Motion of a particle immersed in a two dimensional incompressible perfect fluid and point vortex dynamics

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Abstract. In these notes, we expose some recent works by the author in collaboration with Olivier Glass, Christophe Lacave and Alexandre Munnier, establishing point vortex dynamics as zero-radius limits of motions of a rigid body immersed in a two dimensional incompressible perfect fluid in several inertia regimes.

1. Introduction

The aim of these notes is to present some recent and forthcoming works, in particular [4, 5, 7] and [6], obtained by the author in collaboration with Olivier Glass, Christophe Lacave and Alexandre Munnier establishing point vortex dynamics as zero-radius limits of motions of a rigid body immersed in a two dimensional incompressible perfect fluids in several inertia regimes.

The rigid body is assumed to be only accelerated by the force exerted by the fluid pressure on its boundary according to the Newton equations, the fluid velocity and pressure being given by the incompressible Euler equations. The equations at stake then read:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{and} \quad \operatorname{div} u = 0, \tag{1}$$

$$mh''(t) = \int_{\partial \mathcal{S}(t)} \pi n \, ds \quad \text{and} \quad \mathcal{J} r'(t) = \int_{\partial \mathcal{S}(t)} (x - h(t))^{\perp} \cdot \pi n \, ds. \tag{2}$$

Here

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- $u = (u_1, u_2)$ and π respectively denote the velocity and pressure fields (the fluid is supposed to be homogeneous, of density 1 in order to simplify the notations),
- $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the momentum of inertia of the body,
- $h'(t)$ is the velocity of the center of mass $h(t)$ of the body and $r(t)$ is the angular velocity. The body rigidly moves so that at times t it occupies a domain $\mathcal{S}(t)$ which is isometric to its initial position $\mathcal{S}_0$ which is supposed to be a simply connected smooth compact subset of $\mathbb{R}^2$. Indeed, there exists a rotation matrix
$$R(\theta(t)) := \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \tag{3}$$
such that
$$\mathcal{S}(t) = \{h(t) + R(\theta(t))x, \ x \in \mathcal{S}_0\}.$$
Furthermore, the angle satisfies $\theta'(t) = r(t)$.
- when $x = (x_1, x_2)$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)$,
- n denotes the unit normal vector pointing outside the fluid domain, which of course depends on the solid position.

We assume that the boundary of the solid is impermeable so that the fluid cannot penetrate into the solid and we assume that there is no cavitation as well. The natural boundary condition at the fluid-solid interface is therefore
$$u \cdot n = \Big(h'(t) + r(t)(x - h(t))^\perp\Big) \cdot n \quad \text{for} \quad x \in \partial\mathcal{S}(t). \tag{4}$$

Let us emphasize that this condition extends the usual condition $u \cdot n = 0$ on a fixed boundary and involves only the normal part of the fluid velocity. As usual with perfect fluid no pointwise boundary condition needs to be prescribed for the tangential part of the fluid velocity. However because the domain occupied by the solid is a hole in the fluid domain there is a global condition on the tangential part of the fluid velocity involving the circulation γ defined as
$$\gamma = \int_{\partial\mathcal{S}_0} u \cdot \tau \, ds$$
where u is the fluid velocity and τ the the unit counterclockwise tangential vector so that $n = \tau^\perp$. Indeed when considering a fluid velocity u which has a good enough regularity, what we will always do in these notes, the so-called Kelvin's theorem applies and γ is preserved over time. The circulation somehow encodes the amount of vorticity hidden in the particle from the fluid viewpoint. Indeed by Green Theorem the circulation can be recast as the integral over $\mathcal{S}_0$ of the vorticity $\operatorname{curl} \overline{u} = \partial_1 \overline{u}_2 - \partial_2 \overline{u}_1$ of any smooth vector field $\overline{u}$ in $\mathcal{S}_0$ such that $\overline{u} \cdot \tau = u \cdot \tau$ on $\partial\mathcal{S}_0$. Therefore the limit where the body radius converges to zero corresponds to a singular perturbation problem

(in space) for the fluid velocity when this latter has to accommodate with a condition of nonzero γ around a shrinking solid. Indeed it is well understood since the works [15] and [23] that for a solid obstacle held fixed in a perfect incompressible fluid, with a nonzero given circulation and with possibly nonzero vorticity in the fluid, in the limit where the obstacle shrinks into a fixed pointwise particle, the Euler equation driving the fluid evolution has to be modified: in the Biot-Savart law providing the fluid velocity generated by the fluid vorticity, a Dirac mass at the fixed position of the pointwise obstacle with an amplitude equal to the circulation has to be added to the fluid vorticity. We will refer to the background fluid velocity in the sequel and we will denote it by u_{bd}. The genuine fluid vorticity ω (that is without the Dirac mass) is convected by the background fluid velocity u_{bd}. In the case of a moving body one may wonder if the divergence of the fluid velocity has to be modified in the zero-radius limit as well in order to accommodate with the non-homogeneous condition (4). In the sequel we will consider some cases where the solid radius ε shrinks to 0 with $(h', \varepsilon\theta')$ bounded so that the limit fluid velocity is divergence free including in the region where the solid has disappeared.[1] Still the analysis of the dynamics of immersed rigid particles requires a more precise analysis, in particular because it is driven by the fluid pressure, a quantity which depends in a non linear and non local way on the fluid velocity. Hence to understand the limit dynamics one has to precisely evaluate the pressure field on the boundary of the solid, that is, where the singularity is concentrated.

One main goal of this line of research is prove another derivation of the point vortex dynamics as motions of immersed particles. In particular we will consider, in some appropriate settings, the limit of the dynamics of an immersed rigid body when its size and its mass go to zero and recover the equation of a point vortex. Let us recall that the point vortex system is a classical model which goes back to Helmholtz [12], Kirchhoff [17], Poincaré [28], Routh [29], Kelvin [16], and Lin [21, 22]. In these works it was thought as an idealized fluid model where the vorticity of an ideal incompressible two-dimensional fluid is discrete. Although it does not constitute a solution of the incompressible Euler equations, even in the sense of distributions, point vortices can be viewed as Dirac masses obtained as limits of concentrated smooth vortices which evolve according to the Euler equations. In particular in the case of a single vortex moving in a bounded and simply-connected domain this was proved by Turkington in [32] and an extension to the case

[1] Let us observe that since

$$\int_{\partial \mathcal{S}(t)} \left(h'(t) + r(t)(x - h(t))^{\perp} \right) \cdot n \, ds = 0,$$

see (24) below, should the behaviour of the solid velocity with ε be worse the resulting singularity would be rather a dipole than a Dirac mass.

of several vortices, including in the case where there is also a part of the vorticity which is absolutely continuous with respect to Lebesgue measure (the so-called wave-vortex system), was given by Marchioro and Pulvirenti, see [25]. Let us also mention that Gallay has recently proven in [3] that the point vortex system can also be obtained as vanishing viscosity limits of concentrated smooth vortices evolving according to the incompressible Navier-Stokes equations. Our main goal is to prove that this classical point vortex motion can also be viewed as the limit of the dynamics of a solid, shrinking into a pointwise massless particle with fixed circulation, in free motion. Indeed, our analysis also covers the case where the mass is kept fixed positive in the limit, one then obtains a second-order ordinary differential equation for the particle's position, that we will refer to as a "massive" point vortex system.

Let us precise more the two kinds of inertia regimes we are going to consider in the small radius limit with the following definitions.

Definition 1.1 (Massive and massless particles). We define

- a massive particle as the limit of a rigid body when its radius ε goes to 0 with its mass m^ε and its momentum of inertia $\mathcal{J}^\varepsilon$ respectively satisfying $m^\varepsilon = m$ and $\mathcal{J}^\varepsilon = \varepsilon^2 \mathcal{J}$,
- a massless particle as the limit of a rigid body when its radius ε goes to 0 with its mass m^ε and its momentum of inertia $\mathcal{J}^\varepsilon$ respectively satisfying $m^\varepsilon = \varepsilon^\alpha m$ and $\mathcal{J}^\varepsilon = \varepsilon^{\alpha+2} \mathcal{J}$,

where $\alpha > 0$, $m > 0$ and $\mathcal{J} > 0$ are fixed independent of ε.

Five remarks are in order:

- it is understood that we consider a self-similar shrinking of the rigid body into its center of mass. Choosing the origin 0 of the frame as the center of mass of $\mathcal{S}_0$ it means that we will as initial domain, for every $\varepsilon \in (0, 1]$,
$$\mathcal{S}_0^\varepsilon := \varepsilon \mathcal{S}_0, \tag{5}$$
- one observes, of course, that the case of a massive particle corresponds to the limit case $\alpha = 0$,
- the scaling of momentum of inertia $\mathcal{J}^\varepsilon$ may look surprising at first sight, but it is quite natural since it corresponds to a second order moment whereas the mass is a zeroth order moment of the body's density,
- it is understood that the circulation γ^ε around the body satisfies $\gamma^\varepsilon = \gamma$, where γ is fixed. The amount of circulation is therefore supposed to be independent of the size of the body in our problem. Moreover we assume that $\gamma \neq 0$ in the case of a massless particle.
- the case where $\mathcal{S}_0$ is a homogeneous disk is the most simple whereas the case where $\mathcal{S}_0$ is a non-homogeneous disk involves some adapted tools in particular in order to deal with the case where $\alpha \geq 2$. We refer to [7]

for a detailed treatment of these cases and we will consider only here the case where $\mathcal{S}_0$ is not a disk.

Let us have a deeper look at Newton's equations (2) and anticipate that in the case of a massless particle the prefactor m and $\mathcal{J}$ in front of the second-order time derivative converge to zero in the zero radius limit so that one faces a singular perturbation problem in time of a non linear dynamics. We will make use of geodesic and gyroscopic features of the system in order to overcome this difficulty. Let us mention from now on that the use of the geodesic structure is more subtle that one may expect at first glance. Indeed the full system "fluid + rigid body" is conservative and enjoy a geodesic structure as a whole in the sense that if on a time interval $(0, T)$ the initial and final configurations are prescribed, then the PDE's system "fluid + rigid body" is satisfied on $(0, T)$ if and only if the couple of flow maps associated with the fluid and solid velocities is a critical point of the action obtained by time integration of the total kinetic energy, cf [8]. This gives some credit to the belief that the energy conservation drives the dynamics of the system, still some important transfers of energy from one phase to another may occur and this lead to a lack of bound of the solid velocity in the case of a light body. Since the fluid velocity corresponding to a point vortex is not square integrable a renormalization of the energy is necessary in the zero radius limit. Indeed one main feature of the point vortex equation is that the self-induced velocity of the vortex is discarded, or more precisely the self-induced velocity as if the point vortex was alone in the plane. Let us therefore introduce $K_{\mathbb{R}^2}[\cdot]$ the Biot-Savart law in the full plane that is the operator which maps any reasonable scalar function ω to the unique vector field $K_{\mathbb{R}^2}[\omega]$ vanishing at infinity and satisfying $\operatorname{div} K_{\mathbb{R}^2}[\omega] = 0$ and $\operatorname{curl} K_{\mathbb{R}^2}[\omega] = \omega$ in $\mathbb{R}^2$, which is given by the convolution formula

$$K_{\mathbb{R}^2}[\omega] := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t, y)\, dy. \tag{6}$$

We believe that the following statement is true in a very general setting.

Conjecture ($\mathcal{C}$). A massive particle immersed in a two dimensional incompressible perfect fluid moves according to Newton's law with a gyroscopic force orthogonally proportional to its relative velocity and proportional to the circulation around the body. A massless particle immersed in a two dimensional incompressible perfect fluid with nonzero circulation moves as a point vortex, its vortex strength being given by the circulation. More precisely, the position $h(t)$ of a massive (respectively massless) particle satisfies the equation

$$m h'' = \gamma \big(h' - u_{\mathrm{d}}(h)\big)^\perp \quad (\text{resp. } h' = u_{\mathrm{d}}(h)), \tag{7}$$

with[2] $u_{\mathrm{d}}(h) = \big(u_{\mathrm{bd}} - K_{\mathbb{R}^2}[\gamma\delta_h]\big)(h)$ where u_{bd} denotes the background fluid velocity hinted above. On the other hand the genuine fluid vorticity ω is convected by the background fluid velocity u_{bd}.

Let us mention that in [2, Chapter 3] Friedrichs already evoked a similar conjecture with a massive point vortex system in the case of two point vortices in the whole plane under the terminology of *bound vortices* (as opposed to *free vortices*). The gyroscopic force appearing in the right hand side of the first equation in (7) is a generalisation of the Kutta-Joukowski lift force which attracted a huge interest at the beginning of the 20th century during the first mathematical investigations in aeronautics, and which will be recalled in Section 2.

Observe that the conjecture above does not mention any sensitivity to the body's shape. Indeed it is expected that a more accurate description of the asymptotic behaviour thanks to a multi-scale asymptotic expansion of the body's dynamics in the limit $\varepsilon \to 0$ will reveal an influence of the body's shape on some corrector terms which appears as subprincal in the limit $\varepsilon \to 0$ for coarse topologies. The case where the circulation is assumed to vanish with ε as $\varepsilon \to 0$ is another setting where such a dependence with respect to the body's shape should appear.

Indeed this conjecture has already been proved in a few cases and this is precisely the goal of these notes to give an account of these results.

In Section 2, we will start with a review of the case, well-known since more than one century, of the motion of one single rigid body immersed in an irrotational fluid filling the rest of the plane. In this setting the equations at stake are the incompressible Euler equations (13) on the fluid domain $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$, the Newton equations (14), the interface condition (4) and the following condition of decay at infinity: $\lim_{|x|\to\infty} |u(t,x)| = 0$. Regarding the initial conditions we observe that there is no loss of generality in assuming that the center of mass (respectively rotation angle) of the solid coincides at the initial time with the origin $(0,0)$ (resp. 0) and we therefore prescribe some initial position and velocity of the solid of the form $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$. On the other hand we prescribe the initial value of the velocity $u|_{t=0} = u_0$ in the initial domain $\mathcal{F}_0 = \mathcal{F}(0) = \mathbb{R}^2 \setminus \mathcal{S}_0$ occupied by the fluid. Of course since we aim at considering smooth solutions we assume that the fluid and solid initial data are compatible. We may therefore consider that the solid translation and rotation velocities ℓ_0 and r_0 are arbitrarily given

[2]The index "d" of u_{d} can be either interpreted as drift or desingularized as it is obtained from u_{bd} by removing its orthoradial singular part in h.

and that the fluid velocity u_0 is the unique vector field compatible in the sense of the following definition.

Definition 1.2 (Compatible initial fluid velocity). Given the initial domain $\mathcal{S}_0$ occupied by the body, ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \mathbb{R}^2 \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field satisfying the following div/curl type system:

$$\operatorname{div} u_0 = 0 \text{ and } \operatorname{curl} u_0 = 0 \text{ in } \mathcal{F}_0,$$
$$u_0 \cdot n = \left(\ell_0 + r_0 x^\perp\right) \cdot n \text{ for } x \in \partial\mathcal{S}_0, \quad \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma,$$
$$\lim_{x\to\infty} u_0 = 0.$$

Indeed the zero vorticity condition propagates from $t = 0$ according to Helmholtz's third theorem so that at any time $t > 0$ the fluid velocity $u(t, \cdot)$ can indeed be recovered from the solid's dynamics by an elliptic-type problem similar to the one given above for the initial data. Since time appears only as a parameter rather than in the differential operators, the fluid state may be seen as only solving an auxiliary steady problem rather than an evolution equation. The Newton equations can therefore be rephrased as a second-order differential equation whose coefficients are determined by the auxiliary fluid problem. In particular the prefactor of the translation and angular accelerations is the sum of the inertia of the solid and of the so-called "added inertia" which is a symmetric positive-semidefinite matrix depending only on the body's shape and which encodes the amount of incompressible fluid that the rigid body has also to accelerate around itself. Remarkably enough in the case where the circulation is 0 it turns out that the solid equations can be recast as a geodesic equation associated with the metric given by the total inertia. Unlike the geodesic structure of the full system "fluid + rigid body" hinted above, the configuration manifold only encodes here the solid's dynamics and is therefore of finite dimensions. This echoes that the equations of motion of point vortices embedded in incompressible flow are usually thought as a reduction of an infinite-dimensional dynamical system, namely the incompressible Euler equation, to a finite-dimensional system. Another celebrated feature of the body's dynamics is due to a gyroscopic force, proportional to the circulation around the body, known as the Kutta-Joukowski lift force. In order to make these features appear, cf. Theorem 2.2, and to make as explicit as possible the quantities genuinely involved in this ODE two approaches were followed in the literature: the first one dates back to Blasius, Kutta, Joukowski, Chaplygin and Sedov, cf. for instance [30], and relies on complex analysis whereas the second one is real-analytic and was initiated by Lamb, cf. [20]. We will report here these two methods.[3] A trivial consequence of this reformulation is that a global-in-time smooth solution to the Cauchy problem exists and is unique. Therefore in Section 2 (cf. below Proposition 2.4) we will prove the following classical result.

Theorem 1.3. *Given the initial domain $\mathcal{S}_0$ occupied by the body, the initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$, the circulation γ in $\mathbb{R}$, and u_0 the associated compatible initial fluid velocity (according to Definition 1.2), there exists a unique smooth global-in-time solution to the problem compound of the incompressible Euler equations* (1) *on the fluid domain, of the Newton equations* (2)*, of the interface condition* (4)*, of the condition at infinity: $\lim_{|x|\to\infty} |u(t,x)| = 0$, and of the initial conditions $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$ and $u|_{t=0} = u_0$.*

Moreover the structure of the reduced ODE hinted above allows to investigate the zero-radius limit quite easily and to obtain the following result.

Theorem 1.4. *Let be given a rescaled initial domain $\mathcal{S}_0$ occupied by the body, some initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$ and a circulation γ in $\mathbb{R}$ in the case of a massive particle and in $\mathbb{R}^*$ in the case of a massless particle, all of them independent of ε. Let, for each $\varepsilon > 0$, u_0^ε the associated compatible initial fluid velocity associated (according to Definition 1.2) with the initial solid domain $\mathcal{S}_0^\varepsilon$ defined by* (5)*, (ℓ_0, r_0) and γ; and consider the corresponding solution given by Theorem 1.3. Then in the zero radius limit $\varepsilon \to 0$, with the inertia scaling described in Definition 1.1, one respectively obtains for the position $h(t)$ of the pointwise limit particle the equations $mh'' = \gamma(h')^\perp$ in the massive limit and $h' = 0$ in the massless limit.*

Therefore, in this historical setting, Conjecture $(\mathcal{C})$ is validated with $u_{\mathrm{d}} = 0$. Of course Theorem 1.4 is a quite informal statement put here for

[3]On the one hand the presentation of the complex-analytic method is extracted from the use we made of it in our first investigations of the rotational case, cf. Section 4 which reports the results of [4, 5]. On the other hand the presentation of the real-analytic method is extracted from the use we made of it in our investigation of the case where fluid-solid system occupies a bounded domain, cf. Section 3 which reports the results of [7]. Arguably the length comparison and the temporary occurrence of Archimedes' type quantities (like the volume of the body, its geometric center..) in some intermediate computations leading to Lemma 2.13 (where they cancel out) emphasize the superiority of Lamb's method for our purposes. Indeed in our forthcoming paper [6] Lamb's approach is extended to tackle the general case where several bodies move in a bounded rotational perfect flow when some of the rigid bodies shrink to pointwise particles, some of them with constant mass, the others with vanishing mass. Still the complex-analytic method is known to be useful to deal with the case of a body whose boundary has singularities thanks to conformal mapping. It could be that it appears relevant as well to investigate the motion of a rigid curve resulting from an anisotropic shrinking. In this direction let us mention the paper [18] which deals with the influence of a fixed curve on the fluid around.

sake of exposition, we will provide a rigorous statement in Section 2, cf. Theorem 2.16.

In Section 3, we consider the case where the fluid-solid system occupies a bounded domain Ω, still in the irrotational case. We assume that Ω is a bounded open regular connected and simply connected domain Ω of $\mathbb{R}^2$ and that the center of mass of the solid coincides at the initial time with the origin 0 which is assumed to be in Ω.

Again the fluid velocity can be recovered from the solid's dynamics by an elliptic-type problem for which time is only a parameter and the Newton equations can therefore be rephrased as a second-order differential equation with geodesic and gyroscopic features involving some coefficients determined by this auxiliary fluid problem.

Still some extra difficulties show up in this process. In particular the way the fluid domain depends on the body motion is more intricate and so are the variations of the added inertia and therefore of the metric given by the total inertia. Indeed even in the case of zero circulation (i.e. when $\gamma = 0$) the reformulation of the system as an geodesic equation was proven only recently in [26]. The general case, with nonzero γ is obtained in [4]. One another main new feature with respect to the unbounded case is that the Kutta-Joukowski lift force is superseded by a more sophisticated force term which has the form of the Lorentz force in electromagnetism. Indeed the magnetic part of the Lorentz force, being gyroscopic and proportional to the circulation around the body, is a quite natural extension of the Kutta-Joukowski lift force of the unbounded case. Still it depends on the body position in a more subtle way. On the other hand the electric-type force which may seem very damaging in order to obtain uniform estimates in the zero-radius limit as it does not disappear in an energy estimate.

At least for fixed radius we will be able to infer straightforwardly from this reformulation the local-in-time existence and uniqueness of smooth solution to the Cauchy problem. Unlike the unbounded case of Section 2, cf. Theorem 1.3, the result is only local-in-time since collision of the body with the external may occur in finite time (at least as far as it concerns smooth solutions), see [13, 14] for some examples of collisions of a disk moving in a potential flow (that is in the case where the circulation γ satisfies $\gamma = 0$) with the fixed external boundary of the fluid domain. Indeed an energy argument, cf. Corollary 3.10, proves that the life-time of such a smooth solution can only be limited by a collision. In order to obtain smooth solutions, even for small time, it is necessary to consider some compatible initial data. We therefore adapt the notion of compatible initial fluid velocity introduced in Definition 1.2 to the bounded case.

Definition 1.5 (Compatible initial fluid velocity). Given the open regular connected and simply connected bounded cavity Ω and the initial regular closed domain $\mathcal{S}_0 \subset \Omega$ occupied by the body, ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field satisfying the following div/curl type system:

$$\operatorname{div} u_0 = 0 \text{ and } \operatorname{curl} u_0 = 0 \text{ in } \mathcal{F}_0,$$
$$u_0 \cdot n = \left(\ell_0 + r_0 x^\perp\right) \cdot n \text{ for } x \in \partial\mathcal{S}_0, \quad \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma,$$
$$u_0 \cdot n = 0 \text{ for } x \in \partial\Omega.$$

Therefore in Section 3 we will prove the following.

Theorem 1.6. *Given the open regular connected and simply connected bounded cavity Ω, the initial closed domain $\mathcal{S}_0 \subset \Omega$ occupied by the body, the initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$, the circulation γ in $\mathbb{R}$, and u_0 the associated compatible initial fluid velocity (according to Definition 1.5), there exists a unique smooth local-in-time solution to the problem compound of the incompressible Euler equations* (1) *on the fluid domain, of the Newton equations* (2)*, of the interface condition* (4)*, of the impermeability condition $u \cdot n = 0$ on $\partial\Omega$, and the initial conditions $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$ and $u|_{t=0} = u_0$. Moreover the life-time of such a smooth solution can only be limited by a collision.*

This result by itself belongs to the mathematical folklore.[4] Here it will be easily deduced from the normal form hinted above in order to introduce the solutions which will be tackled in the zero radius limit and because this normal form is precisely the first step of our strategy in order to tackle the singular features of the body's dynamics in the zero radius limit.

Then we expand the coefficients of the previous normal form in the zero-radius limit and repeatedly use Lamb's lemma to reformulate under an asymptotic normal form closer to the one of the unbounded case where the leading terms of the electric-type force are absorbed in the other terms by a modulation of the unknown. Indeed we will consider in particular as new unknown a quantity obtained by subtracting a drift velocity given by the leading terms of the electric-type potential from the translation velocity. This will allow to extend to this case the vanishingly small limit, still in both cases of a limit pointwise particle which is massive or massless. Indeed we will obtain both the massive point vortex system and the classical point vortex system in a cavity as limit equations for respectively a massive and a massless

[4] Indeed a stronger result has been obtained in [11], where the rotational case (with $\operatorname{curl} u_0$ in L^∞) is handled with pure PDE's methods.

particle, that is from the dynamics of a shrinking solid in the inertia regime mentioned above.

Let us recall that the Kirchhoff-Routh velocity u_Ω is defined as $u_\Omega := \nabla^\perp \psi_\Omega$, where $\nabla^\perp := (-\partial_2, \partial_1)$ and where the Kirchhoff-Routh stream function ψ_Ω is defined as $\psi_\Omega(x) := \frac{1}{2}\psi^0(x,x)$, where $\psi^0(h,\cdot)$ is the solution to the Dirichlet problem: $\Delta\psi^0(h,\cdot) = 0$ in Ω, $\psi^0(h,\cdot) = -\frac{1}{2\pi}\ln|\cdot - h|$ on $\partial\Omega$. Let us now introduce the limit equation for the case of a massive particle.

$$mh'' = \gamma\big(h' - \gamma u_\Omega(h)\big)^\perp \text{ on } [0,T), \text{ with } (h,h')(0) = (0,\ell_0). \tag{8}$$

The existence of the maximal solution (h,T) follows from classical ODE theory. Moreover it follows from the conservation of the energy $\frac{1}{2}mh'\cdot h' - \gamma^2\psi_\Omega(h)$ for any $h \in C^\infty([0,T];\Omega)$ satisfying (8), and from the continuity of the Kirchhoff-Routh stream function ψ_Ω in Ω that T is the time of the first collision of h with the outer boundary $\partial\Omega$ of the fluid domain. If there is no collision, then $T = +\infty$.

Let us also recall the point vortex equation:

$$h' = \gamma u_\Omega(h) \text{ for } t > 0, \text{ with } h(0) = 0. \tag{9}$$

It is well-known that the solution h is global in time, and in particular that there is no collision of the vortex point with the external boundary $\partial\Omega$. This follows from the conservation of the energy $\gamma^2\psi_\Omega(h)$ for any $h \in C^\infty([0,T];\Omega)$ satisfying (9), and the fact that $\psi_\Omega(h) \to +\infty$ when h comes close to $\partial\Omega$.

Next result states the convergence of h^ε to the solutions to these equations (8) and (9).

Theorem 1.7. *Let $\mathcal{S}_0 \subset \Omega$, $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, and $(m, \mathcal{J}) \in (0,+\infty) \times (0,+\infty)$. Let, in the case of a massive (respectively massless) particle, γ in $\mathbb{R}$ (resp. in $\mathbb{R}^*$). Let (h,T) be the maximal solution to* (8) *(resp. h be the global solution to* (9)*). For every $\varepsilon \in (0,1]$ small enough to ensure that the set $\mathcal{S}_0^\varepsilon$ defined by* (5) *satisfies $\mathcal{S}_0^\varepsilon \subset \Omega$, we consider u_0^ε the associated compatible initial fluid velocity associated (according to Definition 1.5) with the initial solid domain $\mathcal{S}_0^\varepsilon$ defined by* (5)*, Ω, (ℓ_0, r_0) and γ, and we denote T^ε the life-time of the associated smooth solution $(h^\varepsilon, \theta^\varepsilon, u^\varepsilon)$ given by Theorem 1.6, with the inertia scaling described in Definition 1.1 and the initial conditions $(h^\varepsilon, (h^\varepsilon)', \theta^\varepsilon, (\theta^\varepsilon)')(0) = (0, \ell_0, 0, r_0)$ and $u^\varepsilon|_{t=0} = u_0^\varepsilon$. Then in the zero radius limit $\varepsilon \to 0$, there holds* $\liminf T^\varepsilon \geqslant T$ *(resp.* $\liminf T^\varepsilon \to +\infty$*) and $h^\varepsilon \rightharpoonup h$ in $W^{2,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$ (resp. $W^{1,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$).*

Therefore, in the bounded setting, Conjecture $(\mathcal{C})$ is also true with $u_{\mathrm{d}} = \gamma u_\Omega$ (and a easy byproduct of the analysis is that $u_{\mathrm{bd}} = K_\Omega[\gamma\delta_h]$ where K_Ω denotes the Biot-Savart law associated with the simply connected domain Ω). Theorem 1.7 will be proven as a consequence of Theorem 3.6 and Theorem 3.11. This result was obtained in [7].

In Section 4, we will consider the motion of a rigid body immersed in a two dimensional incompressible perfect fluid with vorticity. In order to focus on the interaction between the rigid body and the fluid vorticity we go back to the unbounded setting of Section 2 so that the fluid-solid system occupies again the whole plane. We first recall a result obtained in [4] establishing a global in time existence and uniqueness result similar to the celebrated result by Yudovich about the case of a fluid alone. Yudovich's theory relies on the transport of the fluid vorticity in particular to the preservation of L^∞-in space bound of the vorticity when time proceeds and we therefore extend Definition 1.2 to this setting.

