Smoothness of the trajectories of ideal fluid particles with Yudovich vorticities in a planar bounded domain

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Abstract

We consider the incompressible Euler equations in a (possibly multiply connected) bounded domain Ω of $\mathbb{R}^2$, for flows with bounded vorticity, for which Yudovich proved in [29] global existence and uniqueness of the solution. We prove that if the boundary $\partial\Omega$ of the domain is C^∞ (respectively Gevrey of order $M \geqslant 1$) then the trajectories of the fluid particles are C^∞ (resp. Gevrey of order $M+2$). Our results also cover the case of "slightly unbounded" vorticities for which Yudovich extended his analysis in [30]. Moreover if in addition the initial vorticity is Hölder continuous on a part of Ω then this Hölder regularity propagates smoothly along the flow lines. Finally we observe that if the vorticity is constant in a neighborhood of the boundary, the smoothness of the boundary is not necessary for these results to hold.

1 Introduction

We consider the initial-boundary-value problem for the 2-D incompressible Euler equations in a regular (possibly multiply connected) bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & \text{in } (0,+\infty) \times \Omega, \\ \text{div } u = 0, & \text{in } [0,+\infty) \times \Omega, \\ u \cdot \hat{n} = 0, & \text{on } [0,+\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{1.1}$$

Here, $u = (u_1, u_2)$ is the velocity field, p is the pressure, $\hat{n}$ denotes the unit outward normal to the boundary $\partial\Omega$ of Ω and ∇ denotes the gradient with respect to the space variable $x = (x_1, x_2) \in \mathbb{R}^2$. A key quantity in the analysis is the vorticity $\omega := \text{curl } u = \partial_1 u_2 - \partial_2 u_1$, which satisfies the transport equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \text{ in } (0,+\infty) \times \Omega,$$

so that, at least formally, the integral over Ω of any function of the vorticity is conserved when time proceeds.

The global existence and uniqueness of classical solutions to (1.1) were obtained by Wolibner [28] and extended to multiply connected domains by Kato in [12]. This result was extended by Yudovich [29] to flows such that the initial vorticity (and hence the vorticity at any moment t) is bounded. The corresponding velocity field u is Log-Lipschitz so that there exists a unique flow map Φ continuous from $\mathbb{R}_+ \times \Omega$ to Ω such that

$$\Phi(t,x) = x + \int_0^t u(s, \Phi(s,x)) ds. \tag{1.2}$$

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Moreover there exists $c > 0$ such that for any $t > 0$, the vector field $\Phi(t,\cdot)$ lies in the Hölder space $C^{0,\exp(-ct\|\omega_0\|_{L^\infty(\Omega)})}(\Omega)$, and an example of Bahouri and Chemin [2] shows that this estimate is optimal. Here and in the sequel we denote $C^{\lambda,r}(\Omega)$, for λ in $\mathbb{N}^0$ and $r \in (0,1)$, the Hölder space endowed with the norm:

$$\|u\|_{C^{\lambda,r}(\Omega)} := \sup_{|\alpha|\leqslant\lambda} \big(\|\partial^\alpha u\|_{L^\infty(\Omega)} + \sup_{x\neq y\in\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^r}\big) < +\infty,$$

and the notation $C^{\lambda,r}_{\mathrm{loc}}(\Omega_0)$ holds for the space of the functions which are in $C^{\lambda,r}(K)$ for any compact subset $K \subset \Omega$.

Above and in the sequel we use the notation $\mathbb{N}^0$ for the set of the non-negative integers and $\mathbb{N}^*$ for the set of the positive integers.

In this paper we prove the following result concerning the smoothness in time of the flow map.

Theorem 1. *Assume that the boundary $\partial\Omega$ is C^∞ (respectively Gevrey of order $M \geqslant 1$). Then there exists $c > 0$ such that for any divergence free vector field u_0 in $L^2(\Omega)$ tangent to the boundary $\partial\Omega$, with $\omega_0 := \operatorname{curl} u_0 \in L^\infty(\Omega)$, the flow map Φ is, for any $r \in (0,1)$, for any $T > 0$, C^∞ (resp. Gevrey of order $M+2$) from $[0,T]$ to $C^{0,\tilde r}(\Omega)$, with $\tilde r := r\exp(-cT\|\omega_0\|_{L^\infty(\Omega)})$.*

Let us be more explicit here about the meaning of the Gevrey smoothness in the claim above. We say that the flow map Φ is, for $M \geqslant 1$, $\tilde r \in (0,1)$ and $T > 0$, Gevrey of order $M+2$ from $[0,T]$ to $C^{0,\tilde r}(\Omega)$ if there exists $L > 0$ such that for any $t \in [0,T]$, for any $k \in \mathbb{N}^0$,

$$\|\partial_t^{k+1}\Phi(t,.)\|_{C^{0,\tilde r}(\Omega)} \leqslant (k!)^{M+2} L^{k+1}.$$

In the proof of Theorem 1 we will establish such a property with a constant L depending only on Ω.

Theorem 1 extends some previous results on the smoothness of the trajectories of the incompressible Euler equations that we now recall. In [6], [7] Chemin proved some similar statements for classical solutions in the full space. More precisely, he proves that the flow map Φ is C^∞ from $[0,T]$, for any $T \in (0,T^*)$ with T^* is the lifetime of the classical solution, to the Hölder space[1] $C^{1,r}(\mathbb{R}^d)$ for $r \in (0,1)$ and for both $d = 2$ and $d = 3$. These results were improved by Gamblin [17] and Serfati [24], [23], [25] who prove that the flow of classical solutions is analytic and that the flow of Yudovich's solutions with bounded vorticity is Gevrey 3, still for fluids filling the whole space.

Their results were extended to the case of classical solutions in bounded domains, in both 2 and 3 dimensions, in [19] by Kato (if the boundary $\partial\Omega$ of the domain is C^∞ then the flow map Φ is C^∞ from $[0,T]$ to the Hölder space $C^{1,r}(\Omega)$ for $r \in (0,1)$) and in [18] (if the boundary $\partial\Omega$ of the domain is analytic then the flow map Φ is analytic from $[0,T]$ to the Hölder space $C^{1,r}(\Omega)$ for $r \in (0,1)$). Actually the main result in [18] is that the motion of a rigid body immersed in an incompressible perfect fluid which occupies a three dimensional bounded domain is at least as smooth as the boundaries (of the body and of the domain) when the initial velocity of the fluid is in the Hölder space $C^{1,r}(\Omega)$ (till the classical solution exists and till the solid does not hit the boundary). One ingredient of the proof was precisely the smoothness of the flow of the incompressible Euler equations. We therefore hope that the analysis of the paper should be applied to the smoothness of the motion of a body immersed in a perfect incompressible fluid with Yudovich vorticities.

The following result bridges Theorem 1 and the earlier results about classical solutions, proving that, for Yudovich solutions, extra local Hölder regularity propagates smoothly along the flow lines.

Theorem 2. *Under the (respective) hypotheses of Theorem 1, and assuming moreover that the restriction $\omega_0|_{\Omega_0}$ is in the Hölder space $C^{\lambda_0,r}_{loc}(\Omega_0)$, where $\lambda_0 \in \mathbb{N}^0$ and $r \in (0,1)$ and Ω_0 an open set such that $\overline{\Omega_0} \subset \Omega$, we have that the flow map Φ is, for any $T > 0$, for any compact set $K \subset \Omega_0$, C^∞ (resp. Gevrey of order $M + 2 + (\lambda_0+1)(r+1)$) from $[0,T]$ to $C^{\lambda_0+1,r}(K)$.*

Actually we will obtain Theorem 1 (resp. Theorem 2) as a particular case of Theorem 5 (resp. Theorem 7) below, which encompasses more general initial vorticities. More precisely we will also consider the "slightly unbounded" vorticities introduced by Yudovich in [30].

[1] One has also to require a decreasing condition at infinity to avoid anomalous solutions, for instance imposing that the velocity field u is in $L^q(\mathbb{R}^3)$ with $1 < q < +\infty$.

2 Yudovich's slightly unbounded vorticities

In this section we recall the setting of Yudovich's paper [30] with a few extra remarks which will be useful in the sequel. We start with the following definition.

Definition 1 (Admissible germs). *A function $\theta : [p_0, +\infty) \to (0, +\infty)$, with $p_0 > 1$, is said admissible if the auxiliary function $T_\theta : [1, +\infty) \to (0, +\infty)$ defined for $a > 1$ by*

$$T_\theta(a) := \inf\{\frac{a^\epsilon}{\epsilon}\theta(\frac{1}{\epsilon}),\ 0 < \epsilon \leqslant 1/p_0\} \textit{ satisfies } \int_1^{+\infty} \frac{da}{aT_\theta(a)} = \infty.$$

Let us denote $\theta_0(p) := 1$, and, for any $m \in \mathbb{N}^*$,

$$\theta_m(p) := \log p \cdot \log^2 p \cdots \log^m p, \tag{2.1}$$

where $\log^m$ is log composed with itself m times.

Examples 1. *For any $m \in \mathbb{N}^0$, the germs θ_m are admissible.*

Proof. Let be given $\theta : [p_0, +\infty) \to (0, +\infty)$, with $p_0 > 1$. For $a > e^{p_0}$,

$$T_\theta(a) \leqslant e \log a \cdot \theta(\log a), \tag{2.2}$$

since $\epsilon = (\log a)^{-1}$ is in $(0, 1/p_0)$. Therefore,

$$\int_1^{+\infty} \frac{da}{aT_\theta(a)} \geqslant e^{-1} \int_{e^{p_0}}^{+\infty} \frac{d\log a}{\log a \cdot \theta(\log a)} = e^{-1} \int_{p_0}^{+\infty} \frac{dp}{p \cdot \theta(p)}.$$

The result follows from a repeated change of variables: for any $m \in \mathbb{N}^0$, for any $p_2 \geqslant p_1 \geqslant \exp^m(1)$, where $\exp^m$ is exp composed with itself m times,

$$\int_{p_1}^{p_2} \frac{dp}{p \cdot \theta_m(p)} = \log^{m+1} p_2 - \log^{m+1} p_1. \tag{2.3}$$

□

Definition 2 (The space $\mathbb{Y}_\theta$). *Given an admissible germ $\theta : [p_0, +\infty) \to (0, +\infty)$, with $p_0 > 1$, we denote $\mathbb{Y}_\theta$ the space of the divergence free vector fields u in $L^2(\Omega)$ tangent to the boundary, such that $\operatorname{curl} u$ belongs in $\cap_{p \geqslant p_0} L^p(\Omega)$, and such that there exists $c_f > 0$ such that*

$$\|f\|_{L^p(\Omega)} \leqslant c_f \theta(p) \textit{ for } p \geqslant p_0. \tag{2.4}$$

It is a Banach space endowed with the following norm:

$$\|f\|_{\mathbb{Y}_\theta} := \|f\|_{L^2(\Omega)} + \inf\{c_f > 0/\ (2.4) \textit{ holds true }\}.$$

Remark 1. *In particular for $\theta = \theta_0$, the space $\mathbb{Y}_\theta$ corresponds to the space of the divergence free vector fields u in $L^2(\Omega)$ tangent to the boundary with $\operatorname{curl} u$ in $L^\infty(\Omega)$.*

Remark 2. *Vorticities with a point singularity at $x_0 \in \Omega$ of type $\log\log \|x - x_0\|^{-1}$ belong to the space $\mathbb{Y}_\theta$, with θ of the form $\theta = c\theta_1$ (where c is a positive constant), which is therefore admissible (cf. [30], Example 3.3). On the other hand thanks to Laplace's method, we have that L^p norms of a vorticity with a point singularity at $x_0 \in \Omega$ of type $\log \|x - x_0\|^{-1}$ is equivalent to cp (where c is a positive constant), and is therefore non-admissible (cf. [30], Example 3.2).*

In this setting existence and uniqueness holds according to the following result.

Theorem 3 (Yudovich [30]). *Assume that the boundary $\partial\Omega$ of the domain is C^2. Given u_0 in $\mathbb{Y}_\theta$, there exists a unique weak solution u of (1.1) in $L^\infty([0, +\infty), \mathbb{Y}_\theta)$.*

We are now going to examine the flow map of these solutions. We first recall the following definition.

Definition 3 (Modulus of continuity)**.** *We will say that, $a > 0$ being given, a function $\mu : [0, a] \to \mathbb{R}_+$ is a modulus of continuity if it is an increasing continuous function such that $\mu(0) = 0$. We will denote by $C_\mu(\Omega)$ the space of continuous functions f over Ω such that the following semi-norm*

$$\|f\|_{C_\mu(\Omega)} := \sup_{0<\|x-y\|\leqslant a} \frac{\|f(x) - f(y)\|}{\mu(\|x-y\|)}$$

is finite.

