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Dynamics of a rigid body in a two dimensional incompressible perfect fluid and the zero-radius limit

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Abstract In this survey we report some recent results on the dynamics of a rigid body immersed in a two dimensional incompressible perfect fluid, with an emphasis on the zero-radius limit.

1 Introduction

We consider the motion of a rigid body in a two dimensional incompressible perfect fluid. This rigid body is supposed to initially occupy a simply connected smooth compact subset $\mathcal{S}_0$ of $\mathbb{R}^2$. At time t it occupies a domain $\mathcal{S}(t)$ which is isometric to $\mathcal{S}_0$ so that there exists a rotation matrix

$$R(\theta(t)) := \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \tag{1.1}$$

such that $\mathcal{S}(t) = \{h(t) + R(\theta(t))x,\ x \in \mathcal{S}_0\}$, where $h(t)$ is the position of the center of mass of the body at time t. We denote by $r(t) := \theta'(t)$ the angular velocity. The rigid body is assumed to be only accelerated by the force exerted by the fluid pressure on its boundary according to the Newton equations, the fluid velocity $u = (u_1, u_2)$ and pressure π being given by the incompressible Euler equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div} u = 0, \tag{1.2}$$

$$mh''(t) = \int_{\partial\mathcal{S}(t)} \pi n \, ds \quad \text{and} \quad \mathcal{J} r'(t) = \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \, ds. \tag{1.3}$$

Here $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the momentum of inertia of the body. When $x = (x_1, x_2)$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)$, n denotes the unit normal vector pointing outside the fluid domain. We assume that the boundary of the solid is impermeable so that the natural boundary condition at the fluid-solid interface is

$$u \cdot n = \Big(h'(t) + r(t)(x - h(t))^\perp\Big) \cdot n \quad \text{for} \quad x \in \partial\mathcal{S}(t). \tag{1.4}$$

Let us emphasize that this condition extends the usual condition $u \cdot n = 0$ on a fixed boundary and involves only the normal part of the fluid velocity. Furthermore, when considering a fluid velocity u which has a good enough regularity, Kelvin's theorem asserts that the circulation γ defined as

$$\gamma = \int_{\partial\mathcal{S}_0} u \cdot \tau \, ds,$$

where τ the the unit counterclockwise tangential vector so that $n = \tau^\perp$, remains constant over time.

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2 Case of an unbounded irrotational flow

We start with a review of the case, which is well-known for more than one century, of the motion of one single rigid body immersed in an irrotational fluid filling the rest of the plane. In this setting the equations at stake are the incompressible Euler equations (2.1) on the fluid domain $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$, the Newton equations (2.2), the interface condition (1.4) and the following condition of decay at infinity: $\lim_{|x|\to\infty} |u(t,x)| = 0$. Regarding the initial conditions we observe that there is no loss of generality in assuming that the center of mass (respectively rotation angle) of the solid coincides at the initial time with the origin 0 (resp. 0) and we therefore prescribe some initial position and velocity of the solid of the form $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$. On the other hand we prescribe the initial value of the velocity $u|_{t=0} = u_0$ in the initial domain $\mathcal{F}_0 = \mathcal{F}(0) = \mathbb{R}^2 \setminus \mathcal{S}_0$ occupied by the fluid. The equations then read :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{for } x \in \mathcal{F}(t), \tag{2.1}$$

$$mh''(t) = \int_{\partial \mathcal{S}(t)} \pi n \, ds \quad \text{and} \quad \mathcal{J}\theta''(t) = \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \, ds, \tag{2.2}$$

$$u \cdot n = \Big(h'(t) + r(t)(x - h(t))^\perp \Big) \cdot n \quad \text{for} \quad x \in \partial \mathcal{S}(t), \tag{2.3}$$

$$\lim_{|x|\to\infty} |u(t,x)| = 0, \tag{2.4}$$

$$u|_{t=0} = u_0 \quad \text{for } x \in \mathcal{F}_0 \quad \text{and} \quad (h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0). \tag{2.5}$$

2.1 Compatible and irrotational initial data

We assume that the solid translation and rotation velocities ℓ_0 and r_0 are arbitrarily given and, to allow the existence of smooth solutions, that the fluid velocity u_0 is the unique vector field compatible in the sense of the following definition.

Definition 2.1. Given the initial domain $\mathcal{S}_0$ occupied by the body, ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \mathbb{R}^2 \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field satisfying the following div/curl type system:

$$\operatorname{div} u_0 = 0 \text{ and } \operatorname{curl} u_0 = 0 \text{ in } \mathcal{F}_0,$$

$$u_0 \cdot n = \Big(\ell_0 + r_0 x^\perp \Big) \cdot n \text{ for } x \in \partial \mathcal{S}_0, \quad \int_{\partial \mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma,$$

$$\lim_{|x|\to\infty} u_0 = 0.$$

Observe in particular that we assume above that the flow is initially irrotational. According to Helmholtz's third theorem this zero vorticity condition propagates to any time $t > 0$.

2.2 Reformulation in the body frame

By the following isometric change of variable:

$$\begin{cases} v(t,x) = R(\theta(t))^T \; u(t, R(\theta(t))x + h(t)), \\ \tilde{\pi}(t,x) = \pi(t, R(\theta(t))x + h(t)), \\ \ell(t) = R(\theta(t))^T \; h'(t), \end{cases} \tag{2.6}$$

the equations (2.1)-(2.5) become

$$\partial_t v + \left[(v - \ell - r x^\perp) \cdot \nabla \right] v + r v^\perp + \nabla \tilde{\pi} = 0 \text{ and } \operatorname{div} v = 0 \text{ for } x \in \mathcal{F}_0, \tag{2.7}$$

$$m\ell'(t) = \int_{\partial \mathcal{S}_0} \tilde{\pi} n \; ds - mr\ell^\perp \text{ and } \mathcal{J}r'(t) = \int_{\partial \mathcal{S}_0} x^\perp \cdot \tilde{\pi} n \; ds, \tag{2.8}$$

$$v \cdot n = \left(\ell + r x^\perp \right) \cdot n \quad \text{for } x \in \partial \mathcal{S}_0, \tag{2.9}$$

$$v(0,x) = v_0(x) \text{ for } x \in \mathcal{F}_0 \text{ and } (\ell, r)(0) = (\ell_0, r_0). \tag{2.10}$$

In addition v satisfies $\operatorname{curl} v = 0$ in $\mathcal{F}_0$.

2.3 Kirchhoff potentials

The following so-called Kirchhoff potentials $\Phi := (\Phi_i)_{i=1,2,3}$ will play a major role. They are defined as the solutions of the following problems:

$$-\Delta\Phi_i = 0 \quad \text{in } \mathcal{F}_0, \quad \Phi_i \to 0 \quad \text{when } x \to \infty, \quad \frac{\partial \Phi_i}{\partial n} = K_i \quad \text{on } \partial\mathcal{F}_0, \tag{2.11}$$

where $(K_1,\, K_2,\, K_3) := (n_1,\, n_2,\, x^\perp \cdot n)$. We have that for all $i = 1, 2, 3$:

$$\Phi_i(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } \nabla\Phi_i(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to +\infty, \tag{2.12}$$

and consequently that $\nabla\Phi_i$ are in $L^2(\mathcal{F}_0)$.