Definition 1.8 (Compatible initial fluid velocity). Given the initial domain $\mathcal{S}_0$ occupied by the body, the initial solid velocities ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, an initial fluid vorticity ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$ and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \mathbb{R}^2 \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field in $C^0(\overline{\mathcal{F}_0}; \mathbb{R}^2)$ satisfying the following div/curl type system:

$$\operatorname{div} u_0 = 0 \text{ and } \operatorname{curl} u_0 = \omega_0 \text{ in } \mathcal{F}_0,$$
$$u_0 \cdot n = \left(\ell_0 + r_0 x^\perp\right) \cdot n \text{ for } x \in \partial\mathcal{S}_0, \quad \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma,$$
$$\lim_{x\to\infty} u_0 = 0.$$

We are now ready to state the existence and uniqueness result with bounded vorticity.

Theorem 1.9. *For any $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, $\omega_0 \in L^\infty_c(\overline{\mathcal{F}_0})$, there exists a unique solution to the problem compound of the incompressible Euler equations* (1) *on the fluid domain, of the Newton equations* (2)*, of the interface condition* (4)*, of the condition at infinity:* $\lim_{|x|\to\infty} |u(t,x)| = 0$*, and of the initial conditions* $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$ *and* $u|_{t=0} = u_0$*, with u_0 the compatible initial velocity associated with ℓ_0, r_0 and ω_0 by Definition 1.8. Moreover* $(h, \theta) \in C^2(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R})$ *and for all $t > 0$,* $\omega(t) := \operatorname{curl} u(t) \in L^\infty_c(\overline{\mathcal{F}(t)})$.

A key of Yudovich's approach is that a L^∞-in space bound of the vorticity is enough to control the log-Lipschitz regularity of the fluid velocity.[5] We will see in Section 4 that this is still true for the case with an immersed body. Moreover this amount of regularity is sufficient in order to insure convenient a priori bounds regarding the solid motion. We refer to [9, 10, 31] for some other results regarding the existence of solutions with less regularity. Indeed we will slightly modify the proof of Theorem 1.9 given in [4] regarding the a priori estimates of the rigid body's acceleration. We will use here an

[5] Indeed the uniqueness part of the result above has to be understood to hold in the class of solutions with such a regularity.

argument from [10, 31] which requires pretty much only a L^2 type a priori estimate of the fluid velocity.

This setting will allow to investigate the zero radius limit. In [4] and [5] we have obtained respectively the following results corresponding to the massive and massless cases.

Theorem 1.10. *Let be given a circulation γ in $\mathbb{R}$ in the case of a massive particle and in $\mathbb{R}^*$ in the case of a massless particle, $(\ell_0, r_0) \in \mathbb{R}^3$, ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$ and consider as initial fluid velocity u_0^ε is then defined as the unique vector field u_0^ε compatible with $\mathcal{S}_0^\varepsilon$, ℓ_0, r_0, γ and $\omega_0^\varepsilon := \omega_{0|\mathcal{F}_0^\varepsilon}$. For any $\varepsilon \in (0,1]$, let us denote $(h^\varepsilon, r^\varepsilon, u^\varepsilon)$ the corresponding solution of the system.Then in the zero radius limit $\varepsilon \to 0$, with the inertia scaling described in Definition 1.1, one respectively obtains the following equation for the position $h(t)$ of the pointwise limit particle:*

$$mh''(t) = \gamma\Big(h'(t) - K_{\mathbb{R}^2}[\omega(t,\cdot)](h(t))\Big)^\perp, \tag{10}$$

in the massive limit and

$$h'(t) = K_{\mathbb{R}^2}[\omega(t,\cdot)](h(t)), \tag{11}$$

in the massless limit. Regarding the fluid state, one obtain at the limit the following transport equation for the fluid vorticity:

$$\frac{\partial \omega}{\partial t} + \operatorname{div}\big(\omega K_{\mathbb{R}^2}[\omega + \gamma\delta_h]\big) = 0. \tag{12}$$

Therefore Conjecture $(\mathcal{C})$ is also true with $u_{\mathrm{d}} = K_{\mathbb{R}^2}[\omega]$ and $u_{\mathrm{bd}} = K_{\mathbb{R}^2}[\omega + \gamma\delta_h]$.

The fluid equation (12) is the same whether the body shrinks to a massive or a massless pointwise particle. Equation (12) describes the evolution of the vorticity of the fluid: it is transported by a velocity obtained by the usual Biot-Savart law in the plane, but from a vorticity which is the sum of the fluid vorticity and of a point vortex placed at the (time-dependent) position $h(t)$ where the solid shrinks, with a strength equal to the circulation γ around the body. Equation (12) and its corresponding initial condition hold in the sense that for any test function $\psi \in C_c^\infty([0,T) \times \mathbb{R}^2)$ we have

$$\int_0^\infty \int_{\mathbb{R}^2} \psi_t \omega \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla_x \psi \cdot K_{\mathbb{R}^2}[\omega + \gamma\delta_h]\omega \, dx \, dt + \int_{\mathbb{R}^2} \psi(0,x)\omega_0(x) \, dx = 0.$$

The uniqueness of the solution to the massive limit system(10)-(12) is an interesting question. Observe that a putative uniqueness result would entail the convergence of the whole sequence. This is the case for the massless limit system (11)-(12) for which uniqueness does hold according to a result due to Marchioro and Pulvirenti, cf. [25] and revisited by Lacave and Miot, cf. [19].

We will take advantage of the approach developed in [6] and exposed in Section 3 to provide a sketch of a more simple proof of the results claimed in Theorem 1.10 than the ones achieved in [4] for the massive case and most of all in [5] for the massless case. In order to do so we will start with an exact reformulation of the body's dynamics for fixed radius into an ODE with a geodesic feature, a Lorentz type force of the same form than the one mentioned above in the irrotational case, but with an extra dependence to the fluid vorticity, and a new term describing a somehow more direct influence of the vorticity but which does not enjoy much structure, cf. Theorem 4.11. Then we expand the coefficients of the previous ODE in the zero-radius limit using in particular an irrotational approximation of the fluid velocity on the body's boundary in order to use Lamb's lemma. This provides in particular the leading terms in the expansion of the Lorentz type force with less effort than by the complex-analytic method used in [4] and [5]. Another simplification comes from the extra term encoding a direct influence of the vorticity to the body's dynamics which is more simple to estimate than its counterpart in [5].[6] We thus obtain an asymptotic normal form once again close to the one of Section 2 where the leading terms of the electric-type force are absorbed in the other terms by a modulation of the solid translation velocity by a drift velocity of the particle under the influence of the fluid vorticity. This will allow to extend to this case the vanishingly small limit, still in both cases of a limit pointwise particle which is massive or massless. Indeed Theorem 1.10 will be proven in Section 3.6 as a consequence of Theorem 4.12.

We aim at extending this analysis to the case where both interactions of a body with an external boundary and with the fluid vorticity are considered in the same time, for several massive and massless particles, which will provide a positive answer to Conjecture $(\mathcal{C})$ in a wide setting, cf. the ongoing work [6].

2. Case of an unbounded irrotational flow

In this section we assume that the system "fluid + solid" is unbounded so that the domain occupied by the fluid at time t is $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$ starting

[6]In particular it avoids again the temporary occurrence of Archimedes' type quantities.

from the initial domain $\mathcal{F}_0 := \mathbb{R}^2 \setminus \mathcal{S}_0$. The equations at stake then read :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{for } x \in \mathcal{F}(t), \tag{13}$$

$$mh''(t) = \int_{\partial \mathcal{S}(t)} \pi n \, ds \quad \text{and} \quad \mathcal{J}\theta''(t) = \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \, ds, \tag{14}$$

$$u \cdot n = \Big(h'(t) + r(t)(x - h(t))^\perp \Big) \cdot n \quad \text{for} \quad x \in \partial \mathcal{S}(t), \tag{15}$$

$$\lim_{|x| \to \infty} |u(t,x)| = 0, \tag{16}$$

$$u|_{t=0} = u_0 \quad \text{for } x \in \mathcal{F}_0 \quad \text{and} \quad (h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0). \tag{17}$$

2.1. Reduction to an ODE. Statement of Theorem 2.2

In the irrotational case, the system above can be recast as an ODE whose unknowns are the degrees of freedom of the solid, namely h and θ. In particular the motion of the fluid is completely determined by the solid position and velocity. In order to state this, let us introduce the variable $q := (h, \theta) \in \mathbb{R}^3$. Since the fluid and solid domains only depend on t through the solid position, we will rather denote them respectively $\mathcal{F}(q)$ and $\mathcal{S}(q)$ instead of $\mathcal{F}(t)$ and $\mathcal{S}(t)$.

Let us gather the mass and moment of inertia of the solid into the following matrix:

$$\mathcal{M}_g := \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix}. \tag{18}$$

Observe that $\mathcal{M}_g$ is diagonal and in the set $S_3^{++}(\mathbb{R})$ of the real symmetric positive definite 3×3 matrices. As already mentioned in the introduction the reformulation relies on the phenomenon of added mass, which, loosely speaking, measures how much the surrounding fluid resists the acceleration as the body moves through it. This will be encoded by a matrix $\mathcal{M}_a$ in the set $S_3^{+}(\mathbb{R})$ of the real symmetric positive-semidefinite 3×3 matrices. The index a refers to "added", by opposition to the genuine inertia $\mathcal{M}_g$. This matrix $\mathcal{M}_a$ depends on the shape of the domain occupied by the solid and therefore on the solid position. Still since the system "fluid + solid" occupies the full plane the added inertia is invariant by translation and therefore only depends on θ. In order to measure its variations let us denote by $\mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$ the space of bilinear mappings from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}^3$.

Definition 2.1 (a-connection). Given a C^∞ mapping $\theta \in \mathbb{R} \mapsto \mathcal{M}_{a,\theta} \in S_3^{+}(\mathbb{R})$, we say that the C^∞ mapping $\theta \in \mathbb{R} \mapsto \Gamma_{a,\theta} \in \mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$ is the associated a-connection if for any $p \in \mathbb{R}^3$,

$$\langle \Gamma_{a,\theta}, p, p\rangle := \left(\sum_{1\leqslant i,j\leqslant 3} (\Gamma_{a,\theta})^k_{i,j} p_i p_j \right)_{1\leqslant k\leqslant 3} \in \mathbb{R}^3, \tag{19a}$$

with for every $i, j, k \in \{1, 2, 3\}$ and for any $q = (h, \theta)$,

$$(\Gamma_{a,\theta})^k_{i,j}(q) := \frac{1}{2}\Big((\mathcal{M}_{a,\theta})^i_{k,j} + (\mathcal{M}_{a,\theta})^j_{k,i} - (\mathcal{M}_{a,\theta})^k_{i,j} \Big)(q), \tag{19b}$$

where $(\mathcal{M}_{a,\theta})^k_{i,j}$ denotes the partial derivative with respect to q_k of the entry of indexes (i, j) of the matrix $\mathcal{M}_{a,\theta}$, that is

$$(\mathcal{M}_{a,\theta})^k_{i,j} := \frac{\partial (\mathcal{M}_{a,\theta})_{i,j}}{\partial q_k}. \tag{19c}$$

As already mentioned in the introduction another celebrated feature of the body's dynamics in the case of an unbounded irrotational flow is the Kutta-Joukowski force. This force also depends on the shape of the domain occupied by the solid and in particular on the solid position through θ only. Since this force is gyroscopic, i.e. orthogonal to the velocity q' (which gathers both translation and rotation velocities), and proportional to the circulation around the body, it will be encoded by a vector B_θ in $\mathbb{R}^3$.

The first main result of this section is the following reformulation of the equations (13)-(16) into an ODE for the degrees of freedom of the solid only.

Theorem 2.2. *There exists a C^∞ mapping $\theta \in \mathbb{R} \mapsto \big(\mathcal{M}_{a,\theta}, B_\theta\big) \in S_3^+(\mathbb{R}) \times \mathbb{R}^3$ depending only on $\mathcal{S}_0$ such that the equations* (13)-(16) *are equivalent to the following ODE for $q = (h, \theta)$:*

$$(\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q'' + \langle \Gamma_{a,\theta}, q', q'\rangle = \gamma q' \times B_\theta, \tag{20}$$

where $\Gamma_{a,\theta}$ denotes the a-connection associated with $\mathcal{M}_{a,\theta}$, the fluid velocity u being given with respect to q and $q' = (h', \theta')$ as the unique solution to the following div/curl type system:

$$\operatorname{div} u = 0 \text{ and } \operatorname{curl} u = 0 \text{ in } \mathcal{F}(q),$$
$$u \cdot n = \Big(h' + \theta'(x - h)^\perp \Big) \cdot n \text{ for } x \in \partial\mathcal{S}(q), \quad \int_{\partial\mathcal{S}(q)} u \cdot \tau \, ds = \gamma,$$
$$\lim_{x\to\infty} u = 0.$$

Indeed the matrix $\mathcal{M}_{a,\,\theta}$, its associated a-connection $\Gamma_{a,\theta}$ and B_θ will be given by precise formulas in the next section, cf. (26), (28) and (35). Let us already mention here that the dependence on θ of $\big(\mathcal{M}_{a,\theta}, B_\theta\big)$ is quite simple since

$$\mathcal{M}_{a,\theta} = \mathcal{R}(\theta)\mathcal{M}_{a,0}\mathcal{R}(\theta)^t \text{ and } B_\theta = \mathcal{R}(\theta) B_0,$$

where we associate the 3×3 rotation matrix

$$\mathcal{R}(\theta) := \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(3) \tag{21}$$

with the 2×2 rotation matrix $R(\theta)$ defined in (3).

Remark 2.3. Let us emphasize that the coefficients $(\Gamma_{a,\theta})^k_{i,j}$ defined in (19b) are the Christoffel symbols of the first kind associated with the metric $\mathcal{M}_g + \mathcal{M}_{a,\,\theta}$ so that, should its right hand side vanish, (20) would be the geodesic equation associated with the metric $\mathcal{M}_g + \mathcal{M}_{a,\,\theta}$.

We will prove Theorem 2.2 in the sequel but we first deduce and prove the following result.

Proposition 2.4. *Given some initial data* $(q, q')(0) = (0, \ell_0, 0, r_0)$ *there exists a unique global solution* $q \in C^\infty([0, +\infty); \mathbb{R}^3)$ *to* (20)*. Moreover the quantity*

$$\frac{1}{2}(\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q' \cdot q' \tag{22}$$

is conserved.

Theorem 1.3 is then a consequence of Theorem 2.2 and Proposition 2.4.

Remark 2.5. The quantity (22) corresponds to the sum of the kinetic energy of the solid associated with its genuine inertia and of the one associated with the added inertia. It will become apparent in the sequel, cf. Section 2.3.2, that the kinetic energy associated with the added inertia of the rigid body can be also interpreted as a renormalization of the kinetic energy of the fluid by retaining only the potential contribution and discarding the term due to the circulation around the body.

Proof. Local existence and uniqueness follow from classical ODE theory. Global existence would be a consequence of the energy conservation. Indeed defining for any θ in $\mathbb{R}$, for any p in $\mathbb{R}^3$, the matrix

$$S_{a,\theta}(p) := \Big(\sum_{1 \leq i \leq 3} (\Gamma_{a,\theta})^k_{i,j} p_i \Big)_{1 \leq k,j \leq 3} \quad \text{so that } \langle \Gamma_{a,\theta}, p, p\rangle = S_{a,\theta}(p)p,$$

then, an explicit computation proves that for any θ in $\mathbb{R}$, for any p in $\mathbb{R}^3$,

$$\frac{1}{2}\frac{\partial \mathcal{M}_{a,\theta}}{\partial q}(\theta) \cdot p - S_{a,\theta}(p) \text{ is skew-symmetric.}$$

Therefore for any θ in $\mathbb{R}$, for any p in $\mathbb{R}^3$,

$$\langle \Gamma_{a,\theta}, p, p\rangle \cdot p = \big(S_{a,\theta}(p)p\big) \cdot p = \frac{1}{2}\Big(\frac{\partial \mathcal{M}_{a,\theta}}{\partial q}(\theta)p\Big) \cdot p,$$

so that taking the inner product of (20) with q' yields on the one hand

$$(\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q'' \cdot q' = \frac{1}{2}\big((\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q' \cdot q'\big)' - \frac{1}{2}\big(\frac{\partial \mathcal{M}_{a,\theta}}{\partial q}(\theta)\, q'\big) \cdot q'$$

and on the other hand

$$\langle \Gamma_{a,\theta}, q', q' \rangle \cdot q' = \frac{1}{2}\big(\frac{\partial \mathcal{M}_{a,\theta}}{\partial q}(\theta) q'\big) \cdot q',$$

Therefore when doing the energy estimate the term coming from the a-connection exactly compensates the term coming from the commutation of one time derivative in the acceleration term, and the conservation of the energy follows by observing that the contribution of the right hand side of (20) is 0.

□

2.2. Explicit definition of the ODE coefficients $\mathcal{M}_{a,\,\theta}$, $\Gamma_{a,\theta}$ and B_θ

In this section we are going to provide some precise formulas for the matrix $\mathcal{M}_{a,\,\theta}$ (and for its associated a-connection $\Gamma_{a,\theta}$ as well) and for the vector B_θ thanks to some elementary flows corresponding respectively to potential and circulatory type flows. Indeed we will see in Section 2.3.2 (in a different frame, moving with the body) that the real flow can be decomposed thanks to these elementary flows.

2.2.1. Kirchhoff potentials. The following so-called Kirchhoff potentials $\Phi := (\Phi_i)_{i=1,2,3}$ will play a major role. They are defined as the solutions of the following problems:

$$-\Delta \Phi_i = 0 \quad \text{in } \mathcal{F}_0, \quad \Phi_i \longrightarrow 0 \quad \text{when } x \to \infty, \quad \frac{\partial \Phi_i}{\partial n} = K_i \quad \text{on } \partial\mathcal{F}_0, \tag{23}$$

where $(K_1,\, K_2,\, K_3) := (n_1,\, n_2,\, x^\perp \cdot n)$. The compatibility condition for this Neumann Problem is satisfied i.e. we check that

$$\int_{\partial \mathcal{S}_0} K_i \, ds = 0, \tag{24}$$

for $i = 1, 2, 3$, which is obvious by the Stokes formula :

$$\int_{\partial \mathcal{S}_0} n_1 \, ds = -\int_{\mathcal{S}_0} \operatorname{div}(e_1)\, dx = 0 \text{ and similarly } \int_{\partial \mathcal{S}_0} n_2 \, ds = 0,$$

$$\text{whereas } \int_{\partial \mathcal{S}_0} x^\perp \cdot n \, ds = -\int_{\mathcal{S}_0} \operatorname{div}(x^\perp)\, dx = 0.$$

Above e_1 and e_2 are the unit vectors of the canonical basis of $\mathbb{R}^2$.

We have that for all $i = 1, 2, 3$:

$$\Phi_i(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } \nabla \Phi_i(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to +\infty, \tag{25}$$

and consequently that $\nabla \Phi_i$ are in $L^2(\mathcal{F}_0)$.

For instance in the case where $\mathcal{S}_0$ is a disk one has $\Phi_1(x) = -\frac{x_1}{|x|^2}$, $\Phi_2(x) = -\frac{x_2}{|x|^2}$ and $\Phi_3(x) = 0$. If $\mathcal{S}_0$ is not a disk then these three functions are linearly independent; this can be easily be proved by using a smooth arc length parameterization of the boundary and the usual Frenet equations, see for instance Lemma 6.1. of [27].

2.2.2. Added inertia. Let us define the matrices

$$\mathcal{M}_a := \left(m_{i,j}\right)_{i,j\in\{1,2,3\}} \text{ and } \mathcal{M}_{a,\theta} := \mathcal{R}(\theta)\mathcal{M}_a\mathcal{R}(\theta)^t, \tag{26}$$

where for $i, j \in \{1, 2, 3\}$

$$m_{i,j} := \int_{\mathcal{F}_0} \nabla\Phi_i \cdot \nabla\Phi_j \, dx, \tag{27}$$

and $\mathcal{R}(\theta)$ is the 3×3 rotation matrix defined in (21). Let us mention from now on that the matrix $\mathcal{M}_{a,\theta}$ is positive definite if and only $\mathcal{S}_0$ is not a disk. When $\mathcal{S}_0$ is a disk then $\mathcal{M}_{a,\theta}$ has the form $diag(m_a, m_a, 0)$ with $m_a > 0$. The case where $\mathcal{S}_0$ is disk is therefore peculiar, indeed by combining the translation and rotation equations on observe that $\mathcal{J}\theta'' = mh'' \cdot (h_c - h)^\perp$, where h_c denotes of the position of the center of the disk $\mathcal{S}(q)$, which can be different from h if the body is not homogeneous. As a consequence, in this case where $\mathcal{S}_0$ is a disk, a particular reduction of the dynamics is possible and is indeed very helpful in order to tackle the case of massless particles. In the sequel we will focus on the case where $\mathcal{S}_0$ is not a disk, and we refer to [7] for a full treatment of the case where $\mathcal{S}_0$ is a disk.

2.2.3. Added inertia connection. Let us define for $p = (\ell, r)^t$,

$$\langle \Gamma_{a,\theta}, p, p\rangle \quad := \quad -\begin{pmatrix} P_{a,\theta} \\ 0 \end{pmatrix} \times p - r\mathcal{M}_{a,\theta}\begin{pmatrix} 0 \\ \ell^\perp \end{pmatrix}, \tag{28}$$

where $P_{a,\theta}$ are the two first coordinates of $\mathcal{M}_{a,\theta}\, p$. A tedious computation reveals that $\Gamma_{a,\theta}$ is the a-connection associated with $\mathcal{M}_{a,\theta}$.

2.2.4. Harmonic field. To take the velocity circulation around the body into account, we introduce the following harmonic field: let H be the unique solution vanishing at infinity of

$$\operatorname{div} H = 0 \text{ and } \operatorname{curl} H = 0 \text{ in } \mathcal{F}_0,\ H \cdot n = 0 \text{ on } \partial\mathcal{S}_0,\ \int_{\partial\mathcal{S}_0} H \cdot \tau \, ds = 1.$$

The vector field H admits a harmonic stream function $\Psi_H(x)$:

$$H = \nabla^\perp \Psi_H,$$

which vanishes on the boundary $\partial\mathcal{S}_0$, and behaves like $\frac{1}{2\pi}\ln|x|$ as x goes to infinity. One way to get more information on the far-field behaviour of H is to use a little bit of complex analysis. We identify $\mathbb{C}$ and $\mathbb{R}^2$ through

$$(x_1, x_2) = x_1 + ix_2 = z. \tag{29}$$

We also use the notation

$$\widehat{f} = f_1 - if_2 \text{ for any } f = (f_1, f_2). \tag{30}$$

The reason of this notation is the following: if f is divergence and curl free if and only if $\widehat{f}$ is holomorphic. In particular the function $\widehat{H}$ is holomorphic (as a function of $z = x_1 + ix_2$), and can be decomposed in Laurent Series :

$$\widehat{H}(z) = \frac{1}{2i\pi z} + \mathcal{O}(1/z^2) \quad \text{as } z \to \infty. \tag{31}$$

Coming back to the variable $x \in \mathbb{R}^2$, the previous decomposition implies

$$H(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } \nabla H = \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{32}$$

Moreover, we deduce from (31) that

$$x^\perp \cdot H = \frac{1}{2\pi} + \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{and} \quad (H)^\perp - x^\perp \cdot \nabla H = \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

2.2.5. Conformal center. The harmonic field H allows to define the following geometric constant, known as the conformal center of $\mathcal{S}_0$:

$$\xi_1 + i\xi_2 := \int_{\partial\mathcal{S}_0} z\widehat{H}\, dz, \tag{33}$$

which depends only on $\mathcal{S}_0$. In the particular case of a disk, the harmonic field H is given by $\frac{1}{2\pi}\frac{x^\perp}{|x|^2}$ so that the conformal center ξ of a disk is obviously 0. In the general case one proves the following real-analytic characterization.

Proposition 2.6. *There holds:*

$$\xi = \int_{\partial\mathcal{S}_0} (H \cdot \tau)x\, ds. \tag{34}$$

In order to prove Proposition 2.6 we will use the following result which relates the integral $\int_{\mathcal{C}} \widehat{f}\, dz$ associated with a tangent vector field f to its flux and its circulation.

Lemma 2.7. *Let $\mathcal{C}$ be a smooth Jordan curve, $f := (f_1, f_2)$ a smooth vector fields on $\mathcal{C}$:*

$$\int_{\mathcal{C}} \widehat{f}\, dz = \int_{\mathcal{C}} f \cdot \tau\, ds - i \int_{\mathcal{C}} f \cdot n\, ds.$$

Proof of Lemma 2.7. Denoting by $\gamma = (\gamma_1, \gamma_2)$ a arc-lenght parametrization of $\mathcal{C}$ then $\tau = (\gamma_1', \gamma_2')/|\gamma'|$, $n = (-\gamma_2', \gamma_1')/|\gamma'|$, $ds = |\gamma'(t)|dt$ and $dz = (\gamma_1'(t) + i\gamma_2'(t))dt$. Hence the conclusion follows from

$$\int_{\mathcal{C}} (f_1 - if_2)\, dz = \int (f_1\gamma_1' + f_2\gamma_2')\, dt - i \int (-f_1\gamma_2' + f_2\gamma_1')\, dt.$$

□

Proof of Proposition 2.6. Observe that $z(H_1 - iH_2) = f_1 - if_2$ with $f_1 = x\cdot H$ and $f_2 = x^\perp \cdot H$ so that applying Lemma 2.7 we have that

$$\int_{\partial\mathcal{S}_0} z\widehat{H}\, dz = \int_{\partial\mathcal{S}_0} g\, ds, \text{ with } g := \begin{pmatrix} x\cdot H \\ x^\perp \cdot H\end{pmatrix}\cdot \tau - i\begin{pmatrix} x\cdot H \\ x^\perp \cdot H\end{pmatrix}\cdot n.$$

Moreover, for $x \in \partial\mathcal{S}_0$, we have

$$\begin{aligned} g &= (x_1H_1 + x_2H_2)\tau_1 + (-x_2H_1 + x_1H_2)\tau_2 \\ &\quad -i(x_1H_1 + x_2H_2)n_1 - i(-x_2H_1 + x_1H_2)n_2 \\ &= x_1(H_1\tau_1 + H_2\tau_2) + x_2(H_2\tau_1 - H_1\tau_2) \\ &\quad -ix_1(H_1n_1 + H_2n_2) - ix_2(H_2n_1 - H_1n_2), \end{aligned}$$

and using that $(n_1, n_2) = (-\tau_2, \tau_1)$, we deduce that $g = z(H\cdot\tau) - iz(H\cdot n)$. It is then sufficient to recall that $H\cdot n = 0$ to conclude. □

2.2.6. Kutta-Joukowski field. Then the vector field B_θ is defined by the following formula:

$$B_\theta := \mathcal{R}(\theta)\begin{pmatrix}\xi^\perp \\ -1\end{pmatrix}. \tag{35}$$

Observe that the corresponding force in the left hand side of (20) is therefore

$$q' \times B_\theta = \begin{pmatrix} (h')^\perp - \theta' R(\theta)\xi^\perp \\ R(\theta)\xi\cdot h'\end{pmatrix}.$$

2.3. Reformulation as an ODE in the body frame. Statement of Theorem 2.8

In order to transfer the equations in the body frame we apply the following isometric change of variable:

$$\begin{cases} v(t,x) = R(\theta(t))^T\ u(t, R(\theta(t))x + h(t)), \\ \tilde\pi(t,x) = \pi(t, R(\theta(t))x + h(t)), \\ \ell(t) = R(\theta(t))^T\ h'(t), \end{cases} \tag{36}$$

where we recall that $R(\theta(t))$ is the 2×2 rotation matrix defined in (3) so that the equations (13)-(17) become

$$\frac{\partial v}{\partial t} + \left[(v - \ell - rx^\perp)\cdot\nabla\right] v + rv^\perp + \nabla\tilde\pi = 0 \text{ and } \operatorname{div} v = 0 \quad \text{for } x\in\mathcal{F}_0, \tag{37}$$

$$m\ell'(t) = \int_{\partial\mathcal{S}_0} \tilde\pi n\, ds - mr\ell^\perp \text{ and } \mathcal{J}r'(t) = \int_{\partial\mathcal{S}_0} x^\perp\cdot\tilde\pi n\, ds, \tag{38}$$

$$v\cdot n = \left(\ell + rx^\perp\right)\cdot n \quad \text{for } x\in\partial\mathcal{S}_0, \tag{39}$$

$$v(0,x) = v_0(x) \text{ for } x\in\mathcal{F}_0 \text{ and } (\ell, r)(0) = (\ell_0, r_0), \tag{40}$$

where $r(t) = \theta'(t)$.

