ENTROPY BOUNDARY LAYERS

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ABSTRACT. We consider the Euler system of compressible and entropic gaz dynamics in a bounded open domain of $\mathbb{R}^d$ with wall boundary condition. We prove the existence and the stability of families of solutions which correspond to a ground state plus a large entropy boundary layer. The ground state is a solution of the Euler system which satisfies some explicit additional conditions on the boundary. These conditions are used in a reduction of the system. We construct BKW expansions at all order. The profile problems are linear thanks to a transparency property. We prove the stability of these expansions by proving ε-conormal estimates for a characteristic boundary value problem.

1. INTRODUCTION

We consider the **Euler system of compressible and entropic gaz dynamics**. We respectively denote by $\mathtt{s}$, $\mathtt{v}$ and $\mathtt{p}$ the entropy, the speed and the pressure. The volumic density, noted ρ , is a function of $\mathtt{p}$ and $\mathtt{s}$. We also assume that $\rho(\mathtt{p},\mathtt{s}) > 0$ for all $\mathtt{p}$ and $\mathtt{s}$. We introduce the function $\boldsymbol{\alpha}(\mathtt{p},\mathtt{s}) := (\rho(\mathtt{p},\mathtt{s}))^{-1}\,\rho'_{\mathtt{p}}(\mathtt{p},\mathtt{s})$. We also assume that $\boldsymbol{\alpha}(\mathtt{p},\mathtt{s}) > 0$ for all $\mathtt{p}$ and $\mathtt{s}$. We denote by t the time variable and by $x = (x_1, ..., x_d)$ the space variable. The derivative with respect to t will be denoted ∂_t. For a $\mathbb{R}$-valued function f which depends on x, $\partial_x f$ means the gradient of f with respect to x. For a $\mathbb{R}^d$-valued function f which depends on x, we will denote by $\operatorname{div} f$ the divergence of f with respect to x. We denote by $\mathbf{X}_{\mathtt{v}}$ the particle derivative $\mathbf{X}_{\mathtt{v}} := \partial_t + \mathtt{v}.\partial_x$. The Euler system reads as follows:

$$\mathbf{X}_{\mathtt{v}}\,\mathtt{v} + \rho^{-1}\,\partial_x\,\mathtt{p} = 0, \quad \mathbf{X}_{\mathtt{v}}\,\mathtt{p} + \boldsymbol{\alpha}^{-1}\operatorname{div}\mathtt{v} = 0, \quad \mathbf{X}_{\mathtt{v}}\,\mathtt{s} = 0,$$

The system above is a nonconservative form of a system of conservation laws. Moreover, this system is hyperbolic symmetrizable. It has three characteristic fields. One of them is linearly degenerate (cf. *section* 4). We consider these equations in a bounded open domain $\Omega \subset \mathbb{R}^d$ lying on one side of its C^∞ boundary Γ. More precisely, since we will need an equation of the boundary Γ, we fix once for all a function $\varphi \in C^\infty(\mathbb{R}^d,\mathbb{R})$ and we assume that $\Omega = \{\varphi > 0\}$, $\Gamma = \{\varphi = 0\}$ and $|\partial_x\,\varphi(x)| = 1$ in an open neighborhood $\mathcal{V}$ of Γ[1]. We consider the natural boundary condition $\mathtt{v}.\mathfrak{n} = 0$, where $\mathfrak{n}$ is the unit outward normal to Γ. For $T > 0$, the **boundary value problem** reads:

$$\left.\begin{array}{l} \mathbf{X}_{\mathtt{v}}\,\mathtt{v} + \rho^{-1}\,\partial_x\,\mathtt{p} = 0 \\ \mathbf{X}_{\mathtt{v}}\,\mathtt{p} + \boldsymbol{\alpha}^{-1}\operatorname{div}\,\mathtt{v} = 0 \\ \mathbf{X}_{\mathtt{v}}\,\mathtt{s} = 0 \end{array}\right\} \quad \text{where}\,(t,x) \in (0,T)\times\Omega, \qquad (1)$$
$$\mathtt{v}.\mathfrak{n} = 0 \qquad \text{where}\,(t,x) \in (0,T)\times\Gamma.$$

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[1]Hence for $x \in \Omega\cap\mathcal{V}$: $\varphi(x) = dist(x,\Gamma)$.

The boundary Γ is characteristic for the linearly degenerate field. We introduce the tangential velocity $\mathtt{v_t}$ and the normal velocity $\mathtt{v_n}$. Thus, we have $\mathtt{v} = \mathtt{v_t} + \mathtt{v_n}$. The choice of the set of thermodynamic variables $\mathtt{v}$, $\mathtt{p}$ and $\mathtt{s}$ is particularly well adapted to boundary problem (cf. [38]). The existence of **local regular solutions** of (1) can be found in the papers by H. Beirão da Veiga [2], [1], by S. Schochet [33] and by O. Guès [13]. If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $H^\infty(\mathcal{O})$ the set of $u \in L^2(\mathcal{O})$ such that all the derivatives of u are in $L^2(\mathcal{O})$. From now on, we will assume that a real $T_\dagger > 0$ and a solution $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger) \in H^\infty((0,T_\dagger) \times \Omega)$ of (1) are given.

An interesting question about the Euler system is the study of the convergence of a more acute model: the Navier-Stokes system, which includes a viscosity term, when the amplitude of the viscosity goes to 0. The difficulty is linked to the existence of a **boundary layer** i.e. of a rapid variation of the solutions of the viscous model near the boundary. There is a sensitivity on the type of boundary conditions imposed for the viscous model. The most delicate is the homogeneous Dirichlet condition. In this case, there are in general some large characteristic boundary layers of large amplitude[2]. In one space dimension, a simple case is the isentropic one since there is no boundary layer and the solutions of the Navier-Stokes system are regular perturbations of the solutions of the Euler system. For the entropic Navier-Stokes equations, an answer was given by F. Rousset in [29] using boundary layers analysis. In several space dimensions, the analysis is quite more complicated even for the incompressible Navier-Stokes equations. There is a huge literature about the tangential velocity boundary layers which appear (see, for example, the papers of Z. Xin and T. Yanagisawa [41], M. Sammartino and R.E. Caflisch [30], [31], E. Grenier [11], [10],...). An attempt of analysis involves Prandtl equations (see the surveys of W.E [40] and E. Grenier [12]).

Here we do not consider the Navier-Stokes equations but only the Euler system. The idea to investigate first the stability of boundary layers type solutions for the Euler equations is a classical approach in fluid mechanics (see the books of P.G. Dranzin and W.H. Reid [7], of C. Marchioro and M. Pulvirenti [19], H. Schlichting and K. Gersten [32]. This idea was followed more recently by E. Grenier for the study of velocity boundary layers [11], [10]. Such a strategy is also possible for entropy boundary layers since they are characteristic, like the velocity ones. Thus our goal in this paper is to study **entropy boundary layers** for the Euler system. As far as we know there was no mathematical study of entropy boundary layer in **several space dimensions**.

2. Overview of the results

We introduce the space

$$\mathcal{N}(T) := H^\infty((0,T) \times \Omega, \mathcal{S}(\mathbb{R}_+)) \tag{2}$$

where we denote by $\mathcal{S}(\mathbb{R}_+)$ the Schwartz space of C^∞ rapidly decreasing functions. Thus a function $U(t,x,X) \in \mathcal{N}(T)$ is C^∞ rapidly decreasing with respect to X. Let us begin to

[2]For other boundary conditions the amplitude of the boundary layers can be weaker or the boundary layers can be noncharacteristic. Then the problem is simpler (see the papers by P. Fabrie and G. Carbou [8], by T. Clopeau, A. Mikelić and R. Robert [5], by F. Sueur [38], R. Temam and X. Wang [39]).

look naively for solutions $\check{u}^\varepsilon := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \check{\mathtt{s}}^\varepsilon)_{0<\varepsilon\leqslant 1}$ of (1) with $\check{\mathtt{s}}^\varepsilon$ of the form

$$\check{\mathtt{s}}^\varepsilon(t,x) := \mathtt{s}^\dagger(t,x) + \tilde{\mathtt{S}}^0(t,x,\frac{\varphi(x)}{\varepsilon}) \tag{3}$$

where the function $\tilde{\mathtt{S}}^0$ is in $\mathcal{N}(T_\dagger)$ and the function $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger)$ is the ground state given above. Replacing in the third equation of the system (1) leads to the following linear transport equation that necessarily the function $\tilde{\mathtt{S}}^0$ verifies[3]:

$$(\mathbf{X}_{\mathtt{v}^\dagger} + \frac{\mathtt{v}^\dagger_{\mathtt{n}}}{\varphi(x)}.X\partial_X)\tilde{\mathtt{S}}^0 = 0 \quad \text{where}\,(t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+. \tag{4}$$

Thanks to the boundary condition (see (1)), the function $\mathtt{v}^\dagger_{\mathtt{n}}/\varphi(x)$ is C^∞. In general these necessary conditions are not sufficient to insure that the functions $(\mathtt{v}^\dagger, \mathtt{p}^\dagger, \check{\mathtt{s}}^\varepsilon)_{0<\varepsilon\leqslant 1}$ are solutions of (1). Indeed, because the functions ρ and $\boldsymbol{\alpha}$ depend on $\mathtt{p}$ and $\mathtt{s}$, the two first equations of (1) are not satisfied. However if in addition we assume that the ground state $(\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger)$ satisfies

$$\mathbf{X}_{\mathtt{v}^\dagger}\,\mathtt{v}^\dagger = 0, \quad \mathrm{div}_x\,\mathtt{v}^\dagger = 0, \quad \partial_{t,x}\,\mathtt{p}^\dagger = 0 \quad \text{where}\,(t,x) \in (0,T_\dagger)\times\Omega, \tag{5}$$

then it is easy to check that the functions $(\mathtt{v}^\dagger, \mathtt{p}^\dagger, \check{\mathtt{s}}^\varepsilon)_{0<\varepsilon\leqslant 1}$ are solutions of (1). *In section* 5, we will prove the existence of ground states $(\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger) \in H^\infty$ solutions of (1) and verifying the conditions (5).

Our goal is to relax the conditions (5) into **conditions localized on the boundary** Γ. In fact the conditions (5) were first introduced in a paper of C. Cheverry, O. Guès and G. Métivier [3] where the existence and the stability of large amplitude high frequency entropy oscillations are shown. Here we look for entropy boundary layers which are local singularities. It seems rather natural that local conditions are sufficient to deal with boundary layers. We reach our goal and claim the following theorem.

Theorem 2.1. *Assume that a solution* $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger) \in H^\infty((0,T_\dagger)\times\Omega)$ *of* (1) *verifying*

$$\boldsymbol{X}_{\mathtt{v}^\dagger}\,\mathtt{v}^\dagger = 0 \text{ and } \boldsymbol{X}_{\mathtt{v}^\dagger}\,\mathtt{p}^\dagger = 0 \quad \text{for } (t,x) \in (0,T_\dagger)\times\Gamma \tag{6}$$

and a solution $\tilde{\mathtt{S}}^0 \in \mathcal{N}(T_\dagger)$ *of* (4) *are given. For* $0 < \varepsilon \leqslant 1$*, we denote by* $\check{u}^\varepsilon$ *the function* $\check{u}^\varepsilon := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \check{\mathtt{s}}^\varepsilon)$ *where* $\mathtt{s}^\varepsilon$ *is given by* (3)*. Then there exists a family of solutions* $(\boldsymbol{u}^\varepsilon)_{0<\varepsilon\leqslant 1}$ *in* $H^\infty((0,T_\dagger)\times\Omega)$ *of* (1) *such that* $\check{u}^\varepsilon - u^\varepsilon$ *tends to* 0 *in* $H^1((0,T_\dagger)\times\Omega)$ *when* $\varepsilon \to 0^+$.

In fact Theorem 2.1 is a corollary of more acute results involving **WKB** (Wentzel-Kramers-Brillouin) expansions (subsection 8.1.4). We will prove the **existence and stability** of families of solutions $(\mathtt{v}^\varepsilon, \mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon)_{0<\varepsilon\leqslant 1}$ of the Euler system with a large amplitude

[3]Notice that for the transport operator $\mathbf{X}_{\mathtt{v}^\dagger} + \frac{\mathtt{v}^\dagger_{\mathtt{n}}}{\varphi(x)}.X\partial_X$ is tangent to $\{x \in \Gamma\}$ and to $\{X = 0\}$. As a consequence, none boundary condition needs to be prescribed for $\tilde{\mathtt{S}}^0$.

entropy boundary layer i.e. of the form

$$\begin{cases} \mathtt{v}^\varepsilon_{\mathtt{t}}(t,x) := \mathtt{v}^\dagger_{\mathtt{t}}(t,x) + \varepsilon(\mathtt{V}_{\mathtt{t}}(t,x) + \tilde{\mathtt{V}}_{\mathtt{t}}(t,x,\frac{\varphi(x)}{\varepsilon})) + O(\varepsilon^2), \\ \mathtt{v}^\varepsilon_{\mathtt{n}}(t,x) := \mathtt{v}^\dagger_{\mathtt{n}}(t,x) + \varepsilon \mathtt{V}_{\mathtt{n}}(t,x) + O(\varepsilon^2), \\ \mathtt{p}^\varepsilon(t,x) := \mathtt{p}^\dagger(t,x) + \varepsilon \mathtt{P}(t,x) + O(\varepsilon^2), \\ \mathtt{s}^\varepsilon(t,x) := \mathtt{s}^\dagger(t,x) + \tilde{\mathtt{S}}^0(t,x,\frac{\varphi(x)}{\varepsilon}) + O(\varepsilon), \end{cases} \tag{7}$$

where $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger)$ is a solution of (1) verifying the conditions (6). This analysis is inspired by [3], where the propagation of large amplitude high frequency entropy waves[4] is shown for ground states $u^\dagger$ solutions of (1) verifying the conditions (5) (in the first equation of section 2.3 of [3], read $\partial_{t,x}\mathtt{p} = 0$ instead of $\nabla_x \mathtt{p} = 0$). In our analysis the condition on the particle derivative is localized on the boundary. Considering directly the Euler system, we use implicitly the general structure conditions of [3]. In particular, the choice of the set of thermodynamic variables $\mathtt{v}$, $\mathtt{p}$ and $\mathtt{s}$ is a key point.

An important feature of the expansion (7) is that there are some boundary layers not only on the entropy but also on the other components, ponderated by some ε. More accurately, boundary layer appears on tangential velocity with an amplitude ε whereas boundary layer appears on the normal velocity and the pressure with an amplitude ε^2. The conditions (6) on the ground state play a main role in the fact that the large boundary layer keeps polarized on the entropy. *In section* 6, we reduce the system, thanks to a change of unknown singular with respect to ε. This **reduction** is inspired by [3]. The conditions (6) on the ground state will be used at this step. Because the localized conditions (6) are weaker than the conditions (5) of [3], our reduction is much more delicate than in [3]. We now describe in more detail the rest of the contents of the paper.