A function in $C_\mu(\Omega)$ extends uniquely to a function in $C_\mu(\overline{\Omega})$. In particular, since Ω is bounded, a function in $C_\mu(\Omega)$ is bounded. Moreover in the case where $a > \operatorname{diam}(\Omega)$, and $\mu(h) := h^r$ with $r \in (0, 1)$, then $C_\mu(\Omega) = C^{0,r}(\Omega)$.

Let us remark the following.

Lemma 1. *Let F be in the Hölder space $C^{0,r}(\Omega)$, with $r \in (0, 1)$, let μ be a modulus of continuity and let $\phi : \Omega \to \Omega$ be in $C_\mu(\Omega)$; then μ^r is a modulus of continuity and $F \circ \phi$ is in $C_{\mu^r}(\Omega)$ with $\|F \circ \phi\|_{C_{\mu^r}(\Omega)} \leqslant \|F\|_{C^{0,r}(\Omega)} \cdot \|\phi\|^r_{C_\mu(\Omega)}$.*

Proof. For x, y in Ω, with $0 < \|x - y\| \leqslant a$, we have

$$\frac{\|F \circ \phi(x) - F \circ \phi(y)\|}{\mu(\|x-y\|)^r} \leqslant \frac{\|F \circ \phi(x) - F \circ \phi(y)\|}{\|\phi(x) - \phi(y)\|^r} \cdot \frac{\|\phi(x) - \phi(y)\|^r}{\mu(\|x-y\|)^r} \leqslant \|F\|_{C^{0,r}(\Omega)} \cdot \|\phi\|^r_{C_\mu(\Omega)},$$

and $\mu^r : h \in [0, a] \mapsto \mu(h)^r$ is a modulus of continuity. ☐

Definition 4 (Osgood modulus of continuity)**.** *We say that a modulus of continuity $\mu : [0, a] \to \mathbb{R}_+$ is an Osgood modulus of continuity if $\int_0^a \frac{dh}{\mu(h)} = +\infty$.*

Remark 3. *For a modulus of continuity (Osgood or not), only the behavior of μ near 0 does really matter for our purposes, not the value of a.*

Theorem 4 (Yudovich [30])**.** *Assume that the boundary $\partial\Omega$ of the domain is C^2. There exists $C > 0$ (depending only on Ω) such that for any admissible germ θ, for any initial velocity u_0 in $\mathbb{Y}_\theta$, the corresponding unique weak solution u of the Euler equations provided by Theorem 3 is in $L^\infty([0, +\infty), C_\mu(\Omega))$, where the modulus of continuity μ satisfies*

$$\mu(h) \leqslant ChT_\theta(h^{-2}), \tag{2.5}$$

where T_θ is the function which appeared in Definition 1. Thus μ is an Osgood modulus of continuity.

Let us stress that the function μ in Theorem 4 is independent of time.

Remark 4. *Theorem 4 applies in particular for $\theta = \theta_0$ (bounded vorticity), and we recover the well-known fact since [29] that we can take an Osgood modulus of continuity of the form*

$$\mu(h) := C\|\omega_0\|_{L^\infty(\Omega)} h \log(h^{-2}), \text{ with } C > 0 \text{ (depending only on } \Omega).$$

It is therefore classical (see for example [3]) that, for u as in Theorem 4, there exists a unique corresponding flow map Φ continuous from $\mathbb{R}_+ \times \Omega$ to Ω such that

$$\Phi(t, x) = x + \int_0^t u(s, \Phi(s, x))ds.$$

This relies on the Osgood Lemma that we recall above under a form appropriated for the sequel.

Lemma 2. *Let ρ be a measurable function from $[0, T]$ into $[0, a]$, μ a modulus of continuous on $[0, a]$, $c \in (0, +\infty)$ and assume that for all $t \in [0, T]$,*

$$\rho(t) \leqslant c + \int_0^t \mu(\rho(s))ds.$$

Then for all $t \in [0, T]$,

$$\int_c^{\rho(t)} \frac{dh}{\mu(h)} \leqslant t.$$

Let $T > 0$. Since the modulus of continuous μ provided by Theorem 4 is an Osgood modulus there exists $\tilde{a} \in (0, a)$ such that

$$\int_{\tilde{a}}^{a} \frac{dh}{\mu(h)} \geqslant \kappa T, \text{ where } \kappa := \|u\|_{L^\infty([0,+\infty), C_\mu(\Omega))}.$$

For any $t \in [0, T]$, for any $h \in (0, \tilde{a}]$ there exists an unique $\Gamma_t(h) \in [h, a]$ such that

$$\int_{h}^{\Gamma_t(h)} \frac{dh}{\mu(h)} = \kappa t. \tag{2.6}$$

In addition, for any $t \in [0, T]$, extended by $\Gamma_t(0) = 0$, the function Γ_t is a modulus of continuity. Furthermore, we have the following.

Lemma 3. *For any $t \in [0, T]$, the flow map $\Phi(t, .)$ at time t belongs to $C_{\Gamma_t}(\Omega)$.*

Proof. Let x, y be in Ω with $0 < \|x - y\| \leqslant \tilde{a}$. Using Theorem 4 we get that for any $t \in [0, T]$,

$$\|\Phi(t, x) - \Phi(t, y)\| \leqslant \|x - y\| + \kappa \int_0^t \mu(\|\Phi(s, x) - \Phi(s, y)\|)ds.$$

Thanks to the Lemma 2, we infer that for any $t \in [0, T]$,

$$\int_{\|x-y\|}^{\|\Phi(t,x)-\Phi(t,y)\|} \frac{dh}{\mu(h)} \leqslant \kappa t, \quad \text{and thus } \|\Phi(t, x) - \Phi(t, y)\| \leqslant \Gamma_t(\|x - y\|).$$

□

At this point we can already say that for any $x \in \Omega$, the curve $t \mapsto \Phi(t, x)$ is absolutely continuous hence differentiable almost everywhere with $\partial_t \Phi(t, x) = u(t, \Phi(t, x))$. In addition uniqueness implies that the flow satisfies the Markov semigroup property. Finally for any $t \in [0, T]$, the flow map $\Phi(t, \cdot)$ at time t is a volume-preserving homeomorphism.

Let us now have a deeper look at the smoothness in space of the flow map. We start with recalling the following.

Remark 5. *In the particular case where the initial vorticity is bounded (when $\theta = \theta_0$) it is well-known that there exists $c > 0$ such that for any $t > 0$, $\Phi(t, \cdot)$ lies in the Hölder space $C^{0,\exp(-ct\|\omega_0\|_{L^\infty(\Omega)})}$. An example by Bahouri and Chemin [2] shows that this estimate is actually optimal.*

Lemma 4. *For $\theta = \theta_m$, as in* (2.1)*, with $m \in \mathbb{N}^*$, the modulus of continuity Γ_t of the flow map satisfies*

$$\Gamma_t(h) \leqslant (\exp^m((\log^m(h^{-2}))^{\exp(-2C\kappa t)}))^{-\frac{1}{2}}. \tag{2.7}$$

Proof. Combining (2.2) and (2.5), we get $\mu(h) \leqslant Ceh\theta_{m+1}(h^{-2})$. Hence, using (2.6),

$$\kappa t = \int_h^{\Gamma_t(h)} \frac{dh}{\mu(h)} \geqslant -\tilde{C}^{-1} \int_{h^{-2}}^{\Gamma_t(h)^{-2}} \frac{dp}{p \cdot \theta_{m+1}(p)} = \tilde{C}^{-1}(-\log^{m+2}(\Gamma_t(h)^{-2}) + \log^{m+2}(h^{-2})),$$

by (2.3), with $\tilde{C} := 2Ce$. The result is then straightforward. □

Kelliher [13] provided some examples of Yudovich's slightly unbounded initial vorticities for which the solution to the Euler equations in the plane has an associated flow which lies in none Hölder space of positive exponent for all positive time. However for $\theta = \theta_m$, with $m \in \mathbb{N}^*$, the flow map is necessarily Dini continuous. Before to prove this, let us recall the following definition.

Definition 5 (Dini modulus of continuity)**.** *We say that a modulus of continuity $\mu : [0, a] \to \mathbb{R}_+$ is a Dini modulus of continuity if $\int_0^a \frac{\mu(h)}{h} dh < +\infty$. A function which belongs to a space C_μ where μ is a Dini modulus of continuity is said Dini continuous.*

Lemma 5. *For $\theta = \theta_m$ with $m \in \mathbb{N}^*$, $r \in (0, 1]$ and $t \geqslant 0$, the modulus of continuity Γ_t^r of the flow map is Dini.*

Proof. Let $m \in \mathbb{N}^*$ and $t \geqslant 0$. We set $u := \log^m(h^{-2})$ so that, using (2.7), we get

$$2 \int_0^a \frac{\Gamma_t^r(h)}{h} dh \leqslant \int_{\log^m(a^{-2})}^{+\infty} \prod_{i=1}^{m-1} \exp^i(u) \cdot \exp(-\frac{r}{2} \exp^{m-1}(u^{\exp(-2Ct)}))du < +\infty.$$

□

3 Statement of the results

Our analysis applies as well to the case of multiply connected domains. Let us therefore assume that Ω has as internal boundaries some piecewise smooth Jordan curves $\mathcal{C}_1$, ..., $\mathcal{C}_d$, and is bounded externally by a closed curve $\mathcal{C}_0$. We choose the positive directions on the curves $\mathcal{C}_0$, $\mathcal{C}_1$..., $\mathcal{C}_d$ such that the domain Ω is always on the left (so that the curves $\mathcal{C}_1$, ..., $\mathcal{C}_d$ are oriented clockwise and the curve $\mathcal{C}_0$ is oriented counter-clockwise). For a smooth enough function f, we denote $\gamma_i(f)$ the circulations of f around the curve $\mathcal{C}_i$, for $1 \leqslant i \leqslant d$.

3.1 Smoothness of the trajectories for bounded or "slightly unbounded" vorticities

We are now ready to state the first general result hinted in the introduction.

Theorem 5. *Let θ be an admissible germ and assume that the initial data u_0 is in $\mathbb{Y}_\theta$.*

1. *Assume that the boundary $\partial\Omega$ is C^∞. Then the flow map Φ is, for any $r \in (0,1)$, for any $T > 0$, C^∞ from $[0,T]$ to $C_{\Gamma_T^r}(\Omega)$.*
2. *Assume that the boundary $\partial\Omega$ is Gevrey of order $M \geqslant 1$. Then there exists $L > 0$ depending only on Ω such that for any $t \in [0,T]$, for any $k \in \mathbb{N}^0$, for any $r \in (0,1)$, the flow map Φ satisfies*

$$\|\partial_t^{k+1}\Phi(t,.)\|_{C_{\Gamma_t^r}(\Omega)} \leqslant \frac{(k!)^{M+1}L^{k+1}}{(1-r)^k}\Big(\frac{k!}{(1-r)^{k+1}}\theta\big(\frac{2(k+1)}{1-r}\big)^{k+1} + \sum_{i=1}^{d}|\gamma_i(u_0)|^{k+1}\Big). \tag{3.1}$$

To deduce Theorem 1 from Theorem 5 it suffices to take into account Remark 1 and Remark 5. Let us stress that Theorem 5 fails to prove that the flow is C^∞ from $[0,T]$ to $C_{\Gamma_T}(\Omega)$, for the estimate (3.1) blows up when r tends to 1. In the particular case where $\theta := \theta_m$, with $m \in \mathbb{N}^0$, the estimate (3.1) and Lemma 5 yield that the flow map is, for any $\varepsilon \in (0,1)$, for any $T > 0$, Gevrey of order $M + 2 + \varepsilon$ from $[0,T]$ to the space $C_D(\Omega)$ of Dini continuous functions.

3.2 Extra local Hölder regularity propagates smoothly

In this section we deal with the case where the initial vorticity is locally Hölder continuous. We will prove first the following result:

Theorem 6. *Assume that the boundary $\partial\Omega$ of the domain is C^2. Assume that the initial data u_0 is in $\mathbb{Y}_{\theta_m}$, with $m \in \mathbb{N}^0$. Assume that Ω_0 is a open subset such that $\overline{\Omega_0} \subset \Omega$. Assume that $\omega_0|_{\Omega_0}$ is in $C^{\lambda_0,r}_{loc}(\Omega_0)$ with λ_0 in $\mathbb{N}^0$ and $r \in (0,1)$. Then for any $t \geqslant 0$, the restrictions $\omega(t,\cdot)|_{\Omega_t}$ and $u(t,\cdot)|_{\Omega_t}$ of the vorticity and of the velocity to the set $\Omega_t := \{\Phi(t,x),\ x \in \Omega_0\}$ are respectively in $C^{\lambda_0,r}_{loc}(\Omega_0)$ and $C^{\lambda_0+1,r}_{loc}(\Omega_0)$.*

Theorem 6 is a slight extension of Proposition 8.3 of [20] which deals only with the case $\theta = \theta_0$, that is with bounded vorticities. Let us also mention the paper [9] by Danchin about singular vortex patches.