2.4 Harmonic field of circulation 1

Let H be the unique solution vanishing at infinity of

$$\operatorname{div} H = 0 \text{ and } \operatorname{curl} H = 0 \text{ in } \mathcal{F}_0,\ H \cdot n = 0 \text{ on } \partial\mathcal{S}_0,\ \int_{\partial\mathcal{S}_0} H \cdot \tau \, ds = 1.$$

The vector field H is not in $L^2(\mathcal{F}_0)$.

2.5 Decomposition of the velocity

The fluid velocity v is given as the unique solution to the following div/curl type system:

$$\operatorname{div} v = 0 \text{ and } \operatorname{curl} v = 0 \text{ in } \mathcal{F}_0, \tag{2.13}$$

$$v \cdot n = \left(\ell + r x^\perp\right) \cdot n \text{ on } \partial\mathcal{S}_0, \quad \int_{\partial\mathcal{S}_0} v \cdot \tau \, ds = \gamma, \tag{2.14}$$

$$\lim_{x\to\infty} v = 0. \tag{2.15}$$

It can be decomposed as

$$v = \gamma H + \ell_1 \nabla\Phi_1 + \ell_2 \nabla\Phi_2 + r\nabla\Phi_3. \tag{2.16}$$

2.6 Pressure force and torque as distributed integrals

By an integration by parts and the decay properties of v and $\nabla\Phi_i$ at infinity,

$$\left(\int_{\partial\mathcal{S}_0} \tilde{\pi} n \, ds,\ \int_{\partial\mathcal{S}_0} \tilde{\pi} x^\perp \cdot n \, ds\right) = \left(\int_{\mathcal{F}_0} \nabla\tilde{\pi} \cdot \nabla\Phi_i \, dx\right)_{i=1,2,3}. \tag{2.17}$$

Moreover by (2.7),

$$\nabla\tilde{\pi} = -\partial_t v - \frac{1}{2}\nabla(v^2) + \nabla((\ell + r x^\perp) \cdot v). \tag{2.18}$$

2.7 Added inertia

Using an integration by parts, the boundary condition (2.9) and another integration by parts, one observes that the contribution of $\partial_t v$ in (2.17) is

$$\int_{\mathcal{F}_0} \partial_t v \cdot \nabla\Phi_i(x) \, dx = \mathcal{M}_a \begin{pmatrix} \ell \\ r \end{pmatrix}', \tag{2.19}$$

where $\mathcal{M}_a$ is the matrix

$$\mathcal{M}_a := \left(m_{i,j}\right)_{i,j\in\{1,2,3\}}, \tag{2.20}$$

where for $i, j \in \{1, 2, 3\}$

$$m_{i,j} := \int_{\mathcal{F}_0} \nabla\Phi_i \cdot \nabla\Phi_j \, dx. \tag{2.21}$$

The matrix $\mathcal{M}_a$ is added inertia, it is a symmetric positive-semidefinite matrix depending only on the body's shape and which encodes the amount of incompressible fluid that the rigid body has also to accelerate around itself. To compare observe that the left hand sides of (2.8) can be gathered into

$$\begin{pmatrix} m\ell' \\ \mathcal{J}r' \end{pmatrix} = \mathcal{M}_g \begin{pmatrix} \ell \\ r \end{pmatrix}', \tag{2.22}$$

where $\mathcal{M}_g$ denotes the genuine inertia

$$\mathcal{M}_g := \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix}. \tag{2.23}$$

Let us mention from now on that the matrix $\mathcal{M}_a$ is positive definite if and only $\mathcal{S}_0$ is not a disk. When $\mathcal{S}_0$ is a disk then $\mathcal{M}_a$ has the form $diag(m_a, m_a, 0)$ with $m_a > 0$. The case where $\mathcal{S}_0$ is disk is therefore peculiar and will not be treated here. We refer to [3] for this case.

2.8 Reformulation of Newton's equations

Let us introduce $q := (h, \theta) \in \mathbb{R}^3$ and $p := (\ell, r) \in \mathbb{R}^3$. By (2.17), (2.18), (2.19) and (2.22), and using two other integrations by parts and (2.11) to transform the contributions of the two last terms of (2.18) in the right hand side of (2.17), we arrive at the following reformulation of (2.8):

$$(\mathcal{M}_g + \mathcal{M}_a)p' + \langle \Gamma_g, p, p\rangle = -(\frac{1}{2}\int_{\partial\mathcal{S}_0} |v|^2 K_i \, ds - \int_{\partial\mathcal{S}_0} (\ell + r x^\perp)\cdot v K_i \, ds)_i, \tag{2.24}$$

where i runs over the integers $1, 2, 3$ and $\Gamma_g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ the bilinear symmetric mapping defined, for all $p = (\ell,\ r) \in \mathbb{R}^3$, by

$$\langle \Gamma_g, p, p\rangle = mr \begin{pmatrix} \ell^\perp \\ 0 \end{pmatrix}. \tag{2.25}$$

The quadratic mapping Γ_g is gyroscopic in the sense that

$$\forall p \in \mathbb{R}^3, \quad \langle \Gamma_g, p, p\rangle \cdot p = 0. \tag{2.26}$$

2.9 Lamb's lemma

To compute the first term in the right hand side of (2.24) we will use the following lemma where e_1 and e_2 are the unit vectors of the canonical basis of $\mathbb{R}^2$ and $\zeta_1(x) := e_1$, $\zeta_2(x) := e_2$ and $\zeta_3(x) := x^\perp$. are the associated elementary rigid velocities.

Lemma 2.2. *For any pair of vector fields (u, v) in $C^\infty(\overline{\mathbb{R}^2 \setminus \mathcal{S}_0}; \mathbb{R}^2)$ satisfying* $\operatorname{div} u = \operatorname{div} v = \operatorname{curl} u = \operatorname{curl} v = 0$*, $u(x) = O(1/|x|)$ and $v(x) = O(1/|x|)$ as $|x| \to +\infty$, for any $i = 1, 2, 3$,*

$$\int_{\partial\mathcal{S}_0} (u\cdot v) K_i \mathrm{d}s = \int_{\partial\mathcal{S}_0} \zeta_i \cdot \Big((u\cdot n)v + (v\cdot n)u\Big)\mathrm{d}s. \tag{2.27}$$

Proof of Lemma 2.2. By an integration by parts

$$\int_{\partial\mathcal{S}_0} (u\cdot v)K_i \mathrm{d}s = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} \operatorname{div}\big((u\cdot v)\zeta_i\big)\,\mathrm{d}x, = \int_{\mathbb{R}^2\setminus\mathcal{S}_0} \zeta_i \cdot (u\cdot\nabla v + v\cdot\nabla u)\,\mathrm{d}x,$$

using that $\operatorname{div}\zeta_i = \operatorname{curl} u = \operatorname{curl} v = 0$. Then (2.27) follows from another integration by parts, using that $\operatorname{div} u = \operatorname{div} v = 0$. □