2.1. **Formal BKW solutions.** *In section* 7, we look for formal WKB solutions of the problem (1). This means that we construct WKB expansions of infinite order. Let us precise this. We introduce the profile space

$$\begin{aligned} \mathcal{P}(T) := \{&\mathcal{U} \in L^2((0,T)\times\Omega\times\mathbb{R}_+) \ / \text{ there exist } \underline{\mathcal{U}} \in H^\infty((0,T)\times\Omega) \text{ and } \tilde{\mathcal{U}} \in \mathcal{N}(T) \\ &\text{such that for all } (t,x,X) \in (0,T)\times\Omega\times\mathbb{R}_+, \mathcal{U}(t,x,X) = \underline{\mathcal{U}}(t,x) + \tilde{\mathcal{U}}(t,x,X)\}. \end{aligned} \tag{8}$$

The function $\underline{\mathcal{U}}$ is the regular part and $\tilde{\mathcal{U}}$ is the characteristic boundary layer term. We will split $\mathcal{U} \in \mathcal{P}(T)$ into $\mathcal{U} = (\mathtt{V}, \mathtt{P}, \mathtt{S})$ and $\mathtt{V}$ into $\mathtt{V} := \mathtt{V}_{\mathtt{t}} + \mathtt{V}_{\mathtt{n}}$ where $\mathtt{V}_{\mathtt{n}} := (\mathtt{V}.\mathtt{n})\mathtt{n}$. The function $\mathtt{V}$ (respectively $\mathtt{P}$) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}$). The function $\mathtt{V}_{\mathtt{t}}$ (respectively $\mathtt{V}_{\mathtt{n}}$) takes its values in $\mathbb{R}^{d-1}$ (resp. $\mathbb{R}$). By abuse of notations, we will say that $\mathtt{V}$, $\mathtt{V}_{\mathtt{t}}$, $\mathtt{V}_{\mathtt{n}}$ and $\mathtt{P}$ are in $\mathcal{P}(T)$ even if they do not take values in $\mathbb{R}^{d+2}$. We look for formal solutions $(u^\varepsilon)_\varepsilon$ of (1) of the form

$$u^\varepsilon(t,x) = \sum_{n\geqslant 0} \varepsilon^n \, \mathcal{U}^n(t,x,\frac{\varphi(x)}{\varepsilon}) \tag{9}$$

[4]Earlier works about large amplitude high frequency oscillations are due to W. E [40], A. Heibig [17], B. Sévennec [35], A. Corli and O. Guès [6] , A. Museux [25] in one space dimension and to D. Serre [34] in several spaces dimension.

where each $\mathcal{U}^n$ belongs to $\mathcal{P}(T)$ and $\underline{\mathcal{U}}^0 = u^\dagger$. Let us explain what is meant by *formal solutions*. Plugging the expansion (9) into the system, using Taylor expansions and ordering the terms in powers of ε, we get a formal expansion in power series of ε:

$$\sum_{n\geqslant -1} \varepsilon^n \, \Phi^n(t,x,\frac{\varphi(x)}{\varepsilon})$$

where the $(\Phi^n)_{n\geqslant -1}$ are in $\mathcal{P}(T)$. We say that $(u^\varepsilon)_\varepsilon$ is a formal solution when all the resulting Φ^n are identically zero. Theorem 7.1 will sum up the main results of section 7. It states that the system has formal solutions of the form

$$\begin{cases} \mathtt{v}^\varepsilon_{\mathtt{t}}(t,x) = \mathtt{v}^\dagger_{\mathtt{t}}(t,x) + \sum\limits_{j\geqslant 1} \varepsilon^j \, \mathtt{V}^{j-1}_{\mathtt{t}}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{v}^\varepsilon_{\mathtt{n}}(t,x) = \mathtt{v}^\dagger_{\mathtt{n}}(t,x) + \varepsilon \, \mathtt{V}^0_{\mathtt{n}}(t,x) + \sum\limits_{j\geqslant 2} \varepsilon^j \, \mathtt{V}^{j-1}_{\mathtt{n}}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{p}^\varepsilon(t,x) = \mathtt{p}^\dagger(t,x) + \varepsilon \, \mathtt{p}^\dagger(t,x) + \sum\limits_{j\geqslant 2} \varepsilon^j \, \mathtt{P}^{j-1}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{s}^\varepsilon(t,x) = \sum\limits_{j\geqslant 0} \varepsilon^j \, \mathtt{S}^j(t,x,\frac{\varphi(x)}{\varepsilon}), \end{cases} \tag{10}$$

and that we can prescribe arbitrary initial values to the $(\mathtt{S}^j|_{t=0})_{j\in\mathbb{N}}$ and to the $(\mathtt{V}^j_{\mathtt{t}}|_{t=0})_{j\in\mathbb{N}}$. The profiles $\mathtt{V}^0_{\mathtt{t}}$, $\mathtt{V}^0_{\mathtt{n}}$ and $\mathtt{p}^\dagger$ involved in (10) are the profiles $\mathtt{V}_{\mathtt{t}}$, $\mathtt{V}_{\mathtt{n}}$ and $\mathtt{P}$ involved in (7). In (10), we use the index 0 to match with notations of section 7. In (7), we did not write the index in order to avoid heavy notations.

The two first equations of (1) involve the entropy $\mathtt{s}$, through the functions ρ and $\boldsymbol{\alpha}$. A priori the large entropy boundary layer could contaminate the velocity[5]. We prove that since we consider some ground states $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger)$ solutions of (1) verifying the conditions (6), there is no contamination at order ε^0: when looking at the expansion (10), we notice that there is no large pressure boundary layer and no large velocity boundary layer. To use the conditions (6), we reduce the system (cf. section 6). This step is difficult. Let us briefly mention here some key points of our strategy. For sake of clarity, we begin with the first equation of (1) only. We look for solutions $(u^\varepsilon)_\varepsilon$ where $\mathtt{v}^\varepsilon$ and $\mathtt{p}^\varepsilon$ are of the form $\mathtt{v}^\varepsilon := \mathtt{v}^\dagger + \varepsilon \mathtt{V}^\varepsilon$, $\mathtt{p}^\varepsilon := \mathtt{p}^\dagger + \varepsilon \mathtt{P}^\varepsilon$. We split the particle derivative $\mathbf{X}_{\mathtt{v}^\varepsilon} \, \mathtt{v}^\varepsilon$ into

$$\mathbf{X}_{\mathtt{v}^\varepsilon} \, \mathtt{v}^\varepsilon = \mathbf{X}_{\mathtt{v}^\dagger} \, \mathtt{v}^\dagger + \varepsilon \mathbf{X}_{\mathtt{v}^\varepsilon} \, \mathtt{V}^\varepsilon + \varepsilon \mathtt{V}^\varepsilon . \partial_x \, \mathtt{v}^\dagger .$$

Thanks to the conditions (6), there exists a function $\phi \in C^\infty((0,T_\dagger)\times\Omega)$ such that for all $(t,x) \in (0,T_\dagger)\times\Omega$, $(\mathbf{X}_{\mathtt{v}^\dagger} \, \mathtt{v}^\dagger)(t,x) = \varphi(x)\,\phi(t,x)$. Remark that if $\tilde{\mathcal{U}} \in \mathcal{N}(T_\dagger)$, then

$$\mathbf{X}_{\mathtt{v}^\dagger} \, \mathtt{v}^\dagger . \tilde{\mathcal{U}}(t,x,\frac{\varphi(x)}{\varepsilon}) = \varepsilon(\phi(t,x) X \tilde{\mathcal{U}}(t,x,X))|_{X=\frac{\varphi(x)}{\varepsilon}} \quad \text{and} \quad X\tilde{\mathcal{U}} \in \mathcal{N}(T_\dagger).$$

[5]which can create a destabilizing effect (see [11], [10])

To use this remark, we develop the term $\rho(\mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon)$. Thus, we write $\mathtt{s}^\varepsilon$ under the form

$$\mathtt{s}^\varepsilon(t,x) = \mathtt{s}^\dagger(t,x) + \tilde{\mathtt{S}}^0(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon \mathtt{S}^{\varepsilon,\diamond}(t,x), \tag{11}$$

with $\tilde{\mathtt{S}}^0 \in \mathcal{N}(T_\dagger)$. We get for the term $\rho(\mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon)$ the following expansion:

$$\rho(\mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon) \quad = \quad \rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger) + (\rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger + \tilde{\mathtt{S}}^0) - \rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger)) + \varepsilon D\rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger).(\mathtt{P}^\varepsilon, \mathtt{S}^{\varepsilon,\diamond}) + O(\varepsilon^2).$$

Because $u^\dagger$ satisfies the equation (1), we get the equation:

$$\rho(\mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon).\mathbf{X}_{\mathtt{v}^\varepsilon}\, \mathtt{V}^\varepsilon + \partial_x\, \mathtt{P}^\varepsilon + \mathfrak{P}_{\mathtt{v}} = O(\varepsilon) \quad \text{with} \quad \mathfrak{P}_{\mathtt{v}} := \mathfrak{P}_{\mathtt{v}}^a + \mathfrak{P}_{\mathtt{v}}^b + \mathfrak{P}_{\mathtt{v}}^c$$

$$\text{where} \quad \begin{cases} \mathfrak{P}_{\mathtt{v}}^a := \rho(\mathtt{p}^\dagger, \mathtt{s}^\varepsilon).\mathtt{V}^\varepsilon.\partial_x\, \mathtt{v}^\dagger, \\ \mathfrak{P}_{\mathtt{v}}^b := \varepsilon^{-1}\,(\rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger + \tilde{\mathtt{S}}^0) - \rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger)).\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{v}^\dagger, \\ \mathfrak{P}_{\mathtt{v}}^c := D\rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger).(\mathtt{P}^\varepsilon, \mathtt{S}^{\varepsilon,\diamond}).\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{v}^\dagger. \end{cases}$$

With the previous remark, we get

$$\mathfrak{P}_{\mathtt{V}}^b = \{\phi X(\rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger + \tilde{\mathtt{S}}^0) - \rho(\mathtt{p}^\dagger, \mathtt{s}^\dagger))\}|_{X=\frac{\varphi(x)}{\varepsilon}}.$$

Proceed in the same way for the second equation of (1), we get the equation:

$$\boldsymbol{\alpha}(\mathtt{p}^\varepsilon, \mathtt{s}^\varepsilon).\mathbf{X}_{\mathtt{v}^\varepsilon}\, \mathtt{P}^\varepsilon + \operatorname{div} \mathtt{V}^\varepsilon + \mathfrak{P}_{\mathtt{p}} = O(\varepsilon) \quad \text{with} \quad \mathfrak{P}_{\mathtt{p}} := \mathfrak{P}_{\mathtt{p}}^a + \mathfrak{P}_{\mathtt{p}}^b + \mathfrak{P}_{\mathtt{p}}^c$$

$$\text{where} \quad \begin{cases} \mathfrak{P}_{\mathtt{p}}^a := \boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\varepsilon).\mathtt{V}^\varepsilon.\partial_x\, \mathtt{p}^\dagger, \\ \mathfrak{P}_{\mathtt{p}}^b := \varepsilon^{-1}\,(\boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\dagger + \tilde{\mathtt{S}}^0) - \boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\dagger)).\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{p}^\dagger \\ \qquad = \{\phi X(\boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\dagger + \tilde{\mathtt{S}}^0) - \boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\dagger))\}|_{X=\frac{\varphi(x)}{\varepsilon}}, \\ \mathfrak{P}_{\mathtt{p}}^c := D\boldsymbol{\alpha}(\mathtt{p}^\dagger, \mathtt{s}^\dagger).(\mathtt{P}^\varepsilon, \mathtt{S}^{\varepsilon,\diamond}).\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{p}^\dagger. \end{cases}$$

In other words, the unknown $U^\varepsilon := (\mathtt{V}^\varepsilon, \mathtt{P}^\varepsilon, \mathtt{s}^\varepsilon)$ verify the Euler system (1) except the perturbation terms $\mathfrak{P}_{\mathtt{V}}$, $\mathfrak{P}_{\mathtt{P}}$, and $O(\varepsilon)$. We see that the terms $\mathfrak{P}_{\mathtt{v}}^a$, $\mathfrak{P}_{\mathtt{v}}^b$, $\mathfrak{P}_{\mathtt{p}}^a$ and $\mathfrak{P}_{\mathtt{p}}^b$ do not have any singular factor with respect to ε. This is a consequence of (6). These terms are expressed in function of the unknown U^ε, of the ground state $u^\dagger$ and of the boundary layer $\tilde{\mathtt{S}}^0$. The terms $\mathfrak{P}_{\mathtt{v}}^c$ and $\mathfrak{P}_{\mathtt{p}}^c$ involve $\mathtt{S}^{\varepsilon,\diamond}$. If we try to eliminate $\mathtt{S}^{\varepsilon,\diamond}$ via (11), we involve the unknown U^ε in a singular way via the term $\varepsilon^{-1}\, \mathtt{s}^\varepsilon$. We overcome this difficulty in Lemma 7.8 using that the terms $\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{p}^\dagger$ and $\mathbf{X}_{\mathtt{v}^\dagger}\, \mathtt{v}^\dagger$ are respectively in factor of $\mathfrak{P}_{\mathtt{v}}^c$ and $\mathfrak{P}_{\mathtt{p}}^c$. Moreover the terms $\mathfrak{P}_{\mathtt{V}}$ and $\mathfrak{P}_{\mathtt{P}}$ are affine with respect to $(\mathtt{V}^\varepsilon, \mathtt{P}^\varepsilon)$.

The profile equations are linear, thanks to some original **transparency** properties of the Euler system. On one hand, the entropy boundary layer profile $\tilde{\mathtt{S}}^0$ verifies a transport equation which is linear with respect to the entropy (cf. equation (4)). On the other hand, the amplitude of the boundary layer on the tangential velocity is weak (of order ε) and the boundary is characteristic for a linearly degenerate field. Thanks to this, the tangential velocity boundary layer profile $\tilde{\mathtt{V}}_{\mathtt{t}}$ satisfies a linear equation, without Burgers-like nonlinearity (cf. equation (51)). This is a transparency phenomenon analogous to the one observed in [38]. A interesting point is that such transparency phenomena does not occur for large amplitude high frequency entropy waves (see Theorem 3.9 of [3]).

2.2. **Stability.** *In section* 8 we are interested in the **existence** (cf. Theorem 8.2) and the **propagation** (cf. Theorem 8.1) of **exact solutions** of (1) asymptotic to approximate solutions obtained by truncating formal solutions constructed in section 7. Theorem 2.1 given in the introduction is a consequence of Theorem 7.1 and Theorem 8.2. After a reduction (cf. Prop. 8.6, subsection 8.3), we will face a singular perturbation problem because of boundary layers which correspond to variations in $\frac{\varphi(x)}{\varepsilon}$. More precisely we deal with a family of quasi-linear symmetric hyperbolic boundary value problem. As for the originating Euler problem, the boundary is conservative and characteristic of constant multiplicity. To tackle this characteristic problem we get inspired by the paper [13] of O.Guès which uses the notion of conormal regularity and the spaces

$$E^m(T) := \{u \in L^2(T) / \sum_{0 \leqslant 2k+|\alpha| \leqslant m} ||\partial_{\mathfrak{n}}^k Z^\alpha u||_{L^2(T)} < \infty\},$$

with $\alpha := (\alpha_0, ..., \alpha_d) \in \mathbb{N}^{d+1}$, $|\alpha| := \alpha_0 + ... + \alpha_d$, $Z^\alpha := Z_0^{\alpha_0} ... Z_d^{\alpha_d}$ where $(Z_0, ..., Z_d)$ generates the algebra of C^∞ tangent vector to Γ. To simplify, we denote $L^2(T) := L^2((0,T) \times \Omega)$. For these spaces one normal derivative corresponds to two conormal derivatives. We adapt the method of [13] by substituting the derivative $\varepsilon \partial_{\mathfrak{n}}$ to the derivative $\partial_{\mathfrak{n}}$ in order to obtain uniform estimates and will use the following subsets of $L^2(T)^{]0,1]}$:

$$\mathbf{E}^m(T) := \{(u^\varepsilon)_\varepsilon \in L^2(T) / \quad \sup_{0<\varepsilon\leqslant 1} \sum_{0 \leqslant 2k+|\alpha| \leqslant m} ||(\varepsilon \partial_{\mathfrak{n}})^k Z^\alpha u^\varepsilon||_{L^2(T)} < \infty\}.$$

This idea to use some derivatives with ε in factor for some singular perturbation problems is natural and was also used in the papers of [15], [14] with the ε-stratified notion, [3] with the **ε-conormal** notion. Here, this idea is applied to (characteristic) boundary value problem and anisotropic Sobolev spaces.