Under the assumptions of Theorem 6, if the boundary is smooth, local Hölder regularity propagates smoothly along the flow lines.

Theorem 7. *Under the hypotheses of Theorem 6.*

1. *Assume moreover that the boundary $\partial\Omega$ is C^∞, then the flow map Φ is, for any $T > 0$, for any compact set $K \subset \Omega_0$, C^∞ from $[0,T]$ to $C^{\lambda_0+1,r}(K)$.*
2. *Assume moreover that the boundary $\partial\Omega$ is Gevrey of order $M \geqslant 1$, then there exists $L > 0$ such that for any $T > 0$, for any compact set $K \subset \Omega_0$, for any $t \in [0,T]$, for any $k \in \mathbb{N}^0$, the flow map Φ satisfies*

$$\|\partial_t^{k+1}\Phi(t,.)\|_{C^{\lambda_0+1,r}(K)} \leqslant L^{k+1}(k!)^{M+1+(\lambda_0+1)(r+1)}\Big(\|u(t,.)\|_{C^{\lambda_0+1,r}(\underline{K}_t)} + \|u(t,.)\|_{W^{1,\frac{2(k+1)}{1-r}}(\Omega)}\Big)^{k+1}, \tag{3.2}$$

where $\underline{K}_t := \{\Phi(t,x),\ x \in \underline{K}\}$ with $\underline{K}$ a compact set such that $K \subset \mathring{\underline{K}} \subset \underline{K} \subset \Omega_0$.

Above the notation $\mathring{\underline{K}}$ stands for the interior of the set $\underline{K}$.

After the proof of Theorem 7, it would be clear that Theorem 7 yields Theorem 2 when we consider the case $m = 0$ observing that last factor of the right hand side of (3.2) can be therefore estimated by the initial vorticity with a extra factor $(k!)$.

3.3 A few remarks about weaker solutions and the influence of the boundary smoothness

Actually what we really need in the proof of Theorem 5 is, first, of course, the existence of a flow and that the vorticity lies in any $L^p(\Omega)$ for large p. However Theorem 5 does not cover some cases where a flow map can be defined, and even uniquely. In particular in [27] Vishik proves the following result of existence and uniqueness of solutions to the 2D incompressible Euler equations in the full plane in a borderline space of Besov type.

Theorem 8 (Vishik)**.** *Assume that $\Omega := \mathbb{R}^2$ and that $\omega_0 \in L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)$ with $1 < p_0 < 2 < p_1 < +\infty$. Assume moreover that ω_0 is in*

$$B_\Gamma := \{f \text{ in } \mathcal{S}'(\mathbb{R}^2) \text{ s.t. } \textstyle\sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty(\mathbb{R}^2)} = O(\Gamma(N))\}, \tag{3.3}$$

where $\Gamma(N) := \log N$ and the $\Delta_j f$ denote the terms in the Littlewood-Paley decomposition of f. Let u_0 be the velocity associated to ω_0 by the Biot-Savart law. Then there exists $T > 0$ and a solution to the Euler equations (1.1) satisfying

$$\omega \in L^\infty(0,T; L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)) \cap C_{w*}([0,T]; B_{\Gamma_1}),$$

where $\Gamma_1(N) := (N+2)\Gamma(N)$. The corresponding velocity u is in $L^\infty(0,T; C_\mu(\mathbb{R}^2))$ with $\mu(r) := r \cdot \log r^{-1} \cdot \log^2 r^{-1}$, so that the flow map is uniquely defined.

It is proved in [27], Proposition 2.1 that for any $\rho > 1$, there exists f in B_Γ and in $\cap_{1 \leqslant p < \rho} L^p$ but not in $\cap_{p \geqslant \rho} L^p$. Therefore our proof of Theorem 5 based on the scale of the Lebesgue spaces L^p is not adapted to tackle Vishik's solutions. However the smoothness of the flow map in this case can be deduced from Gamblin's work [17].

Theorem 9. *Under the assumptions of Theorem 8, the flow map is for any $\varepsilon \in (0,1)$, Gevrey of order $3+\varepsilon$ from $[0,T]$ to the space $C_D(\mathbb{R}^2)$ of Dini continuous functions.*

Proof. According to [17], estimate (2.3), there holds for any $k \in \mathbb{N}^0$, for any $\varepsilon_1 \in (0, 1/(2(k+1)))$, for any $t \in [0,T]$,

$$\|D^k u\|_{C^{0,1-\varepsilon_1(k+1)}(\mathbb{R}^2)} \leqslant \|u\|_{C^{0,1-\varepsilon_1}(\mathbb{R}^2)} (C\varepsilon_1^{-1} \|u\|_{C^{0,1-\varepsilon_1}(\mathbb{R}^2)})^k \frac{k!}{(k+2)^2}.$$

It then suffices to take $\varepsilon_1 := \varepsilon/(k+1)$, to use the embedding

$$\|\cdot\|_{C^{0,1-\varepsilon_1}} \leqslant C\varepsilon_1^{-1} \log \varepsilon_1^{-1} \|\cdot\|_{C_\mu},$$

Lemma 1 and 5 to conclude. □

Moreover for any initial vorticity in $L^p(\Omega)$, with $p > 2$, one gets a corresponding velocity which is continuous so that Peano's theorem applies and provides the existence of a flow. Furthermore it is well-known that uniqueness is generic in the sense of Baire's category for Peano's continuous vector-fields (see for instance Bernard's paper [4] Theorem 1). Let us also refer here to the renormalization theory by Di Perna-Lions [10] and Ambrosio [1] for some properties of the flow map up to some zero Lebesgue measure sets.

Next Theorem provides some examples of even weaker solutions than in Theorem 5 for which some flow lines are analytic, despite the boundary is only assumed to be C^2.

Theorem 10. *Assume that the boundary $\partial\Omega$ of the domain is C^2. Let be given $(\overline{\gamma}_i)_{1 \leqslant i \leqslant d} \in \mathbb{R}^d$, $N \geqslant 1$ distinct points $x_1, ..., x_N$ in Ω and $(\alpha_l)_{1 \leqslant l \leqslant N} \in \mathbb{R}^N$. Let $T > 0$ and $z(t) := (z_1(t), ..., z_N(t))$ be the unique solution (up to the first collision) in $C^\omega([0,T])$ of the Kirchoff-Routh-Lin equations of point vortices (cf. Lemma 9) with x_l as initial positions, of respective strength α_l, for $1 \leqslant l \leqslant N$, with $\overline{\gamma}_i$ as respective circulation on the inner boundary $\mathcal{C}_i$, for $1 \leqslant i \leqslant d$. Then*

$$t \mapsto \sum_{1 \leqslant l \leqslant N} \alpha_l \delta_{z_l(t)}, \tag{3.4}$$

provides a weak solution of the Euler equation on $[0,T]$ (in the sense of Definition 10) with $\omega_0 := \sum_{1 \leqslant l \leqslant N} \alpha_l \delta_{x_l}$ as initial data.

It should be argued that Theorem 10 belongs to the mathematical folklore. We provide an explicit proof in Appendix B for sake of completeness. We will show in particular in what sense the motions of point vortices can be seen as weak solutions of the Euler equations, adapting the weak vorticity formulation already used by Turkington [26] (in a simply connected domain) and Schochet [22] (in the full plane) to multiply connected domains.

In view of Theorem 5 and Theorem 10, it is natural to wonder to what extent it is possible to get rid of the boundary smoothness assumption. The following result bridges theses two results showing that, for an initial data with a Yudovich vorticity (let say here bounded, in order to simplify the statement) constant near the boundary, the smoothness of the flow map inside the domain can be obtained without assuming that the boundary is smooth.

Theorem 11. *Assume that the boundary $\partial\Omega$ is C^2. Then there exists $c > 0$ such that for any divergence free vector field u_0 in $L^2(\Omega)$ tangent to the boundary $\partial\Omega$, with $\omega_0 := \operatorname{curl} u_0 \in L^\infty(\Omega)$ constant outside of a compact set $\underline{K} \subset \Omega$, for any compact set $K \subset \Omega$, the flow map Φ is, for any $r \in (0,1)$, for any $T > 0$, for any $M > 1$, Gevrey of order $M+2$ from $[0,T]$ to $C^{0,r\exp(-cT\|\omega_0\|_{L^\infty(\Omega)})}(K)$.*

Let us stress that there is a arbitrary small loss of Gevrey order with respect to the result of Gamblin [17] about Yudovich flows in the full plane (and also with respect to Theorem 1 when assuming that the boundary $\partial\Omega$ is C^ω).

For classical flows, with vorticities constant near the boundary, it is possible to localize without any loss. Since this also holds in three dimensions, we prefer to postpone this to Appendix C, in order to avoid any confusion about the setting of these results.

We also plan to investigate this issue of smoothness along the flow lines in the case where the vorticity of the flows has some Dirac masses, in addition to a bounded (or "slightly unbounded") part. This setting was introduced by Marchioro and Pulvirenti, see [21]. Uniqueness is known to hold when the flow occupies the full plane, when the absolutely continuous part of the vorticity is bounded and when initially the point vortices are surrounded by regions of constant vorticity, see also [16]. It is therefore natural to wonder if a strategy with a cut-off could allow to deal with this case, and for extensions to bounded domains, and to the case where the absolutely continuous part is slightly unbounded. An underlying motivation is to prove some property of smoothness along the flow lines for any setting where existence and uniqueness of the incompressible Euler equations are known to hold.

4 Proof of Theorem 5

This section is devoted to the proof of Theorem 5. We will focus on the Gevrey case, the C^∞ case would be a byproduct of the analysis. We therefore assume that the function $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ satisfies the following: there exists $c_\rho > 1$ such that for all $s \in \mathbb{N}$, on a neighborhood $\mathcal{W} \subset \overline{\Omega}$ of the boundary $\partial\Omega$,

$$\|\nabla^s \rho\| \leqslant c_\rho^s \, (s!)^M, \tag{4.1}$$

as a function (on $\mathcal{W}$) with values in the set of symmetric s-linear forms.

We will proceed by regularization, working from now on a smooth flow, with the same notation. We will explain this regularization process in subsection 4.3.

Let us also recall a few basic ingredients.

Definition 6. *A vector field $\mathcal{X}$ from Ω to $\mathbb{R}^2$ is said tangential harmonic if it is $W^{1,2}(\Omega;\mathbb{R}^2)$, satisfies $\operatorname{div}\mathcal{X} = 0$ and $\operatorname{curl}\mathcal{X} = 0$ in Ω, and $\hat{n}\cdot\mathcal{X} = 0$ on $\partial\Omega$.*

Let us first recall the following classical result from the Hodge-De Rham theory.

Theorem 12. *The tangential harmonic vector fields are smooth up to their boundary. Their set is a vector space H of dimension d, orthogonal, in $L^2(\Omega;\mathbb{R}^2)$, to any gradient of smooth functions. There is a unique family $\{\mathcal{X}_1, ..., \mathcal{X}_d\}$ which are a basis of H and satisfy $\gamma_i(\mathcal{X}_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant d$.*

It is a well-known result, let us refer to [21], Theorem 2.1 and to the appendix to introduction of [19] for a detailed proof of the smoothness up to the boundary.

Definition 7. *We will denote by Π be the orthogonal projection of $L^2(\Omega;\mathbb{R}^2)$ onto the space H of tangential harmonic vectors.*

Lemma 6. *There exists $C_\Pi > 0$ (depending only on Ω) such that for any $p > 2$, for any f in $L^p(\Omega)$, $\|\Pi f\|_{L^p(\Omega)} \leqslant C_\Pi \|f\|_{L^p(\Omega)}$.*

Proof. Let us orthonormalize the $\mathcal{X}_i$. We denote by $\tilde{\mathcal{X}}_i$, $1 \leqslant i \leqslant d$, the orthonormal system obtained. Then

$$\Pi f = \sum_{i=1}^d (\int_\Omega f \cdot \tilde{\mathcal{X}}_i)\tilde{\mathcal{X}}_i,$$

and the $\tilde{\mathcal{X}}_i$ are smooth. □

We will use the following elliptic regularity estimate.

Lemma 7. *There exists $c, c_h > 0$ such that for any $p > 2$, for any smooth vector field f from Ω to $\mathbb{R}^2$ such that there exists ϕ in $W^{1,p}(\mathcal{W})$ such that $(\hat{n} \cdot f)|_{\partial\Omega} = \phi|_{\partial\Omega}$,*

$$\|f\|_{W^{1,p}(\Omega)} \leqslant cp(\|\operatorname{div} f\|_{L^p(\Omega)} + \|\operatorname{curl} f\|_{L^p(\Omega)} + \|\phi\|_{W^{1,p}(\mathcal{W})}) + c_h\|\Pi f\|_{L^p(\Omega)}. \tag{4.2}$$

Above the notation $\mathcal{W}$ refers to the neighborhood of the boundary $\partial\Omega$ in $\overline{\Omega}$, defined in the beginning of this section.

