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On 3D Domain Walls for the Landau Lifshitz Equations

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Abstract. We show that the Landau-Lifshitz equations of micromagnetics admit solutions with large variations as the exchange coefficient ε^2 tends to zero, corresponding to a large gradient ($\sim \varepsilon^{-1}$) of the magnetic moment. These solutions are described by an asymptotic expansion involving internal layers of width $O(\varepsilon)$ and amplitude $O(1)$ located in a neighborhood of a smooth fixed hypersurface contained in the domain. The magnetic moment varies fastly accross this hypersurface, called a *wall* in micromagnetism. The evolution of the transition layer is governed by a *nonlinear PDE* and our results apply for interval of times of lenght $O(1)$. As $\varepsilon \to 0$ the solution converges to a *discontinuous* solution of the "hyperbolic" model, with no exchange term.

Contents

1. Introduction

In this paper we are interested in some special materials called since Weiss (1907) *ferromagnets*. For such materials -which have nowadays a huge interest in magnetic storage industry- magnetization is not a linear answer to the magnetic

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field. Landau and Lifshitz (cf. [**19**]) proposed in 1935 the equation:

$$\partial_t M = \mu_0\big(\gamma M \wedge H_{\text{eff}} - \frac{\alpha}{|M|} M \wedge (M \wedge H_{\text{eff}})\big), \tag{1.1}$$

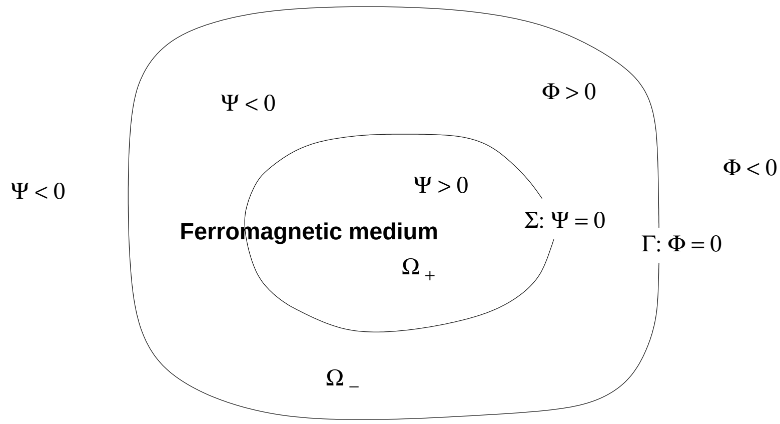
where the *magnetic moment* M is valued in $\mathbb{R}^3$ and saturated, which means of constant length, μ_0 is the magnetic permeability of the vacuum, γ is the Larmor precession factor, α a Gilbert damping constant, H_{eff} denotes the effective magnetic field containing contributions from several terms as a Zeeman field (or exterior field), an anisotropy field (which models the crystalline anisotropies of the material), the demagnetizing (or stray-field) field (which is satisfies the static Maxwell equations in the quasi-stationary regime i.e. when the demagnetizing field relaxes to steady state on a time scale short compared to the time scale for the motion of the local magnetic moments, see [**7**], [**17**], [**29**], [**30**]) and the exchange field (which models the tendency towards parallel alignment of neighboring magnetization vectors in the underlying atomic lattice) see for example [**16**]. The second term in the right hand side of (1.1) have been introduced phenomenologically into the equation in order to modelize the macroscopic effects of various kinds of microscopic relaxation processes (cf. [**27**], [**18**]). Equation (1.1) benefits from deep structure (cf. [**1**], [**20**]).

A well-known ferromagnetic pattern formation is the decomposition of the ferromagnet into almost uniformly magnetized regions (the so-called *magnetic domains*) separated by thin transition layers, called *domain walls*. Some explicit one dimensional domain walls are even known as Bloch walls (named after Felix Bloch who first conceived a continuous wall transition but first proposed and calculated by Landau and Lifshitz cf. [**19**]) and Néel walls cf. [**21**]. They are sometimes called 180 degree wall since they describe the transition between two domains which have magnetic moment of opposite direction. Let us also mention some recent results in thick films in [**23**], in thin films by [**24**] and in nanowires by [**9**].

Our purpose here is to begin a study of such patterns in three dimensions. One difficulty is the non local character of the demagnetizing field. In order to simplify the presentation (and because this does not change our analysis) we will set all the other constants to 1 and we will omit the anisotropy field and the Zeeman field.

2. Geometry setting

The piece of ferromagnet occupies a regular bounded open set Ω in $\mathbb{R}^3$ with a $\mathcal{C}^\infty$ boundary $\Gamma := \partial\Omega$, the set Ω being locally on one side of Γ. Let Σ be a smooth compact embedded hypersurface contained in Ω (thus the hypersurfaces Γ and Σ do not intersect). We assume that there exist a function Φ (respectively Ψ) in $\mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\Omega = \{\Phi > 0\}$, $\Gamma = \{\Phi = 0\}$ (resp. $\Sigma = \{\Psi = 0\}$) and $|\nabla\Phi(x)| = 1$ (resp. $|\nabla\Psi(x)| = 1$) for x in an open neighborhood $\mathcal{V}_\Gamma$ (resp. $\mathcal{V}_\Sigma$) of Γ (resp. Σ). We assume that the neighborhoods $\mathcal{V}_\Gamma$ and $\mathcal{V}_\Sigma$ have been fixed small enough in order that $\mathcal{V}_\Gamma \cap \mathcal{V}_\Sigma = \varnothing$. We denote $\Omega_\pm := \Omega \cap \{\pm\Psi > 0\}$. We consider a C^∞ vector field $\partial_{\mathbf{n}}$ which coincides on $\mathcal{V}_\Gamma$ with $-\nabla_x\Phi \cdot \nabla_x$ and on $\mathcal{V}_\Sigma$ with $-\nabla_x\Psi \cdot \nabla_x$.



For $0 \leq s < \infty$ call $H^s_\Sigma(\Omega)$ the set of functions $u \in L^2(\Omega)$ such that $u_{|\Omega_\pm} \in H^s(\Omega_\pm)$ where $H^s(\Omega_\pm)$ is the usual Sobolev space on L^2. We endow $H^s_\Sigma(\Omega)$ with the norm

$$\|u\|_{H^s_\Sigma} := \|u_{|\Omega_-}\|_{H^s(\Omega_-)} + \|u_{|\Omega_+}\|_{H^s(\Omega_+)}$$

This definition extends to the case when $s = \infty$: the space $H^\infty_\Sigma(\Omega)$ is the natural Fréchet space.

3. Domain wall in the absence of the exchange field

In this section we also omit the exchange field and look for functions $u(t, x)$ valued in the unit sphere of $\mathbb{R}^3$, solutions in Ω of the simplified version of (1.1):

$$\begin{cases} \partial_t u^0 = u^0 \wedge \mathcal{H}(u^0) - u^0 \wedge (u^0 \wedge \mathcal{H}(u^0)) & \text{in } \mathbb{R}_+ \times \Omega \\ u^0_{|t=0} = u_0 \ , \end{cases} \tag{3.1}$$

where no boundary condition is needed. The operator $\mathcal{H} \in \mathcal{L}\big(L^2(\Omega)\big)$ (linear continuous mapping from $L^2(\Omega)$ to $L^2(\Omega)$), is defined by $\mathcal{H}(u) := H_{|\Omega}$ where the demagnetizing field $H \in L^2(\mathbb{R}^3)$ solves the static Maxwell equations:

$$\text{curl } H = 0 \text{ and div } (H + \overline{u}) = 0, \text{ in } \mathbb{R}^3 \tag{3.2}$$

where $\overline{u}$ means the extension of $u \in L^2(\Omega)$ by 0 outside of the set Ω. The action of $\mathcal{H}$ on the function $u^0(t, x)$ in (3.1) is extended by treating t as a parameter: $\mathcal{H}\big(u^0\big)(t, x) = \mathcal{H}\big(u^0(t, .)\big)(x)$.

Since $\mathcal{H} \in \mathcal{L}\big(L^2(\Omega)\big)$ for any initial data $u_0 \in L^2(\Omega)$ such that a.e. $|u_0| = 1$, there exists one corresponding global solution u to the Cauchy problem (3.1) in $C^\infty(\mathbb{R}, L^2(\Omega))$. Moreover Starynkévitch proved in [**29**], Theorem 1.6 that this solution is unique.

On the other hand Carbou, Fabrie and Guès proved (cf. [**6**] Theorem 2.1) that for any initial data u_0 in the Sobolev space $H^2(\Omega)$ such that a.e. $|u_0| = 1$, there exists an unique corresponding global solution u to the equations (3.1) in $C(\mathbb{R}, H^2(\Omega))$.

We fill in the gap between these two results by claiming the following result of global existence of solution of (3.1) discontinuous through the hypersurface Σ.

THEOREM 3.1. *Let $s \in]\frac{3}{2}, \infty]$ and $u_0 \in H^s_\Sigma(\Omega)$. Then there exists a unique $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^s_\Sigma(\Omega)\big)$ solution of the Cauchy problem* (3.1).

Proof. Let us quote that in the closer setting of semilinear symmetric hyperbolic systems, it is well known since the works of Rauch and Reed, and Métivier ([**26**], [**25**]) that there exist local piecewise regular solutions discontinuous across a smooth characteristic hypersurface of constant multiplicity. In the present setting, the proof is in fact simpler since the principal part of the hyperbolic operator is simply ∂_t.

However, the zero-order part of the operator contains the operator $\mathcal{H}$ which is a non local operator which does not act in L^∞. Hence the classical proofs do not apply and need to be modified.

Let us consider the homogeneous symbol $p(\xi) := (-\xi_i\xi_j/|\xi|^2)_{i,j}$ in Hormander's class $S^0_{1,0}$ with values in 3×3 real matrices and let us denote P the associated operator. The key argument is that the P is a pseudo-differential operator which satisfies the transmission property (see [**2**], [**3**], [**11**] and [**12**]). Since $\mathcal{H}(u) := H_{|\Omega}$ where H, as defined in (3.2), satisfies $H = P\overline{u}$, it follows that $\mathcal{H} \in \mathcal{L}\big(H^s_\Sigma(\Omega)\big)$, for all $s \in \mathbb{R}$.

Hence, when $s > 3/2$, the equation (3.1) is of the form $v' = F(v),\ v(0) = u_0 \in H^s_\Sigma(\Omega)$ where F is C^∞ mapping from $H^s_\Sigma(\Omega)$ to itself, and the local existence of the unique solution follows from the Cauchy-Lipschitz theorem.

The global existence is a consequence of the following Yudovitch type inequality as in paper [**6**]:

$$\|\mathcal{H}(u)\|_{L^\infty(\Omega)} \le c\log(1 + \|u\|_{H^s_\Sigma(\Omega)}). \tag{3.3}$$

Let us prove the estimate (3.3) above, for completeness. We begin to recall the classical logarithmic estimate (cf. for example [**4**] Proposition 2.3.5):

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le c\|f\|_{C^0_*(\mathbb{R}^3)} \log(1 + \frac{\|f\|_{C^{s-3/2}_*(\mathbb{R}^3)}}{\|f\|_{C^0_*(\mathbb{R}^3)}}),$$

where C^s_* denotes the Holder space of index s.

By Sobolev imbedding (cf. for example [**4**] Proposition 2.3.3) we infer that

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le c\|f\|_{C^0_*(\mathbb{R}^3)} \log(1 + \frac{\|f\|_{H^s(\mathbb{R}^3)}}{\|f\|_{C^0_*(\mathbb{R}^3)}}).$$

By using the universal extension operators associated to $\Omega_\pm$ (cf. [**32**]) we get:

$$\|H\|_{L^\infty(\Omega_\pm)} \le c\|H\|_{C^0_*(\Omega_\pm)} \log(1 + \frac{\|H\|_{H^s(\Omega_\pm)}}{\|H\|_{C^0_*(\Omega_\pm)}}).$$

Then we use that $P \in \mathcal{L}\big(C^0_*\big)$ and that $L^\infty \subset C^0_*$ to bound $H = P\overline{u}$ in C^0_*.

Since the function $x \mapsto x\log(1 + C/x)$ is increasing and since $\mathcal{H} \in \mathcal{L}\big(H^s_\Sigma(\Omega)\big)$ we get (3.3).

□

This shows that when the initial data is in H^∞_Σ, the solution of (3.1) is in $\mathcal{C}^\infty\big([0,+\infty[, H^\infty_\Sigma(\Omega)\big)$.