In order to state recast the system above as an ODE in the body frame we are going to introduce now a few objects. Let $\Gamma_g : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ and

$\Gamma_a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the bilinear symmetric mappings defined, for all $p = (\ell\ r) \in \mathbb{R}^3$, by

$$\langle \Gamma_g, p, p \rangle = mr \begin{pmatrix} \ell^\perp \\ 0 \end{pmatrix} \text{ and } \langle \Gamma_a, p, p \rangle = \begin{pmatrix} r(\mathcal{M}_\flat \ell)^\perp \\ \ell^\perp \cdot \mathcal{M}_\flat \ell \end{pmatrix} + rp \times \mu, \tag{41}$$

where

$$\mu := \begin{pmatrix} m_{1,3} \\ m_{2,3} \\ 0 \end{pmatrix} \text{ and } \mathcal{M}_\flat := \left(m_{i,j} \right)_{i,j \in \{1,2\}} . \tag{42}$$

Note that

$$\forall p \in \mathbb{R}^3, \quad \langle \Gamma_g, p, p \rangle \cdot p = 0 \text{ and } \langle \Gamma_a, p, p \rangle \cdot p = 0, \tag{43}$$

and that Γ_a is depending only on $\mathcal{S}_0$. Because of (43) we will refer to the quadratic mappings Γ_g and Γ_a as *gyroscopic terms.*

One will deduce Theorem 2.2 from the following result by going back in the original frame.

Theorem 2.8. *The equations* (37)-(38) *are equivalent to the following ODE for* $p := (\ell, r)^T$*:*

$$\left[\mathcal{M}_g + \mathcal{M}_a \right] p' + \langle \Gamma_g, p, p \rangle + \langle \Gamma_a, p, p \rangle = \gamma p \times B, \tag{44}$$

the fluid velocity v being given as the unique solution to the following div/curl type system:

$$\operatorname{div} v = 0 \text{ and } \operatorname{curl} v = 0 \text{ in } \mathcal{F}_0, \tag{45}$$

$$v \cdot n = \left(\ell + r x^\perp \right) \cdot n \text{ on } \partial\mathcal{S}_0, \quad \int_{\partial\mathcal{S}_0} v \cdot \tau \, ds = \gamma, \tag{46}$$

$$\lim_{x \to \infty} v = 0. \tag{47}$$

Let us recall that $\mathcal{M}_g$ and $\mathcal{M}_a$ are respectively the genuine and added inertia, see (26) and (18), and that B is a fixed vector defined in (35).

Observe that one may also obtain from this formulation the conservative feature of the system since it suffices to multiply (44) by p, to use the symmetry of the matrices $\mathcal{M}_g$ and $\mathcal{M}_a$ and the properties (43) to deduce that the total energy $\frac{1}{2} p \cdot \left(\mathcal{M}_g + \mathcal{M}_a \right) p$ is conserved along time.

The rest of the section is devoted to the proof of Theorem 2.8. Indeed after a slight reformulation of the solid equations and the decomposition of the velocity into several pieces corresponding to the various sources in the right hand sides of the system (45)-(46)-(47), we will compare two methods, one based on complex argument and the other one on real analysis only.

2.3.1. Reformulation of the solid equations. The first step of the proof of Theorem 2.8 uses the Euler equations and the Kirchhoff potentials in order to get rid of the pressure and to make appear the added inertia.

Lemma 2.9. *Equations* (38) *can be rewritten in the form*

$$(\mathcal{M}_g+\mathcal{M}_a)p'+\langle\Gamma_g,p,p\rangle = -(\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2K_i\,ds-\int_{\partial\mathcal{S}_0}(\ell+rx^\perp)\cdot vK_i\,ds)_i, \quad (48)$$

where i *runs over the integers* $1,2,3$.

Proof. Using the following equality for two vector fields a and b in $\mathbb{R}^2$:

$$\nabla(a\cdot b) = a\cdot\nabla b + b\cdot\nabla a - (a^\perp\operatorname{curl} b + b^\perp\operatorname{curl} a), \quad (49)$$

the equation (37) can be written as

$$\frac{\partial v}{\partial t} + \frac{1}{2}\nabla(v^2) - \nabla((\ell+rx^\perp)\cdot v) + \nabla\tilde{\pi} = 0. \quad (50)$$

We use this equation do deduce the force/torque acting on the body:

$$\left(\int_{\partial\mathcal{S}_0}\tilde{\pi}n\,ds,\ \int_{\partial\mathcal{S}_0}\tilde{\pi}x^\perp\cdot n\,ds\right) = \left(\int_{\mathcal{F}_0}\nabla\tilde{\pi}\cdot\nabla\Phi_i\,dx\right)_{i=1,2,3}.$$

One can check that the above integration by parts is licit by using the decay properties of v and $\nabla\Phi_i$. Using an integration by parts, the boundary condition (39) and another integration by parts, one observes that the contribution of $\partial_t v$ is

$$\int_{\mathcal{F}_0}\partial_t v\cdot\nabla\Phi_i(x)\,dx = \mathcal{M}_a\begin{pmatrix}\ell\\ r\end{pmatrix}', \quad (51)$$

and one obtains the result. □

2.3.2. Decomposition of the velocity field. Finally, for ℓ in $\mathbb{R}^2$, r and γ in $\mathbb{R}$ given, there exists a unique vector field v verifying (45)-(46)-(47) and it is given by the law:

$$v = \gamma H + \ell_1\nabla\Phi_1 + \ell_2\nabla\Phi_2 + r\nabla\Phi_3, \quad (52)$$

We will denote by

$$\tilde{v} := v - \gamma H. \quad (53)$$

the part without circulation, that we will decomposed sometimes into

$$\tilde{v} = v_\# + r\nabla\Phi_3, \quad (54)$$

with

$$v_\# := \ell_1\nabla\Phi_1 + \ell_2\nabla\Phi_2, \quad (55)$$

in order to analyse separately the effects of the body translation and of the body rotation.

Observe that in the particular case the fluid velocity is assumed to be globally a gradient (the so-called potential case corresponding to $\gamma = 0$) it may be expanded with respect to the Kirchhoff potentials only.

Another crucial observation is that the first term in the right hand side of (48) is quadratic in v and that v is decomposed into a potential part and a circulatory part, cf. (52). Roughly speaking the a-connection (the last term in the right hand side of (44)) will result from the quadratic self-interaction of the potential part and the Kutta-Joukowski term (the left hand side of (44)) from the crossed interaction between the potential part and the circulatory part. There will be a cancellation of the quadratic self-interaction of the circulatory part, cf. (80) and Lemma 2.15. Indeed this cancellation echoes the renormalization hinted in Remark 2.5. The kinetic energy of the fluid should be $\frac{1}{2}\int_{\mathcal{F}_0} v^2\, dx$ but the term $\frac{1}{2}(\mathcal{M}_{a,\theta}\, q')\cdot q'$ included in (22) is equal to $\frac{1}{2}\int_{\mathcal{F}_0} \tilde{v}^2\, dx$. Since $\tilde{v}$ and H are orthogonal in $L^2(\mathcal{F}_0)$ this renormalization therefore formally consists in discarding the contribution to the fluid kinetic energy due to the fluid velocity associated with the circulation around the body. Observe that this contribution is infinite because of the slow decay at infinity of this part of the fluid velocity but depends only on the body's shape, not on its position or velocity.

2.4. A complex-analytic proof of Theorem 2.8

We will follow here a strategy based on complex analysis after Blasius, Kutta, Joukowski, Chaplygin and Sedov. A key lemma is the following Blasius formula about tangent vector fields where we use the identifications (29) and (30).

Lemma 2.10. *Let $\mathcal{C}$ be a smooth Jordan curve, $f := (f_1, f_2)$ and $g := (g_1, g_2)$ two smooth tangent vector fields on $\mathcal{C}$. Then*

$$\int_{\mathcal{C}} (f\cdot g) n\, ds = i\left(\int_{\mathcal{C}} \widehat{f}\widehat{g}\, dz\right)^*, \tag{56}$$

$$\int_{\mathcal{C}} (f\cdot g)(x^\perp \cdot n)\, ds = \mathrm{Re}\left(\int_{\mathcal{C}} z\widehat{f}\widehat{g}\, dz\right). \tag{57}$$

Above$(\cdot)^*$ denotes the complex conjugation.

Proof of Lemma 2.10. Thanks to the polarization identity, it is sufficient to consider the case where $f = g$. Let us consider $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \mathbb{R}^2$ a smooth arc length parameterization of the Jordan curve $\mathcal{C}$. On one side, one has

$$\int_{\mathcal{C}} (f\cdot f) n\, ds = \int_0^1 \left(f_1(\gamma(t))^2 + f_2(\gamma(t))^2\right)\begin{pmatrix} -\gamma_2'(t) \\ \gamma_1'(t)\end{pmatrix} dt. \tag{58}$$

On the other side, one has

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 \, dz$$
$$= \int_0^1 \Big(f_1(\gamma(t)) - if_2(\gamma(t))\Big)\Big[\big(f_1(\gamma(t)) - if_2(\gamma(t))\big)(\gamma_1'(t) + i\gamma_2'(t))\Big] \, dt.$$

But since f is tangent to $\mathcal{C}$, one sees that the expression inside the brackets above is real, and hence is equal to its complex conjugate. It follows that

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 \, dz = \int_0^1 \big|f_1(\gamma(t)) - if_2(\gamma(t))\big|^2 (\gamma_1'(t) - i\gamma_2'(t)) \, dt,$$

and (56) follows.

The proof of (57) is analogous: using again

$$(f_1(\gamma(t)) - if_2(\gamma(t)))(\gamma_1'(t) + i\gamma_2'(t)) = (f_1(\gamma(t)) + if_2(\gamma(t)))(\gamma_1'(t) - i\gamma_2'(t)),$$

we deduce

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 z \, dz$$
$$= \int_0^1 |f_1(\gamma(t)) - if_2(\gamma(t))|^2 (\gamma_1(t) + i\gamma_2(t))(\gamma_1'(t) - i\gamma_2'(t)) \, dt,$$

so that

$$\begin{aligned} & \mathrm{Re}\left(\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 z \, dz\right) \\ = & \int_0^1 \Big(f_1(\gamma(t))^2 + f_2(\gamma(t))^2\Big)\Big(\gamma_1(t)\gamma_1'(t) + \gamma_2(t)\gamma_2'(t)\Big) \, dt \\ = & \int_{\mathcal{C}} (f \cdot f)(x^\perp \cdot n) \, ds. \end{aligned}$$

□

The idea of the complex-analytic approach of the computation of the terms in the right hand side of the equation in Lemma 2.9 is to decompose them in order to make appear some vector fields tangent to the boundary $\partial\mathcal{S}_0$, to use Blasius' lemma and then Cauchy's residue theorem.

In this process of computing the residue we will use the Laurent series of $\widehat{\nabla\Phi_i}$. Because of the decay property at infinity in (23) the Laurent series of $\widehat{\nabla\Phi_i}$ has to start at least with a term in $\mathcal{O}(1/z)$, and this term is

$$\frac{1}{2i\pi} \int_{\partial\mathcal{S}_0} \widehat{\nabla\Phi_i} \, dz.$$

Thanks to Lemma 2.7 we get that $\int_{\partial\mathcal{S}_0} \widehat{\nabla\Phi_i} \, dz = 0$ since the circulation of a gradient around $\partial\mathcal{S}_0$ vanishes and the flux as well according to (24).

Moreover Lemma 2.7 also allows to compute the second term in the Laurent series:

Corollary 2.11. *Let $\mathcal{C}$ be a smooth Jordan curve, $f := (f_1, f_2)$ a smooth vector fields on $\mathcal{C}$:*

$$\begin{aligned}\int_{\mathcal{C}} z\widehat{f}\,dz &= \int_{\mathcal{C}} \begin{pmatrix} x\cdot f \\ x^{\perp}\cdot f\end{pmatrix}\cdot \tau\,ds - i\int_{\mathcal{C}} \begin{pmatrix} x\cdot f \\ x^{\perp}\cdot f\end{pmatrix}\cdot n\,ds \\ &= \int_{\mathcal{C}} (x_1+ix_2)(f\cdot\tau)\,ds - i\int_{\mathcal{C}}(x_1+ix_2)(f\cdot n)\,ds.\end{aligned}$$

Proof. To apply the previous lemma, we have to identify a function g such that $z(f_1 - if_2) = g_1 - ig_2$. Hence, to get the first equality, it is sufficient to check that

$$(x_1+ix_2)(f_1-if_2) = (x_1f_1+x_2f_2) - i(-x_2f_1+x_1f_2) = (x\cdot f) - i(x^{\perp}\cdot f).$$

To obtain the second equality, we simply use $(n_1, n_2) = (-\tau_2, \tau_1)$:

$$\begin{aligned}\begin{pmatrix} x\cdot f \\ x^{\perp}\cdot f\end{pmatrix}\cdot\tau - i\begin{pmatrix} x\cdot f \\ x^{\perp}\cdot f\end{pmatrix}\cdot n &= (x_1f_1+x_2f_2)\tau_1 + (-x_2f_1+x_1f_2)\tau_2 \\ &\quad -i(x_1f_1+x_2f_2)n_1 - i(-x_2f_1+x_1f_2)n_2 \\ &= x_1(f_1\tau_1+f_2\tau_2) + x_2(f_2\tau_1 - f_1\tau_2) \\ &\quad -ix_1(f_1n_1+f_2n_2) - ix_2(f_2n_1 - f_1n_2) \\ &= x_1(f_1\tau_1+f_2\tau_2) + x_2(f_2n_2+f_1n_1) \\ &\quad -ix_1(f_1n_1+f_2n_2) - ix_2(-f_2\tau_2 - f_1\tau_1) \\ &= (x_1+ix_2)(f_1\tau_1+f_2\tau_2) \\ &\quad -i(x_1+ix_2)(f_1n_1+f_2n_2)\end{aligned}$$

which ends the proof. □

Replacing x_2 by $-x_2$ in the previous proof, we note that we obtain

$$\int_{\mathcal{C}} \bar{z}\widehat{f}\,dz = \int_{\mathcal{C}}(x_1-ix_2)(f\cdot\tau)\,ds - i\int_{\mathcal{C}}(x_1-ix_2)(f\cdot n)\,ds. \qquad (59)$$

We apply the previous results to the function $\nabla\Phi_i$:

Lemma 2.12. *One has:*

$$\begin{aligned}\int_{\partial\mathcal{S}_0} z\widehat{\nabla\Phi_i}\,dz &= -(m_{i,2}+|\mathcal{S}_0|\delta_{i,2}) + i(m_{i,1}+|\mathcal{S}_0|\delta_{i,1}), \quad \textit{for } i=1,2; \\ \int_{\partial\mathcal{S}_0} z\widehat{\nabla\Phi_3}\,dz &= -(m_{3,2}+|\mathcal{S}_0|x_{G,1}) + i(m_{3,1}-|\mathcal{S}_0|x_{G,2});\end{aligned}$$

where $m_{i,j}$ is defined in (27).

Proof. We use the previous corollary with $f = \nabla\Phi_i$:

$$\int_{\partial\mathcal{S}_0} z\widehat{\nabla\Phi_i}\, dz \quad = \quad \int_{\partial\mathcal{S}_0} (x_1 + ix_2)\partial_\tau \Phi_i\, ds - i\int_{\partial\mathcal{S}_0} (x_1 + ix_2)\partial_n \Phi_i\, ds.$$

We can integrate by part in the first integral:

$$\begin{aligned}
\int_{\partial\mathcal{S}_0} (x_1 + ix_2)\partial_\tau \Phi_i\, ds \quad &= \quad -\int_{\partial\mathcal{S}_0} \partial_\tau(x_1 + ix_2)\Phi_i\, ds \\
&= \quad -\int_{\partial\mathcal{S}_0} (\tau_1 + i\tau_2)\Phi_i\, ds \\
&= \quad -\int_{\partial\mathcal{S}_0} (n_2 - in_1)\Phi_i\, ds \\
&= \quad -\int_{\mathcal{F}_0} \nabla\Phi_2 \cdot \nabla\Phi_i\, ds + i\int_{\mathcal{F}_0} \nabla\Phi_1 \cdot \nabla\Phi_i\, ds \\
&= \quad -m_{i,2} + im_{i,1}.
\end{aligned}$$

The second integral can be computed thanks to the boundary condition and (65):

$$\begin{aligned}
\int_{\partial\mathcal{S}_0} x_j\partial_n\Phi_i\, ds = \int_{\partial\mathcal{S}_0} x_j n_i\, ds = -\delta_{i,j}|\mathcal{S}_0| \quad \text{for } i, j = 1, 2, \\
\int_{\partial\mathcal{S}_0} x_1\partial_n\Phi_3\, ds = \int_{\partial\mathcal{S}_0} x_1(x^\perp \cdot n)\, ds = |\mathcal{S}_0|x_{G,2} \text{ and} \\
\int_{\partial\mathcal{S}_0} x_2\partial_n\Phi_3\, ds = \int_{\partial\mathcal{S}_0} x_2(x^\perp \cdot n)\, ds = -|\mathcal{S}_0|x_{G,1}.
\end{aligned}$$

□

Using (59), we can reproduce exactly the previous proof to establish that:

$$\begin{aligned}
\int_{\partial\mathcal{S}_0} \bar{z}\widehat{\nabla\Phi_i}\, dz \quad &= \quad (-m_{i,2} + |\mathcal{S}_0|\delta_{i,2}) + i(-m_{i,1} + |\mathcal{S}_0|\delta_{i,1}), \quad \text{for } i = 1, 2; \\
\int_{\partial\mathcal{S}_0} \bar{z}\widehat{\nabla\Phi_3}\, dz \quad &= \quad (-m_{3,2} + |\mathcal{S}_0|x_{G,1}) - i(m_{3,1} + |\mathcal{S}_0|x_{G,2}).
\end{aligned}$$

We are now almost ready to start the proof by itself. The last preparation consists in using the decomposition (53) to deduce from Lemma 2.9 that Equations (38) can be rewritten in the form

$$(\mathcal{M}_g + \mathcal{M}_a)p' + \langle\Gamma_g, p, p\rangle = -(A_i + B_i + C_i)_{i=1,2,3}, \tag{60}$$

where for $i = 1, 2, 3$,

$$A_i := \frac{1}{2}\int_{\partial\mathcal{S}_0} |\tilde{v}|^2 K_i \, ds - \int_{\partial\mathcal{S}_0} (\ell + rx^\perp)\cdot \tilde{v} K_i \, ds,$$

$$B_i := \gamma \int_{\partial\mathcal{S}_0} (\tilde{v} - (\ell + rx^\perp)) \cdot H K_i \, ds, \tag{61}$$

$$C_i := \frac{\gamma^2}{2}\int_{\partial\mathcal{S}_0} |H|^2 K_i \, ds. \tag{62}$$

We start with the computation of the terms $(A_i)_{i=1,2,3}$.

Lemma 2.13.

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = r^2 \begin{pmatrix} -m_{3,2} \\ m_{3,1} \end{pmatrix} + r\Big(\mathcal{M}_\flat \ell\Big)^\perp \tag{63}$$

and

$$A_3 = \ell^\perp \mathcal{M}_\flat \ell - r\ell \cdot \begin{pmatrix} -m_{3,2} \\ m_{3,1} \end{pmatrix}. \tag{64}$$

Proof. We start with the following observation:

$$A_i = \frac{1}{2}\int_{\partial\mathcal{S}_0} |\tilde{v} - (\ell + rx^\perp)|^2 K_i \, ds - \frac{1}{2}\int_{\partial\mathcal{S}_0} |\ell + rx^\perp|^2 K_i \, ds,$$

which makes appear at least one term with the wished tangence property. Since Blasius' lemma is different for the torque, we replace $\tilde{v}$ by the decomposition (54) to get

$$\begin{aligned} A_i &= \frac{1}{2}\int_{\partial\mathcal{S}_0} |v_\# - \ell|^2 K_i \, ds + \frac{1}{2}\int_{\partial\mathcal{S}_0} |r(\nabla\Phi_3 - x^\perp)|^2 K_i \, ds \\ &\quad + \int_{\partial\mathcal{S}_0} r(v_\# - \ell)\cdot(\nabla\Phi_3 - x^\perp) K_i \, ds - \frac{1}{2}\int_{\partial\mathcal{S}_0} |\ell + rx^\perp|^2 K_i \, ds \\ &=: A_{i,a} + A_{i,b} + A_{i,c} + A_{i,d}. \end{aligned}$$

The first three terms have a form appropriated for the strategy mentioned above. One may worry above the last one but it benefits from a special structure resembling the Archimedes' force (despite the absence of gravity in our setting). Let us see first how this term can be simply computed thanks to the Stokes formula so that we will then be serene to implement the complex-analytic approach to the three other terms.

An Archimedes-type term. In order to compute the term $A_{i,d}$ we first expand

$$\begin{aligned} \int_{\partial\mathcal{S}_0} |\ell + rx^\perp|^2 K_i \, ds &= |\ell|^2 \int_{\partial\mathcal{S}_0} K_i \, ds - 2\ell_1 r \int_{\partial\mathcal{S}_0} x_2 K_i \, ds \\ &\quad + 2\ell_2 r \int_{\partial\mathcal{S}_0} x_1 K_i \, ds + r^2 \int_{\partial\mathcal{S}_0} |x|^2 K_i \, ds. \end{aligned}$$

Thanks to the Stokes formula:

$$\int_{\partial\mathcal{S}_0} x_j n_i \, ds = -\int_{\mathcal{S}_0} \operatorname{div}(x_j e_i)\, dx = -\delta_{i,j}|\mathcal{S}_0|, \quad \text{for } i,j=1,2; \tag{65a}$$

$$\int_{\partial\mathcal{S}_0} x_1 (x^\perp \cdot n)\, ds = -\int_{\mathcal{S}_0} \operatorname{div}(x_1 x^\perp)\, dx = -\int_{\mathcal{S}_0} (-x_2)\, dx = |\mathcal{S}_0| x_{G,2}; \tag{65b}$$

$$\int_{\partial\mathcal{S}_0} x_2 (x^\perp \cdot n)\, ds = -\int_{\mathcal{S}_0} \operatorname{div}(x_2 x^\perp)\, dx = -\int_{\mathcal{S}_0} x_1\, dx = -|\mathcal{S}_0| x_{G,1}; \tag{65c}$$

and

$$\int_{\partial\mathcal{S}_0} |x|^2 n_i\, ds = -\int_{\mathcal{S}_0} \operatorname{div}(|x|^2 e_i)\, dx = -\int_{\mathcal{S}_0} 2x_i\, dx$$
$$= -2x_{G,i}|\mathcal{S}_0|, \quad \text{for } i=1,2; \tag{66a}$$

$$\int_{\partial\mathcal{S}_0} |x|^2 x^\perp \cdot n\, ds = -\int_{\mathcal{S}_0} \operatorname{div}(|x|^2 x^\perp)\, dx = 0, \tag{66b}$$

where $|\mathcal{S}_0|$ is the Lebesgue measure of $\mathcal{S}_0$ and $x_G = (x_{G,1}, x_{G,2})$ is the position of the geometrical center of $\mathcal{S}_0$ (which can be different of the mass center 0 if the solid is not homogenous):

$$x_G := \frac{1}{|\mathcal{S}_0|} \int_{\mathcal{S}_0} x\, dx. \tag{67}$$

Then, using also (24), we check easily that

$$\Big(A_{i,d}\Big)_{i=1,2} = -r\ell^\perp |\mathcal{S}_0| + r^2 x_G |\mathcal{S}_0| \text{ and } A_{3,d} = -r(\ell \cdot x_G)|\mathcal{S}_0|. \tag{68}$$

Computation of the three other terms. Recalling the definition (55) and using the notation (30) we first remark that

$$\widehat{v_\# - \ell}(z) = -\ell_1 + i\ell_2 + \ell_1 \widehat{\nabla\Phi_1} + \ell_2 \widehat{\nabla\Phi_2}. \tag{69}$$

Computation of $A_{i,a}$. We compute separately the case where $i = 1,2$ and the case where $i = 3$.

• As $v_\# - \ell$ is tangent to the boundary, we can apply the Blasius formula (see Lemma 2.10), (69), Cauchy's residue theorem, to obtain

$$\Big(A_{i,a}\Big)_{i=1,2} = 0. \tag{70}$$

• We proceed in the same way for $i = 3$:

$$A_{3,a} = \frac{1}{2}\int_{\partial\mathcal{S}_0} |v_\# - \ell|^2 K_3\, ds = \frac{1}{2}\operatorname{Re}\left(\int_{\partial\mathcal{S}_0} z(\widehat{v_\# - \ell})^2\, dz\right)$$
$$= \operatorname{Re}\left(\Big[(-\ell_1) - i(-\ell_2)\Big]\int_{\partial\mathcal{S}_0} z\Big(\ell_1\widehat{\nabla\Phi_1} + \ell_2\widehat{\nabla\Phi_2}\Big)\, dz\right).$$

so that, thanks to Lemma 2.12,

$$A_{3,a} = (-\ell_1)\Big[-\ell_1 m_{1,2} - \ell_2(m_{2,2} + |\mathcal{S}_0|)\Big] + (-\ell_2)\Big[\ell_1(m_{1,1} + |\mathcal{S}_0|) + \ell_2 m_{2,1}\Big]$$

which finally can be simplified as follows:

$$A_{3,a} = \ell^{\perp}\mathcal{M}_{\flat}\ell. \tag{71}$$

Computation of $A_{i,b}$. Once again we distinguish the case where $i = 1,2$ and the case where $i = 3$.

• As $\nabla\Phi_3 - x^{\perp}$ is tangent to the boundary, we can write for $i = 1,2$ by Lemma 2.10 and by the Cauchy's residue theorem:

$$\Big(A_{i,b}\Big)_{i=1,2} = \frac{r^2}{2}\int_{\partial\mathcal{S}_0} |\nabla\Phi_3 - x^{\perp}|^2 n\, ds = \frac{ir^2}{2}\left(\int_{\partial\mathcal{S}_0} (\widehat{\nabla\Phi_3 - x^{\perp}})^2\, dz\right)^* \\ = \frac{ir^2}{2}\left(\int_{\partial\mathcal{S}_0} 2i\bar{z}\widehat{\nabla\Phi_3}\, dz - \int_{\partial\mathcal{S}_0} \bar{z}^2\, dz\right)^*,$$

where we have noted that $-\widehat{x^{\perp}} = i\bar{z}$. Let us observe that

$$\begin{aligned}\int_{\partial\mathcal{S}_0} \bar{z}^2\, dz &= \int (\gamma_1 - i\gamma_2)^2(\gamma_1' + i\gamma_2') \\ &= \int (\gamma_1^2\gamma_1' - \gamma_2^2\gamma_1' + 2\gamma_1\gamma_2\gamma_2') + i\int (\gamma_1^2\gamma_2' - \gamma_2^2\gamma_2' - 2\gamma_1\gamma_2\gamma_1') \\ &= -\int_{\partial\mathcal{S}_0} \begin{pmatrix} 2x_1x_2 \\ x_2^2 \end{pmatrix}\cdot n\, ds - i\int_{\partial\mathcal{S}_0} \begin{pmatrix} x_1^2 \\ 2x_1x_2 \end{pmatrix}\cdot n\, ds \\ &= \int_{\mathcal{S}_0} \operatorname{div}\begin{pmatrix} 2x_1x_2 \\ x_2^2 \end{pmatrix} dx + i\int_{\mathcal{S}_0} \operatorname{div}\begin{pmatrix} x_1^2 \\ 2x_1x_2 \end{pmatrix} dx \\ &= 4|\mathcal{S}_0|x_{G,2} + 4i|\mathcal{S}_0|x_{G,1}.\end{aligned}$$

Therefore thanks to Lemma 2.12 we state that:

$$\begin{aligned}\Big(A_{i,b}\Big)_{i=1,2} &= ir^2\Big(i(-m_{3,2} + |\mathcal{S}_0|x_{G,1}) + (m_{3,1} + |\mathcal{S}_0|x_{G,2}) \\ &\quad -2(|\mathcal{S}_0|x_{G,2} + i|\mathcal{S}_0|x_{G,1})\Big)^* \\ &= r^2\Big((-m_{3,2} - |\mathcal{S}_0|x_{G,1}) + i(m_{3,1} - |\mathcal{S}_0|x_{G,2})\Big) \\ &= r^2\left(\begin{pmatrix} -m_{3,2} \\ m_{3,1} \end{pmatrix} - |\mathcal{S}_0|x_G\right),\end{aligned}$$

and thus

$$\Big(A_{i,b}\Big)_{i=1,2} = 0. \tag{72}$$

• For $i = 3$, we have that:

$$\begin{aligned}
A_{3,b} &= \int_{\partial\mathcal{S}_0} |\nabla\Phi_3 - x^\perp|^2 (x^\perp \cdot n)\, ds \\
&= \int_{\partial\mathcal{S}_0} |\nabla\Phi_3|^2 x^\perp \cdot n\, ds - 2\int_{\partial\mathcal{S}_0} (\nabla\Phi_3 \cdot x^\perp)(x^\perp \cdot n)\, ds + \int_{\partial\mathcal{S}_0} |x|^2 x^\perp \cdot n\, ds \\
&= \int_{\mathcal{F}_0} \operatorname{div}(|\nabla\Phi_3|^2 x^\perp)\, dx - 2\int_{\mathcal{F}_0} \nabla(\nabla\Phi_3 \cdot x^\perp)\cdot\nabla\Phi_3\, dx - \int_{\mathcal{S}_0} \operatorname{div}(|x|^2 x^\perp)\, dx,
\end{aligned}$$

where there is no boundary term at infinity because $\nabla\Phi_3 = \mathcal{O}(1/|x|^2)$. Next we use the general relation (49) to obtain that

$$\begin{aligned}
\nabla(\nabla\Phi_3 \cdot x^\perp)\cdot\nabla\Phi_3 &= \Big[(\nabla\Phi_3\cdot\nabla)x^\perp + (x^\perp\cdot\nabla)\nabla\Phi_3\Big]\cdot\nabla\Phi_3 \\
&= -(\nabla\Phi_3)^\perp\cdot\nabla\Phi_3 + \frac{1}{2}(x^\perp\cdot\nabla)|\nabla\Phi_3|^2 = \frac{1}{2}\operatorname{div}(x^\perp|\nabla\Phi_3|^2).
\end{aligned}$$

Hence,

$$A_{3,b} = 0. \tag{73}$$

Computation of $A_{i,c}$.

