v_-^j + E_4 v_-^j$, the vector v^j is an eigenvalue of $E^{-1}A_n$ associated to λ^j.
2. Get some linearly independent vectors $v_-^{\ell_-+1}, \dots, v_-^{\ell_-+\ell_+}$ such that $v_+^j = v_-^j = 0$ for $\ell_- + 1 \leqslant j \leqslant \ell_+ + \ell_-$. For $\ell_- + 1 \leqslant j \leqslant \ell_+ + \ell_-$, v^j is an eigenvector of $E^{-1}A_n$ associated to the eigenvalue 0.
3. By construction, the vectors $v_-^1, \dots, v_-^{\ell_-+\ell_+}$ are linearly independent and they are in $\ker M$. As $\dim\ker M = \ell_0 + \ell_-$, they generate $\ker M$. Moreover, they are in $E_{\leqslant 0}(E^{-1}A_n)$ so $\ker M \subset E_{\leq 0}(E^{-1}A_n)$. The dimensions equality yields $\ker M = E_{\leq 0}(E^{-1}A_n)$. Thus $E \in \mathcal{R}$. ◻

Let us look at two particular cases.

1. If A_n do not have any nonpositive eigenvalue, Theorem 1.3 is still available for maximal nonpositive boundary conditions. Then, all the symmetric positive definite matrices are convenient i.e., $\mathcal{R} := SDP_n$.
2. Let us now look at the particular case $S = 0$. Then the conditions of Proposition 5.1 can be rewritten $E_-(-E_2) = \mathbb{R}^{\ell_-}$ and $E_4 = 0$. The first condition is automatic because the matrix E_2 is symmetric non positive definite. Then we can detail the topology of $\mathcal{R}$ remarking that for $0 \leqslant r \leqslant n$, we have $SDP_n \simeq SDP_r \times SDP_{n-r} \times B_r$, where B_r is the unit open ball of $M_{r,n-r}$ for the topology induced by the euclidian norms. Therefore, we get

$$\mathcal{R} \simeq SDP_{\ell_+} \times SDP_{\ell_-} \times SDP_{\ell_0} \times B_{\ell_0}.$$

An example of system which lies in this particular case is the vacuum Maxwell's system with "incoming wave condition." In particular, the Laplacian is a Rauch's viscosity for this problem.

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REFERENCES

[1] PHILLIPE DONNAT, *Quelques contributions mathématiques en optique non linéaire*, PhD thesis, Ecole Polytechnique, 1994.

[2] CHARLES L. FEFFERMAN and DUONG H. PHONG, *On positivity of pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 4673–4674. MR507931 (80b:47064)

[3] GERALD B. FOLLAND, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, N.J., 1976. MR0599578 (58 #29031)

[4] MARGUERITE GISCLON and DENIS SERRE, *Étude des conditions aux limites pour un système strictement hyberbolique via l'approximation parabolique*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), 377–382. MR1289315 (95e:35119) (French)

[5] OLIVIER GUÈS, *Problème mixte hyperbolique quasi-linéaire caractéristique*, Comm. Partial Differential Equations **15** (1990), 595–645. MR1070840 (91i:35122) (French)

[6] ______, *Développement asymptotique de solutions exactes de systèmes hyperboliques quasilinéaires*, Asymptotic Anal. **6** (1993), 241–269. MR1201195 (94b:35067) (French, with English summary)

[7] ______, *Perturbations visqueuses de problèmes mixtes hyperboliques et couches limites*, Ann. Inst. Fourier (Grenoble) **45** (1995), 973–1006. MR1359836 (97i:35103) (French)

[8] TOSIO KATO, *Linear and quasilinear equations of hyperbolic type*, Cime (1976).

[9] SERGIU KLAINERMAN and ANDREW MAJDA, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math. **34** (1981), 481–524. MR615627 (84d:35089)

[10] HEINZ-OTTO KREISS and JENS LORENZ, *Initial-boundary Value Problems and the Navier-Stokes Equations*, Pure and Applied Mathematics, vol. 136, Academic Press Inc., Boston, MA, 1989, ISBN 0-12-426125-6. MR998379 (91a:35138)

[11] ANDREW MAJDA and ROBERT L. PEGO, *Stable viscosity matrices for systems of conservation laws*, J. Differential Equations **56** (1985), 229–262, http://dx.doi.org/10.1016/0022-0396(85)90107-X. MR774165(86b:35132)

[12] GUY MÉTIVIER and KEVIN ZUMBRUN, *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, Mem. Amer. Math. Soc. **175** (2005), vi+107. MR2130346

[13] JEFFREY RAUCH, *Boundary value problems as limits of problems in all space*, Séminaire Goulaouic-Schwartz (1978/1979), École Polytech., Palaiseau, 1979, p. Exp. No. 3, 17. MR557514 (81d:35053)

[14] ______, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc. **291** (1985), 167–187. MR797053 (87a:35122)

[15] FRÉDÉRIC ROUSSET, *Inviscid boundary conditions and stability of viscous boundary layers*, Asymptot. Anal. **26** (2001), 285–306. MR1844545 (2003c:35114)

[16] DENIS SERRE, *Systèmes de lois de conservation. II*, Fondations. [Foundations], Diderot Editeur, Paris, 1996, ISBN 2-84134-072-4. MR1459989 (99e:35144) (French, with French summary)

[17] FRANCK SUEUR, *Approche visqueuse de solutions discontinues de systèmes hyperboliques semilinéaires*, Annales de l'Institut Fourier (to appear).

[18] ______, *Couches limites semilinéaires*, Annales de la Faculté de Toulouse (to appear).

[19] ______, *Couches limites: Un problème inverse*, Communications in Partial Differential Equations (to appear).

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