4. Transition layer in the presence of the exchange field

We now look for functions $u(t,x)$ valued in $\mathbb{R}^3$, which belongs to the unit sphere of $\mathbb{R}^3$, solutions in Ω of the more realistic equations:

$$\partial_t u^\varepsilon = u^\varepsilon \wedge (\mathcal{H}(u^\varepsilon) + \varepsilon^2\Delta u^\varepsilon) - u^\varepsilon \wedge \big(u^\varepsilon \wedge (\mathcal{H}(u^\varepsilon) + \varepsilon^2\Delta u^\varepsilon)\big) \quad \text{in } \mathbb{R}\times\Omega, \tag{4.1}$$

where $\varepsilon > 0$ is the exchange coefficient. The equations (4.1) appear as a singular perturbation of (3.1). Equations (4.1) have to supplemented by a boundary condition. We consider here the homogeneous Neumann boundary condition, which are physically meaningful and usual in this context:

$$\partial_{\mathbf{n}} u^\varepsilon = 0 \ \text{ in } \ \mathbb{R} \times \Gamma, \tag{4.2}$$

where $\mathbf{n}$ is the unit outward normal at the boundary Γ. We also add an initial condition:

$$u^\varepsilon_{|t=0} = u_0. \tag{4.3}$$

In the paper [**6**] it is proved that, for *smooth enough solutions* the system (3.1) is a "good approximation" of the full system (4.1)-(4.2)-(4.3) in the sense that the solution u^0 of (3.1) is indeed the limit in $L^2([0,T]\times\Omega)$ of solutions u^ε of (4.1)-(4.2)-(4.3) as $\varepsilon \to 0$. However, this result holds under the assumption that u^0 belongs to the space $\mathcal{C}\big([0,T], H^5(\Omega)\big)$ where $H^5(\Omega)$ is the usual Sobolev space. In particular this assumption excludes the present case where u^0 is *discontinuous* across an hypersurface contained in Ω. The following theorem extends the results of [**6**] to this case.

Theorem 4.1. *Let $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^\infty_\Sigma(\Omega)\big)$ be a solution of* (3.1)*. There exist $T > 0$, $\varepsilon_0 > 0$ and a family of solutions $u^\varepsilon \in W^{1,\infty}(]0,T[\times\Omega)$ for $\varepsilon \in]0,\varepsilon_0]$, of the equation* (4.1) *on $[0,T]\times\Omega$ and of the equation* (4.2) *on $[0,T]\times\Gamma$, such that*

$$||u^\varepsilon - u^0||_{L^2([0,T]\times\Omega)} = \mathcal{O}(\varepsilon^{\frac{1}{2}}),$$

as $\varepsilon \to 0$.

Let us add several comments to this result.

1/ In the main result of paper [**6**] the choice of the time $T > 0$ is arbitrary: it is proved that for all $T > 0$ the solution of the Landau-Lifshitz equation exists on $[0,T]$ if $\varepsilon > 0$ is sufficiently small, and converges in $L^2([0,T]\times\Omega)$ as $\varepsilon \to 0$. Here, the result obtained in theorem 4.1 is different since we only show the existence and the convergence of u^ε for a suitable choice of a small enough $T > 0$. We don't know if the result still holds for all T *arbitrarily large* and ε_0 small enough, depending on T. This is due to the rapid variations of u^ε accross Σ: the profile equation describing the evolution of the large amplitude boundary layers around Σ is *nonlinear* (equation (6.6)), and we have only been able to prove the local existence of regular solutions for it (in Theorem 6.1).

2/ Let us stress that in Theorem 4.1 none initial data (4.3) for the u^ε are prescribed: the traces of the u^ε at $t = 0$ are not equal in general to the trace of u^0 at $t = 0$.

3/ In Theorem 4.1 the space $W^{1,\infty}$ is the usual Sobolev space of bounded functions with bounded first derivatives. However, a stronger version of the theorem 4.1 could be proved, where $u^\varepsilon \in C^\infty([0,T]\times\Omega)$. Indeed, it is sufficient to show that one can built an initial data $u^\varepsilon_{|t=0}$ satisfying an infinity of compatibility conditions. This has already been done in the paper [**31**] in the case of general semilinear hyperbolic systems, and the proof can be adapted to the present case, but we have not developped this aspect of the analysis.

In fact, we will claim a more accurate result in Theorem 4.2 by showing that u^ε can be described with a WKB expansion which involves boundary layers profiles.

This expansion contains two kinds of boundary layers. On one hand, a boundary layer appears near the boundary to compensate the loss of the Neumann condition from the complete model (4.1)-(4.2)-(4.3) to the limit model (3.1) ($\varepsilon = 0$). Such a boundary layer was already studied in paper [**6**]. The amplitude of this layer is $\sim \varepsilon$ and its behaviour is linear. On another hand, and this is the main point of the paper, there are *large amplitude boundary layers* on each side of the hypersurface Σ, whose evolutions are governed by a *nonlinear PDEs*. Their task is to compensate the loss of transmission conditions across Σ from the complete model (4.1)-(4.2)-(4.3) to the limit model (3.1) ($\varepsilon = 0$).

We define the boundary layer profile spaces

$$\mathcal{N}_\pm(T) := H^\infty([0,T] \times \Omega, \mathcal{S}(\mathbb{R}^\pm)),$$

where the letter $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions. In the easier case where u^0 is continuous across the hypersurface Σ, paper [**6**] shows the existence of solutions u^ε, $\varepsilon \in]0,1]$, of the equation (4.1) in Ω, of the equation (4.2) on Γ, of the form

$$u^\varepsilon(t,x) \quad := \quad u^0(t,x) + \varepsilon\Big(\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon}) + \mathbf{w}^\varepsilon(t,x)\Big)$$

where the function $\mathfrak{U}$ is in $\mathcal{N}_+(\infty)$ and satisfies $\mathfrak{U}(t,x,z) = 0$ for $x \notin \mathcal{V}_\Gamma$. The function $\mathfrak{U}$ describes a boundary layer which appears near the boundary to compensate the loss of the Neumann condition from the complete model (4.1)-(4.2)-(4.3) to the limit model (3.1) ($\varepsilon = 0$). The amplitude of this boundary is weak and its behaviour is linear. We will state this in section 6.2. The functions $\mathbf{w}^\varepsilon$ can be seen as remainders.

Here since we deal with a ground state u^0 which is discontinuous across the hypersurface Σ, we look for solutions u^ε, $\varepsilon \in]0,1]$, of the equation (4.1) in Ω, of the equation (4.2) on Γ, of the form

$$u^\varepsilon(t,x) := \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon}) + \varepsilon\Big(\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon}) + \mathbf{w}^\varepsilon(t,x)\Big). \tag{4.4}$$

The function $\mathcal{U}$ describes a large amplitude internal layer profile i.e. a sharp transition in the neighborhood of the hypersurface Σ of width $\sim \varepsilon$. More precisely the function $\mathcal{U}$ is $\mathcal{C}^\infty$ and satisfies

$$\lim_{y\to\pm\infty} \mathcal{U}(t,x,y) \quad = \quad u^0(t,x) \quad \text{for } x \in \mathcal{V}_\Sigma \cap \Omega_\pm \tag{4.5}$$

$$\mathcal{U}(t,x,y) \quad = \quad u^0(t,x) \quad \text{for } x \notin \mathcal{V}_\Sigma \text{ and } y \in \mathbb{R} \tag{4.6}$$

The profile $\mathfrak{U}$, as we have already said above, was constructed in [**6**]. The functions $\mathbf{w}^\varepsilon$ can still be seen as remainders. Let us explain this time more precisely what we mean by remainders. Let us fix a finite set of smooth vectors fields $\mathcal{T}_0 = \{\mathcal{Z}_i(x;\partial_x); i = 1,\cdots,\mu\}$ on $\mathbb{R}^3$, tangent to the surfaces Γ and Σ (that is satisfying $\mathcal{Z}_i(x;\partial_x)\Phi = 0$ on Γ and $\mathcal{Z}_i(x;\partial_x)\Psi = 0$ on Σ, for all $i \in \{1,\cdots,\mu\}$), and generating the algebra of smooth vector fields tangent to $\Gamma \cup \Sigma$. These vector fields can be viewed as vector fields on $\mathbb{R}^4$ tangent to $\mathbb{R} \times \Gamma$ and to $\mathbb{R} \times \Sigma$. By adding the vector field ∂_t to the family, one gets the set $\mathcal{T} := \{\partial_t\} \cup \mathcal{T}_0$ which generates the set of smooth vector fields in $\mathbb{R}^4$ tangent to $(\mathbb{R} \times \Gamma) \cup (\mathbb{R} \times \Sigma)$. We denote $\mathcal{Z}_0 := \partial_t$. For all multi-index $\alpha \in \mathbb{N}^{1+\mu}$ we denote $\mathcal{Z}^\alpha = \partial_t^{\alpha_0}\mathcal{Z}_1^{\alpha_1}.\cdots.\mathcal{Z}_\mu^{\alpha_\mu}$, with

$\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\mu)$. Let us introduce the usual norm:

$$\|u\|_m := \sum_{|\alpha| \leq m\,,\,\alpha \in \mathbb{N}^{1+\mu}} |\mathcal{Z}^\alpha u\|_{L^2(]0,T[\times\Omega)},$$

and denote $H^m_{co}(]0,T[\times\Omega)$ the space of $u \in L^2(]0,T[\times\Omega)$ such that this norm is finite. We will also note

$$|u|_\infty := \|u\|_{L^\infty(]0,[\times\Omega)}.$$

We introduce the set E of the families $(\mathbf{w}^\varepsilon)_{0<\varepsilon\leqslant 1})$ of functions in $L^2(]0,T[\times\Omega)$ such that for all $m \in \mathbb{N}$, there exists $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} ||\mathbf{w}^\varepsilon||_m + ||\varepsilon\partial_{\mathbf{n}}\,\mathbf{w}^\varepsilon||_m + \varepsilon(|\mathbf{w}^\varepsilon|_\infty + |\mathcal{Z}\mathbf{w}^\varepsilon|_\infty + |\varepsilon\partial_{\mathbf{n}}\,\mathbf{w}^\varepsilon|_\infty) < \infty. \tag{4.7}$$

In fact Theorem 4.1 is the straightforward consequence of the following result.

Theorem 4.2. *Let $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^\infty_\Sigma(\Omega)\big)$ be a solution of* (3.1). *There exist $T > 0$, a profile $\mathcal{U}$ in $\mathcal{C}^\infty((0,T)\times\Omega\times\mathbb{R})$ which satisfies* (4.5) − (4.6) *and a family $(\mathbf{w}^\varepsilon)$ in E such that the function u^ε given by the formula (4.4) are solutions of the equation* (4.1) *on $[0,T]\times\Omega$, and of the equation* (4.2) *on $[0,T]\times\Gamma$.*

Theorem 4.2 exhibits large variation solutions of the Landau-Lifshitz equations as the exchange coefficient ε^2 tends to zero, by means of the asymptotic expansions (4.4).

Remark 4.1. Such an analysis is inspired by the paper [**31**] where we show that discontinuous solutions of multidimensional semilinear symmetric hyperbolic systems, which are regular outside of a smooth hypersurface characteristic of constant multiplicity, are limits, when $\varepsilon \to 0$, of solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ of the system perturbated by a viscosity of size ε. In this paper, we adapt the method to the ferromagnetism quasi-static model, where in particular the non local operator $\mathcal{H}$ occurs. We point out that as the principal part of the "hyperbolic" limit operator ($\varepsilon = 0$) is simply ∂_t, the analysis involves only characteristic boundary layers. In contrast [**31**] stresses the occurrence of characteristic and non characteristic boundary layers. It would be also possible -as in [**31**]- to study the case where the singularity is weaker than a jump of the function u^0 as a jump of a derivative of the function u^0. Then we can take T as large as we want and the quality of the approximation is better. We also refer to papers [**15**], [**14**], [**31**] for the use of boundary layers in transmission strategy.

Remark 4.2. It would be interesting to know if it is possible to obtain such a result in the non static case for which the Landau-Lifshitz equation is coupled with the Maxwell system of electromagnetism. For such a model an analysis of the boundary layer induced by the Neumann boundary condition on Γ is performed in [**8**].

Remark 4.3. Again, by adapting the analysis developed in the paper [**31**] it is possible to take $\mathbf{w}^\varepsilon$ in $C^\infty([0,T]\times\Omega)$ in Theorem 4.2.