• We use again the Blasius formula together with (69) and the Cauchy's residue theorem:

$$\begin{aligned}
\Big(A_{i,c}\Big)_{i=1,2} &= \int_{\partial\mathcal{S}_0} r(v_\# - \ell)\cdot(\nabla\Phi_3 - x^\perp) n\, ds \\
&= ir\left(\int_{\partial\mathcal{S}_0} (\widehat{v_\# - \ell})(\widehat{\nabla\Phi_3} + i\bar{z})\, dz\right)^* \\
&= ir\left(i\Big(-\ell_1 + i\ell_2\Big)\int_{\partial\mathcal{S}_0} \bar{z}\, dz + i\ell_1\int_{\partial\mathcal{S}_0} \widehat{\nabla\Phi_1}\bar{z}\, dz + i\ell_2\int_{\partial\mathcal{S}_0} \widehat{\nabla\Phi_2}\bar{z}\, dz\right)^*.
\end{aligned}$$

Let us observe that

$$\begin{aligned}
\int_{\partial\mathcal{S}_0} \bar{z}\, dz &= \int(\gamma_1 - i\gamma_2)(\gamma_1' + i\gamma_2') = \int(\gamma_1\gamma_1' + \gamma_2\gamma_2') + i\int(\gamma_1\gamma_2' - \gamma_2\gamma_1') \\
&= -i\int_{\partial\mathcal{S}_0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\cdot n\, ds \\
&= i\int_{\mathcal{S}_0} \operatorname{div}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\, dx = 2i|\mathcal{S}_0|;
\end{aligned} \tag{74}$$

$$\begin{aligned}\int_{\partial\mathcal{S}_0}|z|^2\,dz &= \int(\gamma_1^2+\gamma_2^2)(\gamma_1'+i\gamma_2')\\ &= \int_{\partial\mathcal{S}_0}\begin{pmatrix}0\\x_2^2\end{pmatrix}\cdot n\,ds - i\int_{\partial\mathcal{S}_0}\begin{pmatrix}x_1^2\\0\end{pmatrix}\cdot n\,ds\\ &= -\int_{\mathcal{S}_0}\mathrm{div}\begin{pmatrix}0\\x_2^2\end{pmatrix}dx + i\int_{\mathcal{S}_0}\mathrm{div}\begin{pmatrix}x_1^2\\0\end{pmatrix}dx\\ &= -2|\mathcal{S}_0|x_{G,2}+2i|\mathcal{S}_0|x_{G,1}.\end{aligned} \tag{75}$$

Therefore, it suffices to write the value obtained in Lemma 2.12 to get:

$$\begin{aligned}\Big(A_{i,c}\Big)_{i=1,2} =& r\Big[(-\ell_2)2|\mathcal{S}_0|-\ell_1 m_{1,2}+\ell_2(-m_{2,2}+|\mathcal{S}_0|)\Big]\\ &+ir\Big[\ell_1 2|\mathcal{S}_0|-\ell_1(-m_{1,1}+|\mathcal{S}_0|)+\ell_2 m_{2,1}\Big],\end{aligned}$$

which can be simplified as

$$\Big(A_{i,c}\Big)_{i=1,2} = r\left[\Big(\mathcal{M}_\flat+|\mathcal{S}_0|\mathrm{I}_2)\ell\Big)^\perp\right]. \tag{76}$$

- For $i=3$, Lemma 2.10, (69) and Cauchy's residue theorem imply that

$$\begin{aligned}A_{3,c} =& \int_{\partial\mathcal{S}_0} r(v_\# -\ell)\cdot(\nabla\Phi_3 - x^\perp)K_3\,ds\\ =& r\mathrm{Re}\left(\int_{\partial\mathcal{S}_0} z(\widehat{v_\#-\ell})(\widehat{\nabla\Phi_3}+i\bar z)\,dz\right)\\ =& r\mathrm{Re}\Bigg[\Big(-\ell_1+i\ell_2\Big)\int_{\partial\mathcal{S}_0}(z\widehat{\nabla\Phi_3}+i|z|^2)\,dz\\ &+\ell_1 i\int_{\partial\mathcal{S}_0}\widehat{\nabla\Phi_1}|z|^2\,dz+\ell_2 i\int_{\partial\mathcal{S}_0}\widehat{\nabla\Phi_2}|z|^2\,dz\Bigg].\end{aligned}$$

Now applying Lemma 2.7 to $(|z|^2\partial_1\Phi_i, |z|^2\partial_2\Phi_i)$ we have

$$\int_{\partial\mathcal{S}_0}|z|^2(\partial_1\Phi_i - i\partial_2\Phi_i)\,dz = \int_{\partial\mathcal{S}_0}|x|^2\partial_\tau\Phi_i\,ds - i\int_{\partial\mathcal{S}_0}|x|^2\partial_n\Phi_i\,ds$$

where we easily verify that

$$\int_{\partial\mathcal{S}_0}|x|^2\partial_\tau\Phi_i\,ds = -\int_{\partial\mathcal{S}_0}\Phi_i 2x\cdot\tau\,ds = -\int_{\partial\mathcal{S}_0}\Phi_i 2(x^\perp\cdot n)\,ds = -2m_{i,3}.$$

The value of $\int_{\partial\mathcal{S}_0}|x|^2\partial_n\Phi_i\,ds$ has already been computed in (66). We therefore obtain:

$$\int_{\partial\mathcal{S}_0}|z|^2\widehat{\nabla\Phi_i}\,dz = -2m_{i,3}+2i|\mathcal{S}_0|x_{G,i} \text{ for } i=1,2 \text{ and } \int_{\partial\mathcal{S}_0}|z|^2\widehat{\nabla\Phi_3}\,dz = -2m_{3,3}.$$

Hence, we deduce from this, (75) and Lemma 2.12 that

$$\begin{aligned} A_{3,c} = & r\Big[- (-\ell_1)(m_{3,2} + |\mathcal{S}_0|x_{G,1} + 2|\mathcal{S}_0|x_{G,1}) \\ & (-\ell_2)(m_{3,1} - |\mathcal{S}_0|x_{G,2} - 2|\mathcal{S}_0|x_{G,2}) \\ & - \ell_1 2|\mathcal{S}_0|x_{G,1} - \ell_2 2|\mathcal{S}_0|x_{G,2}\Big], \end{aligned}$$

so that

$$A_{3,c} = -r\ell \cdot \left(\begin{pmatrix} -m_{3,2} \\ m_{3,1} \end{pmatrix} - |\mathcal{S}_0|x_G \right). \tag{77}$$

Conclusion. Gathering (68), (70), (71), (72), (73), (76) and (77) we obtain (63) and (64). This ends the proof of Lemma 2.13.

□

Let us continue with the term B_i. Let us prove the following.

Lemma 2.14. *One has*

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = -\gamma \ell^\perp + \gamma r \xi \text{ and } B_3 = -\gamma\, \xi \cdot \ell, \tag{78}$$

where ξ was defined in (33).

Proof. Putting the decomposition (54) in the definition of B_i, we write:

$$B_i = \gamma \int_{\partial \mathcal{S}_0} (v_\# - \ell) \cdot HK_i \, ds + \gamma \int_{\partial \mathcal{S}_0} r(\nabla \Phi_3 - x^\perp) \cdot HK_i \, ds.$$

Concerning the second integral, as $v_\# - \ell$ and H are tangent to the boundary, we apply the Blasius formula (see Lemma 2.10), then we compute by (69) and Cauchy's residue theorem and (31):

$$\begin{aligned} \left(\gamma \int_{\partial \mathcal{S}_0} (v_\# - \ell) \cdot HK_i \, ds \right)_{i=1,2} = & \gamma \int_{\partial \mathcal{S}_0} (v_\# - \ell) \cdot Hn \, ds \\ = & i\gamma \left(\int_{\partial \mathcal{S}_0} (\widehat{v_\# - \ell}) \widehat{H} \, dz \right)^* \\ = & i\gamma \left(\Big(- \ell_1 + i\ell_2 \Big) \int_{\partial \mathcal{S}_0} \widehat{H} \, dz \right)^* \\ = & i\gamma \left(\Big(- \ell_1 + i\ell_2 \Big) \right)^* \\ = & - \gamma \ell^\perp. \end{aligned}$$

For $i=3$, we compute by Lemma 2.10 and the Cauchy's residue theorem that

$$\begin{aligned}\gamma\int_{\partial\mathcal{S}_0}(v_\# - \ell)\cdot HK_3\,ds =&\gamma\mathrm{Re}\left(\int_{\partial\mathcal{S}_0} z(\widehat{v_\# - \ell})\widehat{H}\,dz\right)\\ =&\gamma\mathrm{Re}\left(\Big(-\ell_1+i\ell_2\Big)\int_{\partial\mathcal{S}_0} z\widehat{H}\,dz\right)\\ =&\gamma\mathrm{Re}\left(\Big(-\ell_1+i\ell_2\Big)(\xi_1+i\xi_2)\right)\\ =&-\gamma\ell\cdot\xi.\end{aligned}$$

For the last term, we use that $\nabla\Phi_3 - x^\perp$ and H are tangent to the boundary, and we write by Lemma 2.10 and by the Cauchy's residue theorem:

$$\begin{aligned}\left(\gamma r\int_{\partial\mathcal{S}_0}(\nabla\Phi_3 - x^\perp)\cdot HK_i\,ds\right)_{i=1,2} &= i\gamma r\left(\int_{\partial\mathcal{S}_0}(\widehat{\nabla\Phi_3 - x^\perp})\widehat{H}\,dz\right)^*\\ &= i\gamma r\left(i\int_{\partial\mathcal{S}_0}\bar{z}\widehat{H}\,dz\right)^*\\ &= \gamma r\left(\int_{\partial\mathcal{S}_0}\bar{z}\widehat{H}\,dz\right)^* = \gamma r\xi,\end{aligned}$$

where we have used that $-\widehat{x^\perp} = i\bar{z}$ and

$$\left(\int_{\partial\mathcal{S}_0}\bar{z}\widehat{H}\,dz\right)^* = \int_{\partial\mathcal{S}_0} z\widehat{H}\,dz, \tag{79}$$

the latter being easily shown by using a parametrization. For $i=3$, we have that:

$$\begin{aligned}\gamma r\int_{\partial\mathcal{S}_0}(\nabla\Phi_3 - x^\perp)\cdot HK_3\,ds &= \gamma r\mathrm{Re}\left(\int_{\partial\mathcal{S}_0} z(\widehat{\nabla\Phi_3 - x^\perp})\widehat{H}\,dz\right)\\ &= \gamma r\mathrm{Re}\left(i\int_{\partial\mathcal{S}_0} z\bar{z}\widehat{H}\,dz\right) = 0,\end{aligned}$$

which can also be shown by using a parametrization. Gathering the equalities above yields (78). This ends the proof of Lemma 2.14. □

We now turn to C_i (see (62)). From Lemma 2.10, (31) and Cauchy's Residue Theorem, we deduce that

$$C_1 = C_2 = C_3 = 0. \tag{80}$$

Indeed, we can verify that $\int_{\partial\mathcal{S}_0} z(\widehat{H})^2\,dz = -i/(2\pi)$ is purely imaginary.

Plugging (63), (64), (78) and (80) in (60) and using that for any $p_a := (\ell_a,\omega_a)$ and $p_b := (\ell_b,\omega_b)$ in $\mathbb{R}^2\times\mathbb{R}$, $p_a\times p_b = (\omega_a\,\ell_b^\perp - \omega_b\,\ell_a^\perp, \ell_a^\perp\cdot\ell_b)$, yields (44). This ends the complex-analytic proof of Theorem 2.8.

2.5. A real-analytic proof of Theorem 2.8

We now consider another approach which dates back to Lamb. We therefore go back to Lemma 2.9 and provide an alternative real-analytic end of the proof of Theorem 2.8. Of crucial importance is the following identity which we will use instead of Blasius' lemma though to different terms.

Let

$$\zeta_1(x) := e_1, \quad \zeta_2(x) := e_2 \quad \text{and} \quad \zeta_3(x) := x^\perp.$$

denote the elementary rigid velocities. The following lemma seems to originate from Lamb's work.

Lemma 2.15. *For any pair of vector fields* (u, v) *in* $C^\infty(\overline{\mathbb{R}^2 \setminus \mathcal{S}_0}; \mathbb{R}^2)$ *satisfying*

- $\operatorname{div} u = \operatorname{div} v = \operatorname{curl} u = \operatorname{curl} v = 0,$
- $u(x) = O(1/|x|)$ *and* $v(x) = O(1/|x|)$ *as* $|x| \to +\infty,$

one has, for any $i = 1, 2, 3,$

$$\int_{\partial \mathcal{S}_0} (u \cdot v) K_i \mathrm{d}s = \int_{\partial \mathcal{S}_0} \zeta_i \cdot \Big((u \cdot n) v + (v \cdot n) u \Big) \mathrm{d}s. \tag{81}$$

Proof of Lemma 2.15. Let us start with the case where $i = 1$ or 2. Then

$$\int_{\partial \mathcal{S}_0} (u \cdot v) K_i \mathrm{d}s = \int_{\partial \mathcal{S}_0} \big((u \cdot v) \zeta_i \big) \cdot n \mathrm{d}s = \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} \operatorname{div} \big((u \cdot v) \zeta_i \big) \, \mathrm{d}x, \tag{82}$$

by using that $u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ when $|x| \to +\infty$. Therefore

$$\int_{\partial \mathcal{S}_0} (u \cdot v) K_i \mathrm{d}s = \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} \zeta_i \cdot \nabla (u \cdot v) \, \mathrm{d}x = \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} \zeta_i \cdot (u \cdot \nabla v + v \cdot \nabla u) \, \mathrm{d}x, \tag{83}$$

using that $\operatorname{curl} u = \operatorname{curl} v = 0$. Now, integrating by parts, using that $\operatorname{div} u = \operatorname{div} v = 0$ and once again that $u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ as $|x| \to +\infty$, we obtain (81) when $i = 1$ or 2.

We now tackle the case where $i = 3$. We follow the same lines as above, with two precisions. First we observe that there is no contribution at infinity in (82) and (83) when $i = 3$ as well. Indeed ζ_3 and the normal to a centered circle are orthogonal. Moreover there is no additional distributed term coming from the integration by parts in (83) when $i = 3$ since

$$\int_{\mathbb{R}^2 \setminus \mathcal{S}_0} v \cdot (u \cdot \nabla_x \zeta_i) + u \cdot (v \cdot \nabla_x \zeta_i) \, \mathrm{d}x = \int_{\mathbb{R}^2 \setminus \mathcal{S}_0} (v \cdot u^\perp + u \cdot v^\perp) \, \mathrm{d}x = 0.$$

□

As a consequence, using Lamb's lemma and the boundary conditions (39) we obtain for any $i = 1, 2, 3$,

$$\begin{aligned}
\frac{1}{2}\int_{\partial\mathcal{S}_0} |v|^2 K_i\, ds &= \int_{\partial\mathcal{S}_0} (v\cdot n)(v\cdot\zeta_i)\, ds \\
&= \int_{\partial\mathcal{S}_0} \big((\ell + rx^\perp)\cdot n\big)\big(v\cdot\zeta_i\big)\, ds \\
&= \int_{\partial\mathcal{S}_0} \big((\ell + rx^\perp)\cdot n\big)\big(v\cdot n\big) K_i\, ds \\
&\quad + \int_{\partial\mathcal{S}_0} \big((\ell + rx^\perp)\cdot n\big)\big(v\cdot\tau\big)\big(\zeta_i\cdot\tau\big)\, ds
\end{aligned}$$

so that the right hand side of (48) can be recast as follows:

$$\begin{aligned}
&\frac{1}{2}\int_{\partial\mathcal{S}_0} |v|^2 K_i\, ds - \int_{\partial\mathcal{S}_0} (\ell + rx^\perp)\cdot v K_i\, ds \\
&= -\int_{\partial\mathcal{S}_0} \big((\ell + rx^\perp)\cdot\tau\big)\big(v\cdot\tau\big) K_i\, ds + \int_{\partial\mathcal{S}_0} \big((\ell + rx^\perp)\cdot n\big)\big(v\cdot\tau\big)\big(\zeta_i\cdot\tau\big)\, ds \\
&= \sum_k p_k \int_{\partial\mathcal{S}_0} (v\cdot\tau)\big[\big(\zeta_i\cdot\tau\big)K_k - \big(\zeta_k\cdot\tau\big)K_i\big]\, ds,
\end{aligned}$$

for any $i = 1, 2, 3$, where the sum runs for k over $1, 2, 3$.

2.5.1. Computation of the brackets. We now compute the brackets $[(\zeta_i\cdot\tau)K_k - (\zeta_k\cdot\tau)K_i]$, for $i, k = 1, 2, 3$, in order to make explicit the previous integrals.

- For $i = 1$, we therefore obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{\partial\mathcal{S}_0} |v|^2 K_i\, ds - \int_{\partial\mathcal{S}_0} (\ell + rx^\perp)\cdot v K_i\, ds = \\
&p_2 \int_{\partial\mathcal{S}_0} (v\cdot\tau)\big[\big(\zeta_1\cdot\tau\big)K_2 - \big(\zeta_2\cdot\tau\big)K_1\big]\, ds \\
&+ p_3 \int_{\partial\mathcal{S}_0} (v\cdot\tau)\big[\big(\zeta_1\cdot\tau\big)K_3 - \big(\zeta_3\cdot\tau\big)K_1\big]\, ds.
\end{aligned}$$

Using that

$$\begin{aligned}
\big(\zeta_1\cdot\tau\big)K_2 - \big(\zeta_2\cdot\tau\big)K_1 &= \big(\zeta_1\cdot\tau\big)\big(\zeta_2\cdot n\big) - \big(\zeta_2\cdot\tau\big)\big(\zeta_1\cdot n\big) \\
&= \big(\zeta_2\cdot n\big)^2 + \big(\zeta_2\cdot\tau\big)^2 \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\big(\zeta_1\cdot\tau\big)\big(\zeta_3\cdot n\big) - \big(\zeta_3\cdot\tau\big)\big(\zeta_1\cdot n\big) &= \big(\zeta_2\cdot n\big)\big(\zeta_3\cdot n\big) + \big(\zeta_3\cdot\tau\big)\big(\zeta_2\cdot\tau\big) \\
&= \zeta_2\cdot\zeta_3,
\end{aligned}$$

we infer from the decomposition (52) and (34) that

$$\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2K_i\,ds-\int_{\partial\mathcal{S}_0}(\ell+rx^\perp)\cdot vK_i\,ds=\gamma\ell_2+\gamma r\zeta_2\cdot\xi^\perp$$
$$+r\sum_{j=1}^{3}p_j\,\zeta_2\cdot\int_{\partial\mathcal{S}_0}\big(\nabla\Phi_j\cdot\tau\big)x^\perp\,ds.$$

- For $i=2$,

$$\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2K_i\,ds-\int_{\partial\mathcal{S}_0}(\ell+rx^\perp)\cdot vK_i\,ds=$$
$$p_1\int_{\partial\mathcal{S}_0}\big(v\cdot\tau\big)\big[\big(\zeta_2\cdot\tau\big)K_1-\big(\zeta_1\cdot\tau\big)K_2\big]\,ds$$
$$+p_3\int_{\partial\mathcal{S}_0}\big(v\cdot\tau\big)\big[\big(\zeta_2\cdot\tau\big)K_3-\big(\zeta_3\cdot\tau\big)K_2\big]\,ds$$

But

$$\big(\zeta_2\cdot\tau\big)K_1-\big(\zeta_1\cdot\tau\big)K_2=-\big(\zeta_2\cdot\tau\big)^2-\big(\zeta_1\cdot\tau\big)^2=-1,$$

and

$$\big(\zeta_2\cdot\tau\big)K_3-\big(\zeta_3\cdot\tau\big)K_2=-\big(\zeta_1\cdot n\big)\big(\zeta_3\cdot n\big)-\big(\zeta_3\cdot\tau\big)\big(\zeta_1\cdot\tau\big)=-\zeta_1\cdot\zeta_3,$$

so that

$$\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2K_i\,ds-\int_{\partial\mathcal{S}_0}(\ell+rx^\perp)\cdot vK_i\,ds=-\gamma\ell_1-\gamma r\zeta_1\cdot\xi^\perp$$
$$-r\sum_{j=1}^{3}p_j\,\zeta_1\cdot\int_{\partial\mathcal{S}_0}\big(\nabla\Phi_j\cdot\tau\big)x^\perp\,ds.$$

Moreover, by an integration by parts,

$$\int_{\partial\mathcal{S}_0}\big(\nabla\Phi_j\cdot\tau\big)x^\perp\,ds=-\int_{\partial\mathcal{S}_0}\Phi_j\cdot n\,ds=-\Big(\int_{\partial\mathcal{S}_0}\Phi_j\partial_n\Phi_i\,ds\Big)_i=-\big(m_{i,j}\big)_i,$$

where i runs over $1,2$, so that

$$-\Big(\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2K_i\,ds-\int_{\partial\mathcal{S}_0}(\ell+rx^\perp)\cdot vK_i\,ds\Big)_{i=1,2}\quad=\quad-r\Big(\mathcal{M}^\flat\ell+r\begin{pmatrix}m_{1,3}\\ m_{2,3}\end{pmatrix}\Big)^\perp$$
$$+\gamma\big(\ell^\perp-r\xi\big).$$

- Proceeding in the same way for $i=3$ and using the definitions (41) and (42) we finally arrive at (44). This ends the real-analytic proof of Theorem 2.8.

2.6. Zero radius limit

We now assume that, for every $\varepsilon \in (0,1]$, the domain occupies (5) and for every $q=(h,\theta)\in\mathbb{R}^3$,

$$\mathcal{S}^\varepsilon(q) := R(\theta)\mathcal{S}_0^\varepsilon + h \text{ and } \mathcal{F}^\varepsilon(q) = \mathbb{R}^2 \setminus \bar{\mathcal{S}}^\varepsilon(q). \tag{84}$$

We will treat at once both the massive and massless cases. The following statement implies Theorem 1.4.

Theorem 2.16. *We consider a rescaled initial domain $\mathcal{S}_0$ occupied by the body, some initial solid translation and rotation velocities (ℓ_0, r_0) and a circulation γ in $\mathbb{R}$ in the case of a massive particle and in $\mathbb{R}^*$ in the case of a massless particle, all independent of ε. Let, for each $\varepsilon > 0$, the solution h^ε associated with an initial solid domain $\mathcal{S}_0^\varepsilon$ defined by* (5) *with the inertia scaling described in Definition 1.1, and initial data $q(0)=0$ and $q'(0)=(\ell_0,r_0)$, given by Proposition 2.4. Then for all $T>0$, as $\varepsilon\to 0$ one has in the case of massive particle (respectively massless particle) $h^\varepsilon \rightharpoonup h$ in $W^{2,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$ (resp. $W^{1,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$) and $\varepsilon\theta^\varepsilon \rightharpoonup 0$ in $W^{2,\infty}([0,T];\mathbb{R})$ weak-$\star$ (resp. $W^{1,\infty}([0,T];\mathbb{R})$ weak-$\star$) . Moreover the limit time-dependent vector h satisfies the equations $mh''=\gamma(h')^\perp$ (resp. $h'=0$).*

Proof. In order to compare the influence of the circulation and of the solid velocity in the zero-radius limit we first consider the harmonic field H^ε and the Kirchhoff potentials $(\nabla\Phi_i^\varepsilon)_{i=1,2,3}$ associated with the rigid body $\mathcal{S}^\varepsilon(0)$ as the harmonic field H and the Kirchhoff potentials $(\nabla\Phi_i)_{i=1,2,3}$ were associated with the rigid body $\mathcal{S}_0$ in Section 2.2. They satisfy the following scaling law $H^\varepsilon(x)=\varepsilon^{-1}H(x/\varepsilon)$, whereas obey: $\Phi_i^\varepsilon(x)=\varepsilon\Phi_i^1(x/\varepsilon)$ for $i=1,2$, $\Phi_3^\varepsilon(x)=\varepsilon^2\Phi_3^1(x/\varepsilon)$. Therefore the harmonic field H^ε is more singular than the Kirchhoff potentials $\nabla\Phi_i^\varepsilon(x)$ in the vanishingly small limit. On the other hand we deduce that the added inertia is given by the following matrix

$$\mathcal{M}_{a,\theta}^\varepsilon = \varepsilon^2\, I_\varepsilon \mathcal{M}_{a,\theta} I_\varepsilon, \tag{85}$$

where I_ε is the diagonal matrix $I_\varepsilon := \text{diag }(1,1,\varepsilon)$. This has to be compared with the genuine inertia matrix which, according to Definition 1.1, scales as follows:

$$\mathcal{M}_g^\varepsilon := \text{diag }(m^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon) = \varepsilon^\alpha I_\varepsilon \mathcal{M}_g I_\varepsilon, \tag{86}$$

where the matrix $\mathcal{M}_g$ is given in terms of $m>0$ and $\mathcal{J}>0$ by (18). Recall that $\alpha>0$, $m>0$ and $\mathcal{J}>0$ are defined in Definition 1.1 and fixed independent of ε.

Two remarks are in order.

- First we observe from the comparison of (85) and (86) that the physical case $\alpha=2$ appears as critical.
- Secondly because of the matrix I_ε in the right hand sides of the two inertia matrices $\mathcal{M}_g^\varepsilon$ and $\mathcal{M}_{a,\theta}^\varepsilon$, it is natural to introduce the vector

$p^\varepsilon = ((h^\varepsilon)', \varepsilon(\theta^\varepsilon)')^t$. Hence the natural counterpart to $(h^\varepsilon)'$ for what concerns the angular velocity is rather $\varepsilon(\theta^\varepsilon)'$ than $(\theta^\varepsilon)'$. This can also be seen on the boundary condition (90d): when x belongs to $\partial\mathcal{S}^\varepsilon(t)$, the term $(\theta^\varepsilon)'(x-h^\varepsilon)^\perp$ is of order $\varepsilon(\theta^\varepsilon)'$ and is added to $(h^\varepsilon)'$.

Examining how the other terms scale with ε one obtains:

$$(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})(p^\varepsilon)' + \varepsilon\langle \Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle = \gamma p^\varepsilon \times B_{\theta^\varepsilon}. \tag{87}$$

The energy associated with this scaling is twice $(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})p^\varepsilon \cdot p^\varepsilon$ and its conservation provides that p^ε is bounded uniformly with respect to ε on the time interval $[0,+\infty)$, whatever is α. Now our goal is to pass to the limit in (87). Let $T>0$. Computing the right-hand-side of (87) gives

$$(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})(p^\varepsilon)' + \varepsilon\langle \Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle = \gamma \begin{pmatrix} ((h^\varepsilon)')^\perp - \varepsilon(\theta^\varepsilon)'R(\theta^\varepsilon)\xi \\ R(\theta^\varepsilon)\xi \cdot (h^\varepsilon)' \end{pmatrix}. \tag{88}$$

We start with the massive case for which $\alpha = 0$. Using the equation we deduce some uniform $W^{2,\infty}$ bounds on h_ε and $\varepsilon\theta^\varepsilon$ and this entails the existence of a subsequence of $(h^\varepsilon, \varepsilon\theta^\varepsilon)$ converging to (h,Θ) in $W^{2,\infty}$ weak-$\star$. Moreover the left hand side of (88) (with $\alpha=0$) converges to $\mathcal{M}_g(h'',\Theta'')^t$ in L^∞ weak-$\star$, and using that

$$\varepsilon(\theta^\varepsilon)'R(\theta^\varepsilon)\xi = \varepsilon\big(R(\theta^\varepsilon - \pi/2)\xi\big)' \tag{89}$$

converges in $W^{-1,\infty}$ weak-$\star$ up to a subsequence and that the weak-$\star$ convergence in $W^{2,\infty}$ entails the strong $W^{1,\infty}$ one, we get from the two first lines of (88) that $mh'' = \gamma(h')^\perp$ and $\big(h(0),h'(0)\big) = \big(0,\ell_0\big)$. In order to prove that $\Theta = 0$ one may use a stationary phase argument, cf. Lemma 10 in [4] for more on this.