2.10 Application of Lamb's lemma

Using Lamb's lemma and the boundary conditions (2.9) we obtain, after a few computations, that, for any $i = 1, 2, 3$,

$$\frac{1}{2}\int_{\partial\mathcal{S}_0} |v|^2 K_i \, ds - \int_{\partial\mathcal{S}_0} (\ell + r x^\perp)\cdot v K_i \, ds = \sum_{k=1}^{3} p_k \int_{\partial\mathcal{S}_0} (v\cdot\tau)[(\zeta_i\cdot\tau)K_k - (\zeta_k\cdot\tau)K_i]\, ds.$$

2.11 Reduction to an ODE

We then compute the brackets $[(\zeta_i \cdot \tau)K_k - (\zeta_k \cdot \tau)K_i]$, for $i, k = 1, 2, 3$, use the velocity decomposition (2.16) and that by an integration by parts,

$$\int_{\partial \mathcal{S}_0} (\nabla \Phi_j \cdot \tau) x^\perp \, ds = -(m_{i,j})_i,$$

to obtain that

$$-\Big(\frac{1}{2}\int_{\partial \mathcal{S}_0} |v|^2 K_i \, ds - \int_{\partial \mathcal{S}_0} (\ell + r x^\perp) \cdot v K_i \, ds\Big)_{i=1,2} = -r\Big(\mathcal{M}^\flat \ell + r \begin{pmatrix} m_{1,3} \\ m_{2,3} \end{pmatrix}\Big)^\perp + \gamma(\ell^\perp - r\xi),$$

where

$$\mathcal{M}_\flat := (m_{i,j})_{i,j \in \{1,2\}} \text{ and } \xi := \int_{\partial \mathcal{S}_0} (H \cdot \tau) x \, ds. \tag{2.28}$$

Proceeding in the same way for $i = 3$ we finally arrive at the following equation:

$$[\mathcal{M}_g + \mathcal{M}_a] p' + \langle \Gamma_g, p, p \rangle + \langle \Gamma_a, p, p \rangle = \gamma p \times B, \tag{2.29}$$

where $\Gamma_a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the bilinear symmetric mapping, depending only on $\mathcal{S}_0$, defined, for all $p = (\ell \ r) \in \mathbb{R}^3$, by

$$\langle \Gamma_a, p, p \rangle = \begin{pmatrix} r(\mathcal{M}_\flat \ell)^\perp \\ \ell^\perp \cdot \mathcal{M}_\flat \ell \end{pmatrix} + r p \times \mu, \tag{2.30}$$

where involves the vector

$$\mu := \begin{pmatrix} m_{1,3} \\ m_{2,3} \\ 0 \end{pmatrix}, \text{ and } B := \begin{pmatrix} \xi^\perp \\ -1 \end{pmatrix}. \tag{2.31}$$

The force in the right hand side of (2.29) is called the Kutta-Joukowski force, which is also gyroscopic in the sense introduced in Section 2.8. The equation (2.29) dates back to Blasius, Kutta, Joukowski, Chaplygin and Sedov, see for instance [11]. The initial approach relies on complex analysis. The real-analytic proof given above seems to date back to Lamb, cf. [9]. We refer to [12] for more on the comparison.

2.12 Energy conservation

We multiply (2.29) by p, use the symmetry of the matrices $\mathcal{M}_g$ and $\mathcal{M}_a$ the property (2.26) and that Γ_a is also gyroscopic to deduce that the energy

$$\frac{1}{2}(\mathcal{M}_g + \mathcal{M}_a) p \cdot p \tag{2.32}$$

remains constant over time.

2.13 Back to the original frame

By going back to the original frame (that is by combining (2.6) and (2.29)) we obtain that the dynamics of the rigid body is given by the following ODE:

$$(\mathcal{M}_g + \mathcal{M}_{a,\theta}) \, q'' + \langle \Gamma_{a,\theta}, q', q' \rangle = \gamma q' \times B_\theta, \tag{2.33}$$

where

$$\mathcal{M}_{a,\theta} := \mathcal{R}(\theta) \mathcal{M}_a \mathcal{R}(\theta)^t \text{ and } B_\theta := \mathcal{R}(\theta) B,$$

where we associate the 3×3 rotation matrix

$$\mathcal{R}(\theta) := \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(3) \tag{2.34}$$

with the 2×2 rotation matrix $R(\theta)$ defined in (1.1) and for $p = (\ell, r)^t$,

$$\langle \Gamma_{a,\theta}, p, p \rangle := -\begin{pmatrix} P_{a,\theta} \\ 0 \end{pmatrix} \times p - r \mathcal{M}_{a,\theta} \begin{pmatrix} \ell^\perp \\ 0 \end{pmatrix}, \tag{2.35}$$

where $P_{a,\theta}$ are the two first coordinates of $\mathcal{M}_{a,\theta} \, p$,

2.14 Geodesic interpretation

A tedious computation reveals that for any $\theta \in \mathbb{R}$, for any $p \in \mathbb{R}^3$,

$$\langle \Gamma_{a,\theta}, p, p \rangle := \left(\sum_{1 \leqslant i,j \leqslant 3} (\Gamma_{a,\theta})^k_{i,j} p_i p_j \right)_{1 \leqslant k \leqslant 3} \in \mathbb{R}^3, \tag{2.36a}$$

with for every $i, j, k \in \{1, 2, 3\}$ and for any $q = (h, \theta)$,

$$(\Gamma_{a,\theta})^k_{i,j}(q) := \frac{1}{2}\Big((\mathcal{M}_{a,\theta})^i_{k,j} + (\mathcal{M}_{a,\theta})^j_{k,i} - (\mathcal{M}_{a,\theta})^k_{i,j} \Big)(q). \tag{2.36b}$$

Above $(\mathcal{M}_{a,\theta})^k_{i,j}$ is the partial derivative with respect to q_k of the entry of indexes (i, j) of the matrix $\mathcal{M}_{a,\theta}$, that is

$$(\mathcal{M}_{a,\theta})^k_{i,j} := \frac{\partial (\mathcal{M}_{a,\theta})_{i,j}}{\partial q_k}. \tag{2.37}$$

The coefficients $(\Gamma_{a,\theta})^k_{i,j}$ are the Christoffel symbols of the first kind associated with the metric $\mathcal{M}_g + \mathcal{M}_{a,\,\theta}$ so that, should its right hand side vanish, (2.33) would be the geodesic equation associated with the metric $\mathcal{M}_g + \mathcal{M}_{a,\,\theta}$.

2.15 Global well-posedness

Observe that a trivial consequence of the reformulation above, of the Cauchy-Lipschitz theorem and of the conservation in time of the quantity $(\mathcal{M}_g + \mathcal{M}_{a,\theta})\, q' \cdot q'$ (which is twice the quantity in (2.32)) is that a global-in-time smooth solution to the Cauchy problem (2.1)-(2.5) exists and is unique. The fluid velocity u is given with respect to q and $q' = (h', \theta')$ as the unique solution to the following div/curl type system:

$$\begin{gathered} \operatorname{div} u = 0 \text{ and } \operatorname{curl} u = 0 \text{ in } \mathcal{F}(q), \\ u \cdot n = \Big(h' + \theta'(x - h)^\perp \Big) \cdot n \text{ for } x \in \partial\mathcal{S}(q), \quad \int_{\partial\mathcal{S}(q)} u \cdot \tau \, ds = \gamma, \\ \lim_{x \to \infty} u = 0. \end{gathered}$$

Here we denote the fluid and solid domains respectively by $\mathcal{F}(q)$ and $\mathcal{S}(q)$ rather than $\mathcal{F}(t)$ and $\mathcal{S}(t)$, since the fluid and solid domains only depend on t through the solid position.