5. On wall Motion

In this section we briefly deal with 3D wall motion. First it is worth recalling here that the hypersurface $\Sigma = \{\Psi = 0\}$ involved in the asymptotic expansion (4.4) describing the transition layer does not depend on time. As it is said at the end of the introduction a Zeeman field or an anisotropy field -with convenient smoothness assumptions- would not change this fact. However the profile $\mathcal{U}$ involved in the asymptotic expansion (4.4) does depend on time such that it may hide slow motion. Let us illustrate this by an explicit example in which we assume here that $\Omega = \mathbb{R}^3$ so that there is no boundary. We look for solutions of the equation:

$$\partial_t u^\varepsilon = u^\varepsilon \wedge H_{\text{eff}} - u^\varepsilon \wedge \big(u^\varepsilon \wedge H^\varepsilon_{\text{eff}}\big), \tag{5.1}$$

with $H^\varepsilon_{\text{eff}} := \mathcal{H}(u^\varepsilon) + \varepsilon^2 \Delta u^\varepsilon + H^\varepsilon_a + H_e$ where H^ε_a denotes the uniaxial anisotropy along the unit vector e_1: $H^\varepsilon_a := 2(u^\varepsilon_1 e_1 - u^\varepsilon)$ and H_e the Zeeman field: $H_e = ce_1$. Let R be a rotation solution of $\partial_t R = H_e \wedge R$ i.e. such that $(\partial_t R)u = H_e \wedge Ru$ for any $u \in \mathbb{R}^3$. In the following example -inspired by the papers [**10**] and [**22**]- the explicit transition layer profile $\mathcal{U}$ reveals that the wall moves with the slow velocity εc along the axis e_1.

Lemma 5.1. *The function $u^\varepsilon(t,x) = \mathcal{U}(t, \frac{x_1}{\varepsilon})$ with*

$$\mathcal{U}(t,y) := R(t)\big(\text{th}(y+ct), 0, \text{sech}(y+ct)\big), \tag{5.2}$$

is an exact solution in $\mathbb{R}^4$ of the equation (5.1)*.*

Proof. Plugging the expression into the equation (5.1) yields the exact profile equation: $\partial_t \mathcal{U} = \mathcal{B}(\mathcal{U}, \mathcal{H}(\mathcal{U}))$ where $\mathcal{B}$ denotes $\mathcal{B}(u,v) := u \wedge v - u \wedge (u \wedge v)$ and $\mathcal{H}(\mathcal{U}) := -\mathcal{U}_1 e_1 + \partial^2_y \mathcal{U} + 2(\mathcal{U}_1 e_1 - \mathcal{U}) + ce_1$. Let us prove that the profile $\mathcal{U}$ defined by (5.2) verifies this profile equation. We first split $\mathcal{B}(\mathcal{U}, \mathcal{H}(\mathcal{U}))$ into two pieces

$$\mathcal{B}(\mathcal{U}, \mathcal{H}(\mathcal{U})) = \mathcal{B}(\mathcal{U}, \mathcal{H}(\mathcal{U}) - ce_1) + \mathcal{B}(\mathcal{U}, ce_1). \tag{5.3}$$

Let us denote by $\mathcal{V}(y) := \big(\text{th}(y), 0, \text{sech}(y)\big)$. Then we notice that $\mathcal{H}(\mathcal{V})(y) - ce_1 = (2\text{th}^2(y) - 3)\mathcal{V}(y)$. Since $\mathcal{U}(t,y) = R(t)\mathcal{V}(y+ct)$ we infer that $\mathcal{H}(\mathcal{U})(y) - ce_1 = (2\text{th}^2(y) - 3)\mathcal{U}(y)$ hence the first term in (5.3) vanishes. Moreover the second one can be split into

$$\mathcal{B}(\mathcal{U}, ce_1) \quad = \quad cR\mathcal{V} \wedge e_1 - cR\mathcal{V} \wedge (\mathcal{V} \wedge e_1) = (\partial_t R)\mathcal{V} + cR\mathcal{V}' = \partial_t \mathcal{U}.$$

□

In view of this example an interesting question would be to extend Theorem 4.1 for times $\sim O(1/\varepsilon)$ in order to study the wall motion under a Zeeman field, and extend the 1d explicit wall motions of Walker ([**28**], [**33**]) and Slonczewskii [**22**].

6. Proof of Theorem 4.2

As in [**6**], since the magnetic moment u is unimodular, the equation (4.1) is equivalent for smooth solutions to the following one:

$$\mathcal{L}^\varepsilon(u^\varepsilon, \partial)\, u^\varepsilon = \mathbf{F}\big(u^\varepsilon, \varepsilon\partial_x u^\varepsilon, \mathcal{H}(u^\varepsilon)\big) \tag{6.1}$$

where we have denoted

$$\mathcal{L}^\varepsilon(v, \partial) := \partial_t - \varepsilon^2 \Delta_x - \varepsilon^2 v \wedge \Delta_x,$$

and

$$\mathbf{F}(u, V, H) := |V|^2 u + u \wedge H - u \wedge (u \wedge H),$$

for all $u \in \mathbb{R}^3$, $V \in \mathcal{M}(\mathbb{R}^3, \mathbb{R}^3)$, $H \in \mathbb{R}^3$. From now on we will deal with equation (6.1) rather than (4.1). We will proceed in three steps. In subsection 6.1 we will define the profile $\mathcal{U}$ as a local in time solution of a pair of nonlinear equations in $\Omega \times \mathbb{R}_\pm$ coupled by some transmissions conditions on $\{y = 0\}$. In subsection 6.2 we will recall the results of [**6**] about the profile $\mathfrak{U}$. In subsection 6.3 we will prove the existence of some remainders $\mathbf{w}^\varepsilon$ till the lifetime T of the profile $\mathcal{U}$. Finally we will show that the remainders $\mathbf{w}^\varepsilon$ satisfy the uniform estimates (4.7).

6.1. Construction of the internal layers. Even though $\pm\frac{\Psi(x)}{\varepsilon} > 0$ when $x \in \Omega_\pm$ we will define $\mathcal{U}$ for all $(x,z) \in \Omega \times \mathbb{R}_\pm$ since this will not cause any additional difficulty. An Uryshon argument yields the existence of two functions $u^0_\pm$ in $H^\infty((0,\infty)\times\Omega)$ such that $u^0_\pm = u^0$ for all $x \in \Omega_\pm \cup (\Omega_\mp - \mathcal{V}_\Sigma)$. We look for a viscous internal layer profile $\mathcal{U}$ of the form

$$\mathcal{U}(t,x,y) := \begin{cases} u^0_+(t,x) + \mathcal{U}_+(t,x,y) & \text{if } y > 0, \\ u^0_-(t,x) + \mathcal{U}_-(t,x,y) & \text{if } y < 0. \end{cases} \tag{6.2}$$

The functions $\mathcal{U}_\pm$ are in $\mathcal{N}_\pm(T)$. These functions describe internal large amplitude boundary layers, on each side of the hypersurface Σ. To insure that the function $\mathcal{U}$ is in $C^1((0,T)\times\Omega\times\mathbb{R})$ it is necessary to impose the transmission conditions:

$$\left.\begin{array}{c} \mathcal{U}_+ - \mathcal{U}_- = -u^0_+ + u^0_-, \\ \partial_y\mathcal{U}_+ - \partial_y\mathcal{U}_- = 0 \end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}. \tag{6.3}$$

In Theorem 6.1 we will define the profiles $\mathcal{U}_\pm$ as local solutions of nonlinear equations in $\Omega \times \mathbb{R}_\pm$ coupled by some transmission conditions on $\{y = 0\}$. Let us look for convenient equations. We will plug the functions $u^{\varepsilon,0}$ defined by $u^{\varepsilon,0}(t,x) := \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon})$ instead of u^ε in (6.1). In general it is not possible to verify (6.1) but we will try to choose the functions $\mathcal{U}_\pm$ such that the error term is as small as possible. Let us begin to look at the left side of (6.1). With (6.2) we get in L^∞

$$\mathcal{L}^\varepsilon(u^{\varepsilon,0},\partial)u^{\varepsilon,0} = \partial_t\, u^0_\pm + \Big(L(\mathcal{U},\partial_t,\partial_y^2)\mathcal{U}_\pm\Big)| + O(\varepsilon) \quad \text{for } x \in \Omega_\pm, \tag{6.4}$$

where the vertical bar $|$ means that y is evaluated in $y = \frac{\Psi(x)}{\varepsilon}$ and

$$L(U,\partial_t,\partial_y^2) := \partial_t - \partial_y^2 - U \wedge \partial_y^2.$$

We now turn to the right side of (6.1). First

$$\mathcal{H}(u^{\varepsilon,0}) = \mathcal{H}(u^0_\pm) - (\mathcal{U}_\pm.n)|\, n + O(\varepsilon).$$

Then

$$\begin{aligned} \mathbf{F}\big(u^\varepsilon, \varepsilon\partial_x u^\varepsilon, \mathcal{H}(u^\varepsilon)\big) &:= \mathbf{F}\big(u^0_\pm, 0, \mathcal{H}(u^0_\pm)\big) + F_\pm(\mathcal{U}_\pm, \partial_y\mathcal{U}_\pm)| \\ &+ O(\varepsilon) \quad \text{for } x \in \Omega_\pm, \end{aligned} \tag{6.5}$$

with for all $U \in \mathbb{R}^3$, $V \in \mathcal{M}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\begin{aligned} F_\pm(U,V) \;:=\; & |V|^2\,(u^0_\pm + U) + U \wedge \mathcal{H}(u^0_\pm) - (U.n)(u^0_\pm + U)\wedge n \\ & + U \wedge \Big((u^0_\pm + U)\wedge(\mathcal{H}(u^0_\pm) - (U.n)n)\Big) \\ & + u^0_\pm \wedge \Big(U \wedge (\mathcal{H}(u^0_\pm) - (U.n)n)\Big) \\ & - (U.n)u^0_\pm \wedge (u^0_\pm \wedge n) \end{aligned}$$

Thanks to (6.4) and (6.5) we get by looking at the terms at order 0

$$\partial_t\, u^0_\pm + L(\mathcal{U},\partial_t,\partial_y^2)\mathcal{U}_\pm = \mathbf{F}\big(u^0_\pm, 0, \mathcal{H}(u^0_\pm)\big) + F_\pm(\mathcal{U}_\pm, \partial_y\mathcal{U}_\pm).$$

Since for $x \in \Omega_\pm$, the functions $u^0_\pm$ satisfies (3.1) we could simplify and we get the nonlinear equations

$$L(u^0_\pm + \mathcal{U}_\pm, \partial_t, \partial_y^2)\mathcal{U}_\pm = F_\pm(\mathcal{U}_\pm, \partial_y \mathcal{U}_\pm). \tag{6.6}$$

The equations (6.6) are parabolic with respect to t, y, with x as a parameter. The following theorem claims that it is possible to find some solutions $\mathcal{U}_\pm \in \mathcal{N}_\pm(T)$ of these equations even for all $x \in \Omega$.

Theorem 6.1. *There exists $T > 0$ and there exist some functions $\mathcal{U}_\pm \in \mathcal{N}_\pm(T)$ which verify the equations* (6.6) *when $(t, x, y) \in (0, T) \times \Omega \times \mathbb{R}_\pm$ and the transmission conditions* (6.3). *Moreover for all $x \notin \mathcal{V}_\Sigma$ and $y \in \mathbb{R}_\pm$ there holds $\mathcal{U}_\pm(t, x, y) = 0$.*

Proof. We will proceed in four steps.

Step 1. *We begin to reduce the problem to homogeneous boundary conditions.*

We introduce the functions $V_\pm$ and $\mathcal{U}_\pm$ given by the formula

$$\begin{aligned} V_\pm(t,x,y) &:= (1 - \frac{e^{\mp y}}{2})u^0_\pm(t,x) + \frac{e^{\mp y}}{2}u^0_\mp(t,x), \\ \mathbf{W}_\pm(t,x,y) &:= \mathcal{U}_\pm(t,x,y) \pm \frac{1}{2}(u^0_+(t,x) - u^0_-(t,x))e^{\mp y}. \end{aligned}$$

Let us introduce the notation $[W] = W_+ - W_-$ for a couple of functions $W_\pm$. Thus the transmission conditions (6.3) reads:

$$[\mathbf{W}] = [\partial_y \mathbf{W}] = 0 \qquad \text{when } (t,x,y) \in (0,T) \times \Omega \times \{0\}. \tag{6.7}$$

Moreover the equations (6.6)-(6.3) read for $(t, x, y) \in (0, T) \times \Omega \times \mathbb{R}_\pm$:

$$L(V_\pm + \mathbf{W}_\pm, \partial_t, \partial_y^2)\mathbf{W}_\pm = \hat{F}_\pm(t,x,y,\mathbf{W}_\pm, \partial_y \mathbf{W}_\pm), \tag{6.8}$$

where $\hat{F}_\pm$ are C^∞ functions such that the functions $\hat{F}_\pm(t, x, y, 0, 0)$ are rapidly decreasing with respect to y.