Lemma 7 relies on Calderon-Zygmund theory of singular integral operators. A particular case has been used in Yudovich's proof of Theorem 3 and of Theorem 4 (cf. [29], [30]). The dependance on p was crucial in his proof and it would also be crucial in the proof of Theorem 5. Since we did not find as it in the literature, we provide a proof for sake of completeness. We will use the Yudovich result: there exists $c > 0$ such that for any $p > 2$, for any smooth function φ from Ω to $\mathbb{R}$, satisfying $\varphi = 0$ on $\partial\Omega$ or $\partial_n\varphi = 0$ on $\partial\Omega$ and $\int_\Omega \Delta\varphi = 0$,

$$\|\nabla\varphi\|_{W^{1,p}(\Omega)} \leqslant cp\|\Delta\varphi\|_{L^p(\Omega)}. \tag{4.3}$$

Proof. Let us still denote $\hat{n}$ a smooth extension of the unit normal supported in $\mathcal{W}$. There exists only one smooth function (up to a additive constant) φ which satisfies $\Delta\varphi = \operatorname{div}(f - \phi\hat{n})$ in Ω, and $\partial_n\varphi = 0$ on $\partial\Omega$. Using (4.3) yields that there exists $c > 0$ such that for any $p > 2$,

$$\|\nabla\varphi\|_{W^{1,p}(\Omega)} \leqslant cp(\|\operatorname{div} f\|_{L^p(\Omega)} + \|\phi\|_{W^{1,p}(\mathcal{W})}).$$

There exists only one smooth function ψ which satisfies $\Delta\psi = \operatorname{curl}(f - \phi\hat{n})$ in Ω, and $\psi = 0$ on $\partial\Omega$. Using (4.3) yields that there exists $c > 0$ such that for any $p > 2$,

$$\|\nabla\psi\|_{W^{1,p}(\Omega)} \leqslant cp(\|\operatorname{curl} f\|_{L^p(\Omega)} + \|\phi\|_{W^{1,p}(\mathcal{W})}).$$

Now let us observe that

$$(Id - \Pi)f = \phi\hat{n} + \nabla\varphi + \nabla^\perp\psi + \sum_{i=1}^d \beta_i\mathcal{X}_i,$$

with $\beta_i := -\|\mathcal{X}_i\|^{-2}_{L^2(\Omega)} \int_\Omega \phi\hat{n} \cdot \mathcal{X}_i dx$. Hence

$$\|(Id - \Pi)f\|_{W^{1,p}(\Omega)} \leqslant cp(\|\operatorname{div} f\|_{L^p(\Omega)} + \|\operatorname{curl} f\|_{L^p(\Omega)} + \|\phi\|_{W^{1,p}(\mathcal{W})}). \tag{4.4}$$

It only remains to estimate the harmonic part. It suffices to observe that

$$\Pi f = \sum_{i=1}^d (\int_\Omega \Pi f \cdot \tilde{\mathcal{X}}_i)\tilde{\mathcal{X}}_i,$$

to get (4.2), with a constant c_h (where h stands for harmonic) which depends only on Ω (including through the $\tilde{\mathcal{X}}_i$), but not on p. □

Another way to deal with the harmonic part is to consider the circulations, and for $1 \leqslant i \leqslant d$ the function ϕ_i in $C^\infty(\Omega)$ such that $\Delta\phi_i = 0$ in Ω, with $\phi_i = \delta_{i,j}$, on $\mathcal{C}_j$, for $j = 0, \dots, d$. Let us recall the following (cf. [15], [14]).

Lemma 8. *For any smooth vector field f from Ω to $\mathbb{R}^2$,*

$$\Pi f = \sum_{i=1}^{d} \alpha_i(f) \mathcal{X}_i, \tag{4.5}$$

where

$$\alpha_i(f) := \int_\Omega \phi_i \operatorname{curl} f + \gamma_i(f). \tag{4.6}$$

Proof. Thanks to Green's identity we get for $1 \leqslant i \leqslant d$,

$$\alpha_i(f) = -\int_\Omega \nabla^\perp \phi_i \cdot f.$$

In particular, this yields for $1 \leqslant i, j \leqslant d$,

$$\int_\Omega \nabla^\perp \phi_i \cdot \mathcal{X}_j = -\delta_{ij}. \tag{4.7}$$

Moreover $\nabla^\perp \phi_i \in H$, for $1 \leqslant i \leqslant d$, so that

$$\alpha_i(f) = -\int_\Omega \nabla^\perp \phi_i \cdot \Pi f. \tag{4.8}$$

Now let us look lor the coefficients $\beta_i(f)$ such that $\Pi f = \sum_{j=1}^{d} \beta_j(f) \mathcal{X}_j$. Plugging this into (4.8) and taking into account (4.7) we get $\beta_i(f) = \alpha_i(f)$, for $1 \leqslant i \leqslant d$, and therefore (4.6). □

In the sequel we will need the following consequence of Lemma 7 and Lemma 8.

Lemma 9. *There exists $c > 0$ (depending only on Ω) such that for any $p > 2$, for any smooth divergence free vector field f tangent to the boundary, there holds*

$$\|f\|_{W^{1,p}(\Omega)} \leqslant cp \| \operatorname{curl} f\|_{L^p(\Omega)} + c \sum_{i=1}^{d} |\gamma_i(f)|. \tag{4.9}$$

Proof. Thanks to Lemma 8 there exists $c > 0$ such that for any $p > 2$,

$$\|\Pi f\|_{L^p(\Omega)} \leqslant c \| \operatorname{curl} f\|_{L^p(\Omega)} + c \sum_{i=1}^{d} |\gamma_i(f)|.$$

Plugging this in (4.2) therefore yields (4.9). □

It is also useful to have in mind the following form of the Hölder inequality: for any integer k, for any $\theta := (s, \alpha)$ in

$$\mathcal{A}_k := \{\theta \in \mathbb{N}^* \times (\mathbb{N}^0)^s / \ 2 \leqslant s \leqslant k+1 \text{ and } \alpha := (\alpha_1, \ldots, \alpha_s) \in (\mathbb{N}^0)^s / \ |\alpha| = k+1-s\},$$

where the notation $|\alpha|$ stands for $|\alpha| := \alpha_1 + \ldots + \alpha_s$, and for any $p \geqslant 1$,

$$\| \prod_{i=1}^{s} f_i \|_{L^{\frac{p}{k+1}}(\Omega)} \leqslant \prod_{i=1}^{s} \|f_i\|_{L^{\frac{p}{\alpha_i+1}}(\Omega)}. \tag{4.10}$$

We will use some formal identities, obtained in [18], of the iterated material derivatives $(D^k u)_{k \in \mathbb{N}^*}$, where

$$D := \partial_t + u \cdot \nabla.$$

We use the following notations: for $\alpha := (\alpha_1, \ldots, \alpha_s) \in (\mathbb{N}^0)^s$ we will denote $\alpha! := \alpha_1! \ldots \alpha_s!$. We denote by $\operatorname{tr}\{A\}$ the trace of $A \in \mathcal{M}_2(\mathbb{R})$ and by $\operatorname{as}\{A\} := A - A^*$ the antisymmetric part of $A \in \mathcal{M}_2(\mathbb{R})$. We will also identify, for a smooth vector field $v : \omega \to \mathbb{R}^2$ its scalar vorticity $\operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1$ with the antisymmetric matrix $(\partial_i v_j - \partial_j v_i)_{1 \leqslant i,j \leqslant 2}$.

Lemma 10. *For $k \in \mathbb{N}^*$, we have in Ω*

$$\operatorname{div} D^k u = \operatorname{tr}\left\{F^k[u]\right\} \text{ where } F^k[u] := \sum_{\theta \in \mathcal{A}_k} c_k^1(\theta)\, f(\theta)[u], \tag{4.11}$$

$$\operatorname{curl} D^k u = \operatorname{as}\left\{G^k[u]\right\} \text{ where } G^k[u] := \sum_{\theta \in \mathcal{A}_k} c_k^2(\theta)\, f(\theta)[u], \tag{4.12}$$

and on the boundary $\partial\Omega$

$$\hat{n} \cdot D^k u = H^k[u] \text{ where } H^k[u] := \sum_{\theta \in \mathcal{A}_k} c_k^3(\theta)\, h(\theta)[u], \tag{4.13}$$

where

$$f(\theta)[u] := \nabla D^{\alpha_1} u \cdot \ldots \cdot \nabla D^{\alpha_s} u \text{ and } h(\theta)[u] := \nabla^s \rho\{D^{\alpha_1} u, \ldots, D^{\alpha_s} u\}, \tag{4.14}$$

and where, for $i = 1, 2$, the $c_k^i(\theta)$ are integers satisfying $|c_k^i(\theta)| \leqslant \frac{k!}{\alpha!}$, and the $c_k^3(\theta)$ are negative integers satisfying $|c_k^3(\theta)| \leqslant \frac{k!}{\alpha!(s-1)!}$.

Concerning the pressure it is possible to get by induction from (1.1) the following identities (cf. [19, Prop. 3.5]).

Lemma 11. *For $k \geqslant 1$, we have in Ω*

$$D^k u + \nabla D^{k-1} p = K^k[u] \tag{4.15}$$

where $K^1[u] = 0$ and for $k \geqslant 2$,

$$K^k[u] = -\sum_{s=1}^{k-1} \binom{k-1}{s} \nabla D^{s-1} u \cdot D^{k-s} u. \tag{4.16}$$

4.1 Going down the scale, slowly

Let

$$k_0 \in \mathbb{N}^*,\ r \in (0,1),\ p_1 \geqslant \frac{2}{1-r} \text{ and } p_2 \geqslant p_1 \cdot (k_0 + 1). \tag{4.17}$$

Let us introduce for $L > 0$ the following function

$$\gamma(L) := L^{-1} \sup_{k \geqslant 1} \left\{ 3 \sum_{s=2}^{k+1} s^M L^{2-s} c_\rho^s \left(\frac{k+1}{k-s+2} \right)^2 20^s + C_\Pi \sum_{s=1}^{k-1} (\frac{k-s}{ks})^M \left(\frac{k+1}{(k-s+1)s} \right)^2 \right\}, \tag{4.18}$$

where c_ρ (respectively $C_\Pi > 0$) is the constant introduced in (4.1) (resp. Lemma 6). We fix L large enough such that

$$\gamma(L) \leqslant \frac{1}{c_\Omega}, \tag{4.19}$$

where c_Ω is the maximum of 1 and of the constants c and c_h introduced in Lemma 7. We are going to prove recursively that for any integer $k \leqslant k_0$,

$$\|D^k u\|_{W^{1,\frac{p_2}{k+1}}(\Omega)} \leqslant p_2^k \frac{(k!)^M L^k}{(k+1)^2} \|u\|_{W^{1,p_2}(\Omega)}^{k+1}, \tag{4.20}$$

For $k = 0$ there is nothing to prove. Now let us assume that Eq. (4.20) is proved up to $k - 1 \leqslant k_0 - 1$.

4.1.1 Estimate of $F^k[u]$ and $G^k[u]$

Applying the Hölder inequality (4.10) to the definition of $f(\theta)[u]$ in (4.14), for $\theta \in \mathcal{A}_k$, yields that

$$\|f(\theta)[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant \prod_{i=1}^{s} \|D^{\alpha_i} u\|_{W^{1,\frac{p_2}{\alpha_i+1}}(\Omega)}.$$

Using the induction hypothesis and since for $\theta \in \mathcal{A}_k$, $|\alpha| = k+1-s$, we have

$$\|f(\theta)[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant L^k \|u\|^{k+1}_{W^{1,p_2}(\Omega)} (\alpha!)^M L^{1-s} p_2^{|\alpha|} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}.$$

Now thanks to Lemma 10, we obtain

$$\|F^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant k! L^k \|u\|^{k+1}_{W^{1,p_2}(\Omega)} \sum_{s=2}^{k+1} L^{1-s} \sum_{\alpha/\, |\alpha|=k+1-s} (\alpha!)^{M-1} p_2^{|\alpha|} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}.$$

When $\theta \in \mathcal{A}_k$, $2 \leqslant s \leqslant k+1$ and $|\alpha| = k+1-s$, then $|\alpha| \leqslant k-1$ so that

$$\|F^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant p_2^{k-1} (k!)^M L^k \|u\|^{k+1}_{W^{1,p_2}(\Omega)} \sum_{s=2}^{k+1} \frac{L^{1-s}}{k^{M-1}} \sum_{\alpha/\, |\alpha|=k+1-s} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}. \tag{4.21}$$

We now use the following combinatorial lemma (cf. [8, Lemma 7.3.3]).