2.16 Zero radius limit

To investigate the zero radius limit, with a self-similar shrinking of the rigid body into its center of mass, we now assume that, for every $\varepsilon \in (0, 1]$, the rigid body initially occupies the domain

$$\mathcal{S}_0^\varepsilon := \varepsilon \mathcal{S}_0. \tag{2.38}$$

Accordingly, for every $q = (h, \theta) \in \mathbb{R}^3$,

$$\mathcal{S}^\varepsilon(q) := R(\theta)\mathcal{S}_0^\varepsilon + h \text{ and } \mathcal{F}^\varepsilon(q) = \mathbb{R}^2 \setminus \bar{\mathcal{S}}^\varepsilon(q). \tag{2.39}$$

We consider two inertia regimes.

Definition 2.3. A point-mass particle is the limit of a rigid body when ε goes to 0 with its mass m^ε and its momentum of inertia $\mathcal{J}^\varepsilon$ respectively satisfying $m^\varepsilon = m$ and $\mathcal{J}^\varepsilon = \varepsilon^2 \mathcal{J}$, with $m > 0$ and $\mathcal{J} > 0$ independent of ε,

Definition 2.4. A massless point particle is the limit of a rigid body when ε goes to 0 with its mass m^ε and its momentum of inertia $\mathcal{J}^\varepsilon$ respectively satisfying $m^\varepsilon = \varepsilon^\alpha\, m$ and $\mathcal{J}^\varepsilon = \varepsilon^{\alpha+2}\, \mathcal{J}$, with $m > 0$, $\mathcal{J} > 0$ and $\alpha > 0$ independent of ε.

Recall that the momentum of inertia $\mathcal{J}^\varepsilon$ corresponds to a second order moment of the body's density whereas the mass is the zeroth order moment; this motivates the extra factor ε^2 in the scalings above.

We assume that the initial solid translation and rotation velocities (ℓ_0, r_0) and the circulation γ around the body are independent of ε. Moreover we assume that $\gamma \neq 0$ in the case of a massless point particle.

According to what precedes, for each $\varepsilon > 0$, there is a unique corresponding global smooth solution $q^\varepsilon := (h^\varepsilon, \theta^\varepsilon)$.

Theorem 2.5. *For all $T > 0$, as $\varepsilon \to 0$, in the case of point-mass particle (respectively massless point particle), $h^\varepsilon \longrightarrow h$ in $W^{2,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$ (resp. $W^{1,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$) and h satisfies the equations $mh'' = \gamma(h')^\perp$ (resp. $h' = 0$).*

Proof. Examining how the equation scales with ε one obtains:

$$(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})(p^\varepsilon)' + \varepsilon\langle \Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle = \gamma p^\varepsilon \times B_{\theta^\varepsilon}, \tag{2.40}$$

where $p^\varepsilon = ((h^\varepsilon)', \varepsilon(\theta^\varepsilon)')^t$. The energy is twice $(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})p^\varepsilon \cdot p^\varepsilon$ and its conservation provides that p^ε is bounded uniformly with respect to ε on the time interval $[0,+\infty)$, whatever is α. Let $T > 0$. Computing the right-hand-side of (2.40) we arrive at

$$(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta^\varepsilon})(p^\varepsilon)' + \varepsilon\langle \Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle = \gamma \begin{pmatrix} ((h^\varepsilon)')^\perp - \varepsilon(\theta^\varepsilon)' R(\theta^\varepsilon)\xi \\ R(\theta^\varepsilon)\xi \cdot (h^\varepsilon)' \end{pmatrix}. \tag{2.41}$$

We start with the massive case for which $\alpha = 0$. By (2.41) we deduce some uniform $W^{2,\infty}$ bounds on h^ε and $\varepsilon\theta^\varepsilon$ and this entails the existence of a subsequence of $(h^\varepsilon, \varepsilon\theta^\varepsilon)$ converging to (h,Θ) in $W^{2,\infty}$ weak-$\star$. Moreover the term $\varepsilon\langle \Gamma_{a,\theta^\varepsilon}, p^\varepsilon, p^\varepsilon\rangle$ converges to 0 in L^∞ and therefore the left hand side of (2.41) (with $\alpha = 0$) converges to $\mathcal{M}_g(h'', \Theta'')^t$ in L^∞ weak-$\star$, and using that $\varepsilon(\theta^\varepsilon)' R(\theta^\varepsilon)\xi = \varepsilon\big(R(\theta^\varepsilon - \pi/2)\xi\big)'$ converges in $W^{-1,\infty}$ weak-$\star$ up to a subsequence, we get from the two first lines of (2.41) that $mh'' = \gamma(h')^\perp$.

In the massless case, that is when $\alpha > 0$, the first term of (2.41) converges to 0 in $W^{-1,\infty}$ since $\varepsilon\mathcal{M}_{a,\theta^\varepsilon}$ is bounded in $W^{1,\infty}$, $\mathcal{M}_g$ is constant and $(p^\varepsilon)'$ is bounded uniformly with respect to ε in $W^{-1,\infty}$. Then by (2.41) we infer that $(h^\varepsilon)'$ converges weakly-$\star$ to 0 in $W^{-1,\infty}$. By the a priori estimate, we deduce that h^ε converges to some h in $W^{1,\infty}([0,T];\mathbb{R}^2)$ weak-$\star$ such that $h' = 0$. □

3 Case of a bounded irrotational flow

We consider now the case where the fluid-solid system occupies a bounded open regular connected and simply connected domain Ω of $\mathbb{R}^2$. We assume that the body initially occupies the closed domain $\mathcal{S}_0 \subset \Omega$, so that the domain of the fluid is $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$ at the initial time, and (without loss of generality) that the center of mass of the solid coincides at the initial time with the origin and that $0 \in \Omega$. The domain occupied by the fluid at time $t > 0$ is denoted by $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$. The fluid-solid system is governed by the following set of coupled equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{in } \mathcal{F}(t), \tag{3.1a}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{F}(t), \tag{3.1b}$$

$$mh'' = \int_{\partial \mathcal{S}(t)} \pi n \, \mathrm{d}s \quad \text{and} \quad \mathcal{J}\theta'' = \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \pi n \, \mathrm{d}s, \tag{3.1c}$$

$$u \cdot n = \big(\ell + r(\cdot - h)^\perp\big) \cdot n \quad \text{on } \partial \mathcal{S}(t), \tag{3.1d}$$

$$u \cdot n = 0 \quad \text{on } \partial\Omega, \tag{3.1e}$$

$$u_{t=0} = u_0 \quad \text{in } \mathcal{F}_0 \quad \text{and} \quad (h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0). \tag{3.1f}$$

3.1 Compatible and irrotational initial data

To obtain smooth solutions, it is necessary to consider some compatible initial data in the following sense.