Step 2. *We prove the existence of compatible initial data.*

Let us to explain why the initial values $\mathbf{W}_{0,+}$ must satisfy some compatibility conditions at the corner $\{t = y = 0\}$ in order to obtain smooth solutions $\mathbf{W}_\pm$ of the problem (6.8)-(6.7) with $\mathbf{W}_\pm|_{t=0} := \mathbf{W}_{0,\pm}$. First set $t = 0$ in the transmission conditions (6.7) to see that $\mathbf{W}_{0,+}$ must satisfy the relation

$$[\mathbf{W}_0] = [\partial_y \mathbf{W}_0] = 0 \qquad \text{when } (x,y) \in \Omega \times \{0\}. \tag{6.9}$$

Now, for each $k \geqslant 1$, apply the derivative ∂_t^k to the transmission conditions (6.3). We get

$$[\partial_t^k \mathbf{W}] = [\partial_y \partial_t^k \mathbf{W}] = 0 \qquad \text{when } (t,x,y) \in (0,T) \times \Omega \times \{0\}.$$

Now remark that, by iteration, we can estimate $\partial_t^k \mathbf{W}_\pm$ by the interior equations (6.6) in terms of derivatives with respect to y. More precisely there exists some smooth functions $C^k_\pm$ such that

$$\partial_t^k \mathbf{W}_\pm = C^{2k}_\pm((\partial_y^l \mathbf{W}_\pm)_{l \leqslant 2k}) \quad \text{and } \partial_y \partial_t^k \mathbf{W}_\pm = C^{2k+1}_\pm((\partial_y^l \mathbf{W}_\pm)_{l \leqslant 2k+1}).$$

Thus the following compatibility condition must hold for $k \geqslant 2$:

$$[C^k((\partial_y^l \mathbf{W}_0)_{l \leqslant k})] = 0. \qquad \text{when } (x,y) \in \Omega \times \{0\}. \tag{6.10}$$

Lemma 6.1. *There exist some initial values $\mathbf{W}_{0,\pm}$ in $H^\infty(\Omega, \mathcal{S}(\mathbb{R}_\pm))$ which satisfy the relation (6.9) and (6.10) for all $k \geqslant 2$.*

PROOF. As we will follow the method of [**31**], we only sketch the proof. We start by analyzing more accurately the compatibility conditions and more especially the way the functions $C_\pm^k$ depend on the derivatives with respect to y. Indeed there exists some functions $\tilde{C}_\pm^k$ such that

$$C_\pm^k((\partial_y^l \mathbf{W}_\pm)_{l\leqslant k}) = \tilde{C}_\pm^k((\partial_y^l \mathbf{W}_\pm)_{l\leqslant k-1}) + T_\pm \partial_y^k \mathbf{W}_\pm$$

where $T_\pm$ denote the automorphisms of $H^\infty(\Omega)$:

$$T_\pm : \mathbf{W}_\pm \mapsto \mathbf{W}_\pm + (V_\pm + \mathbf{W}_\pm^{(0)}) \wedge \mathbf{W}_\pm.$$

We deduce by iteration that there exists a family $(\mathbf{W}_\pm^{(k)})_{k\in\mathbb{N}}$ in $H^\infty(\Omega)$ such that

$$[\tilde{C}^k((\mathbf{W}^{(l)})_{l\leqslant k-1}) + T_\pm \mathbf{W}^{(k)}] = 0.$$

Like in paper [**31**], the lemma by Borel on Taylor series implies that there exist functions $\mathbf{W}_{0,\pm}$ in $H^\infty(\Omega, \mathcal{S}(\mathbb{R}_\pm))$ such that for all $k \geqslant 0$, $\partial_y^k \mathbf{W}_{0,\pm} = \mathbf{W}_\pm^{(k)}$. □

As a consequence, we will assume in the rest of the proof that the functions $\mathbf{W}_\pm$ vanish for $t \leqslant 0$.

STEP 3. *We look for linear estimates.*

In order to use an iterative scheme, we look at the linear problem

$$L(\mathfrak{W}_\pm, \partial_t, \partial_y^2)\mathbf{W}_\pm = f_\pm \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\mathbb{R}_\pm, \tag{6.11}$$

$$[\mathbf{W}] = [\partial_y \mathbf{W}] = 0 \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}. \tag{6.12}$$

For all real $\lambda \geqslant 1$, the space $L^2((0,T)\times\Omega\times\mathbb{R}_\pm)$ is endowed with the scalar product associated to the Euclidean norm

$$||\mathbf{W}_\pm||_{0,\lambda,T} := ||e^{-\lambda t}\, W_\pm||_{L^2((0,T)\times\Omega\times\mathbb{R}_\pm)}$$

In order to avoid heavy notations, we will denote $W := (\mathbf{W}_+, \mathbf{W}_-)$, $f := (f_+, f_-)$ and $\mathfrak{W} := (\mathfrak{W}_+, \mathfrak{W}_-)$. We endow the space $L^2((0,T)\times\Omega\times\mathbb{R}_+)\times L^2((0,T)\times\Omega\times\mathbb{R}_-)$ with the scalar product associated to the Euclidean norm

$$||W||_{0,\lambda,T} := ||\mathbf{W}_+||_{+,0,\lambda,T} + ||\mathbf{W}_-||_{-,0,\lambda,T}.$$

For $m \in \mathbb{N}$, we introduce the following weighted norms:

$$||W||_{m,\lambda,T} := \sum_{|\alpha|\leqslant m} ||\partial_{t,x}^\alpha\, W||_{0,\lambda,T} \text{ and } |W|_{m,\lambda,T} := \sum_{|\alpha|\leqslant m} ||\partial_{t,x}^\alpha\, \partial_y^{\alpha_4}\, W||_{0,\lambda,T},$$

where $\alpha := (\alpha_0, ..., \alpha_3) \in \mathbb{N}^4$ and $\partial_{t,x}^\alpha := \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

PROPOSITION 6.1. *Let $R > 0$. If $\mathfrak{W}_\pm$ verify the following estimates*

$$||\mathfrak{W}_+||_{Lip((0,T)\times\Omega\times\mathbb{R}_+)} + ||\mathfrak{W}_+||_{Lip((0,T)\times\Omega\times\mathbb{R}_+)} + |\mathfrak{W}|_{m,\lambda,T} < R,$$

and the following boundary conditions

$$[\mathfrak{W}] = [\partial_y \mathfrak{W}] = 0 \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}, \tag{6.13}$$

then there exist $\lambda_m > 0$ and for all $k \in \mathbb{N}$, $\mu_{k,m} > 0$, such that for all $\lambda \geqslant \lambda_m$,

$$|W|_{m,\lambda,T} \leqslant \frac{\lambda_m}{\lambda}|f|_{m,\lambda,T} \tag{6.14}$$

and for all $\mu \geqslant \mu_{k,m}$,

$$|y^k\, W|_{m,\lambda,T} \leqslant \frac{\mu_{k,m}}{\mu} \sum_{j=0}^{k} |y^j\, f|_{m,\mu,T}. \tag{6.15}$$

Proof. We multiply the equation (6.11) by $\mathbf{W}_\pm$ and integrate for $(x,y) \in \Omega \times \mathbb{R}_\pm$. Hence

$$(1/2)\partial_t \int_{\Omega\times\mathbb{R}_\pm} |\mathbf{W}_\pm|^2 - J_{1,\pm} - J_{2,\pm} = \int_{\Omega\times\mathbb{R}_\pm} f_\pm.\mathbf{W}_\pm \tag{6.16}$$

where $J_{1,\pm} := \int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.\partial_y^2 \mathbf{W}_\pm$ and $J_{2,\pm} := \int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.(\mathfrak{W}_\pm \wedge \partial_y^2)\mathbf{W}_\pm$.

Integrating by parts, we get

$$J_{1,\pm} = - \int_{\Omega\times\mathbb{R}_\pm} |\partial_y \mathbf{W}_\pm|^2 - I_{1,\pm}, \text{ and } J_{2,\pm} = - \int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.(\partial_y \mathfrak{W}_\pm \wedge \partial_y)\mathbf{W}_\pm - I_{2,\pm},$$

where

$$I_{1,\pm} := \int_\Omega (\mathbf{W}_\pm.\partial_y \mathbf{W}_\pm)|_{y=0}, \text{ and } I_{2,\pm} := \int_\Omega (\mathbf{W}_\pm.(\mathfrak{W}_\pm \wedge \partial_y \mathbf{W}_\pm))|_{y=0}.$$

Using the boundary conditions (6.12) and (6.13), we get $[I_1] = [I_2] = 0$. Taking that into account we add the two estimates in (6.16). Then we multiply by $e^{-2\lambda t}$ and integrate in time. By a Gronwall lemma we get that there exists $c > 0$ such that for all $\lambda \geqslant c$,

$$|\partial_y W|^2_{0,\lambda,T} + \lambda |W|^2_{0,\lambda,T} \leqslant c| < f, W >_{\lambda,T} |. \tag{6.17}$$

We go on with estimates tangential to $\{y = 0\}$. To do this we apply the derivative $\partial^\alpha_{t,x}$ to the equations (6.11)-(6.12). So we get that $\partial^\alpha_{t,x}\, \mathbf{W}_\pm$ verify

$$L(\mathfrak{W}_\pm, \partial_t, \partial_y^2)\partial^\alpha_{t,x}\, \mathbf{W}_\pm = \tilde{f}_\pm \quad \text{when } (t,x,y) \in (0,T) \times \Omega \times \mathbb{R}_\pm, \tag{6.18}$$

$$[\partial^\alpha_{t,x}\, \mathbf{W}] = [\partial_y \partial^\alpha_{t,x}\, \mathbf{W}] = 0 \qquad \text{when } (t,x,y) \in (0,T) \times \Omega \times \{0\}, \tag{6.19}$$

where

$$\tilde{f}_\pm := \partial^\alpha_{t,x}\, f_\pm + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|, |\alpha_2|<|\alpha|} \partial^{\alpha_1}_{t,x}\, \mathfrak{W}_\pm \wedge \partial_y^2\, \partial^{\alpha_2}_{t,x}\, \mathbf{W}_\pm. \tag{6.20}$$

We apply the tangential derivative $\partial^\alpha_{t,x}$ to the boundary conditions (6.13) and get

$$[\partial^\alpha_{t,x}\, \mathfrak{W}] = [\partial_y \partial^\alpha_{t,x}\, \mathfrak{W}] = 0 \qquad \text{when } (t,x,y) \in (0,T) \times \Omega \times \{0\}, \tag{6.21}$$

Using the estimate (6.17), we get, for all $\lambda \geqslant c$,

$$|\partial_y \partial^\alpha_{t,x}\, W|^2_{0,\lambda,T} + \lambda |\partial^\alpha_{t,x}\, W|^2_{0,\lambda,T} \leqslant c| < \tilde{f}, \partial^\alpha_{t,x}\, W >_{\lambda,T} |.$$

Thanks to (6.20), we get

$$< \tilde{f}, \partial^\alpha_{t,x}\, W >_{\lambda,T} = < \partial^\alpha_{t,x}\, f, \partial^\alpha_{t,x}\, W >_{\lambda,T} + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|, |\alpha_2|<|\alpha|} I_{\alpha_1,\alpha_2}, \tag{6.22}$$

where $I_{\alpha_1,\alpha_2} := I_{+,\alpha_1,\alpha_2} + I_{-,\alpha_1,\alpha_2}$ with

$$I_{\pm,\alpha_1,\alpha_2} := < \partial^{\alpha_1}_{t,x}\, \mathfrak{W}_\pm \wedge \partial_y^2\, \partial^{\alpha_2}_{t,x}\, \mathbf{W}_\pm, \partial^{\alpha_2}_{t,x}\, \mathbf{W}_\pm >_{\lambda,T} .$$

Using Cauchy-Schwarz inequality, we get

$$| < \partial_{t,x}^{\alpha} f, \partial_{t,x}^{\alpha} W >_{\lambda,T} | \leqslant |f|_{0,\lambda,T} .|W|_{0,\lambda,T}.$$