At first look, this system we obtained is singular with respect to ε but a trick allows to overcome this false singularity (see subsection 8.3). We will use a family of iterative schemes. Thus we will supply in subsection 8.5 linear estimates which are the core the proof. We will successively perform L^2 estimates, conormal estimates and normal estimates. A main difficulty lies in the way to deal with commutators (cf. Proposition 8.21). This strategy yields exact solutions till $T_\dagger$. The proof of Theorem 8.2 needs carefulness about the existence of compatible data. Subsection 8.4 is devoted to this question.

It is possible to obtain L^∞ estimates, in spite of the fact that $d \geqslant 1$. We refer to papers [23], [20] of G. Métivier, paper [26] of J. Rauch and M. Reed and paper [13]. This idea is still relevant when adapting ε-conormal regularity to characteristic boundary value problem. Therefore we can weaken the regularity of the solution and prove a propagation result for some solutions admitting only one normal derivative in L^2. We introduce the following subsets of $L^2(T)^{]0,1]}$:

$$\mathbf{A}^m(T) := \{(u^\varepsilon)_\varepsilon / \quad \sup_{0<\varepsilon\leqslant 1} \sum_{0 \leqslant |\alpha| \leqslant m} ||Z^\alpha u^\varepsilon||_{L^2(T)} + \sum_{0 \leqslant |\alpha| \leqslant m-2} ||\varepsilon \partial_{\mathfrak{n}} Z^\alpha u^\varepsilon||_{L^2(T)} < \infty\}.$$

We will also use some norms built on L^∞. Because the boundary is characteristic, we will need not only the Lipschitz norms but higher order L^∞ control, as O. Guès in [13] and G. Métivier in [20]. We will denote by $L^\infty(T)$ the space $L^\infty(T) := L^\infty((0,T) \times \Omega)$. We will introduce the norms

$$||u||^{*}_{\varepsilon,T} \quad := \sum_{0\leqslant|\alpha|\leqslant 2} ||Z^{\alpha}\, u||_{L^{\infty}(T)} + \sum_{0\leqslant|\alpha|\leqslant 1} ||Z^{\alpha}\, \varepsilon\partial_{\mathtt{n}}\, u||_{L^{\infty}(T)},$$

and the following subsets of $L^{\infty}(T)^{]0,1]}$:

$$\mathbf{\Lambda}^{m}(T) := \{(u^{\varepsilon})_{\varepsilon}/ \quad \sup_{0<\varepsilon\leqslant 1} ||u^{\varepsilon}||^{*}_{\varepsilon,T} < \infty\}.$$

Theorem 8.3 states a propagation result in the spaces $\mathbf{A}^{m}(T) \cap \mathbf{\Lambda}^{m}(T)$.

One quality of our method is that we need approximate solutions with only a few profiles. The minimum number of profiles required is linked to the lost of a factor $\varepsilon^{\frac{1}{2}}$ in a Sobolev embedding Lemma (Lemma 8.10). Let us explain one motivation to minimize the number of profiles needed. In this paper, we consider a ground state $u^{\dagger}$ in $H^{\infty}((0,T_{\dagger}) \times \Omega)$ and formal solutions with H^{∞} regularity. It could also be possible to extend to ground states of high but finite regularity.

3. Forthcoming

We plan to show in a further work that for more general ground states $u^{\dagger}$ which are solutions of (1), which verify the condition $\mathbf{X}_{\mathtt{v}^{\dagger}}\, \mathtt{v}^{\dagger} = 0$ for $(t,x) \in (0,T_{\dagger}) \times \Gamma$ but which do not verify the condition $\mathbf{X}_{\mathtt{v}^{\dagger}}\, \mathtt{p}^{\dagger} = 0$ for $(t,x) \in (0,T_{\dagger}) \times \Gamma$, it is still possible to construct, in small time, some nontrivial formal solutions of (1) but of the more general form

$$\begin{cases} \mathtt{v}^{\varepsilon}_{\mathtt{t}}(t,x) = \sum_{j\geqslant 0} \varepsilon^{j}\, \mathtt{V}^{j}_{\mathtt{t}}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{v}^{\varepsilon}_{\mathtt{n}}(t,x) = \mathtt{v}^{\dagger}_{\mathtt{n}}(t,x) + \sum_{j\geqslant 1} \varepsilon^{j}\, \mathtt{V}^{j}_{\mathtt{n}}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{p}^{\varepsilon}(t,x) = \mathtt{p}^{\dagger}(t,x) + \varepsilon \mathtt{p}^{1}(t,x) + \sum_{j\geqslant 2} \varepsilon^{j}\, \mathtt{P}^{j}(t,x,\frac{\varphi(x)}{\varepsilon}), \\ \mathtt{s}^{\varepsilon}(t,x) = \sum_{j\geqslant 0} \varepsilon^{j}\, \mathtt{S}^{j}(t,x,\frac{\varphi(x)}{\varepsilon}). \end{cases}$$

Moreover we will show that we can prescribe arbitrary initial values for the $(\mathtt{V}^{j}_{\mathtt{t}})_{j\in\mathbb{N}}$ and the $(\mathtt{S}^{j})_{j\in\mathbb{N}}$. However these formal solutions can be unstable.

4. Setting of the notations

To simplify and avoid heavy notations, we will consider from now on that the domain Ω is the half-space $\Omega := \{x \in \mathbb{R}^{d}/\ x_{d} \geqslant 0\}$ and the boundary $\Gamma := \{x \in \mathbb{R}^{d}/\ x_{d} = 0\}$. This assumption does not change the mathematical analysis of the problem.

4.1. **Functions splitting.** We denote $u = (\mathtt{v}, \mathtt{p}, \mathtt{s})$, $v := (\mathtt{v}, \mathtt{p})$ and by $w := \mathtt{s}$. As a consequence there holds $u = (v, w)$. In particular we will split the ground state $u^\dagger$ into $u^\dagger = (v^\dagger, w^\dagger)$. We will split the velocity $\mathtt{v}$ into $\mathtt{v} := (\mathtt{v}_1, ..., \mathtt{v}_d)$. We will often split $\mathcal{U} \in \mathcal{P}(T)$ into $\mathcal{U} = (\mathcal{V}, \mathcal{W})$. The function $\mathcal{V}$ (respectively $\mathcal{W}$) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$). We will sometimes split $\mathcal{V}$ into $\mathcal{V} = (\mathtt{V}, \mathtt{P})$ and $\mathtt{V}$ into $\mathtt{V} := (\mathtt{V}_\mathtt{t}, \mathtt{V}_d)$. The velocity function $\mathtt{V}$ (respectively the pressure function $\mathtt{P}$) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}$). The function $\mathtt{V}_\mathtt{t}$ (respectively $\mathtt{V}_d$) takes its values in $\mathbb{R}^{d-1}$ (resp. $\mathbb{R}$). By abuse of notations, we will say that $\mathcal{V}$, $\mathcal{W}$, $\mathtt{V}$, $\mathtt{V}_\mathtt{t}$, $\mathtt{V}_d$ and $\mathtt{P}$ are in $\mathcal{P}(T)$ even if they do not take values in $\mathbb{R}^{d+2}$. A Taylor expansion proves that there exist a C^∞ function $u^{\dagger,\flat}$ such that $u^\dagger = \dot{u}^\dagger + x_d\, u^{\dagger,\flat}$, where $\dot{u}$ denotes the trace of u on $x_d = 0$. By splitting $u^{\dagger,\flat}$ into $u^{\dagger,\flat} = (v^{\dagger,\flat}, w^{\dagger,\flat})$ this means that there exist two C^∞ functions $v^{\dagger,\flat}$ and $w^{\dagger,\flat}$ such that

$$v^\dagger = \dot{v}^\dagger + x_d\, v^{\dagger,\flat} \text{ and } w^\dagger = \dot{w}^\dagger + x_d\, w^{\dagger,\flat}. \tag{12}$$

4.2. **Symmetric form of the Euler system.** We now put into a symmetric form the Euler system (1) . We fix a notation. If A is a $d+2$ by $d+2$ square matrix, we denote by $A^\star$ the $d+1$ by $d+1$ extracted square matrix which contains the $d+1$ first rows of the $d+1$ first lines. The space derivatives will be denoted ∂_1, ...,∂_d such that $\partial_x = (\partial_1, ..., \partial_d)$. The Euler system (1) is of the form

$$\mathbf{X}_\mathtt{v}\, u + \mathcal{M}(u, \partial_x) u = 0 \tag{13}$$

with, for all $\xi \in \mathbb{R}^d$,

$$\mathcal{M}(u,\xi) := \begin{bmatrix} \mathcal{M}^\star(u,\xi) & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{M}^\star(u,\xi) := \begin{bmatrix} 0 & \rho^{-1}\,\xi \\ \boldsymbol{\alpha}^{-1}\, {}^t\xi & 0 \end{bmatrix}.$$

Remember that we assume that $\rho(\mathtt{p}, \mathtt{s}) > 0$ and $\alpha(\mathtt{p}, \mathtt{s}) > 0$ for all $\mathtt{p}$ and $\mathtt{s}$. We denote by

$$S(u) := \begin{bmatrix} S^\star(u) & 0 \\ 0 & 1 \end{bmatrix} \text{ and } S^\star(u) := \begin{bmatrix} \rho I_d & 0 \\ 0 & \boldsymbol{\alpha} \end{bmatrix}. \tag{14}$$

Multiply Eq. (13) by the matrix $S(u)$ to get the equation

$$\mathfrak{L}(u, \partial_{t,x}) u = 0 \tag{15}$$

where $\mathfrak{L}(u, \partial_{t,x}) := S(u)\mathbf{X}_\mathtt{v} + \mathbf{L}(\partial_x)$ and for all $\xi \in \mathbb{R}^d$,

$$\mathbf{L}(\xi) := S(u)\mathcal{M}(u,\xi) = \begin{bmatrix} \mathbf{L}^\star(\xi) & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{L}^\star(\xi) := \begin{bmatrix} 0 & \xi \\ {}^t\xi & 0 \end{bmatrix}.$$

Eq. (15) is equivalent to the set of the two following equations:

$$\mathfrak{L}^\star(u, \partial_{t,x}) v = 0 \text{ and } \mathbf{X}_\mathtt{v}\, w = 0. \tag{16}$$

We introduce for all $\xi \in \mathbb{R}^d - \{0\}$ the subspace $\mathbb{F}(u,\xi) := \ker \mathcal{M}(u,\xi)$ of $\mathbb{R}^{d+2}$. We denote by $(e_1, ..., e_d)$ the canonical basis of $\mathbb{R}^d$ and we denote $\mathbf{L}_d := \mathbf{L}(e_d)$. We denote by Π the orthogonal projector on $\ker \mathbf{L}_d$ i.e.

$$\Pi := \begin{bmatrix} \Pi^\star & 0 \\ 0 & 1 \end{bmatrix}, \text{ where } \Pi^\star \text{ is a } (d+1) \text{ square matrix } \Pi^\star := \begin{bmatrix} I_{d-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Properties 4.1. *The following properties hold.*

(1) The system (15) *is symmetric hyperbolic since the matrix $S(u)$ is symmetric definite nonnegative and depends in a C^∞ way on u.*
(2) The scalar function $\lambda(u,\xi) := \mathtt{v}.\xi$ is a linearly degenerate eigenvalue with constant multiplicity d i.e. for all vector $u = (\mathtt{v},\mathtt{p},\mathtt{s}) \in \mathbb{R}^{d+2}$, for all vector $\xi \in \mathbb{R}^d - \{0\}$, $\dim \mathbb{F}(u,\xi) = d$ and $r.\partial_u \lambda(u,\xi) = 0$ for all *$r \in \mathbb{F}(u,\xi)$.*
(3) The boundary conditions reads $\mathtt{v}_d = 0$ for $x_d = 0$. They are conservative. This means that for all vector $u = (\mathtt{v},\mathtt{p},\mathtt{s}) \in \mathbb{R}^{d+2}$ with $\mathtt{v}_d = 0$, we have $< \boldsymbol{L}_d u, u >= 0$.

5. Overdetermined ground states

In this section, we prove the existence of ground states $u^\dagger := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \mathtt{s}^\dagger) \in H^\infty$ solutions of (1) and verifying the conditions (5).

Theorem 5.1. *Given $h \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $h'(x)$ is nilpotent in a neighborhood of 0 and such that $h_d(x) = 0$ when $x_d = 0$, there exists a local C^1 solution $\mathtt{v}$ of the initial boundary value problem:*

$$\partial_t \mathtt{v} + (\mathtt{v}.\nabla_x)\mathtt{v} = 0, \quad \mathrm{div}_x \mathtt{v} = 0, \quad \mathtt{v}_d|_{x_d=0} = 0, \quad \mathtt{v}|_{t=0} = h.$$

Proof. Local existence of C^1 solution of multidimensional Burgers equation can be achieved with characteristic method. The classical relation $\mathtt{v}(t, x+th(x)) = h(x)$ holds in a neighborhood of 0. Because $h_d(x) = 0$ when $x_d = 0$, we get $\mathtt{v}_d(t,x) = 0$ for $x_d = 0$. The divergence free relation is a consequence of the nilpotence of $h'(x)$ as proved in Theorem 2.6 of [3]. □

Assume that $d = 2$ for a moment and let us give some examples of initial velocity h which satisfy the assumptions of Theorem 5.1. If h satisfies $h_1(x) = x_2$ and $h_2(x) = 0$ in a neighborhood of 0, then h is convenient. We now detail a more general process to get convenient h. Let F be a function in $C^1(\mathbb{R}_+, \mathbb{R})$ such that $F(0) = 0$ and $F'(x) > 0$ for all $x \in \mathbb{R}_+$. Let $a_{\mathrm{init}} \in C^1(\mathbb{R}_+, \mathbb{R})$. There exists a local solution $a \in C^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ of the scalar initial boundary value problem

$$\partial_1 a + \partial_2 F(a) = 0, \quad a|_{x_2=0} = 0, \quad a|_{x_1=0} = a_{\mathrm{init}}.$$

Consider $h(x) := (a(x), F \circ a(x))$. Then h satisfies the assumptions of Theorem 5.1. It remains to notice that for any function $\mathtt{v}$ given by Theorem 5.1 and for any constant functions $\mathtt{p}$ and $\mathtt{s}$, the function $u := (\mathtt{v},\mathtt{p},\mathtt{s})$ is a solution of (1) which verifies the conditions (5).