In the massless case, that is when $\alpha > 0$, a few modifications in the arguments are in order. First, thanks to the energy estimate, $\varepsilon\mathcal{M}_{a,\theta^\varepsilon}$ is bounded in $W^{1,\infty}$ and since $\mathcal{M}_g$ is constant and $(p^\varepsilon)'$ is bounded uniformly with respect to ε in $W^{-1,\infty}$, we can conclude that the left hand side of (88) converges to 0 in $W^{-1,\infty}$ due to the extra powers of ε. Next, concerning the right hand side, the term $\varepsilon\langle\Gamma_{a,\theta^\varepsilon},(p^\varepsilon),(p^\varepsilon)\rangle$ converges to 0 in L^∞ since the terms inside the brackets are bounded. As before the last term in the two first lines of the equation (88), converges weakly to 0 in $W^{-1,\infty}$ Hence we infer that $(h^\varepsilon)'$ converges weakly-$\star$ to 0 in $W^{-1,\infty}$. Due to the a priori estimate, this convergences occurs in L^∞ weak-$\star$. Again this is sufficient to deduce the strong convergence of h^ε towards some h in L^∞, and that $h'=0$ and $h(0)=0$. This concludes the proof of Theorem 2.16.

□

3. Case of a bounded domain

We consider now the case where the system fluid-solid occupies a bounded open regular connected and simply connected domain Ω of $\mathbb{R}^2$. We assume that the body initially occupies the closed domain $\mathcal{S}_0 \subset \Omega$, so that the domain of the fluid is $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$ at the initial time, and (without loss of generality) that the center of mass of the solid coincides at the initial time with the origin and that $0 \in \Omega$. The domain of the fluid is denoted by $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$ at time $t > 0$. The fluid-solid system is governed by the following set of coupled equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{in } \mathcal{F}(t), \tag{90a}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{F}(t), \tag{90b}$$

$$mh'' = \int_{\partial \mathcal{S}(t)} \pi n \, \mathrm{d}s \quad \text{ and } \quad \mathcal{J}\theta'' = \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \, \mathrm{d}s, \tag{90c}$$

$$u \cdot n = \left(\ell + r(\cdot - h)^\perp\right) \cdot n \quad \text{on } \partial \mathcal{S}(t), \tag{90d}$$

$$u \cdot n = 0 \quad \text{on } \partial \Omega, \tag{90e}$$

$$u_{t=0} = u_0 \quad \text{in } \mathcal{F}_0 \quad \text{ and } \quad (h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0). \tag{90f}$$

Above we have denoted by (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$ the initial solid translation and rotation velocities, and by u_0 the compatible initial fluid velocity associated with (ℓ_0, r_0) and with the initial circulation γ in $\mathbb{R}$ according to Definition 1.5. In particular we still consider the case without any initial vorticity that is we assume that the initial velocity u_0 satisfies $\operatorname{curl} u_0 = 0$ in $\mathcal{F}_0$ (cf. Definition 1.5), so that it will remain irrotational for every time, that is

$$\operatorname{curl} u(t, \cdot) = 0 \text{ in } \mathcal{F}(t). \tag{91}$$

On the other hand the circulation around the body is constant in time equal to γ according to Kelvin's theorem. Since the domains $\mathcal{S}(t)$ and $\mathcal{F}(t)$ depend on $q := (h, \theta) \in \mathbb{R}^3$. only, we shall rather denote them $\mathcal{S}(q)$ and $\mathcal{F}(q)$ in the sequel. Since we will not consider any collision here, we introduce: $\mathcal{Q} := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}(q), \partial\Omega) > 0\}$, where $d(A, B)$ denotes for two sets A and B in the plane that is $d(A, B) := \min \{|x - y|_{\mathbb{R}^2}, \ x \in A, \ y \in B\}$. Above the notation stands $| \cdot |_{\mathbb{R}^d}$ for the Euclidean norm in $\mathbb{R}^d$. Since $\mathcal{S}_0$ is a closed subset in the open set Ω the initial position $q(0) = 0$ of the solid belongs to $\mathcal{Q}$.

As in the unbounded case of the previous section, our analysis here will rely on a reformulation of the system above as an second-order differential ODE for q together with an auxiliary div-curl type system for the fluid velocity. Indeed, again, the solid drives the dynamics of the coupled system as a consequence of the added inertia phenomenon. However in the case where the system occupies a bounded domain the matrix $\mathcal{M}_a$ encoding the added inertia depends not only on θ but on the three components of q. We therefore extend the definition 2.1 to this new setting.

Definition 3.1 (a-connection). Given a C^∞ mapping $q \in \mathcal{Q} \mapsto \mathcal{M}_a(q) \in S_3^+(\mathbb{R})$, we say that the C^∞ mapping $q \in \mathcal{Q} \mapsto \Gamma_a(q) \in \mathcal{BL}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$ is the a-connection associated with this mapping if for any $p \in \mathbb{R}^3$,

$$\langle \Gamma_a(q), p, p \rangle := \left(\sum_{1 \leqslant i,j \leqslant 3} (\Gamma_a(q))_{i,j}^k p_i p_j \right)_{1 \leqslant k \leqslant 3} \in \mathbb{R}^3, \tag{92}$$

with for every $i, j, k \in \{1, 2, 3\}$,

$$(\Gamma_a(q))_{i,j}^k(q) := \frac{1}{2} \Big((\mathcal{M}_a(q))_{k,j}^i + (\mathcal{M}_a(q))_{k,i}^j - (\mathcal{M}_a(q))_{i,j}^k \Big)(q), \tag{93}$$

where $(\mathcal{M}_a(q))_{i,j}^k$ denotes the partial derivative with respect to q_k of the entry of indexes (i, j) of the matrix $\mathcal{M}_a(q)$, that is

$$(\mathcal{M}_a(q))_{i,j}^k := \frac{\partial (\mathcal{M}_a(q))_{i,j}}{\partial q_k}. \tag{94}$$

Remark 2.3 is still in order for the definition above.

3.1. Reduction to an ODE in the case where $\gamma = 0$. Munnier's theorem.

Let us start with the case where the circulation γ is zero. Then the initial fluid velocity and therefore the velocity at any time is potential (that is a gradient globally on $\mathcal{F}(q)$). The following result was proven surprisingly only recently, by Munnier in [26]. This result consists in a reformulation of the system (90) in terms of an ordinary differential equation for the motion of the rigid body which corresponds to geodesics associated with the Riemann metric induced on $\mathcal{Q}$ by the matrix $\mathcal{M}_g + \mathcal{M}_a(q)$, where we recall that $\mathcal{M}_g$ is the genuine inertia defined in Definition 18. This reformulation therefore establishes an equivalence of the Newtonian and the Lagrangian points of view in the potential case.

Theorem 3.2. *Let be given the open regular connected and simply connected bounded cavity Ω, the initial closed domain $\mathcal{S}_0 \subset \Omega$ occupied by the body, the initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$. Assume that the circulation γ is 0. Let u_0 be the associated compatible initial fluid velocity according to Definition 1.5. Then there exists a C^∞ mapping $q \in \mathcal{Q} \mapsto \mathcal{M}_a(q) \in S_3^+(\mathbb{R})$, depending only on $\mathcal{S}_0$ and Ω, such that up to the first collision, System (90) is equivalent to the second order differential equation:*

$$(\mathcal{M}_g + \mathcal{M}_a(q))q'' + \langle \Gamma_a(q), q', q' \rangle = 0,$$

with Cauchy data $q(0) = 0 \in \mathcal{Q}$, $q'(0) = (\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, where Γ_a is the a-connection associated with $\mathcal{M}_a$ by Definition 3.1. For any $q \in \mathcal{Q}$ the fluid *velocity* $u(q,\cdot)$ *is the unique solution of the div-curl type system in the doubly-connected domain* $\mathcal{F}(q)$*, constituted of* (90b)*,* (91)*,* (90d)*,* (90e)*, together with the prescription of zero circulation.*

Indeed we are going to provide a quite explicit expression of $\mathcal{M}_a(q)$. Consider the functions ζ_j, for $j = 1,2,3$, defined for $q = (h,\theta) \in \mathcal{Q}$ and $x \in \mathcal{F}(q)$, by the formula $\zeta_j(q,x) := e_j$, for $j = 1,2$ and $\zeta_3(q,x) := (x-h)^\perp$. Above e_1 and e_2 are the unit vectors of the canonical basis. We introduce $\boldsymbol{\Phi} := (\Phi_1, \Phi_2, \Phi_3)^t$ where the Kirchhoff's potentials $\Phi_j(q,\cdot)$, for $j = 1,2,3$, are the unique (up to an additive constant) solutions in $\mathcal{F}(q)$ of the following Neumann problem:

$$\Delta\Phi_j = 0 \text{ in } \mathcal{F}(q), \quad \frac{\partial\Phi_j}{\partial n}(q,\cdot) = n\cdot\zeta_j(q,\cdot) \text{ on } \partial\mathcal{S}(q), \quad \frac{\partial\Phi_j}{\partial n}(q,\cdot) = 0 \text{ on } \partial\Omega. \tag{95}$$

We can now define the added inertia

$$\mathcal{M}_a(q) := \int_{\partial\mathcal{S}(q)} \boldsymbol{\Phi}(q,\cdot)\otimes\frac{\partial\boldsymbol{\Phi}}{\partial n}(q,\cdot)\mathrm{d}s = \Big(\int_{\mathcal{F}(q)} \nabla\Phi_i\cdot\nabla\Phi_j \mathrm{d}x\Big)_{1\leqslant i,j\leqslant 3}. \tag{96}$$

The added inertia matrix $\mathcal{M}_a(q)$ is symmetric positive-semidefinite.

3.2. Proof of Munnier's result: Theorem 3.2

The first step of the proof of Theorem 3.2 consists in a trade of the fluid pressure against the fluid velocity and its first order derivatives in the body's dynamics. Indeed we start with the observation that the equations (90g-h) can be summarized in the variational form:

$$mh''\cdot\ell^* + \mathcal{J}\theta'' r^* = \int_{\partial\mathcal{S}(q)} \pi(r^*(x-h)^\perp + \ell^*)\cdot n\mathrm{d}s, \quad \forall p^* = (\ell^*, r^*) \in \mathbb{R}^3. \tag{97}$$

Let us associate with $(q,p^*) \in \mathcal{Q}\times\mathbb{R}^3$, with $p^* = (p_1^*, p_2^*, p_3^*)$, the potential vector field

$$u^* := \nabla(\boldsymbol{\Phi}(q,\cdot)\cdot p^*) = \nabla(\sum_{j=1}^3 \Phi_j(q,\cdot)p_j^*), \tag{98}$$

which is defined on $\mathcal{F}(q)$. The pressure π can be recovered by means of Bernoulli's formula which is obtained by combining (90a) and (91), and which reads:

$$\nabla\pi = -\left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u^2)\right) \quad \text{in } \mathcal{F}(q). \tag{99}$$

According to Bernoulli's formula (99) and upon an integration by parts, identity (97) can be turned into:

$$mh''\cdot\ell^* + \mathcal{J}\theta'' r^* = -\int_{\mathcal{F}(q)} \left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u^2)\right)\cdot u^*\mathrm{d}x, \quad \forall p^* = (\ell^*, r^*) \in \mathbb{R}^3. \tag{100}$$

So far we have only used that the fluid velocity u is irrotational. Let us now use that it is potential and therefore reads as $u = u_1$ with u_1 as follows:

$$u_1(q,\cdot) := \nabla(\boldsymbol{\Phi}(q,\cdot)\cdot q') = \nabla(\sum_{j=1}^{3}\Phi_j(q,\cdot)q_j'), \tag{101}$$

where $q \in \mathcal{Q}$. For any $q \in \mathcal{Q}$ this vector field $u_1(q,\cdot)$ is the only solution to the div-curl type system in the doubly-connected domain $\mathcal{F}(q)$, constituted of (90b), (91), (90d), (90e), together with the prescription of zero circulation. Observe that besides the dependence with respect to $\mathcal{S}_0$, to Ω and to the space variable, u_1 depends on q and linearly on q'.

Then Theorem 3.2 will follow from the following lemma.

Lemma 3.3. *For any smooth curve $q(t)$ in $\mathcal{Q}$ and every $p^* = (\ell^*, r^*) \in \mathbb{R}^3$, the following identity holds:*

$$\begin{aligned} mh''\cdot\ell^* + \mathcal{J}\theta'' r^* + \int_{\mathcal{F}(q)}\left(\frac{\partial u_1}{\partial t} + \frac{1}{2}\nabla|u_1|^2\right)\cdot u^*\mathrm{d}x \\ = (\mathcal{M}_g + \mathcal{M}_a(q))q''\cdot p^* + \langle\Gamma_a(q), q', q'\rangle\cdot p^*, \end{aligned}\tag{102}$$

where u^* is given by (98), *u_1 is given by* (101), *$\mathcal{M}_a(q)$ and $\Gamma_a(q)$ are defined in* (96) *and Definition 3.1.*

Proof of Lemma 3.3. We start with observing that, under the assumptions of Lemma 3.3,

$$mh''\cdot\ell^* + \mathcal{J}\theta'' r^* = \mathcal{M}_g\, q''\cdot p^*. \tag{103}$$

Now in order to deal with the last term of the left hand side of (102) we use a Lagrangian strategy. For any q in $\mathcal{Q}$ and every $p = (p_1, p_2, p_3)$ in $\mathbb{R}^3$, let us denote

$$\mathcal{E}_1(q,p) := \frac{1}{2}\int_{\mathcal{F}(q)}|\nabla(\boldsymbol{\Phi}(q,\cdot)\cdot p)|^2\mathrm{d}x. \tag{104}$$

Thus $\mathcal{E}_1(q,p)$ denotes the kinetic energy of the potential part of the flow associated with a body at position q with velocity p. It follows from classical shape derivative theory that $\mathcal{E}_1 \in C^\infty\big(\mathcal{Q}\times\mathbb{R}^3; [0,+\infty)\big)$.

Lemma 3.4. *For any smooth curve $t \mapsto q(t)$ in $\mathcal{Q}$, for every $p^* \in \mathbb{R}^3$, we have:*

$$\int_{\mathcal{F}(q)}\left(\frac{\partial u_1}{\partial t} + \frac{1}{2}\nabla|u_1|^2\right)\cdot u^*\mathrm{d}x = \mathcal{EL} \tag{105}$$

where u_1 is given by (127), *u^* is given by* (98) *and $\mathcal{EL}$ denotes the time-dependent smooth real-valued function:*

$$\mathcal{EL} := \Big(\frac{d}{dt}\big(\frac{\partial\mathcal{E}_1}{\partial p}(q(t),q'(t))\big) - \frac{\partial\mathcal{E}_1}{\partial q}(q(t),q'(t))\Big)\cdot p^*. \tag{106}$$

The name of the function $\mathcal{EL}$ refers to Euler and Lagrange. For sake of simplicity below we will simply denote

$$\mathcal{EL} = \left(\frac{d}{dt} \frac{\partial \mathcal{E}_1}{\partial p} - \frac{\partial \mathcal{E}_1}{\partial q} \right) \cdot p^*.$$

Let us also introduce a slight abuse of notations which simplifies the presentation of the proof of Lemma 3.4. For a smooth function $I(q,p)$, where (q,p) is running into $\mathcal{Q} \times \mathbb{R}^3$, and a smooth curve $q(t)$ in $\mathcal{Q}$ let us denote

$$\left(\frac{\partial}{\partial q} \frac{d}{dt} I(q,p) \right)(t) := (\frac{\partial}{\partial q} J)\big(q(t), q'(t), q''(t)\big),$$

where, for (q,p,r) in $\mathcal{Q} \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$J(q,p,r) = p \frac{\partial I}{\partial q}(q,p) + r \frac{\partial I}{\partial p}(q,p). \tag{107}$$

Observe in particular that

$$\frac{d}{dt}\big(I(q(t), q'(t))\big) = J\big(q(t), q'(t), q''(t)\big),$$

and

$$\frac{d}{dt} \left(\frac{\partial I}{\partial q}(q(t), q'(t)) \right) = \left(\frac{\partial}{\partial q} \frac{d}{dt} I(q,p) \right)(t). \tag{108}$$

Below, in such circumstances, it will be comfortable to write

$$\frac{\partial}{\partial q} \left[J\big(q(t), q'(t), q''(t)\big) \right] \text{ instead of } \left(\frac{\partial J}{\partial q} \right) \big(q(t), q'(t), q''(t)\big),$$

and it will be understood that J is extended from $\big(q(t), q'(t), q''(t)\big)$ to general (q,p,r) by (107).

Proof of Lemma 3.4. We start with computing the right hand side of (105). On the one hand the linearity of u_1 with respect to p and then an integration by parts leads to:

$$\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \int_{\mathcal{F}(q)} u_1 \cdot u^* \mathrm{d}x = \int_{\partial \mathcal{S}(q)} (\mathbf{\Phi} \cdot p)(u^* \cdot n) \,\mathrm{d}s.$$

Then, invoking the Reynold transport theorem, we get:

$$\frac{\partial \mathcal{E}_1}{\partial p} \cdot p^* = \frac{\partial}{\partial q} \left(\int_{\mathcal{F}(q)} (\mathbf{\Phi} \cdot p) \,\mathrm{d}x \right) \cdot p^* - \int_{\mathcal{F}(q)} (\frac{\partial \mathbf{\Phi}}{\partial q} \cdot p) \cdot p^* \mathrm{d}x. \tag{109}$$

On the other hand, again using Reynold's formula, we have:

$$\frac{\partial \mathcal{E}_1}{\partial q} \cdot p^* = \int_{\mathcal{F}(q)} \left(\frac{\partial u_1}{\partial q} \cdot p^* \right) \cdot u_1 \,\mathrm{d}x + \frac{1}{2} \int_{\partial \mathcal{S}(q)} |u_1|^2 (u^* \cdot n) \,\mathrm{d}s. \tag{110}$$

Differentiating (109) with respect to t, we obtain:

$$\frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p}\cdot p^* = \frac{d}{dt}\frac{\partial}{\partial q}\left(\int_{\mathcal{F}(q)} (\mathbf{\Phi}\cdot p)\,\mathrm{d}x\right)\cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p)\cdot p^*\mathrm{d}x\right). \tag{111}$$

With the abuse of notations mentioned above we commute the derivatives involved in the first term of the right hand side, so that the identity (111) can be rewritten as follows:

$$\frac{d}{dt}\frac{\partial \mathcal{E}_1}{\partial p}\cdot p^* = \frac{\partial}{\partial q}\frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\mathbf{\Phi}\cdot p)\,\mathrm{d}x\right)\cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p)\cdot p^*\mathrm{d}x\right). \tag{112}$$

Moreover, using again Reynold's formula, we have:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\mathbf{\Phi}\cdot p)\,\mathrm{d}x\right) = \int_{\mathcal{F}(q)} \partial_t(\mathbf{\Phi}\cdot p)\,\mathrm{d}x + \int_{\partial\mathcal{S}(q)} (\mathbf{\Phi}\cdot p)(u_1\cdot n)\,\mathrm{d}s \tag{113}$$

$$= \int_{\mathcal{F}(q)} \partial_t(\mathbf{\Phi}\cdot p)\,\mathrm{d}x + 2\mathcal{E}_1(q,p), \tag{114}$$

by integration by parts.

We infer from (112) and (113), again with the abuse of notations mentioned above, that:

$$\mathcal{E}\mathcal{L} = \frac{\partial \mathcal{E}_1}{\partial q} + \frac{\partial}{\partial q}\left[\int_{\mathcal{F}(q)} \partial_t(\mathbf{\Phi}\cdot p)\,\mathrm{d}x\right]\cdot p^* - \frac{d}{dt}\left(\int_{\mathcal{F}(q)} (\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p)\cdot p^*\mathrm{d}x\right). \tag{115}$$

Thanks to Reynold's formula, we get for the second term of the right hand side

$$\frac{\partial}{\partial q}\left[\int_{\mathcal{F}(q)} \partial_t(\mathbf{\Phi}\cdot p)\,\mathrm{d}x\right]\cdot p^* = \int_{\mathcal{F}(q)} \frac{\partial}{\partial q}(\partial_t(\mathbf{\Phi}\cdot p))\cdot p^*\,\mathrm{d}x + \int_{\partial\mathcal{S}(q)} \partial_t(\mathbf{\Phi}\cdot p)(u^*\cdot n)\,\mathrm{d}s, \tag{116}$$

and for the last one:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)} \left(\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p\right)\cdot p^*\mathrm{d}x\right) = \int_{\mathcal{F}(q)} \partial_t\left(\left(\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p\right)\cdot p^*\right)\,\mathrm{d}x + \int_{\partial\mathcal{S}(q)} \left(\left(\frac{\partial \mathbf{\Phi}}{\partial q}\cdot p\right)\cdot p^*\right)(u_1\cdot n)\,\mathrm{d}s. \tag{117}$$

Using again (108) for the first term and integrating by parts the second one, we obtain:

$$\frac{d}{dt}\left(\int_{\mathcal{F}(q)}\left(\frac{\partial\mathbf{\Phi}}{\partial q}\cdot p\right)\cdot p^*\mathrm{d}x\right)=\int_{\mathcal{F}(q)}\frac{\partial}{\partial q}\big(\partial_t(\mathbf{\Phi}\cdot p)\big)\cdot p^*\,\mathrm{d}x + \int_{\mathcal{F}(q)}\left(\frac{\partial u_1}{\partial q}\cdot p^*\right)\cdot u_1\,\mathrm{d}x. \tag{118}$$

Plugging the expressions (110), (116) and (118) into (115) and simplifying, we end up with:

$$\mathcal{EL}=\int_{\partial\mathcal{S}(q)}\left[\partial_t(\mathbf{\Phi}\cdot p)+\frac{1}{2}|u_1|^2\right](u^*\cdot n)\,\mathrm{d}s.$$

Upon an integration by parts, we recover (105) and the proof is then completed. □

Now, we observe that $\mathcal{E}_1(q,p)$, as defined by (104), can be rewritten as:

$$\mathcal{E}_1(q,p)=\frac{1}{2}\mathcal{M}_a(q)p\cdot p, \tag{119}$$

where $\mathcal{M}_a(q)$ is defined by (96). Indeed this allows us to prove the following result.

Lemma 3.5. *For any smooth curve $q(t)$ in $\mathcal{Q}$, for every $p^*\in\mathbb{R}^3$, we have:*

$$\mathcal{EL}=\mathcal{M}_a(q)q''\cdot p^*+\langle\Gamma_a(q),q',q'\rangle\cdot p^*. \tag{120}$$

Proof of Lemma 3.5. Using (119) in the definition (106) of $\mathcal{EL}$ we have

$$\mathcal{EL}=\mathcal{M}_a(q)q''\cdot p^*+\Big(\big(D\mathcal{M}_a(q)\cdot q'\big)q'\Big)\cdot p^*-\frac{1}{2}\Big(\big(D\mathcal{M}_a(q)\cdot p^*\big)q'\Big)\cdot q'.$$

Then

$$\mathcal{EL}=\mathcal{M}_a\,q''\cdot p^*+\sum(\mathcal{M}_a)^k_{i,j}\,q'_kq'_jp^*_i-\frac{1}{2}\sum(\mathcal{M}_a)^k_{i,j}\,q'_iq'_jp^*_k,$$

where the sums are over $1\leqslant i,j,k\leqslant 3$. Let us recall the notation $(\mathcal{M}_a)^k_{i,j}(q)$ in (94). A symmetrization with respect to j and k of the second term and an exchange of i and k in the last sum of the right hand side above leads to the result. □

Then Lemma 3.3 straightforwardly results from the combination of (103), Lemma 3.4 and Lemma 3.5. □

3.3. Reduction to an ODE in the general case. Statement of Theorem 3.6

Now let us deal with the general case of a nonzero circulation γ. Next result, obtained in [4], extends Theorem 3.2 and establishes a reformulation of the system in terms of an ordinary differential equation in the general case of a circulation $\gamma \in \mathbb{R}$.

Theorem 3.6. *Let be given the open regular connected and simply connected bounded cavity Ω, the initial closed domain $\mathcal{S}_0 \subset \Omega$ occupied by the body, the initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$, the circulation γ in $\mathbb{R}$, and u_0 the associated compatible initial fluid velocity according to Definition 1.5. There exists F in $C^\infty(\mathcal{Q}\times\mathbb{R}^3;\mathbb{R}^3)$ depending only on $\mathcal{S}_0, \gamma$ and Ω, and vanishing when $\gamma = 0$, such that, up to the first collision, System* (90) *is equivalent to the second order ODE:*

$$(\mathcal{M}_g + \mathcal{M}_a(q))q'' + \langle \Gamma_a(q), q', q' \rangle = F(q, q'), \tag{121}$$

with Cauchy data $q(0) = 0 \in \mathcal{Q}$, $q'(0) = (\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, where $\mathcal{M}_a(q)$ and its associated a-connection $\Gamma_a(q)$ are given by Theorem 3.2. For a solid position $q \in \mathcal{Q}$ the fluid velocity $u(q, \cdot)$ is uniquely determined as the solution of a div-curl type system in the doubly-connected domain $\mathcal{F}(q)$, constituted of (90b), (91), (90d), (90e), *together with the prescription of the circulation γ.*

The local-in-time existence and uniqueness of smooth solutions stated in Theorem 1.6 therefore simply follows from Theorem 3.6 and the Cauchy-Lipschitz theorem. That the life-time of such a smooth solution can only be limited by a collision will follow from an energy argument below, cf. Section 3.5.

Indeed we are going to provide a rather explicit definition of the force term $F(q, q')$. Let us first introduce a normalized stream function for the circulation term: for every $q \in \mathcal{Q}$, there exists a unique $C(q)$ in[7] $\mathbb{R}$ such that the unique solution $\psi(q, \cdot)$ of the Dirichlet problem:

$$\Delta\psi(q, \cdot) = 0 \text{ in } \mathcal{F}(q) \quad \psi(q, \cdot) = C(q) \text{ on } \partial\mathcal{S}(q) \quad \psi(q, \cdot) = 0 \text{ on } \partial\Omega, \tag{122a}$$

satisfies

$$\int_{\partial\mathcal{S}(q)} \frac{\partial\psi}{\partial n}(q, \cdot)\mathrm{d}s = -1. \tag{122b}$$

[7] A simple computation shows that the function $C(q)$ is actually the opposite of the inverse of the condenser capacity of $\mathcal{S}(q)$ in Ω.

Observe that for any $q \in \mathcal{Q}$, $C(q) < 0$ and that $C \in C^\infty(\mathcal{Q}; (-\infty, 0))$ and depends on $\mathcal{S}_0$ and Ω. Eventually, we define:

$$B(q) := \int_{\partial \mathcal{S}(q)} \left(\frac{\partial \psi}{\partial n} \left(\frac{\partial \mathbf{\Phi}}{\partial n} \times \frac{\partial \mathbf{\Phi}}{\partial \tau} \right) \right) (q, \cdot) \, \mathrm{d}s, \tag{123}$$

$$E(q) := -\frac{1}{2} \int_{\partial \mathcal{S}(q)} \left(\left| \frac{\partial \psi}{\partial n} \right|^2 \frac{\partial \mathbf{\Phi}}{\partial n} \right) (q, \cdot) \, \mathrm{d}s, \tag{124}$$

and, for (q, p) in $\mathcal{Q} \times \mathbb{R}^3$, the force term

$$F(q, p) := \gamma^2 E(q) + \gamma \, p \times B(q). \tag{125}$$

The notations E and B are chosen on purpose to highlight the analogy with the Lorentz force acting on a charged particle moving under the influence of a couple of electromagnetic fields E and B.

3.4. Proof of Theorem 3.6

As mentioned above (100) only relies on the fact that the fluid velocity is irrotational and is therefore still granted. However the fluid velocity $u(q, \cdot)$ now involves an extra term due to the nonzero circulation. Indeed, for any $q \in \mathcal{Q}$, one obtains, using (122) and (95), that the solution $u(q, \cdot)$ to the div-curl type system in the doubly-connected domain $\mathcal{F}(q)$, constituted of (90b), (91), (90d), (90e), together with the prescription of circulation γ takes the form:

$$u(q, \cdot) = u_1(q, \cdot) + u_2(q, \cdot), \tag{126}$$

where $u_1(q, \cdot)$ is given by (101) as in the potential case and the new contribution $u_2(q, \cdot)$ is defined by

$$u_2(q, \cdot) := \gamma \nabla^\perp \psi(q, \cdot). \tag{127}$$

So besides the dependence with respect to $\mathcal{S}_0$, to Ω and to the space variable, u_2 depends on q and linearly on γ. Therefore plugging the decomposition (126) into (100) leads to

$$m\ell' \cdot \ell^* + \mathcal{J} r' r^* + \int_{\mathcal{F}(q)} \Big(\frac{\partial u_1}{\partial t} + \frac{1}{2} \nabla |u_1|^2 \Big) \cdot u^* \mathrm{d}x = - \int_{\mathcal{F}(q)} \big(\frac{1}{2} \nabla |u_2|^2 \big) \cdot u^* \mathrm{d}x$$
$$- \int_{\mathcal{F}(q)} \big(\frac{\partial u_2}{\partial t} + \frac{1}{2} \nabla (u_1 \cdot u_2) \big) \cdot u^* \mathrm{d}x, \tag{128}$$

for all $p^* := (\ell^*, r^*) \in \mathbb{R}^3$, with u^* given by (98).