We are going to estimate, for all α_1, α_2 such that $|\alpha_1| + |\alpha_2| = |\alpha|, |\alpha_2| < |\alpha|$, the term I_{α_1,α_2}. Integrating by parts, we get $I_{\pm,\alpha_1,\alpha_2} := \sum_{l=1}^{3} I^l_{\pm,\alpha_1,\alpha_2}$, with

$$\begin{aligned} I^1_{\pm,\alpha_1,\alpha_2} &:= - < \partial_{t,x}^{\alpha_1} \partial_y \mathfrak{W}_\pm \wedge \partial_y \partial_{t,x}^{\alpha_2} \mathbf{W}_\pm, \partial_{t,x}^{\alpha} \mathbf{W}_\pm >_{\lambda,T}, \\ I^2_{\pm,\alpha_1,\alpha_2} &:= - < \partial_{t,x}^{\alpha_1} \mathfrak{W}_\pm \wedge \partial_y \partial_{t,x}^{\alpha_2} \mathbf{W}_\pm, \partial_{t,x}^{\alpha} \partial_y \mathbf{W}_\pm >_{\lambda,T}, \\ I^3_{\pm,\alpha_1,\alpha_2} &:= \mp << \{(\partial_{t,x}^{\alpha_1} \mathfrak{W}_\pm \wedge \partial_y \partial_{t,x}^{\alpha_2} \mathbf{W}_\pm)\}|_{y=0}, \{\partial_{t,x}^{\alpha} \partial_y \mathbf{W}_\pm\}|_{y=0} >>_{\lambda,T}, \end{aligned}$$

where $<< .,. >>_{\lambda,T}$ denotes the scalar product of $L^2((0,T) \times \Omega)$ associated to the mesure $e^{-\lambda t} dtdx$. Thanks to the boundary conditions (6.19)-(6.21), we get $I^3_{+,\alpha_1,\alpha_2} - I^3_{-,\alpha_1,\alpha_2} = 0$. Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} |I^1_{\pm,\alpha_1,\alpha_2}| &\leqslant |\partial_{t,x}^{\alpha_1} \partial_y \mathfrak{W}_\pm \wedge \partial_y \partial_{t,x}^{\alpha_2} \mathbf{W}_\pm|_{0,\lambda,T} .||\mathbf{W}_\pm||_{m,\lambda,T}, \\ |I^2_{\pm,\alpha_1,\alpha_2}| &\leqslant |\partial_{t,x}^{\alpha_1} \mathfrak{W}_\pm \wedge \partial_y \partial_{t,x}^{\alpha_2} \mathbf{W}_\pm|_{0,\lambda,T} .||\partial_y \mathbf{W}_\pm||_{m,\lambda,T}, \end{aligned}$$

Using Gargliardo-Nirenberg inequalities, we get

$$\begin{aligned} &|I^1_{\pm,\alpha_1,\alpha_2}| \leqslant \\ &c(||\partial_y \mathfrak{W}_\pm||_{m,\lambda,T}.||\mathbf{W}_\pm||_{Lip} + ||\mathfrak{W}_\pm||_{Lip}.||\partial_y \mathbf{W}_\pm||_{m,\lambda,T}).||\mathbf{W}_\pm||_{m,\lambda,T}, \\ &|I^2_{\pm,\alpha_1,\alpha_2}| \leqslant \\ &c(||\mathfrak{W}_\pm||_{m-1,\lambda,T}.||\mathbf{W}_\pm||_{Lip} + |\mathfrak{W}_\pm|_{Lip}.||\partial_y \mathbf{W}_\pm||_{m-1,\lambda,T}).||\partial_y \mathbf{W}_\pm||_{m,\lambda,T}. \end{aligned}$$

Hence we get

$$|I_{\alpha_1,\alpha_2}| \leqslant \frac{1}{2} ||\partial_y \mathbf{W}_\pm||^2_{m,\lambda,T} + C(||\mathbf{W}_\pm||^2_{m,\lambda,T} + ||\partial_y \mathbf{W}_\pm||^2_{m-1,\lambda,T}).$$

We deduce that there exists $\lambda_m > 0$ such that for all $\lambda \geqslant \lambda_m$, there holds $||W||_{m,\lambda,T} \leqslant \frac{\lambda_m}{\lambda} ||f||_{m,\lambda,T}$.

To prove the estimates (6.14), it remains to get normal estimates. The cases $\alpha_4 = 0$ or 1 are already treated in the tangential estimates. If $\alpha_4 \geqslant 2$, we proceed by iteration, estimating $\partial_y^2 \mathbf{W}_\pm$ from the equations.

It remains to get the estimates (6.15). First we notice that for $p \geqslant 1$ the function $y^p \mathbf{W}_\pm$ verify the initial boundary value problem

$$\begin{aligned} L(\mathfrak{W}_\pm, \partial_t, \partial_y^2) \mathbf{W}^{[p]}_\pm = f^{[p]}_\pm \quad &\text{when } (t,x,y) \in (0,T) \times \Omega \times \mathbb{R}_\pm, \\ [\mathbf{W}^{[p]}] = [\partial_y \mathbf{W}^{[p]}] = 0 \quad &\text{when } (t,x,y) \in (0,T) \times \Omega \times \{0\}, \\ \mathbf{W}^{[p]}_\pm = 0 \quad &\text{when } (t,x,y) \in \{0\} \times \Omega \times \mathbb{R}_\pm, \end{aligned}$$

where

$$f^{[p]}_\pm = y^p f_\pm + \sum_{j=0}^{p-1} (q^1_j \partial_y \mathbf{W}^{[j]}_\pm + q^2_j \mathfrak{W}_\pm \wedge \partial_y \mathbf{W}^{[j]}_\pm),$$

where the q^1_j and the q^2_j are in $\mathbb{N}$. Thus we prove, by iteration on p and thanks to the inequality (6.14), the estimate

$$\sqrt{\mu} ||\partial_y (y^p W)||_{m,\mu,T} + \mu ||y^p W||_{m,\mu,T} \leqslant \sum_{j=0}^{p} ||y^j f||_{m,\mu,T}$$

which implies the estimate (6.15).

□

STEP 4. *We use an iterative scheme.*

We define the iterative scheme $(\mathbf{W}^\nu_\pm)_{\nu\in\mathbb{N}}$ by setting $\mathbf{W}^0_\pm$ equal to zero and, by iteration, when $\mathbf{W}^\nu_\pm$ is defined, we take $\mathbf{W}^{\nu+1}_\pm$ as solution of

$$\begin{aligned}
&L(V_\pm + \mathbf{W}^\nu_\pm, \partial_t, \partial_y^2)\mathbf{W}^{\nu+1}_\pm = \hat{F}(t,x,y,\mathbf{W}^\nu_\pm, \partial_y \mathbf{W}^\nu_\pm) \\
&\quad \text{when } (t,x,y) \in (0,\infty)\times\Omega\times\mathbb{R}_\pm, \\
&\quad [\mathbf{W}^{\nu+1}] = [\partial_y \mathbf{W}^{\nu+1}] = 0 \qquad \text{when } (t,x,y)\in(0,T)\times\Omega\times\{0\}, \\
&\mathbf{W}^{\nu+1}_\pm = 0 \quad \text{when } (t,x,y)\in\{0\}\times\Omega\times\mathbb{R}_\pm.
\end{aligned}$$

Thanks to the linear estimates, to a Sobolev embedding and to some Gargliardo-Nirenberg inequalities, we show that the iterative scheme $(\mathbf{W}^\nu_\pm)_{\nu\in\mathbb{N}}$ converge, when $\nu\to+\infty$ toward some solutions $\mathbf{W}_\pm \in \mathcal{N}_\pm(T)$ of the problem (6.8)-(6.7). By going back to the original problem (6.6)-(6.3), the first sentence of Theorem 6.1 is now proved. When $x\notin\mathcal{V}_\Sigma$, the function $u^0_+ - u^0_-$ in the right hand side of (6.6) vanishes and so do the functions $\mathcal{U}_\pm$.

□

REMARK 6.1. Notice that the possibility of a blow-up can be controlled with Lipschitz norm in a very classical way. However we do not know whether the solutions $\mathcal{U}$ actually blow-up or exist globally.

6.2. Construction of $\mathfrak{U}$. In this section we define the boundary layer profile $\mathfrak{U}$ as a solution of a linear boundary value problem. Let us recall that this function describes a boundary layer which appears near the boundary to compensate the loss of the Neumann condition from the complete model (4.1)-(4.2)-(4.3) to the limit model (3.1) ($\varepsilon = 0$). Such a boundary layer was already mentioned in paper [**6**]. Let Θ be a C^∞ function on Ω such that $\Theta = 1$ in a neighborhood $\mathcal{W}_\Gamma$ of Γ such that $\mathcal{W}_\Gamma \subset\subset \mathcal{V}_\Gamma$ and $\Theta = 0$ in $\Omega - \mathcal{V}_\Gamma$.

THEOREM 6.2. *There exists $\mathfrak{U}\in\mathcal{N}_+(T)$ which verifies*

$$\begin{aligned}
L(u^0,\partial_t,\partial_z^2)\mathfrak{U} \quad = \quad & -(\mathfrak{U}.n)u^0\wedge n + \mathfrak{U}\wedge\mathcal{H}(u^0) \\
& +\mathfrak{U}\wedge(u^0\wedge\mathcal{H}(u^0)) - (\mathfrak{U}.n)u^0\wedge(u^0\wedge n) + u^0\wedge(\mathfrak{U}\wedge\mathcal{H}(u^0)),
\end{aligned}$$

when $(t,x,z)\in(0,T)\times\Omega\times\mathbb{R}_+$,

$$\partial_z\mathfrak{U} = \Theta(x)\partial_{\mathfrak{n}} u^0 \quad \text{when } (t,x,z)\in(0,T)\times\Omega\times\{0\}. \tag{6.23}$$

Moreover there holds $\mathfrak{U}(t,x,z) = 0$ for $x\notin\mathcal{V}_\Sigma$.

PROOF. Proceeding as in the proof of Theorem 6.1, we prove the existence of compatible initial data. Then we follow the proof of Proposition 4.2 of [**6**]. □

6.3. Construction of $\mathbf{w}^\varepsilon$. In this section, we look at the remainder $\mathbf{w}^\varepsilon$. We will proceed in four steps. First in section 6.3.1 we will reduce the initial problem (4.1)-(4.2)-(4.3) for the unknown u^ε to a problem for $\mathbf{w}^\varepsilon$. Indeed in order to get a homogeneous boundary problem, we will add a corrector to $\mathbf{w}^\varepsilon$ and rather work with the resulting term w^ε. Like in paper [**31**], the lemma by Borel on Taylor series insure the existence of convenient initial data for the resulting reduced problem which means that compatibility conditions either on Γ and on Σ are satisfied. We will prove that the solutions of this nonlinear problems exist not only for a common non trivial time, in fact till the time T of the profiles $\mathcal{U}$. Moreover these solutions satisfy some estimates uniform with respect to ε. The method relies on a simple Picard iterative scheme (cf. section 6.3.2) and on linear estimates (cf. section 6.3.3). More precisely we will use L^2-type conormal estimates of only the two first normal derivatives, and some Lipschitz estimates. A bit of care reveals that the presence of the operator $\mathcal{H}$ does not cause any loss of factor ε or any loss of derivatives.

6.3.1. *A reduced problem.* Since we look for solutions u^ε of (4.1)-(4.2)-(4.3) of the form (4.4) where the functions

$$a^\varepsilon(t,x) \quad := \quad \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon}) + \varepsilon\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon})$$

have been constructed above, we look for a problem in term of the remainder $\mathbf{w}^\varepsilon$. In fact, in order to get a homogeneous boundary problem, we choose a function $\rho(t,x) \in H^\infty$ such that

$$\partial_{\mathbf{n}}\rho\,|_\Gamma \quad = \quad -\partial_{\mathbf{n}}\mathfrak{U}(t,x,0)|_\Gamma. \tag{6.24}$$

and will look for remainders $\mathbf{w}^\varepsilon$ of the form $\mathbf{w}^\varepsilon = \rho + w^\varepsilon$. Let us explain why. On the boundary Γ, the function a^ε satisfies:

$$\partial_{\mathbf{n}}a^\varepsilon|_\Gamma = \varepsilon\,\partial_{\mathbf{n}}\mathfrak{U}(t,x,0)|_\Gamma, \tag{6.25}$$

Hence in general a^ε does not satisfy the homogeneous Neumann boundary condition on Γ. We define the function $\tilde{a}^\varepsilon := a^\varepsilon + \varepsilon\rho$. Thus we look for solutions u^ε of (4.1)-(4.2)-(4.3) of the form $u^\varepsilon = a^\varepsilon + \varepsilon\,\mathbf{w}^\varepsilon = \tilde{a}^\varepsilon + \varepsilon\, w^\varepsilon$. Combine (4.2), (6.24) and (6.25) to find a homogeneous Neumann boundary condition on Γ for w^ε:

$$\partial_{\mathbf{n}}w^\varepsilon \quad = \quad 0 \quad \text{on }]0,T[\times\Gamma. \tag{6.26}$$

We now look for an equation on the unknown w^ε. The function $\tilde{a}^\varepsilon$ belongs to $\mathcal{C}^1((0,T)\times\Omega)$ and to $H^\infty_\Sigma(\Omega)$. Moreover, $\tilde{a}^\varepsilon$ satisfies the equation

$$\mathcal{L}^\varepsilon(\tilde{a}^\varepsilon,\partial)\,\tilde{a}^\varepsilon = \mathbf{F}\big(\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon)\big) + \varepsilon r^\varepsilon \tag{6.27}$$

where the family $(r^\varepsilon)_\varepsilon$ lies in the set E (defined above Theorem 4.2). The system for the unknown $w^\varepsilon(t,x)$ writes

$$\begin{aligned}&\mathcal{L}^\varepsilon(\tilde{a}^\varepsilon+\varepsilon w^\varepsilon,\partial)w^\varepsilon = \\ &K(\varepsilon,\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon),w^\varepsilon,\varepsilon\partial_x w^\varepsilon,\mathcal{H}(w^\varepsilon)) + r^\varepsilon \quad \text{in }]0,T[\times\Omega\end{aligned} \tag{6.28}$$

where K is a smooth function of its arguments. Let us use more concise notations, and denote

$$A^\varepsilon := \big(\,\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon)\big) \quad \text{and} \quad W^\varepsilon := \big(\,w^\varepsilon,\varepsilon\partial_x w^\varepsilon,\mathcal{H}(w^\varepsilon)\,\big). \tag{6.29}$$

Then Taylor's formula shows that the function K has the following form:

$$K(\varepsilon,A^\varepsilon,W^\varepsilon) = G(\varepsilon,A^\varepsilon,\varepsilon W^\varepsilon)W^\varepsilon$$

where G depends smoothly on its arguments (including ε), which will be useful in the sequel.