Lemma 12. *For any couple of positive integers (s, m) we have*

$$\sum_{\substack{\alpha \in (\mathbb{N}^0)^s \\ |\alpha|=m}} \Upsilon(s,\alpha) \leqslant \frac{20^s}{(m+1)^2}, \ \textit{where } \Upsilon(s,\alpha) := \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}. \tag{4.22}$$

We deduce from (4.21) and from the above lemma that

$$\|F^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant p_2^{k-1} \frac{(k!)^M L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2}(\Omega)} \sum_{s=2}^{k+1} \frac{L^{1-s}}{k^{M-1}}\, 20^s \, \frac{(k+1)^2}{(k-s+2)^2}. \tag{4.23}$$

We have the same bound on $\|G^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)}$ using (4.12) instead of (4.11).

4.1.2 Estimate of $H^k[u]$

Applying (4.10) and using (4.1) yields, for $\theta \in \mathcal{A}_k$, that

$$\|h(\theta)[u]\|_{W^{1,\frac{p_2}{k+1}}(\mathcal{W})} \leqslant c_\rho^s \,(s!)^M \prod_{i=1}^{s} \|D^{\alpha_i} u\|_{W^{1,\frac{p_2}{\alpha_i+1}}(\Omega)}.$$

By using the induction hypothesis, we have

$$\|h(\theta)[u]\|_{W^{1,\frac{p_2}{k+1}}(\mathcal{W})} \leqslant L^k \|u\|^{k+1}_{W^{1,p_2}(\Omega)} (\alpha!)^M (s!)^M L^{1-s} c_\rho^s p_2^{|\alpha|} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}.$$

Thanks to Lemma 10 and Lemma 12 we obtain

$$\|H^k[u]\|_{W^{1,\frac{p_2}{k+1}}(\mathcal{W})} \leqslant p_2^{k-1} \frac{(k!)^M L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2}(\Omega)} \sum_{s=2}^{k+1} s^M L^{1-s}\, c_\rho^s \, 20^s \, \frac{(k+1)^2}{(k-s+2)^2}. \tag{4.24}$$

4.1.3 Estimate of $K^k[u]$

Applying the Hölder inequality (4.10) to the definition (4.16) of $K^k[u]$ yields for any $k \geqslant 2$,

$$\|K^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant \sum_{s=1}^{k-1} \binom{k-1}{s} \|D^{s-1}u\|_{W^{1,\frac{p_2}{s}}(\Omega)} \|D^{k-s}u\|_{W^{1,\frac{p_2}{k+1-s}}(\Omega)}.$$

By using the induction hypothesis we get

$$\|K^k[u]\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant p_2^{k-1} \frac{(k!)^M L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2}(\Omega)} L^{-1} \sum_{s=1}^{k-1} (\frac{k-s}{ks})^M \left(\frac{k+1}{s(k-s+1)} \right)^2.$$

Finally using Lemma 11 and Theorem 12 we have $\Pi D^k u = \Pi K^k[u]$ so that, thanks to Lemma 6, we get

$$\|\Pi D^k u\|_{L^{\frac{p_2}{k+1}}(\Omega)} \leqslant C_\Pi p_2^{k-1} \frac{(k!)^M L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2}(\Omega)} L^{-1} \sum_{s=1}^{k-1} (\frac{k-s}{ks})^M \left(\frac{k+1}{s(k-s+1)} \right)^2. \tag{4.25}$$

4.1.4 Conclusion

We now apply Lemma 7 to $f = D^k u$ (observing that, thanks to (4.17), we have $\frac{p_2}{k+1} > 2$) and we use (4.18)-(4.19)-(4.23)-(4.24)-(4.25) to get Eq. (4.20) at rank k.

4.2 Going down the scale, from high enough

We now apply Eq. (4.20) with $p_2(k+1)$ instead of p_2, and we use Stirling's formula to obtain that for any $k \in \mathbb{N}^0$, for any $p_2 \geqslant \frac{2}{1-r}$,

$$\|D^k u\|_{W^{1,p_2}(\Omega)} \leqslant (p_2(k+1))^k \frac{(k!)^M L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2(k+1)}(\Omega)} \leqslant p_2^k (k!)^{M+1} \tilde{L}^k \|u\|^{k+1}_{W^{1,p_2(k+1)}(\Omega)}. \tag{4.26}$$

So far time has intervened only as a parameter, and the inequality (4.26) holds for any time. We will now estimate its right hand side with respect to the initial data. First thanks to (4.9) there exists $c > 0$ such that for any k,

$$\|u\|_{W^{1,p_2(k+1)}(\Omega)} \leqslant c p_2(k+1) \|\operatorname{curl} u\|_{L^{p_2(k+1)}(\Omega)} + c \sum_{i=1}^{d} |\gamma_i(u)|.$$

Now conservation of the L^p norms of the vorticity and Kelvin's circulation theorem yields

$$\begin{aligned} \|u\|_{W^{1,p_2(k+1)}(\Omega)} &\leqslant c p_2(k+1) \|\omega_0\|_{L^{p_2(k+1)}(\Omega)} + c \sum_{i=1}^{d} |\gamma_i(u_0)|, \\ &\leqslant c p_2(k+1) \theta(p_2(k+1)) + c \sum_{i=1}^{d} |\gamma_i(u_0)|, \end{aligned}$$

since u_0 in $\mathbb{Y}_\theta$. Plugging this into (4.26) and using again Stirling's formula, we obtain that there exists $L > 0$ depending only on Ω such that for any k,

$$\|D^k u\|_{W^{1,p_2}(\Omega)} \leqslant p_2^k (k!)^{M+1} L^{k+1} \big(k! p_2^{k+1} \theta(p_2(k+1))^{k+1} + \sum_{i=1}^{d} |\gamma_i(u_0)|^{k+1} \big).$$

Thanks to Morrey's inequality, there exists $C > 0$ such that for any smooth function u on $\overline{\Omega}$, for any $r \in (0,1)$, $\|f\|_{C^{0,r}(\Omega)} \leqslant C \|f\|_{W^{1,p_2}(\Omega)}$, where $p_2 = 2/(1-r)$. This allows to bound $\|D^k u\|_{C^{0,r}(\Omega)}$ thanks to $\|D^k u\|_{W^{1,p_2}(\Omega)}$. Then we differentiate Eq. (1.2) to get $\partial_t^{k+1} \Phi(t,x) = D^k u(t, \Phi(t,x))$. We consider $T > 0$ and we use Lemma 1 to get the estimate (3.1).

4.3 A regularization process

So far we were considering a smooth flow (cf. the beginning of the section 4). Let us now explain how to deduce the estimate (3.1) for the irregular flows tackled in Theorem 5: let us consider an initial data u_0 in $\mathbb{Y}_\theta$, where θ is an admissible germ. By convolution we obtain a sequence $(u_0^n)_{n\in\mathbb{N}_0}$ of smooth vector fields in $\mathbb{Y}_\theta$ and such that $(u_0^n)_{n\in\mathbb{N}_0}$ converges to u_0 in $L^2(\Omega)$. This implies in particular that the circulations $\gamma_i(u_0^n)$, for $i=1,...,d$, converge respectively to $\gamma_i(u_0)$. Now let us consider P the Leray projector associated to the domain Ω, that is the orthogonal projector from $L^2(\Omega)$ onto the vector fields of $L^2(\Omega)$ which are divergence free and tangent to the boundary. It is classical that the Leray projector is continuous for the L^2 topology and that the difference between a function u in $L^2(\Omega)$ and its Leray projection Pu is a gradient. Therefore the sequence $(Pu_0^n)_{n\in\mathbb{N}_0}$ also lies in $\mathbb{Y}_\theta$. Moreover it is classical (cf. [28], [12]) that these regularized initial data $(u_0^n)_n$ launch some global and unique respective smooth solutions of (1.1). Then the previous analysis holds for these flows, in particular the estimate (3.1) holds true for the flow maps Φ^n respectively associated to u_0^n. Passing to the limit for $n\to+\infty$ in these estimates concludes the proof of Theorem 5.

5 Proof of Theorem 6

In order to obtain the propagation of local smoothness we will use some interior elliptic regularity, instead of Lemma 7 and 9. Let us first recall the following Schauder estimate (cf. [11]).

Lemma 13. *Let $\mathcal{D}$ be an open set such that $\overline{\mathcal{D}}\subset\Omega$. Let u be a continuous vector field on $\overline{\Omega}$ such that* $\operatorname{div} u=0$ *in Ω, $\hat{n}\cdot u=0$ on $\partial\Omega$ and* $\operatorname{curl} u|_{\mathcal{D}}$ *in $C^{\lambda,r}_{loc}(\mathcal{D})$. Then u is in $C^{\lambda+1,r}_{loc}(\mathcal{D})$.*

Lemma 13 extends as follows to Dini continuous vorticities.

Lemma 14. *Let $\mathcal{D}$ be an open set such that $\overline{\mathcal{D}}\subset\Omega$. Let u be a continuous vector field on $\overline{\Omega}$ such that* $\operatorname{div} u=0$ *in Ω, $\hat{n}\cdot u=0$ on $\partial\Omega$ and* $\operatorname{curl} u|_{\mathcal{D}}$ *in $C_{\mu,loc}(\mathcal{D})$, with μ is a Dini modulus of continuity. Then u is in $C^1(\mathcal{D})$.*

Above the space $C_{\mu,\mathrm{loc}}(\Omega_0)$ denotes the set of functions which are in $C_\mu(K)$ for any compact subset $K\subset\Omega_0$. We provide a proof for sake of completeness.

Proof. Let us introduce Ψ_0 the unique solution of $\Delta\Psi_0=\operatorname{curl} u$ in Ω and $\Psi_0=0$ on $\partial\Omega$. We then denote $v=\nabla^\perp\Psi_0$ and observe that $u-v$ is in the space H of tangential harmonic vector fields of Ω. Next we introduce

$$\Psi(x):=\frac{1}{2\pi}\int_{\mathcal{D}}\log\|x-y\|\cdot(\operatorname{curl} u)(y)dy,$$

which is in $C^2(\mathcal{D})$ and satisfies $\Delta\Psi=\operatorname{curl} u$ in $\mathcal{D}$, according to Lemma 16 in Appendix A. It suffices to observe that $\Psi-\Psi_0$ is harmonic in $\mathcal{D}$ to conclude the proof. □

We are now equipped to start the proof of Theorem 6. Let us recall that we assume that the boundary $\partial\Omega$ of the domain is C^2, that the initial data u_0 is in $\mathbb{Y}_{\theta_m}$, with $m\in\mathbb{N}^*$, that Ω_0 is a open subset such that $\overline{\Omega_0}\subset\Omega$ and that $\omega_0|_{\Omega_0}$ is in $C^{\lambda,r}_{\mathrm{loc}}(\Omega_0)$ with λ in $\mathbb{N}^0$ and $r\in(0,1)$. Since $\Phi(t,.)^{-1}$ satisfies (1.2) with $-u$ instead of u, arguing as in Lemma 3, we have that $\Phi(t,.)^{-1}$ is in $C^0([0,T],C_{\Gamma_t}(\Omega))$. Let us denote $\Omega_t:=\{\Phi(t,x),\ x\in\Omega_0\}$. Proceeding as in Lemma 1, we get that $\omega(t,\cdot):=\omega_0(\Phi(t,.)^{-1})\in C^0([0,T],C_{\Gamma_t^r,\mathrm{loc}}(\Omega_t))$ (this notation is slightly improper but does not lead here to any confusion). Now thanks to Lemma 5, the modulus of continuity Γ_t^r is Dini. Applying Lemma 14 yields that u is in $C^0([0,T],C^1(\Omega_t))$. By integration, we infer that Φ and $t\mapsto\Phi(t,.)^{-1}$ are in $C^0([0,T],C^1(\Omega_0))$. Proceeding again as in Lemma 1, we get that $\omega(t,\cdot)\in C^{0,r}_{\mathrm{loc}}(\Omega_t)$. Then Lemma 13 yields that $u(t,\cdot)$ is in $C^{1,r}_{\mathrm{loc}}(\Omega_t)$. We can now repeat the bootstrapping arguments exactly as in Proposition 8.3 of [20] to end the proof.

6 Proof of Theorem 7

This section is devoted to the proof of Theorem 7. Let us start by repeating two preliminary remarks of the proof of Theorem 5. First we will focus on the Gevrey case, the C^∞ case being a byproduct of the analysis. Secondly we will work from now on a smooth flow, the result following by a regularization argument.