Of course, because conditions (5) imply conditions (6), what precedes proves the existence of $T_\dagger > 0$ and of a ground state $u^\dagger := (v^\dagger, w^\dagger) \in H^\infty((0,T_\dagger)\times\Omega)$ solution of (1) and verifying the conditions (6). From now on, we will assume that such a ground state is given.

6. Reduction of the system

We look for solutions of (1) of the form $u^\varepsilon = (v^\varepsilon, w^\varepsilon := W^\varepsilon)$ with

$$v^\varepsilon := v^\dagger + \varepsilon V^\varepsilon \tag{17}$$

We are going to realize a singular change of unknown, by looking for an equation for $U^\varepsilon := (V^\varepsilon, W^\varepsilon)$. We split V^ε into $V^\varepsilon := (\mathtt{V}^\varepsilon, \mathtt{P}^\varepsilon)$. So take care that $\mathtt{V}^\varepsilon$ is the velocity part of V^ε whereas $\mathtt{v}^\varepsilon$ is the velocity part of the original unknown v^ε.

Proposition 6.1. *The function u^ε is solution of the equation* (13) *if and only if the function U^ε verifies the equation*

$$\mathfrak{L}(u^\varepsilon, \partial_{t,x})U^\varepsilon + K^\varepsilon = 0, \tag{18}$$

where

$$K^\varepsilon := \frac{1}{\varepsilon}\begin{bmatrix} K_v(v^\dagger, \partial v^\dagger, u^\varepsilon) \\ 0 \end{bmatrix} \quad \text{with } K_v(v^\dagger, \partial v^\dagger, u^\varepsilon) = \mathfrak{L}^\star(u^\varepsilon, \partial_{t,x})v^\dagger. \tag{19}$$

Proof. The equation (13) is equivalent to the two equations in (16). Expand v^ε thanks to (17) into the first one and divide by ε to end the proof. □

The underlying idea of the previous proposition is that because $v^\dagger$ is given as a ground state, U^ε is the real unknown. For each ε, we obtain that U^ε satisfies a hyperbolic system (see Properties 4.1, (1)). However in order to achieve an asymptotic analysis for $\varepsilon \to 0^+$, we can be *a priori* worried about the singular factor ε^{-1} within K^ε. Let us recall the way [3] deals with this term, by expliciting their calculus for this particular case of the Euler system. As a consequence of (17), there holds: $\mathtt{v}^\varepsilon = \mathtt{v}^\dagger + \varepsilon \mathtt{V}^\varepsilon$. Then use the definition of the particle derivative to see that

$$\mathbf{X}_{\mathtt{v}^\varepsilon}\, v^\dagger = \mathbf{X}_{\mathtt{v}^\dagger}\, v^\dagger + \varepsilon \mathtt{V}^\varepsilon.\partial_x\, v^\dagger.$$

Thus we get

$$K^\varepsilon := \begin{bmatrix} \frac{1}{\varepsilon} S^\star(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\dagger}\, v^\dagger + \frac{1}{\varepsilon}\mathbf{L}^\star(\partial_x)v^\dagger + S^\star(u^\varepsilon)\mathtt{V}^\varepsilon.\partial_x\, v^\dagger \\ 0 \end{bmatrix}.$$

Remember that [3] uses the condition (5) on the ground state $u^\dagger$ which are stronger than the condition (6) we use in the present paper. With the notations of section 4.2, the condition (5) reads $\mathbf{X}_{\mathtt{v}^\dagger}\, v^\dagger = \mathbf{L}^\star(\partial_x)v^\dagger = 0$. As a consequence, under condition (5)

$$K^\varepsilon = \begin{bmatrix} S^\star(u^\varepsilon)\mathtt{V}^\varepsilon.\partial_x\, v^\dagger \\ 0 \end{bmatrix}$$

does not contain any singular factor ε^{-1} anymore. The following proposition will show that under condition (6) the term K^ε does not contain any singular factor ε^{-1} too. In order to help the reader, we already give some hints about our strategy. We will search W^ε of the form

$$W^\varepsilon(t,x) = \mathcal{W}^0(t,x,\frac{x_d}{\varepsilon}) + \varepsilon W^{\varepsilon,\diamond}(t,x). \tag{20}$$

with $\mathcal{W}^0 \in \mathcal{P}(T)$ (cf. (8) for the definition) and $\underline{\mathcal{W}}^0 = w^\dagger$. We will expand W^ε, into $W^\varepsilon = w^\dagger + \tilde{\mathcal{W}}^0 + \varepsilon W^{\varepsilon,\diamond}$ within $S^\star(u^\varepsilon)$. Imagine in a first time that the boundary layer profile $\tilde{\mathcal{W}}^0$ is identically zero. Then from (19) we obtain by a Taylor first order expansion that

$$K^\varepsilon = \begin{bmatrix} \frac{1}{\varepsilon}\mathfrak{L}(u^\dagger, \partial_{t,x})v^\dagger + \text{terms of order } \varepsilon^0 \\ 0 \end{bmatrix}.$$

Because the ground state $u^\dagger$ satisfies (1), the first term in the right member is equal to zero and K^ε does not appear singular with respect to ε anymore. Of course we want to deal with some nonvanishing function $\tilde{\mathcal{W}}^0$. A key difference with paper [3] is that here $\tilde{\mathcal{W}}^0$ denotes a boundary layer and we will see in the following proposition that its singular contribution within K^ε contains the trace of $\mathbf{X}_{\mathtt{v}^\dagger} v^\dagger$ on the boundary $\Gamma := \{x_d = 0\}$ as a factor. Therefore the localized conditions (6) allow to conclude. In prevision of the following sections, we will be careful with the way the term K^ε depends of V^ε and $W^{\varepsilon,\diamond}$.

In order to avoid heavy notations, we will omit the two first arguments and write $K_v^\varepsilon(u^\varepsilon)$ instead of $K_v^\varepsilon(v^\dagger, \partial v^\dagger, u^\varepsilon)$. Furthermore, we want now to use the special form of the solutions u^ε we are looking for. Thus we will write $K_v^\varepsilon(v^\varepsilon, w^\varepsilon)$. Remember that a Taylor expansion proves that there exist two C^∞ functions $v^{\dagger,\flat}$ and $w^{\dagger,\flat}$ such that $v^\dagger = \dot{v}^\dagger + x_d\, v^{\dagger,\flat}$ and $w^\dagger = \dot{w}^\dagger + x_d\, w^{\dagger,\flat}$, where $\dot{u}$ is the trace of u on $x_d = 0$ (cf. (12)).

Proposition 6.2. *Assume that W^ε is of the form $W^\varepsilon(t,x) = \mathcal{W}^0(t, x, \frac{x_d}{\varepsilon}) + \varepsilon W^{\varepsilon,\diamond}(t,x)$ where $\mathcal{W}^0 \in \mathcal{P}(T)$ with $\underline{\mathcal{W}}^0 = w^\dagger$. Then there is a C^∞ matrix $K_v^\flat$ such that*

$$K_v(u^\varepsilon) = \varepsilon K_v^\flat(\varepsilon, x_d, \dot{v}^\dagger, v^{\dagger,\flat}, V^\varepsilon, \varepsilon V^\varepsilon, \dot{w}^\dagger, w^{\dagger,\flat}, \tilde{\mathcal{W}}^0, \frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0, W^{\varepsilon,\diamond}, \varepsilon W^{\varepsilon,\diamond}) \tag{21}$$

where $K_v^\flat$ is affine with respect to its fifth argument V^ε and affine with respect to its eleventh argument $W^{\varepsilon,\diamond}$ with $\boldsymbol{X}_{\mathtt{v}^\dagger} v^\dagger$ as factor in the leading coefficient. In (21), to avoid heavy notations we denote $\tilde{\mathcal{W}}^0$ instead of $\tilde{\mathcal{W}}^0(t, x, \frac{x_d}{\varepsilon})$.

Proof. First we give a technical lemma.

Lemma 6.3. *There exist some $d+1$ square matrices $K_{1,1}^\flat(u_1, u_2)$ and some vectors $K_{1,2}^\flat(u_1, u_2)$ in $\mathbb{R}^{d+1}$ both C^∞ with respect to their arguments $u_1 := (v_1, w_1)$ and $u_2 := (v_2, w_2)$ which belong to $\mathbb{R}^{d+1} \times \mathbb{R}$, such that $K_{1,2}^\flat$ has $\boldsymbol{X}_{\mathtt{v}_1} v^\dagger$ as factor and that*

$$K_v(u_2 + u_2) = K_v(u_1) + K_{v,1}^\flat(u_1, u_2).v_2 + w_2\, K_{v,2}^\flat(u_1, u_2). \tag{22}$$

Proof. We will proceed in two steps. We will split any vector $v_2 \in \mathbb{R}^{d+1}$ into $v_2 := (\mathtt{V}_2, \mathtt{P}_2)$ where $\mathtt{V}_2$ is in $\mathbb{R}^d$ and $\mathtt{P}_2$ is in $\mathbb{R}$.

1. Thanks to the definition of the matrix S in (14) a Taylor first order expansion yields the existence of $d+1$ square matrices $S_1^{\star,\flat}(u_1, u_2)$ and $S_2^{\star,\flat}(u_1, u_2)$ such that for all (u_1, u_2),

$$S^\star(v_1 + v_2, w_1 + w_2) = S^\star(u_1) + \mathtt{P}_2.S_1^{\star,\flat}(u_1, u_2) + w_2.S_2^{\star,\flat}(u_1, u_2). \tag{23}$$

2. We introduce, for all (u_1, u_2), the $d+1$ by $d+1$ matrix $K_{v,1}^\flat(u_1, u_2)$ such that for all $z = (\mathtt{z}, z_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$,

$$K_{v,1}^\flat(u_1, u_2).z \quad := \quad z_{d+1}\,.S_1^{\star,\flat}(u_1, u_2).\mathbf{X}_{\mathtt{v}_1+\mathtt{v}_2}\, v^\dagger + (S^\star(u_1) + w_2.S_2^{\star,\flat}).\mathtt{z}.\partial_x\, v^\dagger,$$

and the vector

$$K_{v,2}^\flat(u_1, u_2) := S_2^{\star,\flat}(u_1, u_2).\mathbf{X}_{\mathtt{v}_1}\, v^\dagger \in \mathbb{R}^{d+1}.$$

As by definition, for all $u := (v, w) \in \mathbb{R}^d \times \mathbb{R}$,

$$K_v(u) := S^\star(u)\mathbf{X}_{\mathtt{v}}\, v^\dagger + \mathbf{L}^\star(\partial_x)v^\dagger, \tag{24}$$

we get for all (u_1, u_2),

$$K_v(u_1 + u_2) - K_v(u_1) = S^\star(u_1 + u_2)\mathbf{X}_{\mathtt{v}_1+\mathtt{v}_2}\, v^\dagger - S^\star(u_1)\mathbf{X}_{\mathtt{v}_1}\, v^\dagger.$$

Thanks to Eq. (23) and because $\mathbf{X}_{\mathtt{v}_1+\mathtt{v}_2} v^\dagger = \mathbf{X}_{\mathtt{v}_1} v^\dagger + \mathtt{v}_2.\partial_x\, v^\dagger$, we get (22).

The proof of Lemma 6.3 is now completed. □

Thanks to Lemma 6.3 (and its proof), we get the following result.

Lemma 6.4. *There exists a C^∞ function $K_v^{\flat,2}$ such that*

$$\begin{aligned} K_v(v^\dagger,\mathcal{W}^0) - K_v(u^\dagger) &= \big(S^\star(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) - S^\star(\dot u^\dagger)\big)\boldsymbol{X}_{\dot{\mathrm{v}}^\dagger}\dot v^\dagger \\ &\quad +\varepsilon K_v^{\flat,2}(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal{W}}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0). \end{aligned} \tag{25}$$

Proof. We will proceed in four steps.

1. We use Lemma 7.6 and apply respectively (22) to

$$(u_1,u_2) = (\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0, x_d\, u^{\dagger,\flat}) \text{ and to } (u_1,u_2)=(\dot u^\dagger, x_d\, u^{\dagger,\flat}).$$

Thanks to (12) we get

$$\begin{cases} K_v(v^\dagger,\mathcal{W}^0) = K_v(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) + x_d K_v^a(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal{W}}^0), \\ K_v(u^\dagger) = K_v(\dot u^\dagger) + x_d\, K_v^b(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat}), \end{cases} \tag{26}$$

with

$$\begin{aligned} K_v^a(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal{W}}^0) &:= v^{\dagger,\flat}K_{v,1}^\flat(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0,x_d\,u^{\dagger,\flat}) \\ &\quad + w^{\dagger,\flat}K_{v,2}^\flat(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0,x_d\,u^{\dagger,\flat}), \\ K_v^b(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat}) &:= v^{\dagger,\flat}K_{v,1}^\flat(\dot u^\dagger,x_d\,u^{\dagger,\flat}) + w^{\dagger,\flat}K_{v,2}^\flat(\dot u^0,x_d\,u^{\dagger,\flat}). \end{aligned}$$

2. By a first order Taylor expansion, we obtain that there exist some $(d+1)$ square matrices $G_{1,1}^\flat$ and some $G_{1,2}^\flat\in\mathbb{R}^{d+1}$, C^∞ with respect to their arguments

$$u_1 := (v_1,w_1)\ ,u_2:=(v_2,w_2)\in\mathbb{R}^{d+1}\times\mathbb{R} \text{ and } w_3\in\mathbb{R}$$

such that

$$\begin{aligned} K_{v,1}^\flat(v_1,w_1+w_3,u_2) - K_{v,1}^\flat(u_1,u_2) &= w_3.G_{1,1}^\flat(u_1,u_2,w_3), \\ K_{v,2}^\flat(v_1,w_1+w_3,u_2) - K_{v,2}^\flat(u_1,u_2) &= w_3.G_{1,2}^\flat(u_1,u_2,w_3). \end{aligned}$$

3. We refer to (19) and use (12) to get

$$K_v(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) - K_v(\dot u^0) = T_1+T_2$$

where

$$\begin{aligned} T_1 &:= (S^\star(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) - S^\star(\dot u^0))\mathbf{X}_{\dot{\mathrm{v}}^\dagger}\,\dot v^\dagger, \\ T_2 &:= (S^\star(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) - S^\star(\dot u^0))\mathbf{X}_{\dot{\mathrm{v}}^\dagger}\,(x_d\, v^{\dagger,\flat}) \\ &= x_d(S^\star(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal{W}}^0) - S^\star(\dot u^0))\mathbf{X}_{\dot{\mathrm{v}}^\dagger}\, v^{\dagger,\flat}. \end{aligned}$$

Apply (23) to $u_1:=\dot u^\dagger$ and $u_2:=\dot u^\dagger+(0,\tilde{\mathcal{W}}^0)$ to get

$$T_2 = \varepsilon\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0.S_2^{\star,\flat}(\dot u^\dagger,\dot u^\dagger+(0,\tilde{\mathcal{W}}^0))\mathbf{X}_{\dot{\mathrm{v}}^\dagger}\,v^{\dagger,\flat}$$

4. Thanks the previous points, we obtain (25) with

$$K_v^{\flat,2}(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal W}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal W}^0)=\frac{T_2}{\varepsilon}+\frac{x_d}{\varepsilon}\tilde{\mathcal W}^0\{v^{\dagger,\flat}.G^\flat_{1,1}(\dot u^\dagger,x_d\,u^{\dagger,\flat},\tilde{\mathcal W}^0)+w^{\dagger,\flat}G^\flat_{1,2}(\dot u^0,x_d\,u^{\dagger,\flat},\tilde{\mathcal W}^0)\}.$$

Because $G^\flat_{1,1}$, $G^\flat_{1,2}$ and $S^{\star,2}$ are C^∞, so is the function $K_v^{\flat,2}$.
The proof of Lemma 6.4 is now completed. □

This technical results given, we will proceed in three steps.