Following [**31**] there exist a family $(w^\varepsilon_{\text{init}})_\varepsilon$ of compatible initial conditions for the problem (6.28)-(6.26) which verifies suitable uniform estimates with respect to ε. We choose such a family.

6.3.2. *The iterative scheme.* We want to solve the problem (6.28),(6.26). We use a simple Picard(-Banach-Caccioppoli) iterative scheme defining a sequence $w^{\varepsilon,\nu}$ which will converge to the solution of the problem. For clarity, we adopt the following more concise notations

$$A^\varepsilon := \big(\tilde{a}^\varepsilon, \varepsilon\partial_x \tilde{a}^\varepsilon, \mathcal{H}(\tilde{a}^\varepsilon)\big) \quad \text{and} \quad W^{\varepsilon,\nu} := \big(w^{\varepsilon,\nu}, \varepsilon\partial_x w^{\varepsilon,\nu}, \mathcal{H}(w^{\varepsilon,\nu})\big).$$

With these notations, the iterative scheme writes

$$\mathcal{L}^\varepsilon(\tilde{a}^\varepsilon + \varepsilon w^{\varepsilon,\nu}, \partial) w^{\varepsilon,\nu+1} = f^{\varepsilon,\nu} \quad \text{in }]0,T[\times\Omega \tag{6.30}$$

where

$$f^{\varepsilon,\nu} := G(\varepsilon, A^\varepsilon, \varepsilon W^{\varepsilon,\nu}) W^{\varepsilon,\nu} + r^\varepsilon \tag{6.31}$$

This equation is coupled with the initial and boundary conditions:

$$\partial_{\mathbf{n}} w^{\varepsilon,\nu+1} = 0 \text{ on }]0,T[\times\Gamma \tag{6.32}$$

$$w^{\varepsilon,\nu+1}|_{t=0} = w^\varepsilon_{\text{init}}. \tag{6.33}$$

The iterative scheme is initialized with $w^{\varepsilon,0}(t,x) := w^\varepsilon_{\text{init}}(x)$.

6.3.3. *Estimates for a linear parabolic system.* Consider the linear problem

$$\mathcal{L}^\varepsilon(\tilde{a}^\varepsilon + \varepsilon\mathbf{b}, \partial)\mathbf{u} = f \text{ on }]0,T[\times\Omega \tag{6.34}$$

$$\partial_{\mathbf{n}}\mathbf{u} = 0 \text{ on }]0,T[\times\Gamma, \tag{6.35}$$

We endow the space $H^m_{co}(]0,T[\times\Omega)$ with the usual weighted norm with $\lambda \geq 1$:

$$\|\mathbf{u}\|_{m,\lambda} := \sum_{|\alpha|\leq m\,,\,\alpha\in\mathbb{N}^{1+\mu}} \lambda^{m-|\alpha|} \|e^{-\lambda t}\mathcal{Z}^\alpha \mathbf{u}\|_{L^2(]0,T[\times\Omega)}.$$

In order to estimate the initial data, we introduce the similar norms built with the set $\mathcal{T}_0$ instead of $\mathcal{T}$, integrating on Ω instead of $[0,T]\times\Omega$:

$$|\mathbf{u}|_{m,\lambda} := \sum_{|\alpha|\leq m\,,\,\alpha_0=0\,,\,\alpha\in\mathbb{N}^{1+\mu}} \lambda^{m-|\alpha|} \|\mathcal{Z}^\alpha \mathbf{u}\|_{L^2(\Omega)}.$$

We will use the following classical Gagliardo-Moser-Nirenberg estimates for conormal derivatives (see [**13**]).

LEMMA 6.2. *Let $m \in \mathbb{N}$. There is $c_m > 0$ such that, for any $a_1, \dots, a_k \in H^m_{co}(]0,T[\times\Omega) \cap L^\infty(]0,T[\times\Omega)$, for all multi-index $\alpha_1 \in \mathbb{N}^{\mu+1}, \dots, \alpha_k \in \mathbb{N}^{\mu+1}$, with $|\alpha_1| + \cdots + |\alpha_k| \leq m$, for all $\lambda \geq 1$:*

$$\|\mathcal{Z}^{\alpha_1} a_1 \dots \mathcal{Z}^{\alpha_k} a_k\|_{0,\lambda} \;\leq\; c_m \sum_{1\leq j\leq k} \Big(\|a_j\|_{m,\lambda} \prod_{i\neq j} |a_i|_\infty \Big). \tag{6.36}$$

The following proposition gives some ε-conormal estimates for the two first normal derivatives of the solutions of the problem (6.34)-(6.35).

PROPOSITION 6.2. *Let $R > 0$ be an arbitrary constant and $m \geqslant 3$. There exist $C_m(R) > 0$ and $\lambda_m > 0$ such that for σ fixed constant large enough, depending only on the choices of the vector fields $\mathcal{Z}_j$, the following holds true. Assume that*

$$\varepsilon\,(\,|\mathbf{b}|_\infty + \sum_{0\leq j\leq\mu} |\mathcal{Z}_j \mathbf{b}|_\infty + |\varepsilon\partial_x \mathbf{b}|_\infty\,) \;\leq\; R, \tag{6.37}$$

then, for all $\lambda \geq \lambda_m$, the following estimates hold:

$$\begin{aligned} \|\varepsilon\partial_x \mathbf{u}\|_{m,\lambda} + \lambda\|\mathbf{u}\|_{m,\lambda} \;\leq\; & C_m(R)\,\big[\,\lambda^{-1}\,\|f\|_{m,\lambda} + \,I_{m,\lambda}(\mathbf{u}) \\ & +\;\varepsilon\,(\,\|\varepsilon\partial_x \mathbf{b}\|_{m,\lambda} + \|\mathbf{b}\|_{m,\lambda}\,)\,(\,|\mathbf{u}|_\infty + |\varepsilon\partial_x \mathbf{u}|_\infty\,)\,\big], \end{aligned} \tag{6.38}$$

where

$$I_{m,\lambda}(\mathbf{u}) := \sum_{0\le k\le m} |(\partial_t^k \mathbf{u})_{|t=0}|_{m-k,\lambda}.$$

and

$$\begin{aligned} \|(\varepsilon\partial_{\mathbf{n}})^2\mathbf{u}\|_{m,\lambda} \ \le\ & C_m(R)\big[\ \|f\|_{m,\lambda} \\ +\|\mathbf{u}\|_{m+1,\lambda} + \varepsilon\|\mathbf{b}\|_{m+1,\lambda}\,&(|\mathbf{u}|_\infty + |f|_\infty) \\ & +\varepsilon\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m+1,\lambda} + \varepsilon^2\|\mathbf{u}\|_{m+2,\lambda}\ \big]. \end{aligned} \tag{6.39}$$

PROOF. *Step 1.* Let us denote $\mathbf{v} := e^{-\lambda t}\,\mathbf{u}$, which satisfies

$$\mathcal{L}^\varepsilon(\tilde a^\varepsilon_{app} + \varepsilon\mathbf{b}, \partial)\mathbf{v} + \lambda\mathbf{v} = e^{-\lambda t} f \text{ on }]0,T[\times\Omega \tag{6.40}$$

$$\partial_{\mathbf{n}}\mathbf{v} = 0 \text{ on }]0,T[\times\Gamma. \tag{6.41}$$

$$\mathbf{v} = w^\varepsilon_{\text{init}} \text{ on } t=0. \tag{6.42}$$

Let us denote $\|.\|_{L^2}$ the L^2 norm in $[0,T]\times\Omega$, and $|.|_{L^2}$ the L^2 norm in Ω. Multiplying (6.40) by $\mathbf{v}$ and integrating on $]0,T[\times\Omega$ gives the following estimate, integrating by parts the $\varepsilon^2\Delta_x$ with Green's formula in Ω:

$$\varepsilon^2\|\nabla_x\mathbf{v}\|^2_{L^2} + \lambda\|\mathbf{v}\|^2_{L^2} \le 2\ |((e^{-\lambda t}f, \mathbf{v}))_{L^2}| + |\mathbf{v}(0)|_{L^2}, \tag{6.43}$$

for all $\lambda \ge \lambda_0$ if λ_0 is fixed large enough, and for all $\varepsilon > 0$. In terms of $\mathbf{u}$ it writes

$$\varepsilon^2\|\nabla_x\mathbf{u}\|^2_{0,\lambda} + \lambda\|\mathbf{u}\|^2_{0,\lambda} \le 2\ |((f, \mathbf{u}))_{L^2_\lambda}| + |\mathbf{u}(0)|_{L^2}, \tag{6.44}$$

where L^2_λ is the Hilbert space $L^2(]0,T[\times\Omega, d\mu)$ with the measure $d\mu := e^{-2\lambda t}dtdx$.

Using now the Cauchy-Schwarz inequality in the right hand side, and absorbing in the left hand side the term in $\|v\|^2_{L^2}$ yields the desired estimate for $m = 0$ and some constant $c_0 > 0$.

Step 2. We show the inequality by induction on m. Assume it for $m-1$. We apply a tangential operator $\mathcal{Z}^\alpha$ with fields $\mathcal{Z}_i \in \mathcal{T}$ to the system, and $|\alpha| = m$. The function $\mathcal{Z}^\alpha\mathbf{u}$ satisfies the same boundary conditions. The L^2 estimate (6.44) gives, for $\lambda \ge \lambda_0$:

$$\begin{aligned} &\varepsilon^2\|\nabla_x\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} + \lambda\|\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} \le 2\,|((e^{-\lambda t}\mathcal{Z}^\alpha f \\ &\quad + [(\tilde a^\varepsilon_{app} + \varepsilon\mathbf{b})\varepsilon^2\Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u}, \mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}| + I_{m,\lambda}(\mathbf{u})^2. \end{aligned} \tag{6.45}$$

where $[.,.]$ denotes the commutator. Using Cauchy-Schwarz inequality and $2ab \le 2\lambda^{-1}a^2 + \lambda b^2/2$ yields:

$$\begin{aligned} \varepsilon^2\|\nabla_x\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} + \frac{\lambda}{2}\|\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} \ \le\ & \frac{2}{\lambda}\ \|e^{-\lambda t}\mathcal{Z}^\alpha f\|^2_{L^2} \\ & + 2\ |((([(\tilde a^\varepsilon_{app} + \varepsilon\mathbf{b})\varepsilon^2\Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u}, \mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}| + I_{m,\lambda}(\mathbf{u})^2. \end{aligned} \tag{6.46}$$