6.1 Shrinking the compact set

Let us first introduce a notation: when K is a compact set and $\varepsilon > 0$ we denote $K_\varepsilon := \{x \in K / \ \text{dist}\,(x, K^c) \geqslant \varepsilon\}$. Let us also introduce for $L > 0$ the following function

$$\tilde{\gamma}(L) := 2L^{-1} \sup_{k\geqslant 1} \sum_{s=2}^{k+1} k^{-M} L^{2-s} \left(\frac{k+1}{k-s+2}\right)^2 20^s. \tag{6.1}$$

We fix L large enough (depending on r) such that for any integer $k \geqslant 1$,

$$\|D^k u\|_{C^{0,r}(\Omega)} \leqslant c_\Omega\ \gamma(L) \frac{(k!)^{M+1} L^k}{(k+1)^2} \|u\|^{k+1}_{W^{1,p_2(k+1)}(\Omega)} \tag{6.2}$$

with $p_2 := \frac{2}{1-r}$ (what is possible according to the analysis of section 4.2) and such that

$$c_\Omega\, c_i^{\lambda_0+1}(\gamma(L) + \tilde{\gamma}(L)) \leqslant 1, \tag{6.3}$$

where c_i will appear in Lemma 15.

Let $k_0 \in \mathbb{N}^*$ and $\varepsilon > 0$ such that $\ \text{diam}\, K > k_0\varepsilon$. We are going to prove recursively for any integer $1 \leqslant \lambda \leqslant \lambda_0 + 1$, and then recursively for any integer k such that $1 \leqslant k \leqslant k_0$ that

$$\|D^k u\|_{C^{\lambda,r}(K_{k\varepsilon})} \leqslant c_\Omega\, c_i^\lambda (\gamma(L) + \tilde{\gamma}(L)) \frac{(k!)^{M+1} L^k \varepsilon^{-k\lambda(1+r)}}{(k+1)^2} N^{k+1}_{\lambda,r,K}, \tag{6.4}$$

where

$$N_{\lambda,r,K} := \|u\|_{C^{\lambda,r}(K)} + \|u\|_{W^{1,p_2(k+1)}(\Omega)}.$$

Let us assume that Eq. (6.4) is proved up to $k - 1 \leqslant k_0 - 1$.

Looking forward to the definition of $f(\theta)[u]$ in (4.14), we have that, for $\theta \in \mathcal{A}_k$,

$$\|f(\theta)[u]\|_{C^{\lambda-1,r}(K_{(k-1)\varepsilon})} \leqslant \prod_{i=1}^{s} \|D^{\alpha_i} u\|_{C^{\lambda,r}(K_{(k-1)\varepsilon})} \leqslant \prod_{i=1}^{s} \|D^{\alpha_i} u\|_{C^{\lambda,r}(K_{\alpha_i\varepsilon})}.$$

Using the induction hypothesis, that (6.3) and $c_i > 1$ imply $c_\Omega\, c_i^\lambda(\gamma(L) + \tilde{\gamma}(L)) \leqslant 1$, we therefore obtain:

$$\|f(\theta)[u]\|_{C^{\lambda-1,r}(K_{(k-1)\varepsilon})} \leqslant \varepsilon^{-(k-1)\lambda(1+r)} L^k N^{k+1}_{\lambda,r,K} (\alpha!)^{M+1} L^{1-s} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}.$$

Now thanks to Lemma 10, we obtain

$$\|F^k[u]\|_{C^{\lambda-1,r}(K_{(k-1)\varepsilon})} \leqslant (k!)^{M+1} \varepsilon^{-(k-1)\lambda(1+r)} L^k N^{k+1}_{\lambda,r,K} \sum_{s=2}^{k+1} \frac{L^{1-s}}{k^M} \sum_{\alpha/\ |\alpha|=k+1-s} \prod_{i=1}^{s} \frac{1}{(1+\alpha_i)^2}.$$

Using Lemma 12 we obtain

$$\|F^k[u]\|_{C^{\lambda-1,r}(K_{(k-1)\varepsilon})} \leqslant \frac{(k!)^{M+1} L^k}{(k+1)^2} \varepsilon^{-(k-1)\lambda(1+r)} N^{k+1}_{\lambda,r,K} \sum_{s=2}^{k+1} \frac{L^{1-s}}{k^M}\, 20^s\, \frac{(k+1)^2}{(k-s+2)^2}.$$

We have the same bound on $\|G^k[u]\|_{C^{0,r}(K_{(k-1)\varepsilon})}$.

In order to obtain (6.4) it then suffices to apply the following lemma to $f = D^k u$ and $\tilde{\varepsilon} = (k-1)\varepsilon$ using that (6.3) implies $c_i\tilde{\gamma}(L) \leqslant 1$, the inequality (6.3), and the inequality (6.2) (respectively the inequality (6.4) with $\lambda - 1$ instead of λ) if $\lambda = 1$ (resp. if $\lambda > 1$).

Lemma 15. *There exists $c_i > 1$ such that for any $\varepsilon, \tilde{\varepsilon}$ in $(0,1)$, for any $f \in C^{\lambda-1,r}(K_{\tilde{\varepsilon}})$, such that $\operatorname{div} f$ and $\operatorname{curl} f$ are also in $C^{\lambda-1,r}(K_{\tilde{\varepsilon}})$, then $f \in C^{\lambda,r}(K_{\varepsilon+\tilde{\varepsilon}})$ and*

$$\|f\|_{C^{\lambda,r}(K_{\varepsilon+\tilde{\varepsilon}})} \leqslant c_i\, \varepsilon^{-(1+r)} (\|f\|_{C^{\lambda-1,r}(K_{\tilde{\varepsilon}})} + \|\operatorname{curl} f\|_{C^{\lambda-1,r}(K_{\tilde{\varepsilon}})} + \|\operatorname{div} f\|_{C^{\lambda-1,r}(K_{\tilde{\varepsilon}})}). \tag{6.5}$$

Proof. Let us first recall that there exists $C_1 > 0$, which only depends on r, such that for any v in $C^{0,r}(\mathbb{R}^2)$ such that $\operatorname{div} v$ and $\operatorname{curl} v$ are also in $C^{0,r}(\mathbb{R}^2)$, then $v \in C^{1,r}(\mathbb{R}^2)$ and

$$\|v\|_{C^{1,r}(\mathbb{R}^2)} \leqslant C_1(\|v\|_{C^{0,r}(\mathbb{R}^2)} + \|\operatorname{curl} v\|_{C^{0,r}(\mathbb{R}^2)} + \|\operatorname{div} v\|_{C^{0,r}(\mathbb{R}^2)}). \tag{6.6}$$

On the other hand there exists $C_2 > 0$, which only depends on r, such that for any $\varepsilon, \tilde{\varepsilon} \in (0,1)$, there exists $\phi \in C^\infty(\mathbb{R}^2)$ such that $\phi|_{K^c_{\tilde{\varepsilon}}} = 0$ and $\phi|_{K_{\varepsilon+\tilde{\varepsilon}}} = 1$ and $\|\phi\|_{C^{1,r}(\mathbb{R}^2)} \leqslant C_2\, \varepsilon^{-(1+r)}$. Thus it is sufficient to apply (6.6) to the function $v := \phi\, \partial^\alpha f$, for $|\alpha| = \lambda - 1$ to conclude. □

6.2 Shrinking the compact set, slower

Let K be a compact subset of Ω_0. Let $\underline{K}$ be a compact set such that $K \subset \mathring{\underline{K}} \subset \underline{K} \subset \Omega_0$. Then we apply (6.4) with $\lambda = \lambda_0 + 1$, $\varepsilon :=$ dist $(K, \underline{K}^c)/k$, with $\underline{K}$ instead of K, and using the inequality (6.3), to obtain that there exists $L > 0$ such that for any integer $k \in \mathbb{N}^0$,

$$\|D^k u\|_{C^{\lambda_0+1,r}(K)} \leqslant (k!)^{M+1+(\lambda_0+1)(1+r)} L^k N^{k+1}_{\lambda_0+1,r,\underline{K}}. \tag{6.7}$$

We then conclude as in the proof of Theorem 5.

7 Proof of Theorem 11

Let $M > 1$ and $T > 0$. There exists a compact set K' such that $K \subset K' \subset \Omega$ such that for any t in $[0,T]$, the vorticity is constant outside of K'. There exists $\chi : \Omega \to [0,1]$ Gevrey of order M which vanishes in a neighborhood of the boundary $\partial\Omega$ and which is equal to one on K'. We consider the vector field

$$D_\chi := \partial_t + \chi u \cdot \nabla.$$

The idea is then to proceed as in section 4 estimating recursively the $D^k_\chi u$, for k in $\mathbb{N}^*$, instead of the $D^k u$. The motivation for introducing the cut-off χ is that the identity (4.13) becomes: on the boundary $\partial\Omega$, for k in $\mathbb{N}^*$, $\hat{n} \cdot D^k_\chi u = 0$. Moreover the circulations $\gamma_i(D^k_\chi u)$, for $1 \leqslant i \leqslant d$ and $k \geqslant 1$, vanish. The quantities $\operatorname{div} D^k_\chi u$ and the $\operatorname{curl} D^k_\chi u$ can be estimated thanks to the $D^j_\chi u$, with $j < k$, and with some extra factors involving χ and its derivatives, by using the following identities:

$$D_\chi(\psi_1\psi_2) = (D_\chi\psi_1)\psi_2 + \psi_1(D_\chi\psi_2), \tag{7.1}$$
$$\nabla(D_\chi\psi) - D_\chi(\nabla\psi) = (\nabla(\chi u)) \cdot (\nabla\psi), \tag{7.2}$$
$$\operatorname{div} D_\chi\psi - D_\chi \operatorname{div}\psi = \operatorname{tr}\{(\nabla(\chi u)) \cdot (\nabla\psi)\}, \tag{7.3}$$
$$\operatorname{curl} D_\chi\psi - D_\chi \operatorname{curl}\psi = \operatorname{as}\{(\nabla(\chi u)) \cdot (\nabla\psi)\}. \tag{7.4}$$

The assumption that the vorticity is constant near the boundary is useful to tackle the $\operatorname{curl} D^k_\chi u$. Let us stress in particular that

$$\begin{aligned}
\operatorname{curl} D_\chi u &= D_\chi \operatorname{curl} u + \operatorname{as}\{(\nabla(\chi u)) \cdot (\nabla u)\}, \\
&= D \operatorname{curl} u + \operatorname{as}\{(\nabla(\chi u)) \cdot (\nabla u)\}, \\
&= \operatorname{as}\{(\nabla(\chi u)) \cdot (\nabla u)\}.
\end{aligned}$$

The proof of Theorem 11 then goes as in the proof of Theorem 5. The details are left to the reader.

APPENDIX A

Let $\mathcal{D}$ be an open and bounded subset of $\mathbb{R}^2$. Let us denote by

$$\Gamma(x) := \frac{1}{2\pi} \log \|x\|$$

the fundamental solution of the Poisson problem in $\mathbb{R}^2$, and for a given function $f \in L^\infty(\mathcal{D})$ by

$$\Psi(x) := (\Gamma * f)(x) = \int_{\mathcal{D}} \Gamma(x-y) f(y) dy$$

the Newton potential of f. It is well-known (cf. for instance [11], Lemma 4.1) that $\Psi \in C^1(\mathbb{R}^2)$ with

$$\nabla\Psi(x) = \int_{\mathcal{D}} (\nabla\Gamma)(x-y)f(y)dy.$$

It is well-known (cf. for example [11], Exercice 4.1) that in general $f \in C^0(\mathcal{D})$ does not imply that $\Psi \in C^2(\mathcal{D})$. However we have the following result by Burch [5].

Lemma 16. *If $f \in C_{\mu,loc}(\mathcal{D})$ with μ a Dini modulus of continuity, then $\Psi \in C^2(\mathcal{D})$ and $\Delta\Psi = f$ in $\mathcal{D}$.*

Proof. Let $\mathcal{D}_0 \supset \mathcal{D}$ be a bounded open set with smooth boundary $\partial\mathcal{D}_0$, and for $1 \leqslant i, j \leqslant 2$, let

$$u_{ij}(x) := \int_{\mathcal{D}_0} (\partial_{ij}\Gamma)(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\mathcal{D}_0} (\partial_i\Gamma)(x-y)\nu_j(y)ds(y),$$

where ν is the outward normal unit to $\partial\mathcal{D}_0$, and where f is extended by zero outside $\mathcal{D}$. The function u_{ij} is well-defined for $x \in \mathcal{D}$: the integrands are smooth except when y is in a neighborhood of x in the first integral, say in an open ball $B(x,R)$ such that $\overline{B(x,R)} \subset \mathcal{D}$. Since Γ satisfies the bound $|\partial_{ij}\Gamma(x)| \leqslant \frac{1}{2\pi\|x\|^2}$ and since $f \in C_{\mu,\mathrm{loc}}(\mathcal{D})$, the contribution of the ball $B(x,R)$ to the integral is $\int_0^R \frac{\mu(r)}{r}dr < +\infty$, since μ is a Dini modulus of continuity.