By a simple integration by parts, on obtains that the first term in the right hand side above satisfies:

$$- \int_{\mathcal{F}(q)} \left(\frac{1}{2} \nabla |u_2|^2 \right) \cdot u^* \mathrm{d}x = \gamma^2 E(q) \cdot p^*, \tag{129}$$

where $E(q)$ defined in (124).

Then the reformulation of Equations (90g-h) mentioned in Theorem 3.6 will follow from (128), (129), Lemma 3.3 and from the following identity:

$$-\int_{\mathcal{F}(q)} \left(\frac{\partial u_2}{\partial t} + \nabla(u_1 \cdot u_2) \right) \cdot u^* \mathrm{d}x = \gamma\big(q' \times B(q)\big) \cdot p^*, \tag{130}$$

where $B(q)$ is defined in (123). We refer to [7] for the proof of (130).

3.5. The role of the energy

An important feature of the system (121) is that it is conservative. Let us denote for any (q,p) in $\mathcal{Q} \times \mathbb{R}^3$,

$$\mathcal{E}(q,p) := \frac{1}{2}(\mathcal{M}_g + \mathcal{M}_a(q))p \cdot p - \frac{1}{2}\gamma^2 C(q), \tag{131}$$

with $C(q)$ given by (122). Indeed one can prove that for any $q \in \mathcal{Q}$,

$$E(q) = \frac{1}{2} DC(q), \tag{132}$$

where the notation $DC(q)$ stands for the derivative of $C(q)$ with respect to q, cf. Lemma 2.4 in [7] for a proof, so that the second term in the right-hand-side of (131) can be seen as a potential energy related to the first term in the right-hand-side of (125). Observe that $\mathcal{E}(q,p)$ is the sum of two positive terms and that in addition to its dependence on q and p, the energy $\mathcal{E}$ depends on $\mathcal{S}_0, m, \mathcal{J}, \gamma$ and Ω. Next result proves that $\mathcal{E}(q,q')$ is indeed the natural total kinetic energy of the "fluid+solid" system.

Proposition 3.7. *For any $q = (h,\theta) \in C^\infty([0,T];\mathcal{Q})$ satisfying* (121)*, as far as there is no collision,*

$$\mathcal{E}(q,q') = \frac{1}{2}\int_{\mathcal{F}(q)} u(q,\cdot)^2 \,\mathrm{d}x + \frac{1}{2} m (h')^2 + \frac{1}{2}\mathcal{J}(\theta')^2.$$

Proof. First we have by integrations by parts that

$$\frac{1}{2}\int_{\mathcal{F}(q)} u_2^2 \,\mathrm{d}x = -\frac{1}{2}\gamma^2 C(q) \text{ and } \int_{\mathcal{F}(q)} u_1 \cdot u_2 \,\mathrm{d}x = 0.$$

Then we use (119) and the decomposition (126) to conclude. □

The following result is therefore very natural.

Proposition 3.8. *For any $q \in C^\infty([0,T];\mathcal{Q})$ satisfying* (121)*, as far as there is no collision, $\mathcal{E}(q,q')$ is constant in time.*

Proof. Let us give a proof of Proposition 3.8 which uses the ODE formulation (121). We start with the observation that the energy $\mathcal{E}(q,q')$ as defined in (131) has for time derivative

$$\big(\mathcal{E}(q,q')\big)' = (\mathcal{M}_g + \mathcal{M}_a(q))q'' \cdot q' + \frac{1}{2}(D\mathcal{M}_a(q) \cdot q')q' \cdot q' - \frac{1}{2}\gamma^2 DC(q) \cdot q'. \tag{133}$$

Now, thanks to (121) and (125), we have

$$(\mathcal{M}_g + \mathcal{M}_a(q))q'' \cdot q' = -\langle \Gamma_a(q), q', q' \rangle \cdot q' + F(q, q') \cdot q', \tag{134}$$

and

$$F(q, q') \cdot q' = \gamma^2 E(q) \cdot q'. \tag{135}$$

We introduce the matrix

$$S_a(q, q') := \left(\sum_{1 \leqslant i \leqslant 3} (\Gamma_a)^k_{i,j}(q) q'_i \right)_{1 \leqslant k,j \leqslant 3}, \tag{136}$$

so that

$$\langle \Gamma_a(q), q', q' \rangle = S_a(q, q')q'. \tag{137}$$

Combining (133), (134), (135), (136) and (137) we obtain

$$\big(\mathcal{E}(q, q')\big)' = \gamma^2 \big(E(q) - \frac{1}{2} DC(q)\big) \cdot q' + \big(\frac{1}{2} D\mathcal{M}_a(q) \cdot q' - S_a(q, q')\big) q' \cdot q'.$$

The first term of the right hand side vanishes thanks to (132) and the proof of Proposition 3.8 then follows from the following result.

Lemma 3.9. *For any $(q, p) \in \mathcal{Q} \times \mathbb{R}^3$, the matrix $\frac{1}{2} D\mathcal{M}_a(q) \cdot p - S_a(q, p)$ is skew-symmetric.*

Proof of Lemma 3.9. We start with the observation that $D\mathcal{M}_a(q) \cdot p$ is the 3×3 matrix containing the entries

$$\sum_{1 \leqslant k \leqslant 3} (\mathcal{M}_a)^k_{i,j}(q)\, p_k, \text{ for } 1 \leqslant i, j \leqslant 3.$$

On the other hand, the 3×3 matrix $S_a(q, p)$ contains the entries

$$\frac{1}{2} \sum_{1 \leqslant k \leqslant 3} \Big((\mathcal{M}_a)^k_{i,j} + (\mathcal{M}_a)^j_{i,k} - (\mathcal{M}_a)^i_{k,j} \Big)(q)\, p_k,$$

for $1 \leqslant i, j \leqslant 3$. Therefore, the 3×3 matrix $D\mathcal{M}_a(q) \cdot p - S_a(q, p)$ contains the entries

$$c_{ij}(q, p) = -\frac{1}{2} \sum_{1 \leqslant k \leqslant 3} \Big((\mathcal{M}_a)^j_{i,k} - (\mathcal{M}_a)^i_{k,j} \Big)(q)\, p_k,$$

for $1 \leqslant i, j \leqslant 3$. Using that the matrix $\mathcal{M}_a(q)$ is symmetric, we get that $c_{ij}(q, p) = -c_{ji}(q, p)$ for $1 \leqslant i, j \leqslant 3$, which ends the proof. □

This ends the proof of Proposition 3.8.[8] □

[8] It is also possible to achieve an alternative proof of Proposition 3.8 thanks to the original PDE formulation of the "fluid+solid" system, relying on the equivalence between the ODE and PDE formulations obtained in Theorem 3.6 and on the reformulation of the energy obtained in Proposition 3.7.

If we assume that the body stays at distance at least $\delta > 0$ from the boundary we may infer from Proposition 3.8 a bound of the body velocity depending only on the data and on δ. Indeed we have the following immediate corollary of Proposition 3.8 and of the regularity properties of the functions $C(q)$ and $\mathcal{M}_a(q)$. We denote $\mathcal{Q}_\delta := \{q \in \mathbb{R}^3 \,:\, d(\mathcal{S}(q), \partial\Omega) > \delta\}$.

Corollary 3.10. *Let $\mathcal{S}_0 \subset \Omega$, $p_0 \in \mathbb{R}^3$ and $(\gamma, m, \mathcal{J}) \in \mathbb{R} \times (0, +\infty) \times (0, +\infty)$; $\delta > 0$; $q \in C^\infty([0, T]; \mathcal{Q}_\delta)$ satisfying (121) with the Cauchy data $(q, q')(0) = (0, p_0)$. Then there exists $K > 0$ depending only on $\mathcal{S}_0, \Omega, p_0, \gamma, m, \mathcal{J}, \delta$ such that $|q'|_{\mathbb{R}^3} \leqslant K$ on $[0, T]$.*

This entails in particular that the life-time of a smooth solution to (121) can only be limited by a collision and therefore completes the proof of Theorem 1.6.

3.6. Zero radius limit

Let us now turn our attention to the limit of the dynamics when the size of the solid goes to 0 that is considering an initial domain for the body of the form (5) with the inertia scaling described in Definition 1.1. This aims to extend the analysis performed in Section 2.6 to the case where the "fluid+solid" system occupies a bounded domain rather than the whole plane.

Below, we will use the following notation: for $\mathcal{S}_0 \subset \Omega$; $p_0 = (\ell_0, r_0) \in \mathbb{R}^3$, $(m, \mathcal{J}) \in (0, +\infty) \times (0, +\infty)$, γ in $\mathbb{R}$ (resp. in $\mathbb{R}^*$) in the case of a massive (respectively massless) particle, for every $\varepsilon \in (0, 1]$ small enough to ensure that the set $\mathcal{S}_0^\varepsilon$ defined by (5) satisfies $\mathcal{S}_0^\varepsilon \subset \Omega$, we denote $(q^\varepsilon, T^\varepsilon)$ the maximal solution to (121) associated with the coefficients $\mathcal{M}^\varepsilon$, Γ_a^ε and F^ε which are themselves associated with $\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon$ and γ (as $\mathcal{M}, \Gamma_a$ and F were associated with $\mathcal{S}_0, m, \mathcal{J}$ and γ) where $m^\varepsilon, \mathcal{J}^\varepsilon$ are given in Definition 1.1, and with the initial data $(q^\varepsilon, (q^\varepsilon)')(0) = (0, p_0)$.

Theorem 3.11. *Let $\mathcal{S}_0 \subset \Omega$; $p_0 = (\ell_0, r_0) \in \mathbb{R}^3$, $(m, \mathcal{J}) \in (0, +\infty) \times (0, +\infty)$, γ in $\mathbb{R}$ (resp. in $\mathbb{R}^*$) in the case of a massive (respectively massless) particle. Let (h, T) be the maximal solution to (8) (resp. h be the global solution to (9)). Then, as $\varepsilon \to 0$, $\liminf T^\varepsilon \geqslant T$ (resp. $T^\varepsilon \longrightarrow +\infty$) and $h^\varepsilon \rightharpoonup h$ in $W^{2,\infty}([0, T']; \mathbb{R}^2)$ (resp. in $W^{1,\infty}([0, T']; \mathbb{R}^2)$) weak-$\star$ for all $T' \in (0, T)$ (resp. for all $T' > 0$). Furthermore in the case a massive particle, one also has that $\varepsilon\theta^\varepsilon \rightharpoonup 0$ in $W^{2,\infty}([0, T']; \mathbb{R})$ weak-$\star$ for all $T' \in (0, T)$.*

In the statement above it is understood that q^ε was decomposed into $q^\varepsilon = (h^\varepsilon, \theta^\varepsilon)$. It follows from Theorem 3.6 that Theorem 3.11 implies Theorem 1.7.

Let us provide a scheme of proof of Theorem 3.11.

Scheme of proof of Theorem 3.11. Using that for the inertia regimes considered in Definition 1.1 the genuine inertia matrix scales as in (86), the equation for q^ε reads:

$$\Big(\varepsilon^\alpha I_\varepsilon \mathcal{M}_g I_\varepsilon + \mathcal{M}_a^\varepsilon(q^\varepsilon)\Big)(q^\varepsilon)'' + \langle \Gamma_a^\varepsilon(q^\varepsilon), (q^\varepsilon)', (q^\varepsilon)' \rangle = \gamma^2 E^\varepsilon(q^\varepsilon) + \gamma (q^\varepsilon)' \times B^\varepsilon(q^\varepsilon), \tag{138}$$

where the added inertia matrix $\mathcal{M}_a^\varepsilon$, the a-connection Γ_a^ε and the electric and magnetic type terms E^ε and B^ε are associated with the body of size ε as mentioned above. Here one crucial step in passing to the limit in (138) is to find some uniform bounds in ε. The energy is of course a natural candidate in order to get such estimates. In particular one may turn toward an appropriate modification of Corollary 3.10 in the zero radius limit. A difficulty is that the potential part of the energy (corresponding to the second term in (131)) diverges logarithmically as $\varepsilon \to 0^+$. However such a contribution can be discarded from the energy conservation since it does not depend on the solid position and velocity. Indeed an appropriate renormalization of the energy provides an uniform estimate of $\varepsilon^{\min(1,\frac{\alpha}{2})}\, |(h^\varepsilon)', \varepsilon(\theta^\varepsilon)')|_{\mathbb{R}^3}$ at least till the solid stays away from the external boundary. Unfortunately in the massless case the coefficient α satisfies $\alpha > 0$ and the previous estimate is not sufficient.[9]

One then turns toward the search for an asymptotic normal form of (138) with the hope that more structure shows up in the zero radius limit and reveals another candidate in order to obtain some uniform bounds in ε. In order to do so we first establish some expansions in the limit $\varepsilon \to 0$ of $\mathcal{M}_a^\varepsilon$, Γ_a^ε, E^ε and B^ε. These expansions are obtained by a multi-scale analysis of the Kirchhoff potentials and of the stream functions and repeated use of Lamb's lemma, that is Lemma 2.15. More precisely these expansions involve two scales corresponding respectively to variations over length $O(1)$ and $O(\varepsilon)$ respectively on $\partial\Omega$ and $\partial\mathcal{S}^\varepsilon(q)$. The profiles appearing in these expansions are obtained by successive corrections, considering alternatively at their respective scales the body boundary from which the external boundary seems remote and the external boundary from which the body seems tiny, so that good approximations are given respectively by the case without external boundary and without the body. We refer to [7] for more details on this intricate process and sum up the results below. The leading term of the expansions of $\mathcal{M}_a^\varepsilon$, Γ_a^ε and B^ε in the zero-radius limit are given, up to an appropriate scaling, by the terms obtained in the case where the rigid body is of size $\varepsilon = 1$ and is immersed in a fluid filling the whole plane, that is in the case tackled in Theorem 2.2. On the other hand the leading term of the expansion of E^ε in the zero-radius limit is given, up to an appropriate

[9] Indeed the case where $\alpha \geq 2$ is the most delicate and we will focus on it.

scaling, by the field

$$\mathsf{E}_{\mathit{0}}(q) := -\begin{pmatrix} u_\Omega(h)^\perp \\ u_\Omega(h)\cdot R(\theta)\xi \end{pmatrix}, \text{ where } q=(h,\theta). \tag{139}$$

We recall that u_Ω and ξ were defined respectively above (8) and in (33). Given $\delta>0$ and ε_0 in $(0,1)$, we define the bundle of shrinking body positions at distance δ from the boundary for a radius of order ε with $0<\varepsilon<\varepsilon_0$:

$$\mathfrak{Q}_{\delta,\varepsilon_0} := \{(\varepsilon,q)\in(0,\varepsilon_0)\times\mathbb{R}^3 \;/\; d(\mathcal{S}^\varepsilon(q),\partial\Omega)>\delta\}.$$

Proposition 3.12. *Let $\delta>0$. There exists ε_0 in $(0,1)$, $\mathsf{E}_{\mathit{1}}(q)$ and $\mathsf{B}_{\mathit{1}}(q)$ in $L^\infty(\mathcal{Q}_\delta;\mathbb{R}^3)$, $\mathcal{M}_r$ in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^{3\times3})$, Γ_r in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathcal{BL}(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}^3))$, and E_r and B_r in $L^\infty(\mathfrak{Q}_{\delta,\varepsilon_0};\mathbb{R}^3)$ such that, for all (ε,q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$, with $q=(h,\theta)$,*

$$\mathcal{M}_a^\varepsilon(q) = \varepsilon^2 I_\varepsilon\Big(\mathcal{M}_{a,\theta}+\varepsilon^2\mathcal{M}_r(\varepsilon,q)\Big)I_\varepsilon, \tag{140}$$

$$\langle\Gamma_a^\varepsilon(q),\cdot,\cdot\rangle = \varepsilon I_\varepsilon\big(\langle\Gamma_{a,\theta},I_\varepsilon\cdot,I_\varepsilon\cdot\rangle+\varepsilon^2\langle\Gamma_r(\varepsilon,q),I_\varepsilon\cdot,I_\varepsilon\cdot\rangle\big), \tag{141}$$

$$E^\varepsilon(q) = I_\varepsilon\Big(\mathsf{E}_{\mathit{0}}(q)+\varepsilon\mathsf{E}_{\mathit{1}}(q)+\varepsilon^2E_r(\varepsilon,q)\Big), \tag{142}$$

$$B^\varepsilon(q) = \varepsilon I_\varepsilon^{-1}\Big(B_\theta+\varepsilon\mathsf{B}_{\mathit{1}}(q)+\varepsilon^2B_r(\varepsilon,q)\Big). \tag{143}$$

We recall that $\mathcal{M}_{a,\theta}$ and B_θ are given by Theorem 2.2 as associated with the rigid body of size $\varepsilon=1$ and as if the body was immersed in a fluid filling the whole plane, $\Gamma_{a,\theta}$ denotes the a-connection associated with $\mathcal{M}_{a,\theta}$, I_ε is the diagonal matrix $I_\varepsilon := \text{diag}\,(1,1,\varepsilon)$ and let us avoid any confusion by highlighting that the $\cdot$ in (141) stands for the application to any p in $\mathbb{R}^3$ (which determines completely the bilinear symmetric mapping). Let us also recall that quite explicit expressions of $\mathcal{M}_{a,\theta}$, B_θ and $\Gamma_{a,\theta}$ are given in Section 2.2.

Therefore, using (142), (143), (139) and (35), one obtains that the leading part of the expansion of the right hand side of (138) is

$$I_\varepsilon\Big(\gamma^2\mathsf{E}_{\mathit{0}}(q^\varepsilon)+\gamma\big(I_\varepsilon(q^\varepsilon)'\big)\times B_{\theta^\varepsilon}\Big) = I_\varepsilon\Big(\gamma\hat{p}^\varepsilon\times B_{\theta^\varepsilon}\Big), \tag{144}$$

for any $\hat{p}^\varepsilon$ of the form

$$\hat{p}^\varepsilon = ((h^\varepsilon)'-\gamma u_\Omega(h^\varepsilon),\varepsilon(\theta^\varepsilon)')^t+\eta B_{\theta^\varepsilon}, \tag{145}$$

with η in $\mathbb{R}$.

An instructive digression. The identities (144) and (145) remind a well-known modulation strategy used by Berkowitz and Gardner, cf. [1], in order the tackle the zero-mass limit of the following dynamics of a light particle in a smooth electro-magnetic field:

$$\varepsilon^2q'' = E(q)+q'\times B(q) \text{ with the condition } E(q)\cdot B(q)=0. \tag{146}$$

Here we have dropped the index ε of q for sake of clarity and we will assume that the fields $E(q)$ and $B(q)$ (which actually stand here for electric and magnetic fields) smoothly depend on its argument q but not on ε otherwise. The setting of [1] is slightly more general but the toy-system above will be sufficient for the exposition of the gain obtained by modulation in the analysis. The starting point is that a naive application of the Cauchy-Lipshitz theorem does only provide existence of a solution over a time which may vanish as ε converges to 0. The difficulty resides within the lack of sign or structure of the $E(q)$ term which prevents from obtaining straightforwardly some uniform estimates by energy. To overcome this difficulty Berkowitz and Gardner introduced the modulated variable:

$$\tilde{p} = q' - u(q) \text{ where } u(q) \text{ satisfies } E(q) + u(q) \times B(q) = 0. \tag{147}$$

Observe that the existence for any q of such a vector $u(q)$ is guaranteed by the condition $E(q) \cdot B(q) = 0$ and that the set of such vectors is an one-dimensional affine space. Indeed in [1] Berkowitz and Gardner makes use of the following explicit field

$$u(q) := |B(q)|^{-2}\, E(q) \times B(q), \tag{148}$$

which satisfies the condition in (147) and which turns to be the actual physical drift velocity for this system.

Using the chain-rule, one obtains $\tilde{p}' = q'' - q' \cdot \nabla u(q)$, and then, by using (146) and (147),

$$\varepsilon^2 \tilde{p}' = E(q) + q' \times B(q) - \varepsilon^2 q' \cdot \nabla u(q) = \tilde{p} \times B(q) - \varepsilon^2 (\tilde{p} + u(q)) \cdot \nabla u(q).$$

Therefore, one obtains the following gyroscopic normal form:

$$\tilde{p}' = \frac{1}{\varepsilon^2}\, \tilde{p} \times B(q) - (\tilde{p} + u(q)) \cdot \nabla u(q) \tag{149}$$

Now that the $E(q)$ has been absorbed by the choice of the modulated variable, the only factor with a singular (i.e. negative) power of ε is in front of the $B(q)$ term and this term disappears when taking the inner product of (149) with p in an energy-type estimate. Some Gronwall estimates on (149) and (147) then provide uniforms bound of q and q'. In particular the second Gronwall estimate allows to estimate q and q' from p thanks to (147) and therefore relies on the fact that the modulation $u(q)$ involves one less time derivative than q'.

Let us now go back to our search for an asymptotic normal form of (138) and let see how to extend the analysis performed above. We first observe that the drift velocity (naively) computed as in (148) with $\mathsf{E}_0(q^\varepsilon)$ and B_{θ^ε} instead of $E(q)$ and $B(q)$ corresponds to a nonzero η in (145). Still in the case of (144) one observes as we already did in the proof of Theorem 2.16 that the natural counterpart to $(h^\varepsilon)'$ for what concerns the angular velocity is rather $\varepsilon(\theta^\varepsilon)'$ than $(\theta^\varepsilon)'$. Moreover we will benefit from the fact that the contribution due to $\varepsilon(\theta^\varepsilon)'$ in the first two coordinates of the result of the cross product in the right hand side of (144) provides the term (89) whose special structure somehow allows to gain one factor ε. It turns out that the leading part of the relevant drift velocity in order to pass to the limit in (138) is given by (145) with $\eta = 0$, that is by

$$\hat{p}^\varepsilon = ((h^\varepsilon)' - \gamma u_\Omega(h^\varepsilon), \varepsilon(\theta^\varepsilon)')^t. \tag{150}$$

Still the leading terms of the inertia matrix $\varepsilon^\alpha I_\varepsilon \mathcal{M}_g I_\varepsilon + \mathcal{M}_a^\varepsilon(q^\varepsilon)$ in front of $(q^\varepsilon)''$ in (138) is[10] $I_\varepsilon(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta})I_\varepsilon$, and therefore, in order to cover the case where $\alpha \geq 2$, one has to investigate further the structure of the other terms of the equation (138), and to hope that a recombination as nice as in (144) occurs at the next order. This is actually why we had expanded up to order ε^2 in Proposition 3.12. One observes in particular from (141) that at order ε the a-connection Γ_a^ε comes into play. Indeed combining the previous expansions of Γ_a^ε, E^ε and B^ε one obtains

$$\begin{aligned}\gamma^2 E^\varepsilon(q^\varepsilon) + \gamma(q^\varepsilon)' \times B^\varepsilon(q^\varepsilon) - \langle \Gamma_a^\varepsilon(q^\varepsilon), (q^\varepsilon)', (q^\varepsilon)' \rangle \\ = I_\varepsilon \Big[\gamma \hat{p}^\varepsilon \times B_{\theta^\varepsilon} \\ + \varepsilon \Big(\gamma^2 \mathsf{E}_1(q^\varepsilon) + \gamma I_\varepsilon (q^\varepsilon)' \times \mathsf{B}_1(q^\varepsilon) - \langle \Gamma_{a,\theta^\varepsilon}, I_\varepsilon(q^\varepsilon)', I_\varepsilon(q^\varepsilon)' \rangle \Big) + O(\varepsilon^2) \Big],\end{aligned} \tag{151}$$

Above and thereafter the notation $O(\varepsilon^2)$ holds for a term of the form $\varepsilon^2 F(\varepsilon, q^\varepsilon, \hat{p}^\varepsilon)$ where F is a vector field which is weakly nonlinear in the sense that there exists $\delta > 0$, $\varepsilon_0 \in (0,1)$ and $K > 0$ such that for any (ε, q, p) in $\mathfrak{Q}_{\delta,\varepsilon_0} \times \mathbb{R}^3$, $|F(\varepsilon, q, p)|_{\mathbb{R}^3} \leq K(1 + |p|_{\mathbb{R}^3} + \varepsilon |p|^2_{\mathbb{R}^3})$. Indeed the way (151) has to be understood is even more intricate because among the terms hidden in the $O(\varepsilon^2)$ there is a term for which one obtains such an order only when performing a Gronwall estimate for an energy-type method. More precisely one term abusively included in the notation $O(\varepsilon^2)$ in (151) is of the form $O(\varepsilon)F(q^\varepsilon)$, where F is a vector field in $C^\infty(\mathbb{R} \times \Omega; \mathbb{R}^3)$ weakly gyroscopic in the sense that for any $\delta > 0$ and $\varepsilon_0 \in (0,1)$ there exists $K > 0$ depending on $\mathcal{S}_0$, Ω, γ and δ such that for any smooth curve $q(t) = (h(t), \theta(t))$ in $\{x \in \Omega \ / \ d(x, \partial\Omega) > \delta\} \times \mathbb{R}$, we have, for any $t \geq 0$ and any $\varepsilon \in (0, \varepsilon_0)$, $| \int_0^t \tilde{p} \cdot F(q)| \leq \varepsilon K(1 + t + \int_0^t |\tilde{p}|^2_{\mathbb{R}^3})$, with $\tilde{p} = (h' - \gamma u_\Omega(h), \varepsilon\theta')^t$.

A striking and crucial phenomenon is that some subprincipal contributions (that is, of order ε) of the right hand side of (151) can be gathered into an a-connection term involving the bilinear mapping Γ_a^ε obtained in the case where the rigid body is of size $\varepsilon = 1$ and is immersed in a fluid filling the

[10] Observe that one recovers the same inertia for the leading terms (for which the hierarchy depends on whether $\alpha \geq 2$ or $\alpha \leq 2$) than in Section 2.6 for the case where the "fluid+solid" system occupies the full plane, cf. (85).

whole plane, but applied to the modulated variable as follows[11]:

$$\gamma^2 \mathsf{E}_1(q^\varepsilon) + \gamma I_\varepsilon(q^\varepsilon)' \times \mathsf{B}_1(q^\varepsilon) - \langle \Gamma_{a,\theta^\varepsilon}, I_\varepsilon(q^\varepsilon)', I_\varepsilon(q^\varepsilon)' \rangle \\ = -\langle \Gamma_{a,\theta^\varepsilon}, \hat{p}^\varepsilon, \hat{p}^\varepsilon \rangle + \gamma (u_c(q^\varepsilon), 0)^t \times B_{\theta^\varepsilon} + O(\varepsilon), \quad (152)$$

where $\hat{p}^\varepsilon$ is given by (150), u_c is a smooth vector field on $\mathcal{Q}$ with values in $\mathbb{R}^2$ which depends on Ω and $\mathcal{S}_0$. Indeed a quite explicit expression can be given by $u_c := \nabla_h^\perp \big(D_h \psi_\Omega(h) \cdot R(\theta)\xi \big)$, where D_h denotes the derivative with respect to h. We refer here again to [7] for a proof of (152); it relies on explicit computations of the profiles $\mathsf{E}_1(q)$ and $\mathsf{B}_1(q)$ thanks to geometric quantities and some tedious algebraic computations.

Next the second term in the right hand side of (152) can be absorbed by the principal term in the right hand side of (151) up to a modification of size ε of the arguments that is, thanks to the following second order modulation:

$$\tilde{p}_\varepsilon := \big(h'_\varepsilon - \gamma [u_\Omega(h_\varepsilon) + \varepsilon u_c(q_\varepsilon)], \varepsilon \vartheta'_\varepsilon \big)^t. \quad (153)$$

Observe also that, as long as the solid does not touch the boundary, the drift term in the velocity of the center of mass is bounded. Indeed one may easily proves that there exists $\delta > 0$, ε_0 in $(0,1)$ and $K > 0$ such that for any (ε, q) in $\mathfrak{Q}_{\delta,\varepsilon_0}$ with $q = (h, \theta)$, $|u_\Omega(h) + \varepsilon u_c(q)|_{\mathbb{R}^3} \leq K$.

Thus we deduce from (151) and (152) that

$$\gamma^2 E^\varepsilon(q^\varepsilon) + \gamma (q^\varepsilon)' \times B^\varepsilon(q^\varepsilon) - \langle \Gamma_a^\varepsilon(q^\varepsilon), (q^\varepsilon)', (q^\varepsilon)' \rangle \\ = I_\varepsilon \Big[\gamma \tilde{p}^\varepsilon \times B_{\theta^\varepsilon} - \varepsilon \langle \Gamma_{a,\theta^\varepsilon}, \tilde{p}^\varepsilon, \tilde{p}^\varepsilon \rangle + O(\varepsilon^2) \Big], \quad (154)$$

Using now (140) and a few further tedious manipulations the equation (138) can now be recast into the following geodesic-gyroscopic normal form:

$$\Big(\varepsilon^\alpha M_g + \varepsilon^2 M_{a,\theta_\varepsilon} \Big) \tilde{p}'_\varepsilon + \varepsilon \langle \Gamma_{a,\theta_\varepsilon}, \tilde{p}_\varepsilon, \tilde{p}_\varepsilon \rangle = \gamma \tilde{p}^\varepsilon \times B_{\theta^\varepsilon} + O(\varepsilon^{\min(2,\alpha)}). \quad (155)$$

Observe how (155) is close to (87): the only two differences are the modulation of p^ε into $\tilde{p}^\varepsilon$ and the remainder $O(\varepsilon^{\min(2,\alpha)})$, which actually suffers from the same abuse of notation than the term $O(\varepsilon^2)$ described below (151). At least till the solid stays away from the external boundary one may take advantage of this normal form to obtain an estimate of the modulated energy

$$\frac{1}{2} \Big(\varepsilon^\alpha M_g + \varepsilon^2 M_{a,\theta_\varepsilon} \Big) \tilde{p}_\varepsilon \cdot \tilde{p}_\varepsilon,$$

thanks to a Gronwall estimate. This provides uniform bounds of $|((h^\varepsilon)', \varepsilon(\theta^\varepsilon)')|_{\mathbb{R}^3}$. This estimate in turn allows to pass to the limit proceeding as in the proof of Theorem 2.16. The issue of a possible collision is then tackled in a bootstrapping argument thanks to the behavior of the limit systems.