We need to control the second term in the right hand side of (6.46). The commutator $[\ \tilde a^\varepsilon_{app}\,\varepsilon^2\Delta_x, \mathcal{Z}^\alpha]$ writes as a finite sum

$$\varepsilon^2 \sum_{|\beta|\le m+1} a^\varepsilon_\beta(t,x)\mathcal{Z}^\beta + \varepsilon\sum_{|\gamma|\le m} b^\varepsilon_\gamma(t,x)\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^\gamma + \sum_{|\delta|\le m-1} c^\varepsilon_\delta(t,x)(\varepsilon\partial_{\mathbf{n}})^2\mathcal{Z}^\delta \tag{6.47}$$

where the coefficients a^ε_β , b^ε_γ, c^ε_δ are bounded functions satisfying

$$\sup_{\varepsilon\in]0,1]} \|\varepsilon\partial_{\mathbf{n}} a^\varepsilon_\beta\|_{L^\infty([0,T]\times\Omega)} + \|\varepsilon\partial_{\mathbf{n}} b^\varepsilon_\gamma\|_{L^\infty([0,T]\times\Omega)} + \|\varepsilon\partial_{\mathbf{n}} c^\varepsilon_\delta\|_{L^\infty([0,T]\times\Omega)} < \infty \tag{6.48}$$

for all β,γ,δ, because (6.48) holds clearly if we replace $L^\infty([0,T]\times\Omega)$ by $L^\infty([0,T]\times\Omega_+)$ or by $L^\infty([0,T]\times\Omega_-)$, and because $\tilde{a}^\varepsilon_{app}$ is in $H^1(\Omega)$ for all $\varepsilon>0$. Hence we are led to control the corresponding three sort of terms:

$$\varepsilon^2((\, a^\varepsilon_\beta \mathcal{Z}^\beta \mathbf{u}\,,\, \mathcal{Z}^\alpha \mathbf{u}\,))_{L^2_\lambda},\ \varepsilon((\, b^\varepsilon_\gamma(\varepsilon\partial_\mathbf{n})\mathcal{Z}^\gamma \mathbf{u}\,,\, \mathcal{Z}^\alpha \mathbf{u}\,))_{L^2_\lambda},$$

$$((\, c^\varepsilon_\delta(\varepsilon\partial_\mathbf{n})^2\mathcal{Z}^\delta \mathbf{u}\,,\, \mathcal{Z}^\alpha \mathbf{u}\,))_{L^2_\lambda}, \tag{6.49}$$

where $|\beta|\leq m+1$, $|\gamma|\leq m$, $|\delta|\leq m-1$. The first two terms in (6.49) are simply controlled by $\delta\|\varepsilon\nabla_x\mathbf{u}\|^2_{m,\lambda}+C_\delta\,\delta^{-1}\,\|\mathbf{u}\|^2_{m,\lambda}$ for δ arbitrarily small, and C_δ being a constant depending on δ, but independent of ε. For the third term one uses an integration by parts (by Green's formula) of the field $\partial_\mathbf{n}$ to show that this term writes as a sum of terms of the form

$$d^\varepsilon\ \varepsilon^{2-j-j'}(((\varepsilon\partial_\mathbf{n})^j\mathcal{Z}^\delta\mathbf{u},(\varepsilon\partial_\mathbf{n})^{j'}\mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}$$

where $|\delta|\leq m-1$, $j,j'\in\{0,1\}$, and d^ε is a bounded function (uniformly in ε) since all the boundary terms terms vanishes: $\partial_\mathbf{n}\mathcal{Z}^\alpha\mathbf{u}|_{\partial\Omega}=0$, for all $\alpha\in\mathbb{R}^\mu$. It follows that the third term in (6.49) is controlled by $C\lambda^{-1}\|\varepsilon\nabla_x\mathbf{u}\|^2_{m,\lambda}+C\|\mathbf{u}\|^2_{m,\lambda}$ for a constant C independent of ε, and all $\lambda\geq 1$. Hence, by choosing a $\delta>0$ arbitrarily small, and $\lambda_1>0$ large enough, there holds

$$|\ ((\ [\tilde{a}^\varepsilon_{app}\varepsilon^2\Delta_x,\mathcal{Z}^\alpha]\wedge\mathbf{u}\,,\,\mathcal{Z}^\alpha\mathbf{u}\,))_{L^2_\lambda}\ |\ \leq\ \delta\|\varepsilon\nabla_x\mathbf{u}\|^2_{m,\lambda}+c_m\|\mathbf{u}\|^2_{m,\lambda}$$

for all $\lambda\geq\lambda_1$, and for all $\varepsilon\in]0,1]$, with a constant c_m independent of ε.

We need now to estimate the term

$$(([\ \varepsilon\mathbf{b}\,\varepsilon^2\Delta_x,\mathcal{Z}^\alpha]\wedge\mathbf{u},\mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}. \tag{6.50}$$

The commutator $[\ \mathbf{b}\,\varepsilon^2\Delta_x,\mathcal{Z}^\alpha]$ writes as a finite sum

$$\begin{aligned}&\varepsilon^2\sum_{|\beta|\leq m,|\beta'|\leq m+1,|\beta|+|\beta'|\leq m+2} a_{\beta,\beta'}(\mathcal{Z}^\beta\mathbf{b})\mathcal{Z}^{\beta'}\\ &+\varepsilon\sum_{|\gamma|\leq m,|\gamma'|\leq m,|\gamma|+|\gamma'|\leq m+1} b_{\gamma,\gamma'}(\mathcal{Z}^\gamma\mathbf{b})(\varepsilon\partial_\mathbf{n})\mathcal{Z}^{\gamma'}\\ &+\sum_{|\delta|\leq m,|\delta'|\leq m-1,|\delta|+|\delta'|\leq m} c_{\delta,\delta'}(\mathcal{Z}^\delta\mathbf{b})(\varepsilon\partial_\mathbf{n})^2\mathcal{Z}^{\delta'}\end{aligned}$$

where $a_{\beta,\beta'},b_{\gamma,\gamma'},c_{\delta,\delta'}$ are smooth fonctions on $\overline{\Omega}$. Hence to control the term (6.50) we are led to estimate tri-linear terms in $(\mathbf{b},\mathbf{u},\mathbf{u})$ of the following form (where $d\mu:=e^{-2\lambda t}dtdx$):

$$\varepsilon^2\int_{]0,T[\times\Omega} a_{\beta,\beta'}\mathcal{Z}^\beta\mathbf{b}.\mathcal{Z}^{\beta'}\mathbf{u}_i.\mathcal{Z}^\alpha\mathbf{u}_j\ d\mu,$$

$$|\beta|\leq m,|\beta'|\leq m+1,|\beta|+|\beta'|\leq m+2 \tag{6.51}$$

$$\varepsilon\int_{]0,T[\times\Omega} b_{\gamma,\gamma'}\ \mathcal{Z}^\gamma\mathbf{b}\ .\ \varepsilon\partial_\mathbf{n}\mathcal{Z}^{\gamma'}\mathbf{u}_i\ .\ \mathcal{Z}^\alpha\mathbf{u}_j\ d\mu,$$

$$|\gamma|\leq m,|\gamma'|\leq m,|\gamma|+|\gamma'|\leq m+1 \tag{6.52}$$

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'}\ \mathcal{Z}^\delta\mathbf{b}.\,(\varepsilon\partial_\mathbf{n})^2\mathcal{Z}^{\delta'}\mathbf{u}_i\,.\mathcal{Z}^\alpha\mathbf{u}_j\ d\mu,$$

$$|\delta|\leq m,|\delta'|\leq m-1,|\delta|+|\delta'|\leq m, \tag{6.53}$$

where the $\mathbf{u}_i$ are the components of the vector $\mathbf{u}$. Let us treat the term (6.53). By Green's formula, the integral can be written as a sum of integrals of the form

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'}\ \mathcal{Z}^{\delta}\varepsilon\partial_{\mathbf{n}}\mathbf{b}.\mathcal{Z}^{\delta'}\varepsilon\partial_{\mathbf{n}}\mathbf{u}_i.\mathcal{Z}^{\alpha}\mathbf{u}_j\ d\mu \tag{6.54}$$

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'}\ \mathcal{Z}^{\delta}\mathbf{b}.\,\mathcal{Z}^{\delta'}\varepsilon\partial_{\mathbf{n}}\mathbf{u}_i.\,\mathcal{Z}^{\alpha}\varepsilon\partial_{\mathbf{n}}\mathbf{u}_j\ d\mu, \tag{6.55}$$

$$\varepsilon\int_{]0,T[\times\Omega} d_{\delta,\delta'}\ \mathcal{Z}^{\delta}\mathbf{b}.\,\mathcal{Z}^{\delta'}\varepsilon\partial_{\mathbf{n}}\mathbf{u}_i\,.\mathcal{Z}^{\alpha}\mathbf{u}_j\ d\mu, \tag{6.56}$$

and other terms involving lower order derivatives easy to control. The term (6.54) is controlled by

$$c\|\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^{\delta}\mathbf{b}\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^{\delta'}\mathbf{u}_i\|_{0,\lambda}\ \|\mathbf{u}_j\|_{m,\lambda},$$

which is bounded by using the Gagliargo-Nirenberg-Moser estimate by

$$c\big(\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_\infty+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{b}|_\infty\ \big)\ \|\mathbf{u}\|_{m,\lambda}$$

and hence by

$$c(1+R)\big(\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_\infty+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}\big)\ \|\mathbf{u}\|_{m,\lambda}.$$

For the term (6.55) there are two cases. The first case is when $\delta=0$. In that case the integral is bounded by

$$c\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{u}_i\|_{m-1,\lambda}\ \|\mathbf{u}_j\|_{m,\lambda}\ \le\ \lambda^{-1}\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|^2_{m,\lambda}.$$

The second case is when $|\delta|\ge 1$. In that case we write $\mathcal{Z}^{\delta}\mathbf{b}=\mathcal{Z}^{\delta''}\mathcal{Z}_k\mathbf{b}$ and apply the Gagliardo-Nirenberg-Moser inequality with $\mathcal{Z}\mathbf{b}$ in L^∞. The term in bounded by

$$c\big(\ \|\mathcal{Z}\mathbf{b}\|_{m-1,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_\infty+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m-1,\lambda}|\mathcal{Z}\mathbf{b}|_\infty\ \big)\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}$$

and hence by

$$c\ \|\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_\infty\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}+cR\lambda^{-1}\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|^2_{m,\lambda}$$

The next terms like (6.56) are easier to treat in the same way, and are bounded by the same terms. The term (6.53) was the more delicate to estimate. The terms (6.52) and (6.51) are simpler and can be treated in a similar way. Replacing in the right hand side of (6.45) and summing over all the possible operators $\mathcal{Z}^\alpha$ gives the desired estimate, and the proposition is proved. □

6.3.4. *Iteration.* Now classical arguments show the convergence of the iterative scheme if $\varepsilon\in]0,\varepsilon_0]$ and ε_0 is small enough. We describe the main lines (see [**31**]). Let us fix an integer $m>4$, and let us denote

$$R:=1+\sup_{0<\varepsilon<1}\{\varepsilon\,(\ |w^{\varepsilon,0}|_\infty+\sum_{0\le j\le\mu}|\mathcal{Z}_j w^{\varepsilon,0}|_\infty+|\varepsilon\partial_x w^{\varepsilon,0}|_\infty\ )\}.$$

PROPOSITION 6.3. *Let be given an integer $m\ge 5$ and $\lambda>1$. Then there exists $h>1$ such that for $\varepsilon_0>0$ small enough, for all $\nu\in\mathbb{N}$, for all $\varepsilon\in]0,\varepsilon_0]$, there hold*

$$|w^{\varepsilon,\nu}|_\infty+\sum_{0\le j\le\mu}|\mathcal{Z}_j w^{\varepsilon,\nu}|_\infty+|\varepsilon\partial_x w^{\varepsilon,\nu}|_\infty<R\varepsilon^{-1} \tag{6.57}$$

and

$$\|w^{\varepsilon,\nu}\|_{m,\lambda}+\|\varepsilon\partial_{\mathbf{n}}w^{\varepsilon,\nu}\|_{m,\lambda}<h. \tag{6.58}$$

Proof. For h large enough, the inequalities (6.57) and (6.58) are satisfied for $\nu = 0$. Now suppose that $w^{\varepsilon,\nu}$ satisfies (6.57), (6.58). We want to prove that $w^{\varepsilon,\nu+1}$ also satisfies (6.57), (6.58). The proposition 6.2 gives a constant $C_m(R)$ and the inequality (6.38) holds with $\mathbf{u} = w^{\varepsilon,\nu+1}$, $\mathbf{b} = w^{\varepsilon,\nu}$, and $f = f^{\varepsilon,\nu}$ defined in (6.31). In order to control the right hand side of (6.30), we need a control of $\|\mathcal{H}(w^{\varepsilon,\nu})\|_\infty$ and of $\|\mathcal{H}(w^{\varepsilon,\nu})\|_{m,\lambda}$, which is a consequence of the following lemma.