Step 1 (First order expansion)**.** *We are going to prove that there exists a C^∞ function $K_v^{\flat,1}$, affine with respect to its third variable and affine with respect to its sixth variable with $\boldsymbol{X}_{\mathrm{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient such that*

$$K_v(u^\varepsilon)=K_v(v^\dagger,\mathcal W^0)+\varepsilon K_v^{\flat,1}(\varepsilon,v^\dagger,V^\varepsilon,\varepsilon V^\varepsilon,\mathcal W^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond}). \tag{27}$$

We use Lemma 7.6 by applying (22) to $(u_1,u_2)=\chi$ where we denote

$$\chi:=(v^\dagger,\mathcal W^0,\varepsilon V^\varepsilon,\varepsilon W^{\varepsilon,\diamond}),$$

we get (27) with

$$K_v^{\flat,1}(\varepsilon,v^\dagger,V^\varepsilon,\varepsilon V^\varepsilon,\mathcal W^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond}) \;:=\; V^\varepsilon K^\flat_{v,1}(\chi)+W^{\varepsilon,\diamond}\,K^\flat_{v,2}(\chi).$$

It is clear that the function $K_v^{\flat,1}$ is affine with respect to its third variable and affine with respect to its sixth variable with $\mathbf{X}_{\mathrm{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient.

Step 2 (Do $x_d=0$)**.** *We are going to prove that there exists a C^∞ function $K_v^\flat$, affine with respect to its fifth argument and affine with respect to its eleventh argument with $\boldsymbol{X}_{\mathrm{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient, such that*

$$\begin{aligned}K_v(u^\varepsilon)\;&=\;K_v(u^\dagger)+(S^\star(\dot v^\dagger,\dot w^\dagger+\tilde{\mathcal W}^0)-S^\star(\dot u^\dagger))\boldsymbol{X}_{\dot{\mathrm{v}}^\dagger}\,\dot v^\dagger\\&\quad+\varepsilon K_v^\flat(\varepsilon,x_d,\dot v^\dagger,v^{\dagger,\flat},V^\varepsilon,\varepsilon V^\varepsilon,\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal W}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal W}^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond}).\end{aligned} \tag{28}$$

Thanks to (27) and to Lemma 6.4, we get (28) where

$$K_v^\flat(\varepsilon,x_d,\dot v^\dagger,v^{\dagger,\flat},V^\varepsilon,\varepsilon V^\varepsilon,\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal W}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal W}^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond}):=$$
$$K_v^{\flat,2}(x_d,\dot v^\dagger,v^{\dagger,\flat},\dot w^\dagger,w^{\dagger,\flat},\tilde{\mathcal W}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal W}^0)+K_v^{\flat,1}(\varepsilon,v^\dagger,V^\varepsilon,\varepsilon V^\varepsilon,\mathcal W^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond})$$

Because the first term does not involve V^ε and $W^{\varepsilon,\diamond}$ and the function $K_v^{\flat,1}$ is affine with respect to its third variable V^ε and affine with respect to its sixth variable $W^{\varepsilon,\diamond}$ with $\mathbf{X}_{\mathrm{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient, the function $K_v^\flat$ is affine with respect to its fifth variable V^ε and affine with respect to its eleventh argument $W^{\varepsilon,\diamond}$ with $\mathbf{X}_{\mathrm{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient. Note that as the profile $\tilde{\mathcal W}_0$ is rapidly decreasing in X, the profile $X\tilde{\mathcal W}_0$ is also rapidly decreasing.

Step 3. *Use* (24) *and the ground state properties* (1) *and* (6) *to get that the two first terms of the right side of* (28) *vanish.*

The proof of Proposition 6.2 is now completed. □

7. WKB expansions

7.1. **Formal solutions.** We look for formal solutions $(u^\varepsilon)_\varepsilon$ of (1) of the form $u^\varepsilon = (v^\dagger + \varepsilon V^\varepsilon, W^\varepsilon)$ where $U^\varepsilon := (V^\varepsilon, W^\varepsilon)$ is an expansion

$$\sum_{n\geqslant 0} \varepsilon^n \,\mathcal{U}^n(t, x, \frac{x_d}{\varepsilon}) \tag{29}$$

where each $\mathcal{U}^n$ belongs to $\mathcal{P}(T)$ (cf. (8) for the definition). We rewrite $\mathcal{U}^n$ as $\mathcal{U}^n = (\mathcal{V}^n, \mathcal{W}^n)$ and we suppose that the regular part $\underline{\mathcal{W}}^0$ of the profile $\mathcal{W}^0$ is equal to the ground state $w^\dagger$ i.e. $\underline{\mathcal{W}}^0 := w^\dagger$. Let us explain what is meant by *formal solutions*. Plugging the expansion (29) into the system, using Taylor expansions and ordering the terms in powers of ε, we get a formal expansion in power series of ε:

$$\sum_{n\geqslant -1} \varepsilon^n \,\Phi^n(t, x, \frac{x_d}{\varepsilon}) \tag{30}$$

where the $(\Phi^n)_{n\geqslant -1}$ are in $\mathcal{P}(T)$. We say that $(u^\varepsilon)_\varepsilon$ is a formal solution when all the resulting Φ^n are identically zero. The following theorem states that the system has formal solutions and that we can prescribe arbitrary initial values to the $(\Pi\,\mathcal{U}^j|_{t=0})_{j\in\mathbb{N}}$ (see section 4 for the definition of Π). We introduce $\mathcal{N}_{\text{init}} := H^\infty(\mathbb{R}^d_+, \mathcal{S}(\mathbb{R}_+))$ and the profile space

$$\begin{aligned}\mathcal{P}_{\text{init}} := \{\mathcal{U} \in L^2(\mathbb{R}^d_+ \times \mathbb{R}_+) \text{ such that there exist } \underline{\mathcal{U}} \in H^\infty(\mathbb{R}^d_+),\ \tilde{\mathcal{U}} \in \mathcal{N}_{\text{init}}\\ \text{such that for all } (x, X) \in \mathbb{R}^d_+ \times \mathbb{R}_+, \quad \mathcal{U}(x, X) = \underline{\mathcal{U}}(x) + \tilde{\mathcal{U}}(x, X)\}.\end{aligned}$$

Theorem 7.1. *Assume that some profiles $(\mathcal{U}^j_{init})_{j\in\mathbb{N}}$ such that $(Id-\Pi)\mathcal{U}^j_{init} = 0$ and $\underline{\mathcal{W}}^0_{init} = w^\dagger|_{t=0}$ are given, then there exists a formal solution (u^ε) of (1) on $(0, T_\dagger)$ with some profiles $(\mathcal{U}^j)_{j\in\mathbb{N}}$ in $\mathcal{P}(T_\dagger)$ such that $\underline{\mathcal{W}}^0 := w^\dagger$ and that for all $j \in \mathbb{N}$, $\Pi\,\mathcal{U}^j|_{t=0} = \mathcal{U}^j_{init}$. Moreover the profile $\mathcal{U}^0$ is polarized in the sense that $(Id - \Pi)\tilde{\mathcal{U}}^0 = 0$ for all $(t, x) \in (0, T_\dagger) \times \Omega$.*

Notice that Theorem 7.1 gives the existence of a formal solution till $T_\dagger$. At first sight, this may seems strange because Theorem 7.1 deals with large boundary layers and in some similar settings large boundary layers have in general a nonlinear behavior (see for example paper [24] of G. Métivier and K. Zumbrun about large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems and paper [11] of E.Grenier about large velocity boundary layers for the Euler equations). It is also different from the result on entropy waves of C. Cheverry, O. Guès and G. Métivier [3]. In [3], Theorem 3.4 gives the local existence of formal solutions with large amplitude oscillations on the entropy. More precisely Theorem 3.4 of [3] gives the existence of the profile $\mathcal{U}^0$ for small time and we see on the system (3.36) of [3] that this is not because of a lack of analysis but because of an actual nonlinear effect. At the opposite in the proof of Theorem 7.1 we will construct a formal solution thanks to linear profile problems. This shows that *a system can reveals additional transparency properties when looking at boundary layers instead of high frequency oscillations.* The transparency property of [38] supports this remark.[6]

Since the profile $\mathcal{U}^0$ is polarized, we get that the normal velocity and the pressure boundary layer profiles vanish on the boundary i.e. $\tilde{\mathtt{V}}^0_d = \tilde{\mathtt{P}}^0 = 0$ as we have mentioned it in the

[6]This phenomenon can also be observed when looking at boundary layers of "strong amplitude", following the terminology of [4]. Instead of nonlinear modulation equation as in [3], we obtain linear profile equation.

introduction. This means that the amplitude of normal velocity and pressure boundary layers is weaker than the amplitude of tangential boundary layer. The fact that the boundary layer profile $\tilde{\mathtt{V}}_d^0$ vanishes is a consequence of the conditions (6) and more precisely of the fact that the particle derivative of the ground state pressure vanishes on the boundary i.e. $\mathbf{X}_{\mathtt{v}^\dagger}\,\mathtt{p}^\dagger = 0$ for $(t,x) \in (0,T_\dagger)\times\Gamma$. Without this last condition, it is still possible to construct some nontrivial formal solutions of the form (29) but with a nonvanishing profile $\tilde{\mathtt{V}}_d^0$ (cf. section 3). However such formal solutions can be unstable. We do not consider such formal solutions in this paper.

Next subsection is devoted to the proof of Theorem 7.1. We outline here some key tools of the proof. First we use the reduction of the system of section 3. We use in a essential way Proposition 6.2. For example because K_v contains a factor ε, the term K^ε is not singular with respect to ε. This is crucial because we look for formal solutions without large boundary layers for v^ε but only with large boundary layers for V^ε. In order to find the expansion (30), we use several Taylor expansions with respect to ε and to x_d. The underlying idea is to obtain an expansion (30) inside which the profiles $(\mathcal{U}^j)_{j\in\mathbb{N}}$ appear at the highest possible order. We notably use the fact that $\mathtt{v}_d^\dagger$ and $\mathbf{X}_{\mathtt{v}^\dagger}\,v^\dagger$ vanish on the boundary and the fact that $K_v^\flat$ is affine with respect to its eleventh argument with $\mathbf{X}_{\mathtt{v}^\dagger}\,v^\dagger$ as factor in the leading coefficient (cf. Proposition 6.2 and Lemma 7.8). In order to find some profiles $(\mathcal{U}^j)_{j\in\mathbb{N}}$ for which the profiles $(\Phi^j)_{j\in\mathbb{N}}$ of the resulting expansion (30) are identically zero, we follow the strategy of [16], [37], [36], [38] and exhibit a sequence of profile problem. Theorem 7.10 gives the existence of profiles $(\mathcal{U}^j)_{j\in\mathbb{N}}$ by solving this sequence of problem. Theorem 7.1 will be proved as a consequence of theorem 7.10. One key point in the proof of Theorem 7.10 is that profile problems are linear, thanks to transparency properties. One manifestation of this property is that the right side of (21) is affine with respect to its fifth argument V^ε. The profiles problems are linearized Euler equations for the regular part of the profiles and linear totally characteristic transport equations for the boundary layer part of the profiles.

7.2. **Proof of Theorem** 7.1**.** We will proceed in two steps. First we explicit the expansion (30). Then we look for profiles $(\mathcal{U}^j)_{j\in\mathbb{N}}$ for which the profiles $(\Phi^j)_{j\in\mathbb{N}}$ of the resulting expansion (30) are identically zero by solving a sequence of profile problems.

7.2.1. *Expliciting the expansion* (30)*.* We introduce the operator $\mathcal{H}^0$ which, for any function $W^\diamond \in H^\infty((0,T)\times\mathbb{R}^d_+)$, with real values, for any boundary layer profile $\tilde{\mathcal{W}}^0 \in \mathcal{N}(T)$ (cf. (2) for a definition), maps a function $U=(V,W)\in H^\infty((0,T)\times\mathbb{R}^d_+)$, where the function V (respectively W) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$), to the function

$$\mathcal{H}^0(U,W^\diamond,\tilde{\mathcal{W}}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0,\partial_{t,x})U := \mathfrak{L}(v^\dagger,W,\partial_{t,x})U + \begin{bmatrix} K^0(V,\tilde{\mathcal{W}}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0,W^\diamond) \\ 0 \end{bmatrix} \tag{31}$$

in $H^\infty((0,T)\times\mathbb{R}^d_+)$ with

$$K^0(V,\tilde{\mathcal{W}}_0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0,W^\diamond) := K_v^\flat(0,x_d,\dot{v}^\dagger,v^{\dagger,\flat},V,0,\dot{w}^\dagger,\dot{w}^{0,\flat},\tilde{\mathcal{W}}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0,W^\diamond,0).$$

Remember that the term $K_v^\flat$ is defined in Proposition 6.2. As in (21), we denote $\tilde{\mathcal{W}}^0$ instead of $\tilde{\mathcal{W}}^0(t,x,\frac{x_d}{\varepsilon})$.

Lemma 7.2. *The function $K^0(V,\tilde{W},X\tilde{W},W^\diamond)$ is affine with respect to its first argument V and affine with respect to its fourth argument $W^\diamond$ with $\boldsymbol{X}_{\mathrm{v}^\dagger}\, v^\dagger$ as factor in the leading coefficient.*

Proof. It is a straightforward consequence of Prop 6.2. □

Next proposition explicites the profiles $(\Phi^j)_{j\in\mathbb{N}}$.