[11]As for (144), this relation is algebraic, in the sense that it does not rely on the fact that q^ε satisfies (121).

More precisely we first prove that the lifetime T^ε of the solution q^ε, which can be only limited by a possible encounter between the solid and the boundary $\partial\Omega$, satisfies the following: there exist $\varepsilon_0 > 0$, $\underline{T} > 0$ and $\underline{\delta} > 0$, such that for any ε in $(0, \varepsilon_0)$, we have $T^\varepsilon \geq \underline{T}$ and moreovoer on $[0, \underline{T}]$, one has $(\varepsilon, q^\varepsilon) \in \mathfrak{Q}_{\underline{\delta},\varepsilon_0}$.

Then, using again the uniform estimates obtained thanks to the asymptotic geodesic-gyroscopic normal form (155) one establishes the desired convergence on any time interval during which we have a minimal distance between $\mathcal{S}_\varepsilon(q)$ and $\partial\Omega$, uniform for small ε. This consists in passing to the weak limit, with the help of all a priori bounds, in the two first components of each term of (155). It finally only remains to extend the time interval on which the above convergences are valid to any time interval. □

4. Case of an unbounded flow with vorticity

In this section we investigate the case of a rigid body immersed in an unbounded flow with vorticity.

4.1. Statement of a theorem à la Yudovich in the body frame

For the Cauchy problem it is more convenient to consider the body frame which does not depend on time, as we did in Section 2. We will therefore start back from the equations (37)-(38). In the sequel we will use an abuse of notation and still denote by ω the vorticity in the body frame given by $\omega(t,x) := \operatorname{curl} v(t,x)$. Taking the curl of the equation (37) we get

$$\partial_t \omega + \left[(v - \ell - r x^\perp) \cdot \nabla\right] \omega = 0 \text{ for } x \in \mathcal{F}_0. \tag{156}$$

Due to the equation of vorticity (156) the following quantities are conserved as time proceeds, at least for smooth solutions: for any $t > 0$, for any p in $[1, +\infty]$,

$$\|\omega(t,\cdot)\|_{L^p(\mathcal{F}_0)} = \|\omega_0\|_{L^p(\mathcal{F}_0)}. \tag{157}$$

In the case of a fluid alone, the conservation laws (157) allowed Yudovich, and DiPerna and Majda to construct some global-in-time solutions of the 2d Euler equations in the case of a velocity with finite local energy and L^p initial vorticity, with $p > 1$. In the case $p = +\infty$ Yudovich also obtained a uniqueness result using in particular that the corresponding fluid velocity is in the space $\mathcal{LL}(\mathcal{F}_0)$ of log-Lipschitz $\mathbb{R}^2$-valued vector fields on $\mathcal{F}_0$, that is the set of functions $f \in L^\infty(\mathcal{F}_0)$ such that

$$\|f\|_{\mathcal{LL}(\mathcal{F}_0)} := \|f\|_{L^\infty(\mathcal{F}_0)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|(x-y)(1 + \ln^- |x-y|)|} < +\infty. \tag{158}$$

These results can be adapted to the case where there is a rigid body. In theses notes we will focus on a result of global in time existence and uniqueness similar to the celebrated result by Yudovich about a fluid alone.

Let us first give a global weak formulation of the problem by considering (for the solution as well as for test functions) a velocity field on the whole plane, with the constraint to be rigid on $\mathcal{S}_0$. We introduce the following space

$$\mathcal{H} := \left\{ \Psi \in L^2_{loc}(\mathbb{R}^2) \Big/ \text{ div } \Psi = 0 \text{ in } \mathbb{R}^2 \text{ and } D\Psi = 0 \text{ in } \mathcal{S}_0 \right\},$$

where $D\Psi := \nabla \Psi + (\nabla \Psi)^T$. It is classical that the space $\mathcal{H}$ can be recast thanks to the property:

$$\exists (\ell_\Psi, r_\Psi) \in \mathbb{R}^2 \times \mathbb{R},\ \forall x \in \mathcal{S}_0,\ \Psi(x) = \ell_\Psi + r_\Psi x^\perp. \tag{159}$$

More precisely, $\mathcal{H} = \left\{ \Psi \in L^2_{loc}(\mathbb{R}^2) \Big/ \text{ div } \Psi = 0 \text{ in } \mathbb{R}^2 \text{ and satisfies (159)} \right\}$, and the ordered pair (ℓ_Ψ, r_Ψ) above is unique. Let us also introduce

$$\tilde{\mathcal{H}} := \left\{ \Psi \in \mathcal{H} \Big/ \Psi|_{\overline{\mathcal{F}_0}} \in C^1_c(\overline{\mathcal{F}_0}) \right\},$$

where $\Psi|_{\overline{\mathcal{F}_0}}$ denotes the restriction of Ψ to the closure of the fluid domain. We also introduce for $T > 0$, $\tilde{\mathcal{H}}_T := C^1([0,T]; \tilde{\mathcal{H}})$. When $(\overline{u}, \overline{v}) \in \mathcal{H} \times \tilde{\mathcal{H}}$, we denote by

$$< \overline{u}, \overline{v} >:= m\, \ell_u \cdot \ell_v + \mathcal{J} r_u\, r_v + \int_{\mathcal{F}_0} u \cdot v\, dx,$$

where we use the notations u and v for the restrictions of $\overline{u}$ and $\overline{v}$ to $\overline{\mathcal{F}_0}$. Our definition of a weak solution is the following.

Definition 4.1 (Weak Solution). Let us be given $\overline{v}_0 \in \mathcal{H}$ and $T > 0$. We say that $\overline{v} \in C([0,T]; \mathcal{H} - w)$ is a weak solution to (37)–(40) in $[0,T]$ if for any test function $\Psi \in \tilde{\mathcal{H}}_T$,

$$\begin{aligned} < \Psi(T,\cdot), \overline{v}(T,\cdot) > - < \Psi(0,\cdot), \overline{v}_0 >_\rho = \int_0^T < \frac{\partial \Psi}{\partial t}, \overline{v} > \, dt \\ + \int_0^T \int_{\mathcal{F}_0} v \cdot \left(\left(v - \ell_v - r_v x^\perp \right) \cdot \nabla \right) \Psi \, dx\, dt - \int_0^T \int_{\mathcal{F}_0} r_v\, v^\perp \cdot \Psi \, dx\, dt \\ - \int_0^T m r_v\, \ell_v^\perp \cdot \ell_\Psi \, dt. \end{aligned} \tag{160}$$

We say that $\overline{v} \in C([0,+\infty); \mathcal{H} - w)$ is a weak solution to (37)–(40) in $[0,+\infty)$ if it satisfies (160) for all $T > 0$.

Definition 4.1 is legitimate since a classical solution to (37)–(40) in $[0,T]$ is also a weak solution. This follows easily from an integration by parts in space which provides the identity on $[0,T]$:

$$< \partial_t \overline{v}, \Psi >= \int_{\mathcal{F}_0} v\cdot\big(\big(v-\ell_v-r_v x^\perp\big)\cdot\nabla\big)\,\Psi\,dx - \int_{\mathcal{F}_0} r_v\, v^\perp\cdot\Psi\,dx - m r_v\,\ell_v^\perp\cdot\ell_\Psi, \tag{161}$$

and then from an integration by parts in time.

In the sequel we will often drop the index of ℓ_v and r_v and we will therefore rather write ℓ and r. We will equivalently say that (ℓ, r, v) is a weak solution to (37)–(40).

One has the following result of existence of weak solutions for the above system, the initial position of the solid being given.

Theorem 4.2. *For any $\overline{v}_0 \in \mathcal{H}$ such that the restriction of $\operatorname{curl}\overline{v}_0$ to $\overline{\mathcal{F}_0}$ is in $L^\infty_c(\overline{\mathcal{F}_0})$, there exists a unique weak solution $\overline{v}\in C([0,+\infty);\mathcal{H}-w)$ to (37)–(40) in $[0,+\infty)$. Moreover (ℓ,r) is in $C^1(\mathbb{R}^+;\mathbb{R}^2\times\mathbb{R})$, v is in $L^\infty(\mathbb{R}^+;\mathcal{LL}(\mathcal{F}_0))$ and $\operatorname{curl} v$ is in $L^\infty(\mathbb{R}^+;L^\infty_c(\overline{\mathcal{F}_0}))$.*

Going back to the original frame Theorem 4.2 implies Theorem 1.9. Regarding the initial data, let us observe that with any $(\ell_0, r_0)\in\mathbb{R}^2\times\mathbb{R}$, $\omega_0\in L^\infty_c(\overline{\mathcal{F}_0})$, one may associate $\overline{v}_0\in\mathcal{H}$ by setting $\overline{v}_0 = \ell_0 + r_0 x^\perp$ in $\mathcal{S}_0$ and $\overline{v}_0 = u_0$, where u_0 is the compatible initial velocity associated with ℓ_0, r_0 and ω_0 by Definition 1.8.

4.2. Proof of Theorem 4.2

In order to take into account the velocity contribution due to the vorticity we consider the Green's function $G(x,y)$ of $\mathcal{F}_0$ with Dirichlet boundary conditions. We also introduce the function $K(x,y)=\nabla^\perp G(x,y)$ known as the kernel of the Biot-Savart operator $K[\omega]$ which therefore acts on $\omega\in L^\infty_c(\overline{\mathcal{F}_0})$ through the formula

$$K[\omega](x) = \int_{\mathcal{F}_0} K(x,y)\omega(y)\,dy.$$

It is classical that $K[\omega]$ is in $\mathcal{LL}(\mathcal{F}_0)$, divergence-free, tangent to the boundary and satisfies $\operatorname{curl} K[\omega]=\omega$ and $K[\omega](x)=\mathcal{O}\left(|x|^{-2}\right)$ as $x\to\infty$, (so that it is square-integrable), and its circulation around $\partial\mathcal{S}_0$ is given by $\int_{\partial\mathcal{S}_0} K[\omega]\cdot\tau\,ds = -\int_{\mathcal{F}_0}\omega\,dx$. Then, given ω in $L^\infty_c(\mathcal{F}_0)$, ℓ in $\mathbb{R}^2$, r and γ in $\mathbb{R}$, there is a unique solution v in $\mathcal{LL}(\mathcal{F}_0)$ to the following system:

$$\begin{gathered}
\operatorname{div} v = 0 \text{ and } \operatorname{curl} v = \omega \quad \text{for } x\in\mathcal{F}_0,\\
v\cdot n = \left(\ell + r x^\perp\right)\cdot n \quad \text{for } x\in\partial\mathcal{S}_0 \text{ and } \int_{\partial\mathcal{S}_0} v\cdot\tau\,ds = \gamma,\\
v \longrightarrow 0 \quad \text{as } x\to\infty.
\end{gathered}$$

Moreover v is given by

$$v = \tilde{v} + \beta H, \tag{162}$$

$$\text{with } \tilde{v} := K[\omega] + \ell_1 \nabla\Phi_1 + \ell_2 \nabla\Phi_2 + r\nabla\Phi_3 \text{ and } \beta := \gamma + \int_{\mathcal{F}_0} \omega \, dx.$$

We start with looking for some a priori estimates that is to some estimates satisfied by smooth solutions to (37)–(40). We already mentioned above the a priori estimates (157) regarding the vorticity. From Kelvin's theorem and the vorticity equation (156), one also has, at least for smooth solutions, the following:

$$\gamma = \int_{\partial\mathcal{S}_0} v_0 \cdot \tau \, ds \quad \text{and} \quad \int_{\mathcal{F}_0} \omega(t,x)\, dx = \int_{\mathcal{F}_0} \omega_0(x)\, dx.$$

In particular it follows from these two conservation laws that the coefficient β in (162) is constant in time. Regarding the energy observe that $\tilde{v}$ is in $L^2(\mathcal{F}_0)$ whereas v is not[12] unless $\beta = 0$. Still we have the following result.

Proposition 4.3. *There exists a constant $C > 0$ (depending only on $\mathcal{S}_0$, m and $\mathcal{J}$) such that for any smooth solution (ℓ, r, v) of the problem (37)–(40) on the time interval $[0,T]$, with compactly supported fluid vorticity, the energy-like quantity defined by:*

$$\tilde{\mathcal{E}}(t) := \frac{1}{2}\left(m|\ell(t)|^2 + \mathcal{J} r(t)^2 + \int_{\mathcal{F}_0} \tilde{v}(t,\cdot)^2 dx \right),$$

satisfies the inequality $\tilde{\mathcal{E}}(t) \leqslant \tilde{\mathcal{E}}(0) e^{C|\beta| t}$.

Remark 4.4. In the case where $\beta = 0$, that is, when the solution is of finite energy, the energy is conserved.

Proof. We start by recalling that a classical solution satisfies (161), and we use the decomposition (162) in the left hand side. One has for all t:

$$\tilde{v} = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ and } \nabla\tilde{v} = \mathcal{O}\left(\frac{1}{|x|^3}\right) \text{ as } x \to \infty. \tag{163}$$

We use an integration by parts for the first term of the right hand side of (161) to get that for any test function $\Psi \in \tilde{\mathcal{H}}_T$,

$$m\,\ell' \cdot \ell_\Psi + \mathcal{J} r' r_\Psi + \int_{\mathcal{F}_0} \partial_t \tilde{v} \cdot \Psi \, dx = -\int_{\mathcal{F}_0} \Psi \cdot \left((v - \ell - r x^\perp) \cdot \nabla \right) v \, dx$$
$$- \int_{\mathcal{F}_0} r v^\perp \cdot \Psi \, dx - m r \ell^\perp \cdot \ell_\Psi.$$

Then, using a standard regular truncation process, we obtain that the previous identity is still valid for the test function Ψ defined by $\Psi(t,x) = \tilde{v}(t,x)$

[12] It is interesting to compare the decomposition above with the one used in Section 2, cf. (53).

for (t,x) in $[0,T] \times \mathcal{F}_0$ and $\Psi(t,x) = \ell(t) + r(t)x^\perp$ for (t,x) in $[0,T] \times \mathcal{S}_0$. Hence we get:

$$\begin{aligned}
\tilde{\mathcal{E}}'(t) &= -\int_{\mathcal{F}_0} \tilde{v} \cdot \left((v - \ell - rx^\perp) \cdot \nabla \right) v \, dx - \int_{\mathcal{F}_0} rv^\perp \cdot \tilde{v} \, dx \\
&= -\int_{\mathcal{F}_0} \tilde{v} \cdot \Big((v - \ell - rx^\perp) \cdot \nabla \tilde{v} \Big) \, dx - \beta \int_{\mathcal{F}_0} \tilde{v} \cdot ((\tilde{v} \cdot \nabla) H) \, dx \\
&\quad + \beta \int_{\mathcal{F}_0} \tilde{v} \cdot ((\ell \cdot \nabla) H) \, dx - \beta r \int_{\mathcal{F}_0} \tilde{v} \cdot (H^\perp - (x^\perp \cdot \nabla) H) \, dx \\
&\quad - \beta^2 \int_{\mathcal{F}_0} \tilde{v} \cdot ((H \cdot \nabla) H) \, dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Integrating by parts we infer that $I_1 = 0$, since $v - \ell - rx^\perp$ is a divergence free vector field, tangent to the boundary. Let us stress that there is no contribution at infinity because of the decay properties of the various fields involved, see (163).

On the other hand, using the smoothness and decay at infinity of H, we get that there exists $C > 0$ depending only on $\mathcal{F}_0$ such that

$$|I_2| + |I_3| + |I_4| \leqslant C|\beta| \left(\int_{\mathcal{F}_0} \tilde{v}^2 \, dx + |\ell|^2 + r^2 \right).$$

Let us now turn our attention to I_5. We first use that H being curl free, we have

$$\int_{\mathcal{F}_0} \tilde{v} \cdot ((H \cdot \nabla) H) \, dx = \frac{1}{2} \int_{\mathcal{F}_0} (\tilde{v} \cdot \nabla) |H|^2 \, dx,$$

and then an integration by parts and (39) to obtain

$$\begin{aligned}
\int_{\mathcal{F}_0} \tilde{v} \cdot ((H \cdot \nabla) H) \, dx &= \frac{1}{2} \int_{\partial \mathcal{S}_0} (\tilde{v} \cdot n) |H|^2 \, ds \\
&= \frac{1}{2} \ell \cdot \int_{\partial \mathcal{S}_0} |H|^2 n \, ds + \frac{1}{2} r \int_{\partial \mathcal{S}_0} |H|^2 x^\perp \cdot n \, ds.
\end{aligned}$$

We make use of Blasius' lemma and Cauchy's residue Theorem to obtain that $I_5 = 0$.

Collecting all these estimates it only remains to use Gronwall's lemma to conclude. □

Proposition 4.3 provides in particular some a priori estimates of the solid velocity. We aim now at finding an a priori bound of the body acceleration.

Proposition 4.5. *There exists a constant $C > 0$ depending only on $\mathcal{S}_0$, m, $\mathcal{J}$, β and $\tilde{\mathcal{E}}(0)$ such that any classical solution to* (37)–(40) *satisfies the estimate* $\|(\ell', r')\|_{L^\infty(0,T)} \leqslant C$.

Proof. Again, after a regular truncation procedure, we can use (161) with, as test functions, the functions $(\Psi_i)_{i=1,2,3}$ defined by $\Psi_i = \nabla\Phi_i$ in $\mathcal{F}_0$ and $\Psi_i = e_i$, for $i = 1, 2$ and $\Psi_3 = x^\perp$ in $\mathcal{S}_0$. We observe that the left hand side of (161) can be recast in terms of the acceleration of the body only thanks to the added mass phenomenon:

$$\begin{aligned}\big(< \partial_t \overline{v}, \Psi_i > \big)_{i=1,2,3} &= \mathcal{M}_g \begin{pmatrix} \ell \\ r \end{pmatrix}'(t) + (\int_{\mathcal{F}_0} \partial_t v \cdot \nabla\Phi_i \, dx)_{i=1,2,3} \\ &= \big(\mathcal{M}_g + \mathcal{M}_a\big) \begin{pmatrix} \ell \\ r \end{pmatrix}',\end{aligned}$$

using (51) (observe that the new contribution in the velocity due to the vorticity does not modify this identity). We recall that $\mathcal{M}_g$ and $\mathcal{M}_a$ were respectively given by (18) and (26). Therefore we infer from (161) that

$$\begin{aligned}(\mathcal{M}_g + \mathcal{M}_a) \begin{pmatrix} \ell \\ r \end{pmatrix}' &= \begin{pmatrix} -mr\ell^\perp \\ 0 \end{pmatrix} \\ &+ \big(\textstyle\int_{\mathcal{F}_0} v \cdot \big[((v - \ell - rx^\perp)\cdot\nabla)\nabla\Phi_i\big] \, dx - \int_{\mathcal{F}_0} rv^\perp \cdot \nabla\Phi_i \, dx\big)_{i\in\{1,2,3\}} .\end{aligned} \tag{164}$$

It then suffices to use the decomposition (162), Proposition 4.3 and the decay properties of $\nabla\Phi_i$ and H to obtain a bound of ℓ' and r'. □

With these a priori estimates in hand there are several classical ways to infer the local in time existence of a weak solution to (37)-(40) as promised in the statement of Theorem 4.2. Since we do not have mentioned the existence of smooth solutions, one method to produce directly weak solutions is to apply the following Schauder's fixed point theorem.

Theorem 4.6. *Let E denotes a Banach space and let C be a nonempty closed convex set in E. Let $F : C \mapsto C$ be a continuous map such that $F(C) \subset K$, where K is a compact subset of C. Then F has a fixed point in K.*

Theorem 4.6 is applied to an operator F which maps (ω, ℓ, r) to $(\tilde\omega, \tilde\ell, \tilde r)$ as follows:

$$\partial_t \tilde\omega + \big[(v - \ell - rx^\perp)\cdot\nabla\big]\,\tilde\omega = 0 \text{ in } \mathcal{F}_0,$$

$$\begin{aligned}(\mathcal{M}_g + \mathcal{M}_a) \begin{pmatrix} \tilde\ell \\ \tilde r \end{pmatrix}' &= \Big(\textstyle\int_{\mathcal{F}_0} \Big(v \cdot \big[((v - \ell - rx^\perp)\cdot\nabla)\nabla\Phi_i\big] - rv^\perp \cdot \nabla\Phi_i\Big) \, dx\Big)_{i\in\{1,2,3\}} \\ &+ \begin{pmatrix} -mr\ell^\perp \\ 0 \end{pmatrix},\end{aligned}$$

where v is given by (162), with some appropriate sets C and K of functions (ω, ℓ, r) defined on a time interval $(0, T)$ with T small enough. We thus observe that a fixed point of F verifies (156) and (164). Moreover the previous a priori bounds can be adapted to the system above and this allows to apply

Schauder's fixed point theorem. In particular the compactness for the (ℓ, r)-part is given by an appropriate modification of Proposition 4.5.

The global in time existence follows then from global a priori estimates in particular of the vorticity.

On the other hand the uniqueness part of Theorem 4.2 relies on Yudovich's method for the case of a fluid alone. Suppose that we have two solutions (ℓ_1, r_1, v_1) and (ℓ_2, r_2, v_2) with the same initial data (observe that in this part of the proof the indices do not stand for the components.) In particular, they share the same circulation γ and initial vorticity w_0. As a consequence, despite the fact that v_1 and v_2 are not necessarily in $L^2(\mathcal{F}_0)$, their difference $v_1 - v_2$ does belong to $L^\infty(0,T; L^2(\mathcal{F}_0))$ with[13]

$$v_1 - v_2 = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ and } \nabla(v_1 - v_2) = \mathcal{O}\left(\frac{1}{|x|^3}\right) \text{ as } |x| \to +\infty. \quad (165)$$

Moreover ℓ_1, ℓ_2, r_1, r_2 belong to $W^{1,\infty}(0,T)$. As a consequence, one can prove that ∇q_1 and ∇q_2 belong to $L^\infty(0,T; L^2(\mathcal{F}_0))$. Then defining $\breve{\ell} := \ell_1 - \ell_2$, $\breve{r} := r_1 - r_2$, $\breve{v} := v_1 - v_2$ and $\breve{q} = q_1 - q_2$, we deduce from (37) that

$$\frac{\partial \breve{v}}{\partial t} + \left[(v_1 - \ell_1 - r_1 x^\perp) \cdot \nabla\right] \breve{v} + \left[(\breve{v} - \breve{\ell} - \breve{r} x^\perp) \cdot \nabla\right] v_2 + r_1 \breve{v}^\perp + \breve{r} v_2^\perp + \nabla \breve{q} = 0.$$

We multiply by $\breve{v}$, integrate over $\mathcal{F}_0$ and integrate by parts (which is permitted by (165) and by the regularity of the pressure), and deduce:

$$\frac{1}{2}\frac{d}{dt}\|\breve{v}\|^2_{L^2} + \int_{\mathcal{F}_0} \breve{v} \cdot \left[(\breve{v} - \breve{\ell} - \breve{r} x^\perp) \cdot \nabla v_2\right] dx + \breve{r} \int_{\mathcal{F}_0} \breve{v} \cdot v_2^\perp \, dx + \int_{\partial \mathcal{F}_0} \breve{q} \breve{v} \cdot n = 0.$$

For what concerns the last term,

$$\begin{aligned} \int_{\partial\mathcal{F}_0} \breve{q}\breve{v} \cdot n &= \breve{\ell} \cdot \int_{\partial\mathcal{F}_0} \breve{q} n + \breve{r} \int_{\partial\mathcal{F}_0} \breve{q} x^\perp \cdot n \\ &= m\breve{\ell} \cdot \left(\breve{\ell}' + \breve{r}\ell_1^\perp + r_2 \breve{\ell}^\perp\right) + \mathcal{J}\breve{r}\breve{r}' \\ &= m\breve{r}\,\breve{\ell} \cdot \ell_1^\perp + m\breve{\ell} \cdot \breve{\ell}' + \mathcal{J}\breve{r}\breve{r}'. \end{aligned}$$

Using $(x^\perp \cdot \nabla) v_2 = \nabla(x^\perp \cdot v_2) - v_2^\perp - x^\perp \omega_2$, and an integration by parts, one has

$$\int_{\mathcal{F}_0} \breve{v} \cdot [(x^\perp \cdot \nabla) v_2] \, dx = \int_{\mathcal{S}_0} (x^\perp \cdot v_2)[(\breve{\ell} + \breve{r} x^\perp) \cdot n] \, ds + \int_{\mathcal{F}_0} \breve{v} \cdot (-v_2^\perp - x^\perp \omega_2) \, dx.$$

Hence using the boundedness of v_2 and ω_2 in $L^\infty(0,T; L^\infty(\mathcal{F}_0))$, the boundedness of ℓ^1 and the one of Supp (ω_2), we arrive to

$$\frac{d}{dt}\left(\|\breve{v}\|^2_{L^2} + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\right) \leqslant C\Big(\|\breve{v}\|^2_{L^2} + \|\breve{\ell}\|^2 + \|\breve{r}\|^2 + \|\nabla v_2\|_{L^p} \|\breve{v}^2\|_{L^{p'}}\Big),$$

[13]Recall that both v_1 and v_2 are harmonic for $|x|$ large enough and converge to 0 at infinity.

for $p>2$. Here, the various constants C may depend on $\mathcal{S}_0$ and on the solutions (ℓ_1, r_1, v_1) and (ℓ_2, r_2, v_2), but not on p. Hence using elliptic regularity and interpolation, we obtain that for p large,

$$\begin{aligned} \frac{d}{dt}\left(\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2\right) &\leqslant C\Big(\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2\Big)+Cp\|\breve{v}^2\|_{L^{p'}} \\ &\leqslant C\Big(\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2\Big)+Cp\|\breve{v}\|_{L^2}^{\frac{2}{p'}}\|\breve{v}^2\|_{L^\infty}^{\frac{1}{p}}. \end{aligned}$$

For some constant $C>0$, we have on $[0,T]$: $\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2 \leqslant K$, so for some $C>0$ one has in particular

$$\frac{d}{dt}\left(\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2\right) \leqslant Cp\Big(\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2\Big)^{1/p'}.$$

Now the unique solution to $y'=Ny^\delta$ and $y(0)=\varepsilon>0$ for $\delta\in(0,1)$ and $N>0$ is given by $y(t)=\Big[(1-\delta)Nt+\varepsilon^{1-\delta}\Big]^{\frac{1}{1-\delta}}$. Hence a comparison argument proves that $\|\breve{v}\|_{L^2}^2+\|\breve{\ell}\|^2+\|\breve{r}\|^2 \leqslant (Ct)^p$. We conclude that $\breve{v}=0$ for $t<1/C$ by letting p converge to $+\infty$.

4.3. Energy conservation

Despite the fact that the energy-type bound obtained in Proposition 4.3 turned out to be sufficient in order to deal with the Cauchy problem, one may wonder if even in the case where $\beta\neq 0$ (for which the kinetic fluid energy is infinite, see the discussion above Proposition 4.3) there is a renormalized energy which is exactly conserved at least for regular enough solutions to the problem (37)–(40). Another motivation is that the constant C which appears in Proposition 4.3 depends on the body geometry in such a way that the corresponding estimate is not uniform in the zero radius limit. One may hope that an exactly conserved quantity overcomes this lack of uniformity. For any p in $\mathbb{R}^3$ and for any $\omega\in L_c^\infty(\overline{\mathcal{F}_0})$ we define

$$\begin{aligned} \mathcal{E}(p,\omega) := \frac{1}{2}p\cdot(\mathcal{M}_g+\mathcal{M}_a)p - \frac{1}{2}\int_{\mathcal{F}_0\times\mathcal{F}_0} G_H(x,y)\omega(x)\omega(y)\,dx\,dy \\ -\gamma\int_{\mathcal{F}_0}\omega(x)\Psi_H(x)\,dx, \end{aligned} \tag{166}$$

where Ψ_H is defined in Section 2.2.4 and G_H is the so-called hydrodynamic Green function defined by

$$G_H(x,y) := G(x,y)+\Psi_H(x)+\Psi_H(y), \tag{167}$$

where G is the standard Dirichlet Green's function defined at the beginning of Section 4.2. We recall that $\mathcal{M}_g$ and $\mathcal{M}_a$ were respectively given by (18) and (26). Observe that in the irrotational case where ω is vanishing on $\overline{\mathcal{F}_0}$ the energy $\mathcal{E}(p,0)$ is equal to the quantity (22) which was proved to be conserved in Proposition 2.4. Indeed the three terms in the right hand side of (166) can therefore be respectively interpreted as the kinetic energy of the rigid body with its total inertia included its genuine inertia and the added inertia due to the incompressible fluid around, the self-interaction energy of the fluid vorticity and the interaction between the fluid vorticity and the circulation around the body.