Lemma 6.3. *Let $m \in \mathbb{N}$. There exists $c > 0$ such that for all $\lambda \geq 1$,*

$$\|\mathcal{H}(v)\|_{m,\lambda} + \|\varepsilon\partial_{\mathbf{n}}\mathcal{H}(v)\|_{m-1,\lambda} \leq c(\|v\|_{m,\lambda} + \|\varepsilon\partial_{\mathbf{n}}v\|_{m-1,\lambda}). \tag{6.59}$$

Proof. We denote $E(\partial) := (\text{div}\ , \text{curl}\ )$ the operator from $[\mathcal{S}'(\mathbb{R}^3)]^3$ to $[\mathcal{S}'(\mathbb{R}^3)]^4$. The range of $E(\partial)$ is the space $\mathcal{R}$ of all $f = (a, b) \in \mathcal{S}'(\mathbb{R}^3) \times [\mathcal{S}'(\mathbb{R}^3)]^3$ such that div $b = 0$. We denote by $E^{-1}(\partial)$ the inverse operator from $\mathcal{R}$ to $[\mathcal{S}'(\mathbb{R}^3)]^3$. Then $u = E^{-1}(\partial)f$, is defined by $\hat{u}(\xi) = -i|\xi|^{-2}\big(\hat{a}(\xi)\xi - \xi \wedge \hat{b}(\xi)\big)$ where $\hat{f}(\xi) = (\hat{a}(\xi), \hat{b}(\xi)) \in \mathbb{R} \times \mathbb{R}^3$. We also extend the action of this operator to the whole space $[\mathcal{S}'(\mathbb{R}^3)]^4$ by using the same formula, and we still denote by $E^{-1}(\partial)$ this extension. Thus $\hat{u}(\xi) = M(\xi)\hat{f}(\xi)$, where $M(\xi)$ is a 3×4 matrix whose entries are rational functions of ξ homogeneous of degree -1.

Let us fix $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\chi(\xi) = 0$ when $|\xi| \leq 1$ and $\chi(\xi) = 1$ when $|\xi| \geq 2$, and call $P(D)$ and $R(D)$ the operators from $[\mathcal{S}'(\mathbb{R}^3)]^4$ to $[\mathcal{S}'(\mathbb{R}^3)]^3$ defined by $P(D)f := \mathcal{F}^{-1}\big(\chi M\hat{f}\,\big)$ and $R(D)f := \mathcal{F}^{-1}\big((1-\chi)\hat{f}\,\big)$ where $\mathcal{F}^{-1}$ means the inverse Fourier transform. In the sequel we will simply denote $\mathcal{S}'(\mathbb{R}^3)$ and $L^2(\tilde{\Omega})$ instead of $\big[\mathcal{S}'(\mathbb{R}^3)\big]^4$ and $\big[L^2(\tilde{\Omega})\big]^4$, meaning that we talk about the *components* of the vector valued functions, the (finite) number of components being understood. We have $E^{-1}(\partial) = P(D) + R(D)$. The operator $P(D)$ is a special case of classical pseudo-differential operator of class $S^{-1}_{1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$, elliptic, and $R(D)$ is a smoothing operator of class $S^{-\infty}_{1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$. The operator $P(D)$ satisfies the *transmission property* (introduced by Boutet de Monvel [**2**], [**3**]) on Ω and on $\mathbb{R}^3 \setminus \Omega$ because its symbol is a rational function of ξ, which is a sufficient condition to satisfy the transmission condition. The transmission property has been also studied and used by Grubb, and we also refer to papers [**11**] and [**12**]. To avoid many repetitions, we will denote in what follows $\Omega_1 := \Omega$ and $\Omega_2 = \mathbb{R}^3 \setminus \Omega$. Since $P(D)$ is elliptic of order 1, the transmission property implies (see [**11**] and [**12**]) that if $v \in H^s(\Omega)$ then for $j = 1, 2$, $\big(P(D)\overline{v}\big)_{|\Omega_j} \in H^{s+1}(\Omega_j)$.

Let us now take into account the t coordinate. Let us denote $\tilde{\Omega} =]0, T[\times\Omega$, $\tilde{\Gamma} =]0, T[\times\Gamma$ and $\tilde{\Sigma} =]0, T[\times\Sigma$. We extend the actions of P and R to the spaces of functions or distributions which depend also on t like $L^2(\tilde{\Omega})$ or $C\big([0, T], \mathcal{S}'(\mathbb{R}^3)\big)$, by considering t as a parameter so that $Pu(t, x) := P(D)u(t, .)(x)$.

Let $v \in H^m_{co}(\tilde{\Omega}; \mathbb{R}^4)$ such that $\partial_{\mathbf{n}}v \in H^{m-1}_{co}(\tilde{\Omega}; \mathbb{R}^4)$. Then $v \in H^1(\tilde{\Omega})$, and using local coordinates patches one sees that $g := (v.\mathbf{n})_{|\tilde{\Gamma}} \in H^{m-\frac{1}{2}}(\tilde{\Gamma})$, the usual Sobolev spaces. Let us denote $f := \overline{E(\partial)v}$, which is in $H^{m-1}_{co}(]0, T[\times\mathbb{R}^3)$. Then $\mathcal{H}(v) = u_{|\tilde{\Omega}}$ where $u \in L^2([0, T] \times \mathbb{R}^3)$ is defined by $E(\partial)u = f + (g \otimes \delta_{\tilde{\Gamma}}, 0)$, where the notation $\overline{V}$ means the extension of V by 0 to $[0, T] \times \mathbb{R}^3$.

Let us denote $u_{(j)} = u_{|\tilde{\Omega}_j}$, for $j = 1, 2$, so that $\mathcal{H}(v) = u_{(1)} \in L^2(\tilde{\Omega})$. Using the notations of [**2**], [**11**], [**12**],

$$u_{(j)} = \big(E^{-1}(D)\overline{v}\big)_{|\tilde{\Omega}_{(j)}} = P(D)^{(j)}f + K^{(j)}_{\Gamma}(g) + R(D)^{(j)}E(\partial)\overline{v}, \tag{6.60}$$

where $P(D)^{(j)}f = (P(D)f)_{|\tilde\Omega_{(j)}}$, $R(D)^{(j)}\overline{v} = (R(D)\overline{v})_{|\tilde\Omega_{(j)}}$ and where $K_\Gamma^{(j)}(g) = \big(P(D)\big(g\otimes\delta_\Gamma\big)\big)_{|\tilde\Omega_{(j)}}$ is the "Poisson operator":

$$K_\Gamma^{(j)} : H^s(\Gamma) \to H^{s+1/2}(\Omega^{(j)}), \tag{6.61}$$

(linear continuous), extended to functions depending on t as a parameter (See theorems 2.4 and 2.5 of [**12**]).

Let us now prove the lemma. First of all, $\partial_t^m \mathcal{H}(v) = \mathcal{H}(\partial_t^m v)$ is in $L^2(\tilde\Omega)$ because $\partial_t^m v \in L^2(\tilde\Omega)$ and $\mathcal{H}$ acts on $L^2(\tilde\Omega)$. It is also easy to show that $\partial_t^{m-1}\mathcal{H}(v) \in H^1(\tilde\Omega)$: by assumption, for any $t\in[0,T]$, $\partial_t^{m-1}v(t,.) \in H^1(\Omega)$, hence $\mathcal{H}(\partial_t^{m-1}\overline{v})(t,.) \in H^1(\Omega)$ because $\partial_t^{m-1}\overline{v}(t,.)$ is piecewise-H^1 and because of the properties of $\mathcal{H}$. Hence $\partial_x\partial_t^{m-1}\mathcal{H}(v) \in L^2(\tilde\Omega)$ and since we already know that $\partial_t^m\mathcal{H}(v) \in L^2(\tilde\Omega)$ we have proved that $\partial_t^{m-1}\mathcal{H}(v) \in H^1(\tilde\Omega)$.

Let us show now that $\mathcal{Z}_j\partial_t^{m-2}\mathcal{H}(v) \in H^1(\tilde\Omega)$ for $j = 1,\dots,\mu$. Since $E(\partial)$ is elliptic (as an operator in $\mathcal{S}'(\mathbb{R}^3)$, but not in $\mathcal{S}'(\mathbb{R}^4)$), we can express the normal derivatives of u in term of tangential derivatives and of $E(\partial)u$, and this implies that the commutator $[E(\partial),\mathcal{Z}_j]u$ writes

$$[E(\partial),\mathcal{Z}_j]u = \sum_1^\mu A_j\mathcal{Z}_j u + A_0 f + Bg\otimes\delta_\Gamma \tag{6.62}$$

where A_j, B are matrices with $\mathcal{C}_b^\infty$ entries (depending on the fields $\mathcal{Z}_j$). It follows that

$$E(\partial)\mathcal{Z}_j u = \sum_{|\alpha|\le 1} M_\alpha \mathcal{Z}^\alpha f + \sum_{|\alpha|\le 1} N_\alpha(\mathcal{Z}^\alpha g)\otimes\delta_\Gamma$$

with $\mathcal{C}_b^\infty(\mathbb{R}^3)$ matrices M_α, N_α, and applying ∂_t^{m-2} gives:

$$E(\partial)\mathcal{Z}_j\partial_t^{m-2}u = \sum_{|\alpha|\le 1} M_\alpha.\mathcal{Z}^\alpha\partial_t^{m-2}f + \sum_{|\alpha|\le 1} N_\alpha.(\mathcal{Z}^\alpha\partial_t^{m-2}g)\otimes\delta_\Gamma \tag{6.63}$$

Now $\mathcal{Z}^\alpha\partial_t^{m-2}f \in L^2(\tilde\Omega)$, because $f = \overline{E(\partial)v}$, and the transmission property implies that for every $t\in[0,T]$, the function $P(D)^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}f\big)(t,.)$ is in $H^1\big(\Omega_{(j)}\big)$. This implies that $\partial_x P(D)^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}f\big) \in L^2(\tilde\Omega)$ and since we already know that $\partial_t P(D)^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}f\big) \in L^2(\tilde\Omega)$ from the previous case, we deduce that for $j = 1, 2$ the function $P(D)^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}f\big)$ is in $H^1\big(\tilde\Omega_{(j)}\big)$.

Concerning the boundary term in (6.63), since $g \in H^{m-\frac12}(\tilde\Gamma)$ we know that $\mathcal{Z}^\alpha\partial_t^{m-2}g \in H^{1/2}(\Gamma)$ and the property (6.61) implies that, for all $t\in[0,T]$, the function $K_\Gamma^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}g\big)(t,.)$ is in $H^1\big(\Omega_{(j)}\big)$. By the same way as before we deduce that for $j = 1, 2$ the function $K_\Gamma^{(j)}\big(\mathcal{Z}^\alpha\partial_t^{m-2}g\big)$ is in $H^1\big(\tilde\Omega_{(j)}\big)$.

Now, applying $E(\partial)^{-1} = P(D)+R(D)$ to the equation (6.63) gives $\mathcal{Z}_j\partial_t^{m-2}u_{(j)} \in H^1\big(\tilde\Omega_{(j)}\big)$ as claimed. Then, the proof can be continued by induction in the same way. □

The lemma 6.3, together with the Gagliardo-Nirenberg-Moser estimates and the induction assumption, implies that (like the majoration of the term (5.25) in paper [**31**]):

$$\|f^{\varepsilon,\nu}\|_{m,\lambda} \le c(R)(\|w^{\varepsilon,\nu}\|_{m,\lambda} + \|\varepsilon\partial_x w^{\varepsilon,\nu}\|_{m,\lambda}) < c(R)\rho(\lambda). \tag{6.64}$$

Hence, the proposition 6.2 implies that

$$\|\varepsilon\partial_x w^{\varepsilon,\nu+1}\|_{m,\lambda} + \lambda\|w^{\varepsilon,\nu+1}\|_{m,\lambda} \leq C_m(R)\big[\lambda^{-1}c(R)\rho(\lambda) + R(|w^{\varepsilon,\nu+1}|_\infty + |\varepsilon\partial_x w^{\varepsilon,\nu+1}|_\infty) + I_{m,\lambda}(w^{\varepsilon,\lambda})\big]. \tag{6.65}$$

We now use the following Sobolev inequalities ([**31**]):

$$\begin{aligned} \varepsilon^{1/2}|\mathbf{u}|_\infty &\leq e^{\sigma\lambda}(\|\mathbf{u}\|_{m,\lambda} + \|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}),\\ \varepsilon^{1/2}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_\infty &\leq e^{\sigma\lambda}(\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda} + \|(\varepsilon\partial_{\mathbf{n}})^2\mathbf{u}\|_{m,\lambda}). \end{aligned}$$

By taking λ large enough and $\varepsilon > 0$ small enough the inequality (6.57) is also satisfied for $w^{\varepsilon,\nu+1}$ and the proof by induction is complete. □

Now by extracting a convergent subsequence it is a classical argument to show the convergence in $L^2(]0,T[\times\Omega)$ of $w^{\varepsilon,\nu}$ to a solution w^ε of the non linear problem which satisfies the same estimates (6.57), (6.58). This concludes the proof of Theorem 4.2. □

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