Definition 3.1. Given ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field satisfying the following div/curl type system:

$$\operatorname{div} u_0 = 0 \text{ and } \operatorname{curl} u_0 = 0 \text{ in } \mathcal{F}_0,$$

$$u_0 \cdot n = \Big(\ell_0 + r_0 x^\perp\Big) \cdot n \text{ for } x \in \partial \mathcal{S}_0, \quad \int_{\partial \mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma,$$

$$u_0 \cdot n = 0 \text{ for } x \in \partial\Omega.$$

In particular we still consider the case without any initial vorticity so that it will remain irrotational for every time, that is

$$\operatorname{curl} u(t, \cdot) = 0 \text{ in } \mathcal{F}(t). \tag{3.2}$$

On the other hand the circulation around the body is constant in time equal to γ according to Kelvin's theorem.

3.2 Some notations

Once again since the domains $\mathcal{S}(t)$ and $\mathcal{F}(t)$ depend on $q := (h, \theta) \in \mathbb{R}^3$ only, we shall rather denote them by $\mathcal{S}(q)$ and $\mathcal{F}(q)$ in the sequel. Since we will not consider any collision here, we introduce: $\mathcal{Q} := \{q \in \mathbb{R}^3 \, : \, d(\mathcal{S}(q), \partial\Omega) > 0\}$, where $d(A, B)$ denotes for two sets A and B in the plane that is $d(A, B) := \min\{|x - y|_{\mathbb{R}^2}, \; x \in A, \; y \in B\}$. Observe that the initial position $q(0) = 0$ of the solid belongs to $\mathcal{Q}$.

3.3 Kirchhoff potentials and added mass

Consider the functions ζ_j, for $j = 1, 2, 3$, defined for $q = (h, \theta) \in \mathcal{Q}$ and $x \in \mathcal{F}(q)$, by the formula $\zeta_j(q, x) := e_j$, for $j = 1, 2$ and $\zeta_3(q, x) := (x - h)^\perp$. We introduce $\mathbf{\Phi} := (\Phi_1, \Phi_2, \Phi_3)^t$ where the Kirchhoff's potentials $\Phi_j(q, \cdot)$, for $j = 1, 2, 3$, are the unique (up to an additive constant) solutions in $\mathcal{F}(q)$ of the following Neumann problem:

$$\Delta\Phi_j = 0 \text{ in } \mathcal{F}(q), \quad \frac{\partial\Phi_j}{\partial n}(q, \cdot) = n \cdot \zeta_j(q, \cdot) \text{ on } \partial\mathcal{S}(q), \quad \frac{\partial\Phi_j}{\partial n}(q, \cdot) = 0 \text{ on } \partial\Omega. \tag{3.3}$$

We can now define the added inertia

$$\mathcal{M}_a(q) := \Big(\int_{\mathcal{F}(q)} \nabla\Phi_i \cdot \nabla\Phi_j \mathrm{d}x \Big)_{1 \leqslant i,j \leqslant 3}. \tag{3.4}$$

To measure its variations, for $q \in \mathcal{Q}$, and $p \in \mathbb{R}^3$, we set

$$\langle \Gamma_a(q), p, p \rangle := \Big(\sum_{1 \leqslant i,j \leqslant 3} (\Gamma_a(q))^k_{i,j} \, p_i \, p_j \Big)_{1 \leqslant k \leqslant 3} \in \mathbb{R}^3, \tag{3.5}$$

with for every $i, j, k \in \{1, 2, 3\}$,

$$(\Gamma_a(q))^k_{i,j}(q) := \frac{1}{2}\Big(\frac{\partial(\mathcal{M}_a(q))_{i,k}}{\partial q_j} + \frac{\partial(\mathcal{M}_a(q))_{j,k}}{\partial q_i} - \frac{\partial(\mathcal{M}_a(q))_{i,j}}{\partial q_k} \Big)(q). \tag{3.6}$$

3.4 Reduction to an ODE in the general case

Next result establishes a reformulation of the system in terms of an ordinary differential equation for the motion of the rigid body and of a div-curl type system for the fluid velocity.

Theorem 3.2. *There exists F in $C^\infty(\mathcal{Q} \times \mathbb{R}^3; \mathbb{R}^3)$ depending only on $\mathcal{S}_0, \gamma$ and Ω, and vanishing when $\gamma = 0$, such that, up to the first collision,* (3.1) *reduces to the second order ODE:*

$$(\mathcal{M}_g + \mathcal{M}_a(q))q'' + \langle \Gamma_a(q), q', q' \rangle = F(q, q'), \tag{3.7}$$

with Cauchy data $q(0) = 0$ and $q'(0) = (\ell_0, r_0)$, the fluid velocity being uniquely determined as the unique solution of the div-curl type system in the doubly-connected domain $\mathcal{F}(q)$, constituted of (3.1b), (3.1d), (3.1e), (3.2), *with circulation γ around $\mathcal{S}(q)$.*

The case where the circulation γ is zero was proven by Munnier in [10]. Then the result consists in a reformulation of the system (3.1) in terms of an ordinary differential equation for the motion of the rigid body which corresponds to geodesics associated with the Riemann metric induced on $\mathcal{Q}$ by the matrix $\mathcal{M}_g + \mathcal{M}_a(q)$. On the other hand the fluid velocity at any time is potential (that is a gradient globally on $\mathcal{F}(q)$).

The general case of a circulation $\gamma \in \mathbb{R}$ is obtained in [3]. A new feature with respect to the unbounded case is that the Kutta-Joukowski lift force is superseded by a more sophisticated force term.

3.5 Local well-posedness

The local-in-time existence and uniqueness of smooth solution to the Cauchy problem simply follows from Theorem 3.2 and the Cauchy-Lipschitz theorem. Unlike the unbounded case of Section 2, the result is only local-in-time (at least as far as it concerns smooth solutions) since collision of the body with the external boundary may occur in finite time, see [7,8] for some examples of collisions of a disk moving in a potential flow (that is in the case where the circulation γ satisfies $\gamma = 0$) with the fixed external boundary of the fluid domain. Indeed an energy argument, see Section 3.7 below, proves that the life-time of such a smooth solution can only be limited by a collision, cf. Section 3.7.

Theorem 3.3. *Given the open regular connected and simply connected bounded cavity Ω, the initial closed domain $\mathcal{S}_0 \subset \Omega$ occupied by the body, the initial solid translation and rotation velocities (ℓ_0, r_0) in $\mathbb{R}^2 \times \mathbb{R}$, the circulation γ in $\mathbb{R}$, and u_0 the associated compatible initial fluid velocity (according to Definition 3.1), there exists a unique smooth local-in-time solution to the problem compound of the incompressible Euler equations* (1.2) *on the fluid domain, of the Newton equations* (1.3)*, of the interface condition* (1.4)*, of the impermeability condition $u \cdot n = 0$ on $\partial\Omega$, and the initial conditions $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$ and $u|_{t=0} = u_0$. Moreover the life-time of such a smooth solution can only be limited by a collision.*

This result by itself belongs to the mathematical folklore and can be proved without using Theorem 3.2. Indeed a stronger result has been obtained in [6], where the rotational case (with $\operatorname{curl} u_0$ in L^∞) is handled with pure PDE's methods.