Proposition 7.3. *We obtain*

$$\begin{cases} \underline{\Phi}^{-1} &= \quad 0, \\ \tilde{\Phi}^{-1} &= \boldsymbol{L}_d\,\partial_X \tilde{\mathcal{U}}^0, \end{cases}, \quad \begin{cases} \underline{\Phi}^{0} &= \quad \mathcal{H}^0(\underline{\mathcal{U}}^0,\underline{\mathcal{W}}^1,0,0,\partial_{t,x})\underline{\mathcal{U}}^0 \\ \tilde{\Phi}^{0} &= \boldsymbol{L}_d\,\partial_X \tilde{\mathcal{U}}^1 + \mathcal{H}'(\mathcal{U}^0,\partial_{t,x,X})\tilde{\mathcal{U}}^0 \end{cases}$$

and, for $j\geqslant 1$,

$$\begin{cases} \underline{\Phi}^{j} &= \quad \mathcal{H}^0(\underline{\mathcal{U}}^0,\underline{\mathcal{W}}^1,0,0,\partial_{t,x})\underline{\mathcal{U}}^j - \underline{\mathcal{Q}}^j, \\ \tilde{\Phi}^{j} &= \boldsymbol{L}_d\,\partial_X \tilde{\mathcal{U}}^{j+1} + \mathcal{H}'(\mathcal{U}^0,\partial_{t,x,X})\tilde{\mathcal{U}}^j - \tilde{\mathcal{Q}}^j. \end{cases}$$

The operator $\mathcal{H}'$ maps a function $\mathcal{U}\in\mathcal{P}(T)$ to

$$\mathcal{H}'(\mathcal{U},\partial_{t,x,X})\tilde{\mathcal{U}} := \Big((X\mathrm{v}_d^{\dagger,\flat} + \underline{\dot{\mathrm{v}}}_d + \tilde{\mathrm{v}}_d)S(v^\dagger,\mathcal{W})\partial_X + \mathfrak{L}(v^\dagger,\mathcal{W},\partial_{t,x})\Big)\tilde{\mathcal{U}}$$
$$+(S(v^\dagger,\mathcal{W}) - S(v^\dagger,\underline{\mathcal{W}}))\boldsymbol{X}_{\mathrm{v}^\dagger}\,\underline{\mathcal{U}} + \begin{bmatrix} K_I^0(\mathcal{V},\tilde{\mathcal{W}},X\tilde{\mathcal{W}}) \\ 0 \end{bmatrix}, \tag{32}$$

which belong to $\mathcal{N}(T)$. Above $K_I^0(V,W,XW)$ is a C^∞ function affine with respect to its first argument V such that $K_I^0(\mathcal{V}^0,\tilde{\mathcal{W}}^0,X\tilde{\mathcal{W}}^0)\in\mathcal{N}(T)$. The terms $\underline{\mathcal{Q}}^j$ depend on (t,x), on the profiles $(\underline{\mathcal{U}}^k)_{k<j}$ and on their derivatives and affinely on the profiles $\underline{\mathcal{U}}^j$ and $\underline{\mathcal{W}}^{j+1}$. We split $\underline{\mathcal{Q}}^j$ into $\underline{\mathcal{Q}}^j := (\underline{\mathcal{Q}}^j_v,\underline{\mathcal{Q}}^j_w)$ where $\underline{\mathcal{Q}}^j_v$ (resp. $\underline{\mathcal{Q}}^j_w$) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$). In particular, $\underline{\mathcal{Q}}^1_w = -\underline{\mathrm{V}}^0.\partial_x\underline{\mathcal{W}}^0$. The terms $\tilde{\mathcal{Q}}^j$ depend of (t,x), of the profiles $(\mathcal{U}^k)_{k<j}$, of their traces and of their derivatives and affinely of the profiles $\mathcal{U}^j$ and $\underline{\mathcal{W}}^{j+1}$.

Proof. We begin with some preliminary remarks which will be useful in the whole proof.

(a). Because the boundary layer terms $\tilde{\mathcal{U}}^j(t,x,\frac{x_d}{\varepsilon})$ contain a factor ε^{-1} in their argument, the normal derivative ∂_d plays a crucial role.

(b). Moreover a key point when calculating the $\tilde{\Phi}^j$ is that when a boundary layer profile is in factor, we can do formally "$x_d = 0$". The underlying idea is that for a C^∞ scalar function $\varrho(x_d)$ and $\tilde{\mathcal{U}}\in\mathcal{N}(T)$ using a Taylor expansion yields the existence of a C^∞ scalar function $\varrho^\flat(x_d)$ such that $\varrho(x_d)=\varrho(0)+x_d\varrho^\flat(x_d)$. Then

$$\varrho(x_d)\tilde{\mathcal{U}}(t,x,\frac{x_d}{\varepsilon}) = \varrho(0)\tilde{\mathcal{U}}(t,x,\frac{x_d}{\varepsilon}) + \varepsilon\varrho^\flat(x_d)(X\tilde{\mathcal{U}}(t,x,X))|_{X=\frac{x_d}{\varepsilon}}.$$

Because $X\tilde{\mathcal{U}}(t,x,X)$ is still in $\mathcal{N}(T)$, the second term can be put with the term of order ε. If $\varrho(0)=0$, then you obtain that $\tilde{\mathcal{U}}$ does not appear in the resulting expansion at the order 0. We are going to apply several times this idea.

Let us now turn to the proof. Motivated by (18) we introduce the operator $\mathcal{H}^\varepsilon$ by the formula

$$\mathcal{H}^\varepsilon(U,\partial_{t,x})U := \mathfrak{L}(v^\dagger+\varepsilon V,W,\partial_{t,x})U + K^\varepsilon. \tag{33}$$

Remember that the term K^ε is defined in Proposition 6.1. We will proceed in 5 steps. We begin to tackle the term $\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})$.

Step 1. *Expansion of* $\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})$.

We are going to expand the term $\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})$ in order to reduce its study to the one of the term $\mathfrak{L}(v^\dagger, W, \partial_{t,x})$ and to the study of a remainder which is introduced in the following lemma. Remember that we denote $v^\dagger = (\mathtt{v}^\dagger, \mathtt{p}^\dagger)$ (cf. section 4). We split the velocity ground state $\mathtt{v}^\dagger$ into $\mathtt{v}^\dagger = (\mathtt{v}^\dagger_1, ..., \mathtt{v}^\dagger_d)$. In the same way, we split V into $V = (\mathtt{V}, \mathtt{P})$ and $\mathtt{V}$ into $\mathtt{V} = (\mathtt{V}_1, ..., \mathtt{V}_d)$. In the following lemma, we will also consider some functions $U \in H^\infty((0,T) \times \mathbb{R}^d_+)$, with values in $\mathbb{R}^{d+2}$. We will split them into $U = (V, W)$ where the function V (respectively W) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$).

Lemma 7.4. *There exist* $(d+2) \times (d+2)$ *matrices* B^ε_i *such that for any function* $U = (V, W) \in H^\infty((0,T) \times \mathbb{R}^d_+)$,

$$\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})U = \mathfrak{L}(v^\dagger, W, \partial_{t,x})U + \sum_{i=0}^{d} B^\varepsilon_i(v^\dagger, U)\varepsilon \partial_i U. \tag{34}$$

Moreover the B^ε_i *depend in a* C^∞ *way on their arguments including* ε *up to* $\varepsilon = 0$. *For* $i = 0$, *the matrices* B^ε_0 *are of the form* $B^\varepsilon_0 = \begin{bmatrix} B^{\star,\varepsilon}_0 & 0 \\ 0 & 0 \end{bmatrix}$ *and, for* $\varepsilon = 0$, $B^0_0 = DS(v^\dagger, W).(V, 0)$. *For* $1 \leqslant i \leqslant d$, *they are of the form* $B^\varepsilon_i := \begin{bmatrix} B^{\star,\varepsilon}_i & 0 \\ 0 & \mathtt{V}_i \end{bmatrix}$, *for* $\varepsilon = 0$,

$$B^0_i(v^\dagger, U) := \mathtt{V}_i\, S(v^\dagger, W) + \mathtt{v}^\dagger_i\, DS(v^\dagger, W).(V, 0). \tag{35}$$

Proof. By definition (cf. section 4),

$$\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x}) = S(v^\dagger + \varepsilon V, W)\mathbf{X}_{\mathtt{v}^\dagger + \varepsilon \mathtt{V}} + \mathbf{L}(\partial_x).$$

Expand to get

$$\begin{aligned}\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x}) = \mathfrak{L}(v^\dagger, W, \partial_{t,x}) &+ S(v^\dagger + \varepsilon V, W)(\mathbf{X}_{\mathtt{v}^\dagger + \varepsilon \mathtt{V}} - \mathbf{X}_{\mathtt{v}^\dagger}) \\ &+ (S(v^\dagger + \varepsilon V, W) - S(v^\dagger, W))\mathbf{X}_{\mathtt{v}^\dagger}\end{aligned}$$

and (34) with

$$\begin{aligned} B^\varepsilon_0 &:= \varepsilon^{-1}(S(v^\dagger + \varepsilon V, W) - S(v^\dagger, W)) \text{ and, for } 1 \leqslant i \leqslant d, \\ B^\varepsilon_i &:= S(v^\dagger + \varepsilon V, W)\mathtt{V}_i + \varepsilon^{-1}(S(v^\dagger + \varepsilon V, W) - S(v^\dagger, W))\mathtt{v}^\dagger_i. \end{aligned}$$

Then use a first order Taylor expansion with respect to ε and the special form of S (cf. section 4) to conclude. □

Therefore we begin to study the first term in the right side of (34) namely $\mathfrak{L}(v^\dagger, W, \partial_{t,x})U$.

Step 2. *Action of* $\mathfrak{L}(v^\dagger, W, \partial_{t,x})$.

We are going to look at the action of $\mathfrak{L}(v^\dagger, W, \partial_{t,x})$ on functions $\mathcal{U}^\varepsilon$ in $\mathcal{P}(T)$. The following lemma describe the resulting function. We will split $\mathcal{U}^\varepsilon$ into $\mathcal{U}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$. The function $\mathcal{V}^\varepsilon$ (respectively $\mathcal{W}^\varepsilon$) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$). We will also split $\mathcal{V}^\varepsilon$ into $\mathcal{V}^\varepsilon = (\mathtt{V}^\varepsilon, \mathtt{P}^\varepsilon)$

and $\mathtt{V}^\varepsilon$ into $\mathtt{V}^\varepsilon := (\mathtt{V}^\varepsilon_1, ..., \mathtt{V}^\varepsilon_d)$. The function $\mathtt{V}^\varepsilon$ (respectively $\mathtt{P}^\varepsilon$) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}$).

Lemma 7.5. *Let $\mathcal{U}^\varepsilon$ be a family of functions in $\mathcal{P}(T)$ and denote by $\mathbf{u}^\varepsilon$ and f^ε the following functions:*

$$\mathbf{u}^\varepsilon(t,x) := \mathcal{U}^\varepsilon(t,x,\frac{x_d}{\varepsilon}) \text{ and } f^\varepsilon := \mathfrak{L}(v^\dagger, \mathcal{W}^\varepsilon(t,x,\frac{x_d}{\varepsilon}), \partial_{t,x})\mathbf{u}^\varepsilon.$$

Then there exists a family of functions F^ε in $\mathcal{P}(T)$ such that

$$f^\varepsilon(t,x) = \frac{1}{\varepsilon}\boldsymbol{L}_d\,\partial_X \tilde{\mathcal{U}}^\varepsilon(t,x,\frac{x_d}{\varepsilon}) + F^\varepsilon(t,x,\frac{x_d}{\varepsilon}).$$

Moreover, the function F^ε splits into $F^\varepsilon = \underline{F}^\varepsilon + \tilde{F}^\varepsilon$, where the interior part and the boundary layer part are respectively

$$\begin{aligned}
\underline{F}^\varepsilon &:= \mathfrak{L}(v^\dagger, \underline{\mathcal{W}}^\varepsilon, \partial_{t,x})\underline{\mathcal{U}}^\varepsilon \\
\tilde{F}^\varepsilon &:= \big(\mathtt{v}_d^{\dagger,\flat} S(v^\dagger, \mathcal{W}^\varepsilon) X\partial_X + \mathfrak{L}(v^\dagger, \mathcal{W}^\varepsilon, \partial_{t,x})\big)\tilde{\mathcal{U}}^\varepsilon + (S(v^\dagger, \mathcal{W}^\varepsilon) - S(v^\dagger, \underline{\mathcal{W}}^\varepsilon))\boldsymbol{X}_{\mathtt{v}^\dagger}\,\underline{\mathcal{U}}^\varepsilon.
\end{aligned}$$

Proof. We will proceed in three steps.

(1) Use the definition of $\mathfrak{L}$ (cf. section 4) to get that $f^\varepsilon = f_1^\varepsilon + f_2^\varepsilon$ where

$$\begin{aligned}
f_1^\varepsilon(t,x) &:= \frac{1}{\varepsilon}\Big(\mathbf{L}_d + S(v^\dagger, \mathcal{W}^\varepsilon)\mathtt{v}_d^\dagger\Big)\partial_X \tilde{\mathcal{U}}^\varepsilon(t,x,\frac{x_d}{\varepsilon}), \\
f_2^\varepsilon(t,x) &:= \Big(\mathfrak{L}(v^\dagger, \mathcal{W}^\varepsilon, \partial_{t,x})\mathcal{U}^\varepsilon\Big)(t,x,\frac{x_d}{\varepsilon}).
\end{aligned}$$

(2) Use that $\dot{\mathtt{v}}_d^\dagger = 0$ and $\mathtt{v}_d^\dagger = x_d\,\mathtt{v}_d^{\dagger,\flat}$ (we use here for the first time the preliminary remark (b) done at the beginning of the proof) to get

$$f_1^\varepsilon(t,x) := \Big(\frac{1}{\varepsilon}\mathbf{L}_d\,\partial_X + S(v^\dagger, \mathcal{W})\mathtt{v}_d^{\dagger,\flat} X\partial_X\Big)\tilde{\mathcal{U}}^\varepsilon(t,x,\frac{x_d}{\varepsilon}).$$

(3) Split the profile $\mathcal{U}$ into $\mathcal{U} = \underline{\mathcal{U}} + \tilde{\mathcal{U}}$ to get

$$\mathfrak{L}(v^\dagger, \mathcal{W}^\varepsilon, \partial_{t,x})\mathcal{U}^\varepsilon = \mathfrak{L}(v^\dagger, \underline{\mathcal{W}}^\varepsilon, \partial_{t,x})\underline{\mathcal{U}}^\varepsilon + \mathfrak{L}(v^\dagger, \mathcal{W}^\varepsilon, \partial_{t,x})\tilde{\mathcal{U}}^\varepsilon + (S(v^\dagger, \mathcal{W}^\varepsilon) - S(v^\dagger, \underline{\mathcal{W}}^\varepsilon))\mathbf{X}_{\mathtt{v}^\dagger}\,\underline{\mathcal{U}}^\varepsilon.$$

By gathering these items, we complete the proof of Lemma 7.5. □

Step 3. *Action of $\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})$.*

We want now to extend Lemma 7.5 to the full term $\mathfrak{L}(v^\dagger + \varepsilon V, W, \partial_{t,x})U$. To do so, we have to proceed with the term $B_i^\varepsilon(v^\dagger, U)\varepsilon\partial_i U$. The most delicate term is $B_d^\varepsilon(v^\dagger, U)\varepsilon\partial_d U$ (cf. remark (a) done at the beginning of the proof). Lemma 7.6 is devoted to this term. As in Lemma 7.5, we consider a family of functions $\mathcal{U}^\varepsilon$ in $\mathcal{P}(T)$ Remember that we denote $\mathtt{V}_d^\varepsilon$ the normal velocity component of the profile $\mathcal{U}^\varepsilon$. As the ground state $\mathtt{v}_d^\dagger$ we split the regular part $\underline{\mathtt{V}}_d^\varepsilon$ into $\underline{\mathtt{V}}_d^\varepsilon = \dot{\underline{\mathtt{V}}}_d^\varepsilon + x_d \underline{\mathtt{V}}_d^{\varepsilon,\flat}$.