The following energy conservation property can therefore be interpreted as an extension of Proposition 2.4 to the rotational case.

Proposition 4.7. *For any smooth solution* (ℓ, r, v) *of the problem* (37)-(40) *with compactly supported vorticity, the quantity* $\mathcal{E}(\ell, r, \operatorname{curl} v)$ *is conserved along the motion.*

Proof. We will proceed in two steps. We first give another form of (166). Let us prove that

$$\mathcal{E}(\ell, r, \operatorname{curl} v) = \frac{1}{2} p \cdot \mathcal{M}_g \, p + \frac{1}{2} \int_{\mathcal{F}_0} (|\tilde{v}|^2 + 2\beta \tilde{v} \cdot H) \, dx, \tag{168}$$

with $\tilde{v}$ and β given by the decomposition (162). Observe that the right hand side above can be obtained formally by expanding the natural total kinetic energy of the "fluid+solid" system $\mathcal{E}_g(p) + \frac{1}{2} \int_{\mathcal{F}_0} |v(t,\cdot)|^2 \, dx$ thanks to the decomposition (162) and discarding the infinite term $\frac{1}{2}\beta^2 \int_{\mathcal{F}_0} |H|^2 \, dx$ associated with the circulation around the body. Note in particular that $\tilde{v}(x) = \mathcal{O}(1/|x|^2)$ as $|x| \to +\infty$, so that the last integral in the right hand side of (168) is well-defined. Let us highlight that a difference with the irrotational case discussed in Section 2.3.2 is that $K[\omega]$ and H being not orthogonal in $L^2(\mathcal{F}_0)$ there is a crossed term, given by the contribution of the last summand of the last term of (168), and which encompasses a dependence on γ, through β.

In order to simplify the proof of (168) we introduce a few notations. Let us denote $\Psi(x) := \int_{\mathcal{F}_0} G(x,y)\omega(y) dy$ which is a stream function of $K[\omega]$ vanishing on the boundary $\mathcal{S}_0$, so that $K[\omega] = \nabla^\perp \Psi$. Let us also denote $\nabla \Phi := \ell_1 \nabla \Phi_1 + \ell_2 \nabla \Phi_2 + r \nabla \Phi_3$, so that $\tilde{v} = \nabla^\perp \Psi + \nabla \Phi$. Then we compute

$$\int_{\mathcal{F}_0} |\tilde{v}|^2 dx = \int_{\mathcal{F}_0} \nabla^\perp \Psi \cdot \tilde{v} + \int_{\mathcal{F}_0} \nabla^\perp \Psi \cdot \nabla \Phi + \int_{\mathcal{F}_0} \nabla \Phi \cdot \nabla \Phi. \tag{169}$$

First, integrating by parts yields

$$\int_{\mathcal{F}_0} \nabla^\perp \Psi \cdot \tilde{v} = - \int_{\mathcal{F}_0 \times \mathcal{F}_0} G(x,y) \omega(x) \omega(y) \, dx \, dy, \tag{170}$$

$$\int_{\mathcal{F}_0} \nabla^\perp \Psi \cdot \nabla \Phi = 0 \text{ and } \int_{\mathcal{F}_0} \tilde{v} \cdot H = - \int_{\mathcal{F}_0} \omega(x) \Psi_H(x) dx. \tag{171}$$

There is no boundary terms since Ψ and Ψ_H vanish on the boundary $\mathcal{S}_0$, and $\nabla \Phi$ and $\tilde{v}$ decrease also like $1/|x|^2$ at infinity.

Also, by definition, we have

$$\frac{1}{2}\int_{\mathcal{F}_0} \nabla\Phi\cdot\nabla\Phi = \frac{1}{2}p\cdot\mathcal{M}_a p. \tag{172}$$

Thus combining (169)-(172) we obtain that

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{F}_0}(|\tilde{v}|^2 + 2\beta\tilde{v}\cdot H)\,dx \quad &= \quad \frac{1}{2}p\cdot\mathcal{M}_a p - \frac{1}{2}\int_{\mathcal{F}_0\times\mathcal{F}_0} G(x,y)\omega(x)\omega(y)\,dx\,dy \\ &\quad -\beta\int_{\mathcal{F}_0}\omega(x)\Psi_H(x)\,dx.\end{aligned}$$

This entails (168) thanks to Fubini's theorem, (162) and (167).

Now by taking the time derivative of (168), using the definition of $\mathcal{M}_g$ given in (18), (162) and that the coefficient β in (162) is constant in time, one obtains:

$$\frac{d}{dt}\Big(\mathcal{E}(\ell, r, \operatorname{curl} v)\Big) = m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) + \int_{\mathcal{F}_0}\partial_t v\cdot v. \tag{173}$$

Using now the fluid equation (37) one deduces from (173) that $\frac{d}{dt}\Big(\mathcal{E}(\ell, r, \operatorname{curl} v)\Big) = I_1 + I_2 + I_3$, where where

$$I_1 := m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) - \int_{\mathcal{F}_0}\nabla q\cdot v,$$

$$I_2 := -\int_{\mathcal{F}_0}(v-\ell)\cdot\nabla v\cdot v \text{ and } I_3 := -r\int_{\mathcal{F}_0}[v^\perp - (x^\perp\cdot\nabla)v]\cdot v.$$

One easily justifies from the decay properties of H and $\tilde{v}$ that each integral above is convergent. This allows to integrate by parts both I_1 and I_2. Using the interface condition (39) and then Newton's equations for the body's dynamics, we obtain that $I_1 = 0$. For what concerns I_2 we get that

$$I_2 = -\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2(v-\ell)\cdot n.$$

For what concerns I_3, we consider $R > 0$ large in order that $\mathcal{S}_0\subset B(0,R)$, and consider the same integral as I_3, over $\mathcal{F}_0\cap B(0,R)$. Integrating by parts we obtain

$$\int_{\mathcal{F}_0\cap B(0,R)}[v^\perp - (x^\perp\cdot\nabla)v]\cdot v = -\int_{\partial\mathcal{S}_0}(x^\perp\cdot n)\frac{|v|^2}{2} - \int_{S(0,R)}(x^\perp\cdot n)\frac{|v|^2}{2},$$

where we denote by n also the unit outward normal on the circle $S(0,R)$. Of course $x^\perp\cdot n = 0$ on $S(0,R)$, so letting $R\to+\infty$, we end up with

$$I_3 = \frac{1}{2}\int_{\partial\mathcal{S}_0}(rx^\perp\cdot n)|v|^2.$$

Using (39) we deduce $I_2 + I_3 = 0$, so in total we get $\frac{d}{dt}\Big(\mathcal{E}(\ell, r, \operatorname{curl} v)\Big) = 0$. □

One difficulty with the quantity $\mathcal{E}$ is that both its form (166) and (168) are not the sum of positive terms. However one may extirpate some information from the conservation of $\mathcal{E}$ thanks to the support of vorticity. The basic idea can be exhibited thanks to the following technical lemma, having in mind that the hydrodynamic Green function $G_H(x,y)$ behaves like $\frac{1}{2\pi}\ln|x-y|$ at infinity.

Lemma 4.8. *Let f in $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. We denote by $\rho_f := \inf\{d > 1 \ / \ \mathrm{Supp}(f) \subset B(0,d)\}$. Then there exists $C>0$ such that or any $y\in B(0,\rho_f)$,*

$$\int_{\mathbb{R}^2}\Big|\ln|x-y|f(x)\Big|\,dx \leqslant C\|f\|_{L^\infty}+\ln(2\rho_f)\|f\|_{L^1}.$$

Proof. It is sufficient to decompose the integral depending on whether $|x-y|\geqslant 1$ or not. □

As a consequence we have the following result.

Corollary 4.9. *One has the following estimate for some positive constant C depending only on m, $\mathcal{J}$, $\|\omega_0\|_{L^1\cap L^\infty}$, $|\ell_0|$, $|r_0|$, $|\gamma|, \rho(0)$, and the geometry: $|\ell(t)|+|r(t)|\leqslant C[1+\ln(\rho(t))]$, where $\rho(t) := \inf\{d>1 \ / \ \mathrm{Supp}(\omega(t,\cdot))\subset B(0,d)\}$.*

4.4. A macroscopic normal form tailored for the zero-radius limit

We define the set

$$\mathcal{B} := \cup_{q\in\mathbb{R}^3}\{q\}\times\mathbb{R}^3\times\mathbb{R}\times L^\infty(\mathcal{F}(q);\mathbb{R}).$$

The following result is deduced, by going back to the original frame, from the existence and uniqueness result established in Section 4.2 for the div/curl type system satisfied by the velocity in the body frame.

Proposition 4.10. *For any (q,p,γ,ω) in $\mathcal{B}$ with $p=(\ell,r)$ in $\mathbb{R}^2\times\mathbb{R}$, there exists a unique $U(q,p,\gamma,\omega)$ in the space $\mathcal{LL}(\mathcal{F}(q))$ such that*

$$\operatorname{div} U(q,p,\gamma,\omega)=0 \ \text{ and } \ \operatorname{curl} U(q,p,\gamma,\omega)=\omega \quad \text{for } x\in\mathcal{F}(q),$$

$$U(q,p,\gamma,\omega)\cdot n=\left(\ell+rx^\perp\right)\cdot n \quad \text{for } x\in\partial\mathcal{S}(q) \text{ and } \int_{\partial\mathcal{S}(q)}U(q,p,\gamma,\omega)\cdot\tau\,ds=\gamma,$$

$$U(q,p,\gamma,\omega)\longrightarrow 0 \quad \text{as } x\to\infty.$$

In order to prepare the asymptotic analysis of the rigid body's dynamics in the zero radius limit, we first establish here an exact normal form of the Newton equations for the solid motion for a fixed radius. It follows the analysis performed in Section 3 so that we will adopt here the real-analytic approach developed in [7] rather than the complex-analytic approach performed in [5]. This will simplify the study of the zero radius limit in the next subsection.

Theorem 4.11. *There exists a mapping F in $C(\mathcal{B};\mathbb{R}^3)$ depending only on γ and $\mathcal{S}_0$ such that the equations* (13)-(16) *are equivalent to the following system:*

$$(\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q'' + \langle \Gamma_{a,\theta}, q', q' \rangle = F(q, q', \gamma, \omega), \tag{174}$$

$$\frac{\partial \omega}{\partial t} + \operatorname{div}\big(\omega U(q, q', \gamma, \omega)\big) = 0 \text{ for } x \in \mathcal{F}(q(t)), \tag{175}$$

where $\mathcal{M}_{a,\theta}$ is given by Theorem 2.2 and $\Gamma_{a,\theta}$ denotes the a-connection associated with $\mathcal{M}_{a,\theta}$ by Definition 2.1.

Above it is understood that the equivalence concerns Yudovich type solutions.

One observes that the left hand side of (174) is the same than the one of (20). On the other hand the right hand side of (174) is more intricate. Indeed we are going to provide a rather explicit definition of the force term F. In order to do so we split, for (q, p, γ, ω) in $\mathcal{B}$, the vector field $U(q, p, \gamma, \omega)$ into

$$U(q, p, \gamma, \omega) = U_1(q, p) + U_2(q, \gamma, \omega), \tag{176}$$

where $U_1(q, p)$ denotes the potential part that is the unique solution in the space $\mathcal{LL}(\mathcal{F}(q))$ to the following system:

$$\operatorname{div} U_1(q,p) = 0 \text{ and } \operatorname{curl} U_1(q,p) = 0 \quad \text{for } x \in \mathcal{F}(q),$$
$$U_1(q,p) \cdot n = \big(\ell + r(x-h)^\perp\big) \cdot n \quad \text{for } x \in \partial\mathcal{S}(q) \text{ and } \int_{\partial\mathcal{S}(q)} U_1(q,p) \cdot \tau \, ds = 0,$$
$$U_1(q,p) \longrightarrow 0 \quad \text{as } x \to \infty,$$

where $q = (h, \theta)$ and $p = (\ell, r)$, and $U_2(q, \gamma, \omega)$ therefore denotes the unique solution in the space $\mathcal{LL}(\mathcal{F}(q))$ to the following system:

$$\operatorname{div} U_2(q,\gamma,\omega) = 0 \text{ and } \operatorname{curl} U_2(q,\gamma,\omega) = \omega \quad \text{for } x \in \mathcal{F}(q),$$
$$U_2(q,\gamma,\omega) \cdot n = 0 \quad \text{for } x \in \partial\mathcal{S}(q) \text{ and } \int_{\partial\mathcal{S}(q)} U_2(q,\gamma,\omega) \cdot \tau \, ds = \gamma,$$
$$U_2(q,\gamma,\omega) \longrightarrow 0 \quad \text{as } x \to \infty.$$

Observe that the vector fields $U_1(q, p)$ and $U_2(q, \gamma, \omega)$ are respectively linear with respect to p and (γ, ω) whereas their dependence on q is encoded into the change of variable (36) (since their counterpart in the body frame do not depend on q).

Eventually, we define for (q,p,γ,ω) in $\mathcal{B}$, three vector in $\mathbb{R}^3$, by

$$B(q,\gamma,\omega) := -\int_{\partial\mathcal{S}(q)} U_2(q,\gamma,\omega)\cdot\tau\Big((U_1(q,e_i)\cdot n)_i \times (U_1(q,e_i)\cdot\tau)_i\Big)\,\mathrm{d}s, \tag{177}$$

$$E(q,\gamma,\omega) := -\frac{1}{2}\Big(\int_{\partial\mathcal{S}(q)} |U_2(q,\gamma,\omega)|^2\, U_1(q,e_i)\cdot n\,\mathrm{d}s\Big)_i, \tag{178}$$

$$D(q,p,\gamma,\omega) := -\Big(\int_{\mathcal{F}(q)} \omega U(q,p,\gamma,\omega)^{\perp}\cdot U_1(q,e_i)\,\mathrm{d}x\Big)_i, \tag{179}$$

where the index i runs over $1,2,3$ and the e_i stands for the canonical basis of $\mathbb{R}^3$. We will prove Theorem 4.11 with the mapping F given, for (q,p,γ,ω) in $\mathcal{B}$, by

$$F(q,p,\gamma,\omega) := \gamma^2 E(q,\gamma,\omega) + \gamma\, p\times B(q,\gamma,\omega) + D(q,p,\gamma,\omega). \tag{180}$$

Observe that the vector fields $B(q,\gamma,\omega)$ and $E(q,\gamma,\omega)$ above have the same form than the vector fields $B(q)$ and $E(q)$ used in Section 3, see (123) and (124) except that they also encompass a contribution due to the vorticity through the vector fields U_2. The last term in (180) is a direct contribution of the vorticity, in the sense that it intervenes explicitly inside an integral over the fluid domain. In particular this term may be non vanishing even if $\gamma = 0$ unlike the two first terms.

Proof. First of all (175) simply recasts the transport of the fluid vorticity by the fluid velocity characterized by Proposition 4.10. The proof of Theorem 4.11 therefore reduces to prove the equivalence of Newton's equations for the solid motion with (174). In a perhaps surprising way it seems more convenient not to use the reformulation (37)-(38) of the system in the body frame. Instead we rather proceed as in Section 3 with a few modifications due to the fact that we now deal with a non-vanishing vorticity $\omega = \operatorname{curl} u$. In particular one has to modify (99) into

$$\nabla\pi = -\left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla|u^2| + \omega u^{\perp}\right) \quad \text{in } \mathcal{F}(q).$$

and therefore (100) becomes

$$mh''\cdot\ell^* + \mathcal{J}\theta'' r^* = -\int_{\mathcal{F}(q)} \left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u^2)\right)\cdot U_1(q,p^*)\mathrm{d}x \\ - \int_{\mathcal{F}(q)} \omega u^{\perp}\cdot U_1(q,p^*)\mathrm{d}x, \quad \text{for all } p^* = (\ell^*,r^*)\in\mathbb{R}^3.$$

Then we use that $u = U(q,q',\gamma,\omega) = U_1(q,q') + U_2(q,\gamma,\omega)$ to obtain, for all $p^* := (\ell^*, r^*) \in \mathbb{R}^3$,

$$\begin{aligned} m\ell' \cdot \ell^* + \mathcal{J} r' r^* + \int_{\mathcal{F}(q)} \Big(\frac{\partial U_1(q,q')}{\partial t} + \frac{1}{2} \nabla |U_1(q,q')|^2 \Big) \cdot U_1(q,p^*) \mathrm{d}x \\ = - \int_{\mathcal{F}(q)} \big(\frac{1}{2} \nabla |U_2(q,\gamma,\omega)|^2 \big) \cdot U_1(q,p^*) \mathrm{d}x \\ - \int_{\mathcal{F}(q)} \big(\frac{\partial U_2(q,\gamma,\omega)}{\partial t} + \frac{1}{2} \nabla (U_1(q,q') \cdot U_2(q,\gamma,\omega)) \big) \cdot U_1(q,p^*) \mathrm{d}x \\ - \int_{\mathcal{F}(q)} \omega U(q,q',\gamma,\omega)^\perp \cdot U_1(q,p^*) \mathrm{d}x. \end{aligned} \tag{181}$$

Using Theorem 2.2 in the case where $\gamma = 0$ yields that the left hand side of (181) is equal to inner product of the left hand side of (174) with p^*. By integration by parts one obtains that the first term in the right hand side of (181) is equal to inner product of $\gamma^2 E(q,\gamma,\omega)$ with p^*. By adapting the proof of (130) one proves that the second term in the right hand side of (181) is equal to inner product of $\gamma\, q' \times B(q,\gamma,\omega)$ with p^*. It follows from the linearity of $U_1(q,p)$ with respect to p that the last term of (181) is equal to inner product of $D(q,q',\gamma,\omega)$ with p^*, and this concludes the proof of Theorem 4.11. □

4.5. Zero radius limit

We now investigate the zero radius limit and therefore assume that, for every $\varepsilon \in (0,1]$, the solid domain occupies (5) where $\mathcal{S}_0$ is a fixed simply connected smooth compact subset of $\mathbb{R}^2$. We consider $p_0 = (\ell_0, r_0) \in \mathbb{R}^3$, $m > 0$, $\mathcal{J} > 0$, γ in $\mathbb{R}$ (respectively in $\mathbb{R}^*$) in the case of a massive (resp. massless) particle. Let ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$ Then for every $\varepsilon \in (0,1]$, combining Theorem 1.9 and Theorem 4.11, we obtain that there exists a unique global solution $(h^\varepsilon, \theta^\varepsilon, \omega^\varepsilon)$ with Yudovich regularity (in particular with bounded vorticity) to the equations (174)-(175) with some coefficients $\mathcal{M}^\varepsilon_g$, $\mathcal{M}^\varepsilon_a$, Γ^ε_a and F^ε associated with $\mathcal{S}^\varepsilon_0$, $m^\varepsilon, \mathcal{J}^\varepsilon$ given in Definition 1.1 and γ, and with the initial data $(q^\varepsilon, (q^\varepsilon)')(0) = (0, p_0)$ and $\omega^\varepsilon|_{t=0} = \omega_{0|\mathcal{F}^\varepsilon_0}$. In the massless case with $\alpha \geq 2$, we will consider only here the case where $\mathcal{S}_0$ is not a disk. As already mentioned in the fifth remark after Definition 1.1 the case where $\mathcal{S}_0$ is a non-homogeneous disk requires a few adaptations and can be tackled as in [7] for the irrotational bounded case (whereas this case was actually omitted in [4, 5]). Our results then read as follows.

Theorem 4.12. *Let be given a circulation γ in $\mathbb{R}$ in the case of a massive particle and in $\mathbb{R}^*$ in the case of a massless particle. Let be given $(\ell_0, r_0) \in \mathbb{R}^3$, ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$. For any $\varepsilon \in (0,1]$, let us denote $(h^\varepsilon, \theta^\varepsilon, \omega^\varepsilon)$ the solution to the system associated with with $\mathcal{S}^\varepsilon_0$, $m^\varepsilon, \mathcal{J}^\varepsilon$, ℓ_0, r_0, γ and $\omega_{0|\mathcal{F}^\varepsilon_0}$ as above. Then* *in the zero radius limit $\varepsilon \to 0$, with the inertia scaling described in Definition 1.1, one has, in the case of a massive (respectively massless) particle, that for any $T > 0$, up to a subsequence (resp. for the whole sequence), h^ε converges to h weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R}^2)$ (resp. in $W^{1,\infty}(0,T;\mathbb{R}^2)$), $\varepsilon\theta^\varepsilon$ converges to 0 weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R})$ ω^ε (extended by 0 inside the solid) converges to ω in $C^0([0,T];L^\infty(\mathbb{R}^2)-$ weak-$\star$). Moreover one has (12) in $[0,T]\times\mathbb{R}^2$, respectively (10) in the massive limit and (11) in the massless limit, with the initial conditions $\omega|_{t=0} = \omega_0,\ h(0)=0,\ h'(0)=\ell_0$ (resp. $\omega|_{t=0}=\omega_0,\ h(0)=0$).*

Remark 4.13. Note that the convergence of h^ε cannot be strong in $W^{1,\infty}(0,T;\mathbb{R}^2)$, in general, as this would entail that

$$\ell_0 = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(h_0-y)^\perp}{|h_0-y|^2}\omega_0(y)\,dy.$$

Theorem 1.10 is a consequence of Theorem 4.12.

Proof. We will proceed as in the proof of Theorem 3.11 with a few modifications. First using $p^\varepsilon = ((h^\varepsilon)', \varepsilon(\theta^\varepsilon)')^t$ we obtain that the solid equations are of the form

$$\begin{aligned}(\varepsilon^\alpha\,\mathcal{M}_g + \varepsilon^2\mathcal{M}_{a,\,\theta^\varepsilon})\,(p^\varepsilon)' + \varepsilon\langle\Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle \\ = \gamma^2\tilde{E}^\varepsilon(q^\varepsilon,\gamma,\omega^\varepsilon) + \gamma\, p^\varepsilon\times\tilde{B}^\varepsilon(q^\varepsilon,\gamma,\omega^\varepsilon) + \tilde{D}^\varepsilon(q^\varepsilon,p^\varepsilon,\gamma,\omega^\varepsilon),\end{aligned} \tag{182}$$

where $\tilde{E}^\varepsilon$, $\tilde{B}^\varepsilon$ and $\tilde{D}^\varepsilon$ are respectively deduced from E, B and D defined in (177), (178) and (179) by some appropriate scalings. Here again the crucial issue is to obtain some bounds uniformly in ε in order to pass to the limit in (182). First we look for an appropriate modification of Corollary 4.9 in the zero radius limit thanks to an appropriate renormalization of the energy (166) as $\varepsilon\to 0^+$ by discarding some terms which are logarithmically divergent in the limit but which do not bear any information on the state of the system.[14] This provides an uniform estimate of $\varepsilon^{\min(1,\frac{\alpha}{2})}\,|(h^\varepsilon)',\varepsilon(\theta^\varepsilon)')|_{\mathbb{R}^3}$ at least till the vorticity is neither too far from the solid nor too close. Unfortunately in the massless case the coefficient α satisfies $\alpha > 0$ and the previous estimate is not sufficient. In order to get some improved estimates, we expand the coefficients in (182) as $\varepsilon\to 0^+$ using in particular an irrotational approximation of the fluid velocity on the body's boundary in order to use Lamb's lemma. Some cancellations similar to (151) and (152) allow in particular to absorb the leading orders of the term $\tilde{E}^\varepsilon$ into the leading part of the expansions of

[14]Observe that the quantity (166) was already obtained from the natural total kinetic energy of the "fluid+solid" system by a renormalization at infinity. Here the renormalization rather tackles some undesired concentrations at the center of mass of the shrinking particle.

the terms involving $\Gamma_{a,\theta^\varepsilon}$ and $\tilde{B}^\varepsilon$ thanks to the following modulation of the velocity:

$$\tilde{\ell}^\varepsilon(t) := (h^\varepsilon)'(t) - K_{\mathbb{R}^2}[\omega^\varepsilon(t,\cdot)](h^\varepsilon) - \varepsilon \nabla K_{\mathbb{R}^2}[\omega^\varepsilon(t,\cdot)](h^\varepsilon) \cdot R(\theta^\varepsilon)\xi,$$

where ξ is the conformal center of $\mathcal{S}_0$, cf. (33). On the other hand the term $\tilde{D}^\varepsilon$ turns out to be smaller at least till the vorticity stays supported at distance of order 1 of the solid. Next we introduce the notation $\tilde{p}^\varepsilon := (\tilde{\ell}^\varepsilon, \varepsilon(\theta^\varepsilon)')$. We thus obtain the following asymptotic normal form.

Proposition 4.14. *Let us fix $\rho > 0$. There exists $C > 0$ such that if for a given $T > 0$ and an $\varepsilon \in (0,1)$ one has for all $t \in [0,T]$:*

$$d(h^\varepsilon(t), \mathit{Supp}\,(\omega^\varepsilon(t))) \geqslant 1/\rho \quad \text{and} \quad \mathit{Supp}\,(\omega^\varepsilon(t)) \subset B(h^\varepsilon(t), \rho), \tag{183}$$

then there exist a function $G = G(\varepsilon,t) : (0,1) \times [0,T] \to \mathbb{R}^3$ satisfying

$$\left| \int_0^t \tilde{p}^\varepsilon(s) \cdot G(\varepsilon,s)\, ds \right| \leqslant \varepsilon C \left(1 + t + \int_0^t |\tilde{p}^\varepsilon(s)|^2\, ds \right), \tag{184}$$

and a function $F = F(\varepsilon,t) : (0,1) \times [0,T] \to \mathbb{R}^3$ satisfying

$$|F(\varepsilon,t)| \leqslant C \left(1 + |\tilde{p}^\varepsilon(t)| + \varepsilon |\tilde{p}^\varepsilon(t)|^2\right), \tag{185}$$

such that one has on $[0,T]$:

$$\begin{aligned} \left(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon}\right)(\tilde{p}^\varepsilon)' + \varepsilon \langle \Gamma_{a,\theta^\varepsilon}, \tilde{p}^\varepsilon, \tilde{p}^\varepsilon \rangle & \\ = \gamma\, \tilde{p}^\varepsilon \times B_{\theta^\varepsilon} + \varepsilon\gamma G(\varepsilon,t) + \varepsilon^{\min(\alpha,2)} F(\varepsilon,t). & \end{aligned} \tag{186}$$

From this normal form, we deduce the following modulated energy estimates.

Lemma 4.15. *Let $\rho > 0$. There exists $C > 0$ such that if for a given $T > 0$ and an $\varepsilon \in (0,1)$ one has that (183) is valid on $[0,T]$, then one has $|(h^\varepsilon)'| + \varepsilon|(\theta^\varepsilon)'| \leqslant C$ on $[0,T]$.*

Proof. Let $\rho > 0$ and let $C > 0$ be given by Proposition 4.14. Let $T > 0$ and an $\varepsilon \in (0,1)$ one has that (183) is valid on $[0,T]$. Then according to Proposition 4.14 one has (186) on $[0,T]$. It is then sufficient to multiply (186) by $\tilde{p}^\varepsilon$, to deal with the right hand side as in Proposition 2.4, to use the assumption on the initial data and finally to apply two Grönwall type estimates[15] to conclude.

□

Let us now tackle the passage to the limit. In a first time, we obtain the convergence stated in Theorem 4.12 on a small interval $[0,T]$, and only in a second time obtain this convergence on any time interval. We consider T^ε the supremum of the positive real number τ for which one has for any

[15]The second one being devoted to deduce some estimates for $(h^\varepsilon)'$ from the ones on the modulated velocities.

$t \in [0, \tau]$, $\mathrm{d}(h^\varepsilon(t), \mathrm{Supp}\,\omega^\varepsilon(t)) > 1/(2\rho_T)$ and $\mathrm{Supp}\,\omega^\varepsilon(t) \subset B(h^\varepsilon(t), 2\rho_T)$. For any $\varepsilon > 0$ small enough such that $\mathrm{d}(\mathrm{Supp}\,\omega_0, \mathcal{S}_0^\varepsilon) > 2\rho_T/3$, we have of course $T^\varepsilon > 0$. Using Proposition 4.15, we deduce that there exists $\varepsilon_0 > 0$ and $\underline{T} > 0$ such that $\inf_{\varepsilon \in (0,\varepsilon_0)} T^\varepsilon \geqslant \underline{T}$.

Thanks to a compactness argument using these estimates, the uniqueness of the solutions in the limit and Proposition 4.14 this allows to prove the convergence claimed in Theorem 4.12 locally in time, that is h^ε converges to h weakly-$\star$ in $W^{1,\infty}(0, \underline{T}; \mathbb{R}^2)$ and ω^ε converges to ω in $C^0([0, \underline{T}]; L^\infty(\mathbb{R}^2) - \text{weak-}\star)$. Finally we obtain the solid part of Theorem 4.12 by a sort of continuous induction argument. Moreover, with the previous uniform estimates, passing to the limit in the fluid equation is routine. □

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Contents