3.6 Explicit expression of the force term

For every $q \in \mathcal{Q}$, there exists a unique $C(q)$ in $\mathbb{R}$ such that the unique solution $\psi(q, \cdot)$ of the Dirichlet problem:

$$\Delta\psi(q, \cdot) = 0 \text{ in } \mathcal{F}(q) \quad \psi(q, \cdot) = C(q) \text{ on } \partial\mathcal{S}(q) \quad \psi(q, \cdot) = 0 \text{ on } \partial\Omega, \tag{3.8a}$$

satisfies

$$\int_{\partial\mathcal{S}(q)} \frac{\partial\psi}{\partial n}(q, \cdot)\mathrm{d}s = -1. \tag{3.8b}$$

Observe that for any $q \in \mathcal{Q}$, $C(q) < 0$ and that $C \in C^\infty(\mathcal{Q}; (-\infty, 0))$ and depends on $\mathcal{S}_0$ and Ω. Then, we define:

$$B(q) := \int_{\partial\mathcal{S}(q)} \left(\frac{\partial\psi}{\partial n} \left(\frac{\partial\mathbf{\Phi}}{\partial n} \times \frac{\partial\mathbf{\Phi}}{\partial\tau} \right) \right) (q, \cdot) \, \mathrm{d}s, \tag{3.9}$$

$$E(q) := -\frac{1}{2} \int_{\partial\mathcal{S}(q)} \left(\left| \frac{\partial\psi}{\partial n} \right|^2 \frac{\partial\mathbf{\Phi}}{\partial n} \right) (q, \cdot) \, \mathrm{d}s, \tag{3.10}$$

Indeed according to the proof of Theorem 3.2 in [3] the force term is given by

$$F(q, p) := \gamma^2 E(q) + \gamma \, p \times B(q), \tag{3.11}$$

for any (q, p) in $\mathcal{Q} \times \mathbb{R}^3$. The notations E and B highlight the analogy with the Lorentz force acting on a charged particle moving under the influence of some electromagnetic fields. Indeed the magnetic part of the Lorentz force, being gyroscopic and proportional to the circulation around the body, is a quite natural extension of the Kutta-Joukowski lift force of the unbounded case. Still it depends on the body position in a more subtle way.

3.7 The role of the energy

An important feature of the system (3.7) is that it is conservative. Let us denote for any (q, p) in $\mathcal{Q} \times \mathbb{R}^3$,

$$\mathcal{E}(q, p) := \frac{1}{2}(\mathcal{M}_g + \mathcal{M}_a(q))p \cdot p - \frac{1}{2}\gamma^2 C(q), \tag{3.12}$$

with $C(q)$ given by (3.8). Indeed one can prove that for any $q \in \mathcal{Q}$, $E(q) = \frac{1}{2}DC(q)$, where the notation $DC(q)$ stands for the derivative of $C(q)$ with respect to q, cf. Lemma 2.4 in [3] for a proof, so that the second term in the right-hand-side of (3.12) can be seen as a potential energy related to the first term in the right-hand-side of (3.11). Observe that $\mathcal{E}(q, p)$ is the sum of two positive terms and that in addition to its dependence on q and p, the energy $\mathcal{E}$ depends on $\mathcal{S}_0, m, \mathcal{J}, \gamma$ and Ω. A few computations prove that for any $q = (h, \theta) \in C^\infty([0, T]; \mathcal{Q})$ satisfying (3.7), as far as there is no collision,

$$\mathcal{E}(q, q') = \frac{1}{2} \int_{\mathcal{F}(q)} u(q, \cdot)^2 \, \mathrm{d}x + \frac{1}{2} m(h')^2 + \frac{1}{2} \mathcal{J}(\theta')^2,$$

which is the natural total kinetic energy of the "fluid+solid" system. Moreover, according to [12, Proposition 3.34], for any $q \in C^\infty([0, T]; \mathcal{Q})$ satisfying (3.7), as far as there is no collision, $\mathcal{E}(q, q')$ is constant in time. Using the regularity properties of the functions $C(q)$ and $\mathcal{M}_a(q)$ we deduce a bound of the body velocity depending only on the data and on the distance between the body and the boundary. This entails in particular that the life-time of a smooth solution to (3.7) can only be limited by a collision and therefore completes the proof of Theorem 3.3.

3.8 Zero radius limit

Let us now turn our attention to the limit of the dynamics when the diameter of the solid goes to 0 that is considering an initial domain for the body of the form (2.38) with the inertia scaling described in Definition 2.3 and Definition 2.4. Below we will establish that the limit dynamics is

$$h' = \gamma u_\Omega(h) \text{ for } t > 0, \text{ with } h(0) = 0, \tag{3.13}$$

in the case of a massless particle, and

$$mh'' = \gamma\big(h' - \gamma u_\Omega(h)\big)^\perp \text{ on } [0, T), \text{ with } (h, h')(0) = (0, \ell_0), \tag{3.14}$$

in the case of a point-mass particle. Above u_Ω is the so-called Kirchhoff-Routh velocity given by $u_\Omega := \nabla^\perp \psi_\Omega$, where the Kirchhoff-Routh stream function ψ_Ω is defined as $\psi_\Omega(x) := \frac{1}{2}\psi^0(x, x)$, where $\psi^0(h, \cdot)$ is the solution to the Dirichlet problem: $\Delta\psi^0(h, \cdot) = 0$ in Ω, $\psi^0(h, \cdot) = -\frac{1}{2\pi}\ln|\cdot - h|$ on $\partial\Omega$.

Indeed (3.13) is a well-known equation describing the motion of a point vortex in Ω. In particular Turkington proved in [13] that it can be viewed as a limit of concentrated smooth vortices evolving according to the Euler equations. He also observed that the solution h is global in time, and in particular that there is no collision of the vortex point with the external boundary $\partial\Omega$. This follows from the conservation of the energy $\gamma^2\psi_\Omega(h)$ for any $h \in C^\infty([0, T]; \Omega)$ satisfying (3.13), and the fact that $\psi_\Omega(h) \to +\infty$ when h comes close to $\partial\Omega$.

On the other hand the existence of a maximal solution (h, T) to (3.14) follows from classical ODE theory. Moreover it follows from the conservation of the energy $\frac{1}{2}mh' \cdot h' - \gamma^2\psi_\Omega(h)$ for any $h \in C^\infty([0, T]; \Omega)$ satisfying (3.14), and from the continuity of the Kirchhoff-Routh stream function ψ_Ω in Ω that T is the time of the first collision of h with the outer boundary $\partial\Omega$ of the fluid domain. If there is no collision, then $T = +\infty$.