Lemma 7.6. *There exist some matrices $B_d^{\varepsilon,\flat}$ such that for all family $\mathcal{U}^\varepsilon$ of functions in $\mathcal{P}(T)$, with the following notations:*

$$\mathbf{u}^\varepsilon(t,x) := \mathcal{U}^\varepsilon(t,x,\frac{x_d}{\varepsilon}) \text{ and } g^\varepsilon := B_d^\varepsilon(v^\dagger, \mathbf{u}^\varepsilon)\varepsilon\partial_d \mathbf{u}^\varepsilon.$$

we get

$$g^\varepsilon = g_1^\varepsilon + \varepsilon g_2^\varepsilon,\ \ g_1^\varepsilon(t,x) := G_1^\varepsilon(t,x,\frac{x_d}{\varepsilon}),\ \ g_2^\varepsilon(t,x) := G_2^\varepsilon(t,x,\frac{x_d}{\varepsilon}),$$

where the functions:

$$\begin{aligned} G_1^\varepsilon &:= (\dot{\underline{\mathtt{V}}}_d^\varepsilon + \tilde{\mathtt{V}}_d^\varepsilon) S(v^\dagger, \mathcal{W}^\varepsilon)\partial_X \tilde{\mathcal{U}}^\varepsilon, \\ G_2^\varepsilon &:= \Big(B_d^{\varepsilon,\flat}(\eta) + X \underline{\mathtt{V}}_d^{\varepsilon,\flat} S(v^\dagger, \mathcal{W}^\varepsilon)\Big)\partial_X \tilde{\mathcal{U}}^\varepsilon + B_d^\varepsilon(v^\dagger,\mathcal{U}^\varepsilon)\partial_d \mathcal{U}^\varepsilon \end{aligned}$$

are respectively in $\mathcal{N}(T)$ and $\mathcal{P}(T)$, where we denote $\eta := (x_d, X, \dot{v}^\dagger, v^{\dagger,\flat}, \mathcal{U}^\varepsilon)$. Moreover the matrices $B_d^{\varepsilon,\flat}$ are C^∞ with respect to its arguments η and ε up to 0.

Proof. A naive computation yields

$$g^\varepsilon(t,x) = \Big(B_d^\varepsilon(v^\dagger,\mathcal{U}^\varepsilon)\partial_X \tilde{\mathcal{U}}^\varepsilon + \varepsilon B_d^\varepsilon(v^\dagger,\mathcal{U}^\varepsilon)\partial_d \mathcal{U}^\varepsilon\Big)(t,x,\frac{x_d}{\varepsilon}).$$

We are going to expand the first term of the right side above with respect to ε. The underlying reason to do that is a need for a subtle understanding of the $O(1)$ contribution of g^ε (see Lemma 7.13 and the arguments in (52)). Our method lies on remark (b), some Taylor expansions and the structure of the B_d^ε obtained in Lemma 7.4. We will proceed in four steps.

1. First by a Taylor expansion there exist some matrices $B_d^{\varepsilon,\flat,1}$, C^∞ with respect to its arguments, including ε up to 0, such that for all $(v,U) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+2}$,
$$B_d^\varepsilon(v,U) = B_d^0(v,U) + \varepsilon B_d^{\varepsilon,\flat,1}(v,U).$$
2. By using again a Taylor expansion, there exist some C^∞ matrices $B_d^{0,\flat,2}$, such that for all $U \in \mathbb{R}^{d+2}$,
$$B_d^0(v^\dagger,U) = B_d^0(\dot{v}^\dagger,U) + x_d v^{\dagger,\flat}\, B_d^{0,\flat,2}(\dot{v}^\dagger, x_d v^{\dagger,\flat}, U).$$
3. Moreover, for all $U \in \mathbb{R}^{d+2}$, as $\dot{\mathtt{v}}_d^\dagger = 0$, thanks to (35) we have $B_d^0(\dot{v}^\dagger,U) = \mathtt{V}_d\, S(\dot{v}^\dagger, W)$.
4. Split the normal velocity component $\mathtt{V}_d^\varepsilon$ of the profile $\mathcal{U}^\varepsilon$ into $\mathtt{V}_d^\varepsilon = \dot{\underline{\mathtt{V}}}_d^\varepsilon + x_d \underline{\mathtt{V}}_d^{\varepsilon,\flat} + \tilde{\mathtt{V}}_d^\varepsilon$ and define
$$\eta := (x_d, X, \dot{v}^\dagger, v^{\dagger,\flat}, \mathcal{U}^\varepsilon) \text{ and } B_d^{\varepsilon,\flat}(\eta) := B_d^{\varepsilon,\flat,1}(v^\dagger,\mathcal{U}^\varepsilon) + X v^{\dagger,\flat} B_d^{0,\flat,2}(\dot{v}^\dagger, x_d v^{\dagger,\flat}, \mathcal{U}^\varepsilon).$$
to end the proof of Lemma 7.6.

□

For $0 \leqslant i \leqslant d-1$, with the notations above,

$$B_i^\varepsilon(v^\dagger, \mathbf{u}^\varepsilon)\varepsilon\partial_i \mathbf{u}^\varepsilon = \varepsilon\Big(B_i^\varepsilon(v^\dagger,\mathcal{U}^\varepsilon)\partial_d \mathcal{U}^\varepsilon\Big)(t,x,\frac{x_d}{\varepsilon}).$$

By gathering the previous results, we prove the following Lemma.

Lemma 7.7. *Let, as in Lemma 7.5, $\mathcal{U}^\varepsilon$ a family of functions in $\mathcal{P}(T)$ and denote $\mathbf{u}^\varepsilon(t,x) := \mathcal{U}^\varepsilon(t,x,\frac{x_d}{\varepsilon})$. We split $\mathcal{U}^\varepsilon$ into $\mathcal{U}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$, where $\mathcal{V}^\varepsilon$ (resp. $\mathcal{W}^\varepsilon$) takes its values in*

$\mathbb{R}^{d+1}$ *(resp.* $\mathbb{R}$*). In the same way, we split* $\mathbf{u}^\varepsilon$ *into* $\mathbf{u}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{w}^\varepsilon)$*. We denote* $h^\varepsilon := \mathfrak{L}(v^\dagger + \varepsilon \mathbf{v}^\varepsilon, \mathbf{w}^\varepsilon, \partial_{t,x})\mathbf{u}^\varepsilon$*. Then there exist* $F^{\varepsilon,'}$ *and* $F^{\varepsilon,''}$ *in* $\mathcal{P}(T)$ *such that*

$$h^\varepsilon(t,x) = \frac{1}{\varepsilon}\boldsymbol{L}_d\,\partial_X \tilde{\mathcal{U}}^\varepsilon(t,x,\frac{x_d}{\varepsilon}) + F^{\varepsilon,'}(t,x,\frac{x_d}{\varepsilon}) + \varepsilon F^{\varepsilon,''}(t,x,\frac{x_d}{\varepsilon}).$$

Moreover, the function $F^{\varepsilon,'}$ *splits into* $F^{\varepsilon,'} = \underline{F}^{\varepsilon,'} + \tilde{F}^{\varepsilon,'}$*, where*

$$\begin{aligned} \underline{F}^{\varepsilon,'} &:= \underline{F}^{\varepsilon}, \\ \tilde{F}^{\varepsilon,'} &:= \tilde{F}^{\varepsilon} + (\dot{\underline{\mathtt{V}}}^\varepsilon_d + \tilde{\mathtt{V}}^\varepsilon_d) S(v^\dagger, \mathcal{W}^\varepsilon)\partial_X \tilde{\mathcal{U}}^\varepsilon, \end{aligned}$$

where F^ε *is as in Lemma 7.5. The profile* $F^{\varepsilon,''}$ *depends on* $x_d, X, \dot{v}^\dagger, v^{\dagger,\flat}, \mathcal{U}^\varepsilon$*, and its first derivatives (with respect to* t, x_d, X*, and on* $\underline{\mathtt{V}}^{\varepsilon,\flat}_d$*).*

Step 4. *Study of* K^ε*.*

We now turn to the term K^ε. By a first order Taylor expansion we obtain that there exists a C^∞ function $\mathbf{M}^{\star,\varepsilon}$ such that for any function $U = (V, W) \in H^\infty((0,T)\times\mathbb{R}^d_+)$, where the function V (respectively W) takes its values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$), for any real values function $W^\diamond \in H^\infty((0,T)\times\mathbb{R}^d_+)$, for any real values boundary profile profile $\tilde{\mathcal{W}}^0 \in \mathcal{N}(T)$,

$$K^\flat_v(\varepsilon, x_d, \dot{v}^\dagger, v^{\dagger,\flat}, V, \varepsilon V, \dot{w}^\dagger, w^{\dagger,\flat}, \tilde{\mathcal{W}}^0, \frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0, W^\diamond, \varepsilon W^\diamond) =$$

$$K^0(V, \tilde{\mathcal{W}}_0, \frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0, W^\diamond) + \varepsilon \mathbf{M}^{\star,\varepsilon}(\varepsilon, x_d, \dot{v}^\dagger, v^{\dagger,\flat}, V, \varepsilon V, \dot{w}^\dagger, w^{\dagger,\flat}, \tilde{\mathcal{W}}^0, \frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0, W^\diamond, \varepsilon W^\diamond) \quad (36)$$

Moreover the last term of the right side above is affine with respect to its fifth argument V and affine with respect to its eleventh argument $W^\diamond$ with $\mathbf{X}_{\mathtt{v}^\dagger}\, v^\dagger$ as factor in the leading coefficient. We begin to study the first term of the right side of (36).

Lemma 7.8. *There exist two* C^∞ *functions* K^0_I *and* K^0_{II} *such that for any profile* $\mathcal{U} = (\mathcal{V}, \mathcal{W}^0) \in \mathcal{P}(T)$*, where the function* $\mathcal{V}$ *(resp.* $\mathcal{W}$*) takes its values in* $\mathbb{R}^{d+1}$ *(resp.* $\mathbb{R}$*), for any profile* $\mathcal{W}^\flat \in \mathcal{P}(T)$*,*

$$K^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \mathcal{W}^\flat) = T_I + \varepsilon T_{II}$$

where T_I *is in* $\mathcal{P}(T)$ *with*

$$\underline{T_I} = K^0(\underline{\mathcal{V}}, 0, 0, \underline{\mathcal{W}}^\flat),\ \tilde{T}_I = K^0_I(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0)$$

and

$$T_{II} := K^0_{II}(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \tilde{\mathcal{W}}^\flat, X\tilde{\mathcal{W}}^\flat)$$

is in $\mathcal{N}(T)$*. Moreover the functions* $K^0_I(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0)$ *and* $K^0_{II}(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \tilde{\mathcal{W}}^\flat, X\tilde{\mathcal{W}}^\flat)$ *are affine with respect to their first argument* $\mathcal{V}$*.*

Proof. We denote $T := K^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \mathcal{W}^\flat) - K^0(\underline{\mathcal{V}}, 0, 0, \underline{\mathcal{W}}^1)$. We will proceed in six steps.

1. We begin to split the term T into three parts: $T = T_1 + T_2 + T_3$ where
$$\begin{aligned} T_1 &:= K^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \mathcal{W}^1) - K^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat), \\ T_2 &:= K^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat) - K^0(\mathcal{V}, 0, 0, \underline{\mathcal{W}}^\flat), \\ T_3 &:= K^0(\mathcal{V}, 0, 0, \underline{\mathcal{W}}^\flat) - K^0(\underline{\mathcal{V}}, 0, 0, \underline{\mathcal{W}}^\flat). \end{aligned}$$
2. Thanks to Lemma 7.2 there exist some C^∞ functions K_1^0 and K_2^0 affine with respect to their first argument such that for all $(V, \tilde{\mathcal{W}}, W) \in \mathbb{R}^{d+1} \times \mathbb{R} \times \mathbb{R}$,
$$K^0(V, \tilde{\mathcal{W}}, X\tilde{\mathcal{W}}, W) := K_1^0(V, \tilde{\mathcal{W}}, X\tilde{\mathcal{W}}) + \mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger . K_2^0(V, \tilde{\mathcal{W}}, X\tilde{\mathcal{W}}).W. \tag{37}$$
3. To deal with the term T_1, we use twice the equality (37), first with $(V, \tilde{\mathcal{W}}, W) = (\mathcal{V}, \tilde{\mathcal{W}}^0, \mathcal{W}^\flat)$ and then with $(V, \tilde{\mathcal{W}}, W) = (\mathcal{V}, \tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat)$ to find
$$T_1 = \mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger . K_2^0(V, \tilde{\mathcal{W}}, X\tilde{\mathcal{W}}).\tilde{\mathcal{W}}^\flat \tag{38}$$
4. To deal with the term T_2, we use twice the equality (37), first with $(V, \tilde{\mathcal{W}}, W) = (\mathcal{V}, \tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat)$ and then with $(V, \tilde{\mathcal{W}}, W) = (\mathcal{V}, 0, \underline{\mathcal{W}}^\flat)$ to find
$$\begin{aligned} T_2 &= K_1^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0) - K_1^0(\mathcal{V}, 0, 0) \\ &\quad + \mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger .(K_2^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0) - K_2^0(\mathcal{V}, 0, 0)).\underline{\mathcal{W}}^1. \end{aligned} \tag{39}$$
5. To deal with the term T_3, we use twice the equality (37), first with $(V, \tilde{\mathcal{W}}, W) = (\mathcal{V}, 0, \underline{\mathcal{W}}^\flat)$ and then with $(V, \tilde{\mathcal{W}}, W) = (\underline{\mathcal{V}}, 0, \underline{\mathcal{W}}^\flat)$ to find
$$T_3 = K_1^0(\mathcal{V}, 0, 0) - K_1^0(\underline{\mathcal{V}}, 0, 0) + \mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger .(K_2^0(\mathcal{V}, 0, 0) - K_2^0(\underline{\mathcal{V}}, 0, 0)).\underline{\mathcal{W}}^\flat. \tag{40}$$
6. By using (38), (39), (40), we get the result with
$$\begin{aligned} K_I^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0) &:= K_1^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0) - K_1^0(\underline{\mathcal{V}}, 0, 0) \\ K_{II}^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat, X\tilde{\mathcal{W}}^\flat) &:= \frac{\mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger}{x_d} X\{K_2^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0).\tilde{\mathcal{W}} \\ &\quad +(K_2^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0) - K_2^0(\underline{\mathcal{V}}, 0, 0)).\underline{\mathcal{W}}^1\}. \end{aligned}$$

   We have used that $x_d = \varepsilon X$. Because the ground state particle derivative $\mathbf{X}_{\mathbf{v}^\dagger}\, v^\dagger$ vanishes on the boundary, it contains a factor x_d. Therefore the functions K_I^0 and K_{II}^0 are C^∞, affine with respect to their first argument. Moreover, the function
$$K_{II}^0(\mathcal{V}, \tilde{\mathcal{W}}^0, X\tilde{\mathcal{W}}^0, \underline{\mathcal{W}}^\flat, X\tilde{\mathcal{W}}^\flat)$$
   is in the boundary layer profile space $\mathcal{N}(T)$.

The proof of Lemma 7.8 is now completed. □

Step 5. *Summary of previous steps and conclusion.*

Next lemma gathers and sums up what we have done till here (by using (33), (36) and Proposition 6.2).