Let $\eta \in C^\infty([0,\infty))$ satisfying $\eta(s) = 0$ for $0 \leqslant s \leqslant 1$, $0 \leqslant \eta'(s) \leqslant 2$ for $1 \leqslant s \leqslant 2$ and $\eta(s) = 1$ for $s \geqslant 2$. For $i = 1, 2$, the functions

$$v_{i,\varepsilon}(x) := \int_{\mathcal{D}} \Gamma_{i,\varepsilon}(x-y)f(y)dy, \text{ where } \Gamma_{i,\varepsilon}(x) := (\partial_i\Gamma)(x)\eta(\|x\|/\varepsilon),$$

converges uniformly on the compact subsets of $\mathcal{D}$ to $\partial_i\Psi$ when $\varepsilon > 0$ tends to 0, since for any $x \in \mathcal{D}$ and for $2\varepsilon \in (0, d(x,\partial\mathcal{D}))$,

$$v_{i,\varepsilon}(x) - \partial_i\Psi(x) = \int_{B(x,2\varepsilon)} (\partial_i\Gamma)(x-y)(\eta(\|x-y\|/\varepsilon)-1)f(y)dy$$

hence

$$|v_{i,\varepsilon}(x) - \partial_i\Psi(x)| \leqslant \int_{B(x,2\varepsilon)} \frac{1}{2\pi\|x-y\|}2|f(y)|dy \leqslant 2\varepsilon\|f\|_{L^\infty(\mathcal{D})}.$$

Now, for any $x \in \mathcal{D}$,

$$\partial_j v_{i,\varepsilon}(x) = \int_{\mathcal{D}_0} (\partial_j\Gamma_{i,\varepsilon})(x-y)(f(y)-f(x))dy + f(x)\int_{\mathcal{D}_0} (\partial_j\Gamma_{i,\varepsilon})(x-y)dy.$$

Moreover, thanks to Green's identity, we have

$$\begin{aligned}\int_{\mathcal{D}_0} (\partial_j\Gamma_{i,\varepsilon})(x-y)dy &= -\int_{\partial\mathcal{D}_0} \Gamma_{i,\varepsilon}(x-y)\nu_j(y)ds(y) \\ &= -\int_{\partial\mathcal{D}_0} (\partial_i\Gamma)(x-y)\nu_j(y)ds(y)\end{aligned}$$

for $2\varepsilon \in (0, d(x,\partial\mathcal{D}))$. Therefore

$$\partial_j v_{i,\varepsilon}(x) = \int_{\mathcal{D}_0} (\partial_j\Gamma_{i,\varepsilon})(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\mathcal{D}_0} (\partial_i\Gamma)(x-y)\nu_j(y)ds(y).$$

Then

$$u_{ij}(x) - \partial_j v_{i,\varepsilon}(x) = \int_{B(x,2\varepsilon)} \tilde{\Gamma}_{ij,\varepsilon}(x-y)(f(y)-f(x))dy,$$

with

$$\tilde{\Gamma}_{ij,\varepsilon}(x) := \partial_{ij}\Gamma(x)(1-\eta(\|x\|/\varepsilon)) - \partial_i\Gamma(x)\eta'(\|x\|/\varepsilon)\frac{x_j}{\varepsilon\|x\|}.$$

Since $|\tilde{\Gamma}_{ij,\varepsilon}(x)| \leqslant \frac{1}{\pi}(\frac{1}{\|x\|^2} + \frac{1}{\varepsilon\|x\|})$, we obtain

$$|u_{ij}(x) - \partial_j v_{i,\varepsilon}(x)| \leqslant 6\int_0^{2\varepsilon} \frac{\mu(r)}{r}dr$$

which tends to 0, since μ is a Dini modulus of continuity. We therefore have shown that $\partial_j v_{i,\varepsilon}$ converges to u_{ij} when ε tends to 0 uniformly on the compact subsets of $\mathcal{D}$. Therefore $\Psi \in C^2(\mathcal{D})$ and $\partial_{ij}\Psi = u_{ij}$. It is then sufficient to use that $\Delta\Gamma = \delta_0$ and Green's identity to get $\Delta\Psi = f$ in $\mathcal{D}$. □

APPENDIX B

The goal of this appendix is to provide an explicit proof of Theorem 10. In particular we will show how the motions of isolated point vortices can be considered as weak solutions of the Euler equations, thanks to an appropriated weak vorticity formulation of the Euler equations for multiply connected domains. Since the trajectories of the point vortices are analytic (up to the first collision) this will provide some examples of very singular solutions of the Euler equations for which the flow restricted to the finite collection of the initial positions of the vortices is analytic, despite the boundary of the domain is not analytic. We only assume here that the boundary $\partial\Omega$ of the domain Ω is C^2. The study of the motion of isolated vortices goes back to Helmholtz, Kirchoff, Routh, and to Lin [15] in the case of multiply connected domains that will be considered here. Let us first recall the existence of the hydrodynamic Green function.

Lemma 17. *There exists an unique function $G : (x,y) \in \overline{\Omega} \times \Omega \mapsto G(x,y) \in \mathbb{R}$ satisfying the following properties:*

(i). The function

$$g(x,y) := G(x,y) - \frac{1}{2\pi} \log \|x-y\| \tag{7.5}$$

is harmonic with respect to x on Ω, for any $y \in \Omega$.

(ii). For $1 \leqslant l \leqslant d$, for $y \in \Omega$, the function $G(\cdot, y)$ is constant when x ranges over $\mathcal{C}_l$.

(iii). The function G vanishes over the outer boundary $\mathcal{C}_0$: for $x \in \mathcal{C}_0$, for $y \in \Omega$, $G(x,y) = 0$.

(iv). For $1 \leqslant l \leqslant d$, for $y \in \Omega$, the circulation around $\mathcal{C}_l$ of $\nabla^\perp G(\cdot, y)$ vanishes: $\gamma_l(\nabla^\perp G(\cdot,y)) = 0$.

Moreover G satisfies the reciprocity-symmetry relation: for any $(x,y) \in \Omega \times \Omega$,

$$G(x,y) = G(y,x). \tag{7.6}$$

Proof. We will use again for $1 \leqslant i \leqslant d$ the function ϕ_i in $C^\infty(\Omega)$ such that $\Delta \phi_i = 0$ in Ω, with $\phi_i = \delta_{i,j}$, on $\mathcal{C}_j$, for $j = 0, \dots, d$. We introduce the matrix $M := (m_{i,j})_{1 \leqslant i,j \leqslant d}$ with $m_{i,j} := \gamma_i(\nabla^\perp \phi_j)$. Let us also recall that Green's identity yields for any smooth vector field f from Ω to $\mathbb{R}^2$,

$$\int_\Omega \phi_i \operatorname{curl} f + \gamma_i(f) = -\int_\Omega \nabla^\perp \phi_i \cdot f. \tag{7.7}$$

This yields in particular $m_{i,j} = -\int_\Omega \nabla^\perp \phi_i \cdot \nabla^\perp \phi_j$. Therefore the matrix $M := (m_{i,j})_{1 \leqslant i,j \leqslant d}$ is symmetric definite negative. Let us denote $p_{i,j}$ the entries of its inverse M^{-1}. Let us denote by $G_0(x,y)$ the Green's function associated to the Dirichlet condition. We then set

$$G(x,y) := G_0(x,y) + \sum_{1 \leqslant i,j \leqslant d} p_{i,j} \phi_i(x) \phi_j(y).$$

The conditions (i), (ii), (iii) and the reciprocity-symmetry relation (7.6) are therefore satisfied. Now (7.7) also applies to $f := \nabla^\perp G_0$ and we get for $1 \leqslant l \leqslant d$, for $y \in \Omega$,

$$\gamma_l(\nabla^\perp G_0(\cdot,y)) = -\phi_l(y), \text{ and } \gamma_l\Big(\sum_{1 \leqslant i,j \leqslant d} p_{i,j}\phi_i(\cdot)\phi_j(y)\Big) = \sum_{1 \leqslant i,j \leqslant d} p_{i,j} m_{l,i} \phi_j(y) = \phi_l(y),$$

so that

$$\gamma_l(\nabla^\perp G(\cdot,y)) = 0.$$

The condition (iv) is therefore satisfied. Let us now prove the uniqueness: assume that two functions G_1 and G_2 satisfy the properties (i),..., (iv) then for any $y \in \Omega$, Green's identity yields that $\int_\Omega \|\nabla(G_1 - G_2)\|^2 = 0$, so that $G_1 = G_2$ since they both vanish on $\mathcal{C}_0$. □

We now consider N vortices of respective strength $\alpha_i \in \mathbb{R}^*$, for $1 \leqslant i \leqslant N$, located at N distinct points of Ω and we prescribe some real $\overline{\gamma}_i$ as respective circulations on the inner boundaries $\mathcal{C}_i$, for $1 \leqslant i \leqslant d$. We deduce from Lemma 17 that there exists only one corresponding stream function.

Lemma 18. *Let be given*

$$N \in \mathbb{N}^*,\ \overline{\gamma} := (\overline{\gamma}_l)_{1 \leqslant l \leqslant d} \in \mathbb{R}^d,\ \alpha := (\alpha_l)_{1 \leqslant l \leqslant N} \in \mathbb{R}^N$$

and

$$\overline{x} := (x_1, ..., x_N) \in \Omega_N := \{(x_1, ..., x_N) \in \Omega / \quad x_i \neq x_j \text{ for } 1 \leqslant i \neq j \leqslant N\}.$$

Then there exists a unique function $\psi : \Omega \mapsto \mathbb{R}$ such that

(i). The function

$$\psi - \sum_{i=1}^{N} \frac{\alpha_i}{2\pi} \log \| \cdot - x_i \|$$

is harmonic in Ω.

(ii). For $1 \leqslant l \leqslant d$, the function ψ is constant when x ranges over $\mathcal{C}_l$.

(iii). The function ψ vanishes over the outer boundary $\mathcal{C}_0$: for $x \in \mathcal{C}_0$, $\psi(x) = 0$.

(iv). For $1 \leqslant l \leqslant d$, the circulation around $\mathcal{C}_l$ of $\nabla^\perp \psi$ is $\gamma_l(\nabla^\perp \psi) = \overline{\gamma}_l$.

Moreover

$$\psi := \sum_{l=1}^{N} \alpha_l G(\cdot, x_l) + \psi_0 \text{ with } \psi_0 := \sum_{1 \leqslant i,j \leqslant d} \overline{\gamma}_i p_{i,j} \phi_j. \tag{7.8}$$

The total kinetic energy $\int_\Omega \|\nabla \psi\|^2$ of the flow is infinite (except if all the α_l vanish) so that no information can be derived from its conservation. Nevertheless there exists the following substitute.

Definition 8. *We define the Kirchoff-Routh-Lin function (for $\overline{x} := (x_1, ..., x_N) \in \Omega_N$) by*

$$W(\overline{x}) := \sum_{1 \leqslant l \leqslant N} \alpha_l \psi_0(x_l) + \frac{1}{2} \sum_{1 \leqslant l \leqslant N} \alpha_l^2 r(x_l) + \frac{1}{2} \sum_{1 \leqslant l \neq m \leqslant N} \alpha_l \alpha_m G(x_l, x_m),$$

where the function r is the restriction on its diagonal of the function g appearing in (7.5), *that is the function r defined on Ω by $r(x) := g(x, x)$. The function r is referred as the hydrodynamic Robin function.*

Indeed the Kirchoff-Routh-Lin function W is a renormalized energy of the system, excluding the free part (that is the one which should take place in the absence of boundaries) of the self-interaction of each vortex. The first term in the definition of W corresponds to the energy created by the interaction with vortices outside Ω corresponding to the circulations on the $\mathcal{C}_l$, the second term correspond to the part of the self-interaction of each vortex induced by the presence of boundaries (by symmetry breaking) and the third one corresponds to the interaction between any distinct pair of vortices.

Definition 9 (Lin [15])**.** *The trajectories $z(t) := (z_1(t), ..., z_N(t))$ of N point vortices of respective strength $\alpha_i \in \mathbb{R}^*$, for $1 \leqslant i \leqslant N$, located at initial time at the $(x_1, ..., x_N) \in \Omega_N$ is given by the following Hamiltonian ODE*

$$\frac{d}{dt} z(t) = F(z(t)), \tag{7.9}$$

$$z(0) = (x_1, ..., x_N), \tag{7.10}$$

where $F : z := (z_1, ..., z_N) \in \Omega_N \mapsto F(z) := (F_1(z), ..., F_N(z))$, with $F_i(z) := \frac{1}{\alpha_i} \nabla_{x_i}^\perp W(z)$.

Observing that the vector field F is analytic on Ω_N we have the following.

Lemma 19. *There exists $T > 0$ and a unique solution $z(t)$ in $C^\omega([0, T])$ of* (7.9)-(7.10).

Let us now introduce an appropriated weak vorticity formulation of the Euler equations for multiply connected domains.