Below, we will use the following notation: for $\mathcal{S}_0 \subset \Omega$; $p_0 = (\ell_0, r_0) \in \mathbb{R}^3$, $(m, \mathcal{J}) \in (0, +\infty) \times (0, +\infty)$, γ in $\mathbb{R}$ (resp. in $\mathbb{R}^*$) in the case of a massive (respectively massless) particle, for every $\varepsilon \in (0, 1]$ small enough to ensure that the set $\mathcal{S}_0^\varepsilon$ defined by (2.38) satisfies $\mathcal{S}_0^\varepsilon \subset \Omega$, we denote $(q^\varepsilon, T^\varepsilon)$ the maximal solution to (3.7) associated with the coefficients $\mathcal{M}^\varepsilon$, Γ_a^ε and F^ε corresponding to $\mathcal{S}_0^\varepsilon, m^\varepsilon, \mathcal{J}^\varepsilon$ and γ where $m^\varepsilon, \mathcal{J}^\varepsilon$ are given in Definition 2.3 and Definition 2.4, and with the initial data $(q^\varepsilon, (q^\varepsilon)')(0) = (0, p_0)$. As usually the position q^ε is decomposed into $q^\varepsilon = (h^\varepsilon, \theta^\varepsilon)$. Next result, obtained in [3], states the convergence of h^ε to the solution h to the equation (3.14) and (3.13), depending on whether one considers the point-mass particle regime or the massless particle regime. Thus T will denote the lifetime of the maximal solution to (3.14).

Theorem 3.4. *Then, as $\varepsilon \to 0$, in the case of a point-mass (respectively massless) particle,* $\liminf T^\varepsilon \geqslant T$ *(resp. $T^\varepsilon \to +\infty$) and $h^\varepsilon \longrightarrow h$ in $W^{2,\infty}([0, T']; \mathbb{R}^2)$ (resp. in $W^{1,\infty}([0, T']; \mathbb{R}^2)$) weak-$\star$ for all $T' \in (0, T)$ (resp. for all $T' > 0$).*

3.9 Scheme of proof

Below we give a scheme of proof of Theorem 3.4. We refer to [3] for a complete proof.

Examining how the equation scales with ε we arrive at

$$\Big(\varepsilon^\alpha I_\varepsilon \mathcal{M}_g I_\varepsilon + \mathcal{M}_a^\varepsilon(q^\varepsilon)\Big)(q^\varepsilon)'' + \langle \Gamma_a^\varepsilon(q^\varepsilon), (q^\varepsilon)', (q^\varepsilon)' \rangle = \gamma^2 E^\varepsilon(q^\varepsilon) + \gamma(q^\varepsilon)' \times B^\varepsilon(q^\varepsilon), \tag{3.15}$$

where I_ε is the diagonal matrix $I_\varepsilon := \operatorname{diag}(1, 1, \varepsilon)$ and the matrix $\mathcal{M}_g$ is given in terms of $m > 0$ and $\mathcal{J} > 0$ by (2.23). An appropriate renormalization of the energy (discarding a logarithmic singularity independent on the solid position and velocity) provides an uniform estimate of $\varepsilon^{\min(1, \frac{\alpha}{2})}\, |(h^\varepsilon)', \varepsilon(\theta^\varepsilon)')|_{\mathbb{R}^3}$ in the zero radius limit, at least till the solid stays away from the external boundary. Unfortunately in the massless case the coefficient α satisfies $\alpha > 0$ and the previous estimate is not sufficient to pass to the limit in (3.15). Indeed the case where $\alpha \geqslant 2$ is the most delicate and we will focus on it. First, by a multi-scale analysis of the Kirchhoff potentials and of the stream functions and repeated uses of Lemma 2.2, the terms $\mathcal{M}_a^\varepsilon$, Γ_a^ε, E^ε and B^ε are expanded, in the limit $\varepsilon \to 0$, till the solid stays away from the external boundary, into

$$\mathcal{M}_a^\varepsilon(q) = \varepsilon^2 I_\varepsilon\Big(\mathcal{M}_{a,\theta} + O(\varepsilon^2)\Big)I_\varepsilon, \tag{3.16}$$

$$\langle \Gamma_a^\varepsilon(q), \cdot, \cdot \rangle = \varepsilon I_\varepsilon\big(\langle \Gamma_{a,\theta}, I_\varepsilon \cdot, I_\varepsilon \cdot \rangle + O(\varepsilon^2)\big), \tag{3.17}$$

$$E^\varepsilon(q) = I_\varepsilon\Big(\mathsf{E}_0(q) + \varepsilon\mathsf{E}_1(q) + O(\varepsilon^2)\Big), \tag{3.18}$$

$$B^\varepsilon(q) = \varepsilon I_\varepsilon^{-1}\Big(B_\theta + \varepsilon\mathsf{B}_1(q) + O(\varepsilon^2)\Big), \tag{3.19}$$

where $q = (h, \theta)$; $\mathcal{M}_{a,\theta}$, $\Gamma_{a,\theta}$ and B_θ correspond to the case where the rigid body is of size $\varepsilon = 1$ and is immersed in a fluid filling the whole plane, and

$$\mathsf{E}_0(q) := -\begin{pmatrix} u_\Omega(h)^\perp \\ u_\Omega(h) \cdot R(\theta)\xi \end{pmatrix}, \tag{3.20}$$

with ξ given by (2.28). Indeed the notation $O(\varepsilon^2)$ above refers to the size of the term obtained when performing a Gronwall estimate for an energy-type quantity.

On the one hand, using (3.16) the leading terms of the inertia matrix $\varepsilon^\alpha I_\varepsilon \mathcal{M}_g I_\varepsilon + \mathcal{M}_a^\varepsilon(q^\varepsilon)$ in front of $(q^\varepsilon)''$ in (3.15) is $I_\varepsilon(\varepsilon^\alpha \mathcal{M}_g + \varepsilon^2 \mathcal{M}_{a,\theta})I_\varepsilon$, On the other hand combining the previous expansions of Γ_a^ε, E^ε and B^ε, and observing some striking recombinations which rely on explicit computations of the profiles $\mathsf{E}_1(q)$ and $\mathsf{B}_1(q)$ thanks to geometric quantities and some tedious algebraic computations, see [3], we obtain

$$\gamma^2 E^\varepsilon(q^\varepsilon) + \gamma (q^\varepsilon)' \times B^\varepsilon(q^\varepsilon) - \langle \Gamma_a^\varepsilon(q^\varepsilon), (q^\varepsilon)', (q^\varepsilon)' \rangle = I_\varepsilon \Big[\gamma \tilde{p}^\varepsilon \times B_{\theta^\varepsilon} - \varepsilon \langle \Gamma_{a,\theta^\varepsilon}, \tilde{p}^\varepsilon, \tilde{p}^\varepsilon \rangle + O(\varepsilon^2) \Big],$$

with

$$\tilde{p}_\varepsilon := \big(h_\varepsilon' - \gamma [u_\Omega(h_\varepsilon) + \varepsilon u_c(q_\varepsilon)], \varepsilon \vartheta_\varepsilon' \big)^t, \tag{3.21}$$

where $u_c := \nabla_h^\perp \big(D_h \psi_\Omega(h) \cdot R(\theta)\xi \big)$. Here D_h denotes the derivative with respect to h.