Definition 10. *Let be given ω_N in the space $\mathcal{M}(\Omega)$ of the Radon measures on Ω. We say that ω in $L^\infty(\mathbb{R}_+, \mathcal{M}(\Omega))$ is a weak solution of the Euler equations on $[0,T)$ with ω_0 as initial vorticity and circulations $\overline{\gamma}$ if for any test function $\varphi \in C_c^\infty([0,T) \times \Omega, \mathbb{R})$,*

$$\int_\Omega \varphi(0,x) d\omega_0(x) + \int_{[0,T]} \int_\Omega L_\varphi(t,x)\, d\omega(t,x)dt + \int_{[0,T]} \int_\Omega \int_\Omega H_\varphi(t,x,y)\, d\omega(t,x) d\omega(t,y) dt = 0, \tag{7.11}$$

where L_φ is the function in $C_c^\infty([0,T) \times \Omega, \mathbb{R})$, defined by

$$L_\varphi(t,x) := \partial_t \varphi(t,x) + \mathbf{X}_0(x) \cdot \nabla_x \varphi(t,x), \tag{7.12}$$

and H_φ is the auxiliary function:

$$H_\varphi(t,x,y) := \begin{cases} \frac{1}{2}\Big(\nabla_x \varphi(t,x) \cdot K(x,y) + \nabla_x \varphi(t,y) \cdot K(y,x)\Big), & \text{for } x \neq y \\ \frac{1}{2} \nabla_x \varphi(t,x) \cdot \nabla_x^\perp r(x), & \text{for } x = y, \end{cases} \tag{7.13}$$

with

$$K(x,y) := \nabla_x^\perp G(x,y) \text{ and } \mathbf{X}_0(x) := \nabla_x^\perp \psi_0(x), \tag{7.14}$$

where the function ψ_0 is the one in (7.8).

The three terms in (7.11) makes sense: in particular let us observe that the function H_φ is bounded.

Let us first verify that a smooth solution of the Euler equations is also a weak solution in the sense above.

Lemma 20. *Let be given $u_0 \in C^\infty([0,T] \times \Omega) \cap C([0,T] \times \overline{\Omega})$ satisfying $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot \hat{n} = 0$ on $\partial\Omega$. Let u be the unique solution in $C^\infty([0,+\infty) \times \Omega)$ of the Euler equations (1.1) (cf. [12]). Then for any $T > 0$, $\omega := \operatorname{curl} u$ satisfies the weak vorticity formulation of Definition 10 with $\omega_0 := \operatorname{curl} u_0$ an $\psi_0 := \sum_{1 \leqslant i,j \leqslant d} \gamma_i(u_0) p_{i,j} \phi_j$.*

Proof. We start with the vorticity formulation of the Euler equations:

$$\partial_t \omega + \operatorname{div}(\omega u) = 0. \tag{7.15}$$

We consider $T > 0$ and we multiply by a test function $\varphi \in C_c^\infty([0,T) \times \Omega, \mathbb{R})$, and integrate by parts over $[0,T] \times \Omega$ to get

$$\int_\Omega \varphi(0,x) \omega_0(x) dx + \int_{[0,T]} \int_\Omega \partial_t \varphi(t,x)\, \omega(t,x) dx dt + \int_{[0,T]} \int_\Omega \nabla_x \varphi(t,x)\, \omega(t,x) u(t,x) dx dt = 0, \tag{7.16}$$

where ω_0 is the initial value of ω. Now the velocity can be recovered from the vorticity by using Lemma 17. More precisely we have the following.

Lemma 21. *Let be given $\omega \in C_c^\infty(\Omega)$ and some real $\overline{\gamma}_i$, for $1 \leqslant i \leqslant d$. Then there exists a unique $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} \operatorname{curl} u = \omega, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u \cdot \hat{n} = 0, & \text{on } \partial\Omega, \\ \gamma_i(u) = \overline{\gamma}_i, & \text{for } i = 1, \ldots, d. \end{cases} \tag{7.17}$$

Moreover $u = \mathbf{X}_0 + K[\omega]$, where $\mathbf{X}_0$ is as in Definition 10 and

$$K[\omega](x) := \int_\Omega K(x,y) \omega(y) dy. \tag{7.18}$$

As a consequence we infer that

$$\int_\Omega \varphi(0,x) \omega_0(x)\, dx + \int_{[0,T]} \int_\Omega L_\varphi(t,x)\, d\omega(t,x) dt + \int_{[0,T]} \int_\Omega \nabla \varphi(t,x) \cdot K[\omega](t,x) \omega(t,x)\, dx dt = 0,$$

Now by substituting $K[\omega]$ for its integral expression and subsequently symmetrizing the kernel in the nonlinear term above, we get (7.11) with

$$\frac{1}{2}\Big(\nabla_x\varphi(t,x)\cdot K(x,y)+\nabla_x\varphi(t,y)\cdot K(y,x)\Big) \tag{7.19}$$

instead of $H_\varphi(t,x,y)$. Since the integrand is in $L^1([0,T]\times\Omega\times\Omega)$, modifying (7.19) for H_φ does not modify the value of the integral, so that for any $T>0$, $\omega:=\operatorname{curl}u$ satisfies the weak vorticity formulation of Definition 10. ☐

Let us now start the proof of Theorem 10: we consider the trajectories $z(t):=(z_1(t),...,z_N(t))$ on $[0,T]$ of N isolated point vortices of respective strength $\alpha_l\in\mathbb{R}^*$, for $1\leqslant l\leqslant N$, given by Lemma 9. We denote by ω the following function with measure-values

$$t\mapsto\sum_{1\leqslant l\leqslant N}\alpha_l\delta_{z_l(t)}.$$

Let us consider a test function $\varphi\in C_c^\infty([0,T)\times\Omega,\mathbb{R})$. Thanks to the chain rule, we have for any $t\in[0,T]$, for any $1\leqslant l\leqslant N$,

$$\partial_t(\varphi(t,z_l(t)))=(\partial_t\varphi)(t,z_l(t))+z_l'(t)\cdot\nabla_x\varphi(t,z_l(t)).$$

Let us observe that (7.9) amounts to the equations

$$z_l'(t)=\mathbf{X}_0(z_l(t))+\frac{\alpha_l}{2}\nabla_x^\perp r(z_l(t))+\sum_{m\neq l}\alpha_m\nabla_x^\perp G(z_l(t),z_m(t)).$$

Therefore

$$\partial_t(\varphi(t,z_l(t)))=L_\varphi(t,z_l(t))+\frac{\alpha_l}{2}\nabla_x^\perp r(z_l(t))\cdot\nabla_x\varphi(t,z_l(t))+\sum_{m\neq l}\alpha_m\nabla_x^\perp G(z_l(t),z_m(t))\cdot\nabla_x\varphi(t,z_l(t)).$$

Now integrate on $[0,T]$, multiply by α_l and sum over $1\leqslant l\leqslant N$ to get

$$0=\sum_{1\leqslant l\leqslant N}\alpha_l\varphi(0,x_l)+\sum_{1\leqslant l\leqslant N}\alpha_l\int_{[0,T]}L_\varphi(t,z_l(t))dt$$
$$+\int_{[0,T]}\Big(\frac{1}{2}\sum_{1\leqslant l\leqslant N}\alpha_l^2\nabla_x^\perp r(z_l(t))+\sum_{1\leqslant l\neq m\leqslant N}\alpha_l\alpha_m\nabla_x^\perp G(z_l(t),z_m(t))\Big)\cdot\nabla_x\varphi(z_l(t))dt,$$

what, after symmetrizing the last sum, amouts to say that ω is a weak solution of the Euler equation with $\omega_0:=\sum_{1\leqslant l\leqslant N}\alpha_l\delta_{x_l}$ as initial data and $\overline{\gamma}_i$ as repective circulation around the curve $\mathcal{C}_i$, for $1\leqslant i\leqslant d$.

APPENDIX C

The goal of this appendix is to provide a proof of the statement below Theorem 11 about analyticity of the flow of classical solutions, whose vorticity is constant in a neighborhood of the boundary, which is only assumed to be C^2. To be more general we assume here that the fluid fills a bounded regular domain $\Omega\subset\mathbb{R}^d$, with $d=2$ or 3. We denote, for λ in $\mathbb{N}^0$ and $r\in(0,1)$, the space $C_\sigma^{\lambda,r}(\Omega)$ of divergence free vector fields u in $C^{\lambda,r}(\Omega)$ tangent to the boundary.

Theorem 13. *Assume that the boundary $\partial\Omega$ is C^2. Assume that u_0 in $C_\sigma^{\lambda+1,r}(\Omega)$, where λ in $\mathbb{N}^0$ and $r\in(0,1)$. Assume that $\omega_0:=\operatorname{curl}u_0$ is constant outside of a compact set $\underline{K}\subset\Omega$. Then for any compact set $K\subset\Omega$ the flow Φ is in the space $C^\omega([0,T],C^{\lambda+1,r}(K))$ of real analytic functions from $[0,T]$ to $C^{\lambda+1,r}(K)$.*

Proof. Let us introduce a function χ_0 in $C_c^{\lambda+1,r}(\Omega)$ such that $\chi_0|_{\underline{K}\cup K}=1$. There exists $T>0$ and only one χ in $L^\infty([0,T],C_c^{\lambda+1,r}(\Omega))$ such that

$$D_\chi\chi=0,\quad \chi|_{t=0}=\chi_0,\qquad\text{where } D_\chi:=\partial_t+\chi u\cdot\nabla.$$

As in the proof of Theorem 7, we have, for k in $\mathbb{N}^*$, $\hat{n} \cdot D_\chi^k u = 0$ on the boundary $\partial\Omega$, and $\gamma_i(D_\chi^k u) = 0$, for $1 \leqslant i \leqslant d$. Moreover we obtain, by iteration, using the identities (7.1)-(7.4), that for $k \in \mathbb{N}^*$, in Ω

$$\operatorname{div} D_\chi^k u = \operatorname{tr}\left\{F_\chi^k[u]\right\} \text{ where } F_\chi^k[u] := \sum_{\theta \in \mathcal{A}_k} d_k^1(\theta)\, f_\chi(\theta)[u],$$

where

$$f_\chi(\theta)[u] := \nabla\chi D^{\alpha_1} u \cdot \ldots \cdot \nabla\chi D^{\alpha_{s-1}} u \cdot \nabla D^{\alpha_s} u, \tag{7.20}$$

and where the $d_k^1(\theta)$ are integers satisfying $|d_k^1(\theta)| \leqslant \frac{k!}{\alpha!}$. As in the proof of Theorem 7 the assumption that the vorticity is constant near the boundary is useful to tackle the curl $D_\chi^k u$. For instance, we have that, in Ω,

$$D_\chi \operatorname{curl} u = D \operatorname{curl} u. \tag{7.21}$$

Moreover, using the identities (7.1)-(7.4), we get

$$\operatorname{curl} D_\chi u = D_\chi \operatorname{curl} u + \operatorname{as}\left\{(\nabla(\chi u)) \cdot (\nabla u)\right\}.$$

Since we also that

$$0 = \operatorname{curl} Du = D \operatorname{curl} u + \operatorname{as}\left\{(\nabla u) \cdot (\nabla u)\right\},$$

we infer that

$$\operatorname{curl} D_\chi u = \operatorname{as}\left\{(\nabla((\chi - 1)u)) \cdot (\nabla u)\right\}.$$

Then proceeding by iteration, and using the identities (7.1)-(7.4) and $D_\chi \chi = 0$, we obtain that $k \in \mathbb{N}^*$, in Ω,

$$\operatorname{curl} D_\chi^k u = \operatorname{tr}\left\{G_\chi^k[u]\right\} \text{ where } G_\chi^k[u] := \sum_{\tilde{\theta} \in \tilde{\mathcal{A}}_k} d_k^2(\tilde{\theta})\, g_\chi(\tilde{\theta})[u],$$

where $\tilde{\mathcal{A}}_k$ denotes the set

$$\tilde{\mathcal{A}}_k := \{(s, \varepsilon, \alpha)/\ 2 \leqslant s \leqslant k+1,\ \varepsilon \in \{0,1\}^{s-1} \text{ with } |\varepsilon| = 1, \text{ and } \alpha := (\alpha_1, \ldots, \alpha_s) \in (\mathbb{N}^0)^s /\ |\alpha| = k+1-s\},$$

for $\tilde{\theta} = (s, \varepsilon, \alpha) \in \tilde{\mathcal{A}}_k$, $g_\chi(\tilde{\theta})[u]$ denotes

$$g_\chi(\tilde{\theta})[u] := \nabla\big((\chi - \varepsilon_1) D_\chi^{\alpha_1} u\big) \cdot \ldots \cdot \nabla\big((\chi - \varepsilon_{s-1}) D_\chi^{\alpha_{s-1}} u\big) \cdot \nabla D_\chi^{\alpha_s} u, \tag{7.22}$$

and where the $d_k^2(\tilde{\theta})$ are integers satisfying $|d_k^2(\tilde{\theta})| \leqslant \frac{k!}{\alpha!}$.

Then we estimate recursively the $D_\chi^k u$, for k in $\mathbb{N}^*$, thanks to classical elliptic estimates in Hölder spaces, as in the proof of Theorem 2 of [18].

□

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