Thus the equation (3.15) can be recast under a form which is close to

$$\Big(\varepsilon^\alpha M_g + \varepsilon^2 M_{a,\theta_\varepsilon} \Big) \tilde{p}_\varepsilon' + \varepsilon \langle \Gamma_{a,\theta_\varepsilon}, \tilde{p}_\varepsilon, \tilde{p}_\varepsilon \rangle = \gamma \tilde{p}^\varepsilon \times B_{\theta^\varepsilon} + O(\varepsilon^{\min(2,\alpha)}). \tag{3.22}$$

Let us observe that this equation is quite similar to (2.40) with the differences that (3.22) involves the modulation of p^ε into $\tilde{p}^\varepsilon$ instead of p^ε and the extra remainder term $O(\varepsilon^{\min(2,\alpha)})$. Multiplying by $\tilde{p}^\varepsilon$ and using a Gronwall estimate then provides uniform bounds of $|((h^\varepsilon)', \varepsilon(\theta^\varepsilon)')|_{\mathbb{R}^3}$ till the solid stays away from the external boundary. This estimate in turn allows to pass to the limit in the two first coordinates of (3.22) proceeding as in the proof of Theorem 2.5. The issue of a possible collision is then tackled in a bootstrapping argument thanks to the behavior of the limit systems.

4 Extensions to rotational flows

Let us now discuss the case of a rotational flow when the fluid-solid system occupies the whole plane.

4.1 Compatible initial rotational flows

We first extend Definition 2.1 to this setting.

Definition 4.1. Given the initial domain $\mathcal{S}_0$ occupied by the body, the initial solid velocities ℓ_0 and r_0 respectively in $\mathbb{R}^2$ and $\mathbb{R}$, an initial fluid vorticity ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$ and γ in $\mathbb{R}$, we say that a vector field u_0 on the closure of $\mathcal{F}_0 = \mathbb{R}^2 \setminus \mathcal{S}_0$ with values in $\mathbb{R}^2$ is compatible if it is the unique vector field in $C^0(\overline{\mathcal{F}_0}; \mathbb{R}^2)$ satisfying $\operatorname{div} u_0 = 0$ and $\operatorname{curl} u_0 = \omega_0$ in $\mathcal{F}_0$, $u_0 \cdot n = (\ell_0 + r_0 x^\perp) \cdot n$ for $x \in \partial\mathcal{S}_0$, $\int_{\partial \mathcal{S}_0} u_0 \cdot \tau \, ds = \gamma$ and $\lim_{x\to\infty} u_0 = 0$.

4.2 An existence and uniqueness result

We now state an existence and uniqueness result with bounded vorticity for which we refer to [**?**, 1, 4, 5]

Theorem 4.2. *For any $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, $\omega_0 \in L^\infty_c(\overline{\mathcal{F}_0})$, there exists a unique solution to the problem compound of the incompressible Euler equations* (1.2) *on the fluid domain, of the Newton equations* (1.3)*, of the interface condition* (1.4)*, of the condition at infinity: $\lim_{|x|\to\infty} |u(t,x)| = 0$, and of the initial conditions $(h, h', \theta, \theta')(0) = (0, \ell_0, 0, r_0)$ and $u|_{t=0} = u_0$, with u_0 the compatible initial velocity associated with ℓ_0, r_0 and ω_0 by Definition 4.1. Moreover $(h, \theta) \in C^2(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R})$ and for all $t > 0$, $\omega(t) := \operatorname{curl} u(t) \in L^\infty_c(\overline{\mathcal{F}(t)})$.*

4.3 Zero radius limit

To tackle the zero radius limit, we consider a initial vorticity ω_0 in $L^\infty_c(\mathbb{R}^2 \setminus \{0\})$, a circulation γ in $\mathbb{R}$ (respectively in $\mathbb{R}^*$) in the case of a point-mass particle (resp. in the case of a massless point particle) and $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, all independent of ε. The corresponding initial fluid velocity u_0^ε is then defined as the unique vector field u_0^ε compatible with $\mathcal{S}_0^\varepsilon$, ℓ_0, r_0, γ and $\omega_0^\varepsilon := \omega_{0|\mathcal{F}_0^\varepsilon}$. Then we investigate the limit, as $\varepsilon \to 0$, of the corresponding solutions given by Theorem 4.2 with the inertia scaling described in Definition 2.3 and Definition 2.4. Let us recall that the Biot-Savart law in the full plane is given by the formula

$$K_{\mathbb{R}^2}[\omega] := \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)\,dy.$$

Theorem 4.3. *In the zero radius limit one obtains the following equation for the position $h(t)$ of the limit point particle:*

$$mh''(t) = \gamma\Big(h'(t) - K_{\mathbb{R}^2}[\omega(t,\cdot)](h(t))\Big)^\perp,$$

in the massive limit, and

$$h'(t) = K_{\mathbb{R}^2}[\omega(t,\cdot)](h(t)),$$

in the massless limit. Regarding the fluid state, one obtains at the limit the following transport equation for the fluid vorticity:

$$\frac{\partial\omega}{\partial t} + \mathrm{div}\,\big(\omega K_{\mathbb{R}^2}[\omega + \gamma\delta_h]\big) = 0. \tag{4.1}$$

This result was obtained in [1] and [2], and a simplified proof is given in [12]. Indeed in [1, 2] we have followed a complex-analytic approach whereas the simplified proof in [12] followed the real-analytic method used by Lamb in the case of an unbounded flow (as in Section 2) and extended to the bounded case in [3] (as in Section 3).

Observe that the fluid equation (4.1) is the same whether the body shrinks to a massive or a massless point particle: it describes the evolution of the vorticity of the fluid as transported by a velocity obtained by the usual Biot-Savart law in the plane, but from a vorticity which is the sum of the fluid vorticity and of a point vortex placed at the (time-dependent) position $h(t)$ where the solid shrinks, with a strength equal to the circulation γ around the body.

5 A conjecture about the zero radius limit

We believe that the following statement is true in a very general setting.

Conjecture ($\mathcal{C}$). A point-mass particle immersed in a two dimensional incompressible perfect fluid moves according to Newton's law with a gyroscopic force orthogonally proportional to its relative velocity and proportional to the circulation around the body. A massless point particle immersed in a two dimensional incompressible perfect fluid with nonzero circulation moves as a point vortex, its vortex strength being given by the circulation. More precisely, the position $h(t)$ of a point-mass particle (respectively massless point particle) satisfies the equation $mh'' = \gamma\big(h' - u_{\mathrm{d}}(h)\big)^\perp$ (resp. $h' = u_{\mathrm{d}}(h)$), with $u_{\mathrm{d}}(h) = \big(u_{\mathrm{bd}} - K_{\mathbb{R}^2}[\gamma\delta_h]\big)(h)$ where u_{bd} denotes the background fluid velocity. On the other hand the genuine fluid vorticity ω is convected by the background fluid velocity u_{bd}.

Theorem 2.5 proves that ($\mathcal{C}$) is true in the case of an unbounded irrotational flow with $u_{\mathrm{d}} = 0$, Theorem 3.4 proves that ($\mathcal{C}$) is true in the bounded setting with $u_{\mathrm{d}} = \gamma u_\Omega$ (and a easy byproduct of the analysis is that $u_{\mathrm{bd}} = K_\Omega[\gamma\delta_h]$ where K_Ω denotes the Biot-Savart law associated with the simply connected domain Ω), and Theorem 4.3 proves that ($\mathcal{C}$) is true for a rotational unbounded flow with $u_{\mathrm{d}} = K_{\mathbb{R}^2}[\omega]$ and $u_{\mathrm{bd}} = K_{\mathbb{R}^2}[\omega + \gamma\delta_h]$.

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