Lemma 7.9. *Let, as in Lemma* 7.5*,* $\mathcal{U}^\varepsilon$ *a family of functions of* $\mathcal{P}(T)$ *and denote* $\mathbf{u}^\varepsilon(t,x) := \mathcal{U}^\varepsilon(t, x, \frac{x_d}{\varepsilon})$*. We split* $\mathcal{U}^\varepsilon$ *into* $\mathcal{U}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{W}^\varepsilon)$*, where* $\mathcal{V}^\varepsilon$ *(resp.* $\mathcal{W}^\varepsilon$*) takes its values in* $\mathbb{R}^{d+1}$

(resp. $\mathbb{R}$). In the same way, we split $\mathbf{u}^\varepsilon$ into $\mathbf{u}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{w}^\varepsilon)$. Assume that $\mathcal{W}^\varepsilon$ is of the form $\mathcal{W}^\varepsilon = \mathcal{W}^0 + \varepsilon \mathcal{W}^{\varepsilon,\diamond}$ where the functions $\mathcal{W}^0$, $\mathcal{W}^{\varepsilon,\diamond}$ are in $\mathcal{P}(T)$ and $\underline{\mathcal{W}}^0 = w^\dagger$. Then

$$\mathcal{H}(\mathbf{u}^\varepsilon, \partial_{t,x})\mathbf{u}^\varepsilon(t,x) = \Big((\frac{1}{\varepsilon} \boldsymbol{L}_d \, \partial_X + \mathcal{H}'(\mathcal{U}^\varepsilon, \partial_{t,x,X}) \Big) \tilde{\mathcal{U}}^\varepsilon(t, x, \frac{x_d}{\varepsilon}) + \mathcal{H}^0(\underline{\mathcal{U}}^\varepsilon, \underline{\mathcal{W}}^\varepsilon, 0, 0, \partial_{t,x}) \underline{\mathcal{U}}^\varepsilon(t,x) + \mathbf{q}^\varepsilon(t,x),$$

with $\mathbf{q}^\varepsilon(t,x) := \mathbf{Q}^\varepsilon(t, x, \frac{x_d}{\varepsilon})$ where $\mathbf{Q}^\varepsilon$ is a function in $\mathcal{P}(T)$ such that

1. *$\underline{\mathbf{Q}}^\varepsilon$ depends on (t,x), and affinely on $\underline{\mathcal{U}}^\varepsilon$, on its derivatives and on $\underline{\mathcal{W}}^{\varepsilon,\diamond}$.*
2. *$\tilde{\mathbf{Q}}^\varepsilon$ depend of (t,x), affinely on the profile $\mathcal{U}^\varepsilon$, on its traces and on its derivatives and on $\tilde{\mathcal{W}}^{\varepsilon,\diamond}$.*

The most difficult part of the proof of Proposition 7.3 is done. Let us explain how to conclude. For any $N \in \mathbb{N}^*$, to compute the profiles Φ^{-1},...,Φ^N, apply Lemma 7.9 to the family $\mathcal{U}^\varepsilon$ defined by the formula $\mathcal{U}^\varepsilon := \sum_{j=0}^{N+1} \varepsilon^j \, \mathcal{U}^j$, develop inside the nonlinearities with respect to the powers of ε thanks to Taylor expansion to end the proof of Proposition 7.3 . Let us just detail why the profile $\underline{\mathcal{Q}}^1_w$ is given by the formula $\underline{\mathcal{Q}}^1_w = -\underline{\mathsf{V}}^0 . \partial_x \, \underline{\mathcal{W}}^0$ for the sake of completeness. To do so, notice that the last equation in (18) reads simply $\mathbf{X}_{\mathsf{v}^\dagger + \varepsilon \mathsf{V}} \, \mathcal{W} = 0$, plug the expansion (29) and take the regular part. The proof of Proposition 7.3 is now completed. □

7.2.2. *A sequence of profile problems.* Since the profiles Φ^j are now explicite thanks to Proposition 7.3, we now look for profiles $\mathcal{U}^j$ such that the resulting profiles Φ^j are identically zero by solving a sequence of profiles problems. This will be done in Theorem 7.10 infra. It will prove Theorem 7.1.

We split, for $j \geqslant 0$, $\underline{\Phi}^j$ into $\underline{\Phi}^j := (\underline{\Phi}^j_v, \underline{\Phi}^j_w)$ where the functions $\underline{\Phi}^j_v$ (resp. $\underline{\Phi}^j_w$) take their values in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}$). We define the problem

$$(\mathcal{S}^j(T)) : \quad \begin{cases} (Id - \Pi)\tilde{\Phi}^{j-1} = \Pi \, \tilde{\Phi}^j = \underline{\Phi}^j_v = \underline{\Phi}^{j+1}_w = 0 & \text{where}\, (t,x) \in (0,T) \times \Omega, \\ \mathsf{V}^j_d|_{X=0} = 0 & \text{where}\, (t,x) \in (0,T) \times \Gamma. \end{cases}$$

Remember that some profiles $(\mathcal{U}^j_{\text{init}})_{j \in \mathbb{N}}$ such that $(Id - \Pi)\mathcal{U}^j_{\text{init}} = 0$ and $\mathcal{W}^0_{\text{init}} = w^\dagger|_{t=0}$ are given. Our goal in this section is to prove the following theorem:

Theorem 7.10. *There exist profiles $(\mathcal{U}^j)_{j \geqslant 0}$ in $\mathcal{P}(T_\dagger)$ which verify $\underline{\mathcal{W}}^0 := w^\dagger$, $(\mathcal{S}^j(T_\dagger))_{j \geqslant 0}$ and $\Pi \mathcal{U}^j|_{t=0} = \mathcal{U}^j_{init}$ for all $j \geqslant 0$. Moreover the profile $\mathcal{U}^0$ is polarized in the sense that $(Id - \Pi)\tilde{\mathcal{U}}^0 = 0$ for all $(t,x) \in (0, T_\dagger) \times \Omega$.*

Remember that Π denote the orthogonal projector on $\ker \mathbf{L}_d$ (cf. section 4). We illustrate our strategy with the following table

	$(\mathcal{S}^0(T))$	$(\mathcal{S}^1(T))$	$(\mathcal{S}^2(T))$	...
Φ^{-1}	$(Id-\Pi)\tilde{\Phi}^{-1}$			
Φ^0	$\Pi\,\tilde{\Phi}^0, \underline{\Phi}_v^0$	$(Id-\Pi)\tilde{\Phi}^0$		
Φ^1	$\underline{\Phi}_w^1$	$\Pi\,\tilde{\Phi}^1, \underline{\Phi}_v^1$	$(Id-\Pi)\tilde{\Phi}^1$	
Φ^2		$\underline{\Phi}_w^2$	$\Pi\,\tilde{\Phi}^2, \underline{\Phi}_v^2$	
Φ^3			$\underline{\Phi}_w^3$	
...				

Theorem 7.1 is a consequence of Theorem 7.10 since in the table above, each element of the left column contain the sum of the corresponding line. This is clear for the third line and the following ones i.e. for Φ^j with $j \geqslant 1$. Let us give a few explanations about the two first lines. Thanks to Proposition 7.3 we know that $\Pi\tilde{\Phi}^{-1} = 0$ and $\underline{\Phi}_w^0 = \mathbf{X}_{\mathtt{v}^\dagger}\, w^\dagger$ (see also the definition of $\mathcal{H}^0$ in (31) and the definition of $\mathcal{L}$ in section 4). Moreover because the ground state $(v^\dagger, \mathcal{W}^0)$ is solution of the Euler system, there holds $\underline{\Phi}_w^0 = 0$. We will solve the problems $(\mathcal{S}^j(T))$ for all $j \geqslant 0$, one after another with j increasing. The unknown profiles are $\mathcal{V}^j$, $\underline{\mathcal{W}}^{j+1}$ and $\tilde{\mathcal{W}}^j$ when solving the problem $(\mathcal{S}^j(T))$. We will use during the whole proof Proposition 7.3 to explicit the functions Φ^j. Let us fix some notations. We split the profiles $\mathcal{U}^j_{\text{init}}$ into $\mathcal{U}^j_{\text{init}} := (\mathtt{V}^j_{\text{init}}, \mathtt{P}^j_{\text{init}}, \mathtt{S}^j_{\text{init}})$. We denote $\mathcal{V}^j_{\text{init}} := (\mathtt{V}^j_{\text{init}}, \mathtt{P}^j_{\text{init}})$ and $\mathcal{W}^j_{\text{init}} := \mathtt{S}^j_{\text{init}}$. We also split the velocity profiles $\mathtt{V}^j_{\text{init}}$ into $\mathtt{V}^j_{\text{init}} := (\mathtt{V}^j_{\text{t,init}}, \mathtt{V}^j_{d,\text{init}})$. The profile $\mathtt{V}^j_{\text{t,init}}$ and the profile $\mathtt{V}^j_{d,\text{init}}$ are respectively the tangential and the normal part of the velocity profile $\mathtt{V}^j_{\text{init}}$.

Proof. We will proceed in two steps first studying the problem $(\mathcal{S}^0(T_\dagger))$ then for all $j \geqslant 1$, the problems $(\mathcal{S}^j(T_\dagger))$. We will use during the first step in a fundamental way some transparency properties. To use them we split the problem $(\mathcal{S}^0(T_\dagger))$ into several sub-problems. The second step lies on an iteration and on linearized versions of the sub-problems which are involved in the first step.

Step 1.

The problem $(\mathcal{S}^0(T))$ splits into two sub-problems

$$\begin{cases} \underline{\Phi}_v^0 = \underline{\Phi}_w^1 = 0 & \text{where } (t,x) \in (0,T) \times \Omega \\ \underline{\mathtt{V}}_d^0 = 0 & \text{where } (t,x) \in (0,T) \times \Gamma \end{cases} \tag{41}$$

and

$$(Id-\Pi)\tilde{\Phi}^{-1} = \Pi\,\tilde{\Phi}^0 = 0 \quad \text{where } (t,x) \in (0,T) \times \Omega \tag{42}$$

Indeed notice that since $(Id-\Pi)\tilde{\Phi}^{-1} = 0$ (cf. (42)), there holds $\tilde{\mathtt{V}}_d^0 = 0$. As a consequence, there holds $\mathtt{V}_d^0|_{X=0} = \underline{\mathtt{V}}_d^0 = 0$ for all $(t,x) \in (0,T) \times \Gamma$. Let us begin to look at (41). Thanks to Proposition 7.3 and (31), there holds

$$\begin{aligned} \underline{\Phi}_v^0 &= \mathfrak{L}^\star(u^\dagger, \partial_{t,x})\underline{\mathcal{V}}^0 + K^0(\underline{\mathcal{V}}^0, 0, 0, \underline{\mathcal{W}}^1), \\ \underline{\Phi}_w^1 &= \mathbf{X}_{\mathtt{v}^\dagger}\, \underline{\mathcal{W}}^1 + \underline{\mathtt{V}}^0.\partial_x\, \underline{\mathcal{W}}^0. \end{aligned}$$

In (41), the determinations of $\underline{\mathcal{V}}^0$ and $\underline{\mathcal{W}}^1$ are therefore coupled. The existence of solutions is given by the following lemma.

Lemma 7.11. *There exist some profiles* $(\underline{\mathcal{V}}^0, \underline{\mathcal{W}}^1)$ *in* $H^\infty((0,T_\dagger)\times\Omega)$ *solution of:*

$$\begin{cases} \mathfrak{L}(u^\dagger, \partial_{t,x}) \begin{bmatrix} \underline{\mathcal{V}}^0 \\ \underline{\mathcal{W}}^1 \end{bmatrix} + \begin{bmatrix} K^0(\underline{\mathcal{V}}^0, 0, 0, \underline{\mathcal{W}}^1) \\ \underline{\mathtt{V}}^0 . \partial_x \underline{\mathcal{W}}^0 \end{bmatrix} = 0 & \text{where } (t,x) \in (0,T_\dagger)\times\Omega \\ \underline{\mathtt{V}}^0_d = 0 & \text{where } (t,x) \in (0,T_\dagger)\times\Gamma \end{cases} \tag{43}$$

such that $\Pi^\star \underline{\mathcal{V}}^0|_{t=0} = \underline{\mathcal{V}}^0_{init}$ *and* $\underline{\mathcal{W}}^1|_{t=0} = \underline{\mathcal{W}}^1_{init}$.

Proof. Some compatibility conditions on the corner $\{t = x_d = 0\}$ are necessary to the existence of solutions in H^∞. The existence of initial compatible data $(\hat{\underline{\mathcal{V}}}^0_{\text{init}}, \hat{\underline{\mathcal{W}}}^1_{\text{init}})$ are given in Proposition 3.10 of [37]. Moreover Proposition 3.10 of [37] insures that we can prescribe arbitrarily $\Pi\,\hat{\underline{\mathcal{V}}}^0_{\text{init}}$ and $\hat{\underline{\mathcal{W}}}^1_{\text{init}}$. It is therefore possible to choose $\Pi\,\hat{\underline{\mathcal{V}}}^0_{\text{init}} = \underline{\mathcal{V}}^0_{\text{init}}$ and $\hat{\underline{\mathcal{W}}}^1_{\text{init}} = \underline{\mathcal{W}}^1_{\text{init}}$. Since the system is linear thanks to Lemma 7.2 Theorem 3.10 of [37] yields the existence of a solution $(\underline{\mathcal{V}}^0, \underline{\mathcal{W}}^1)$ in $H^\infty((0,T_\dagger)\times\Omega)$ of (43) such that $\Pi^\star \underline{\mathcal{V}}^0|_{t=0} = \underline{\mathcal{V}}^0_{\text{init}}$ and $\underline{\mathcal{W}}^1|_{t=0} = \underline{\mathcal{W}}^1_{\text{init}}$. The proof of Lemma 7.11 is therefore completed. □

Remark 7.12. *Referring to Proposition* 6.2*, we can compute the term* $K^0(\underline{\mathcal{V}}^0, 0, 0, \underline{\mathcal{W}}^1)$ *and see that the Prob.* (43) *can be restated as a boundary value problem for the linearized Euler system. To see this by another way, plug* $v^\varepsilon = v^\dagger + \varepsilon\underline{\mathcal{V}}^0$ *and* $w^\varepsilon = w^\dagger + \varepsilon\underline{\mathcal{W}}^1$ *in* (13).

Let us attack (42). To solve it we use the following result:

Lemma 7.13. *There is a unique profile* $\tilde{\mathcal{U}}^0$ *in* $\mathcal{N}(T_\dagger)$ *solution of*

$$(Id - \Pi)\tilde{\mathcal{U}}^0 = 0 \qquad \text{where } (t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+, \tag{44}$$

$$\Pi\,\mathcal{H}'(\mathcal{U}^0, \partial_{t,x,X})\tilde{\mathcal{U}}^0 = 0 \qquad \text{where } (t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+, \tag{45}$$

$$\Pi\,\tilde{\mathcal{U}}^0|_{t=0} = \tilde{\mathcal{U}}^0_{init} \qquad \text{where } (x,X) \in \Omega\times\mathbb{R}_+. \tag{46}$$

Proof. If we solve directly the problem, because the system is a priori nonlinear, we can only claim the local existence of a solution $\tilde{\mathcal{U}}^0$. A more acute analysis is possible by using transparency properties. In order to exploit them, we are going to proceed in two steps first looking for $\tilde{\mathcal{W}}^0$ then looking for the tangential velocity $\tilde{\mathtt{V}}^0_{\mathfrak{t}}$. First at all, equation (44) means that $\tilde{\mathcal{U}}^0$ is of the form $\tilde{\mathcal{U}}^0 = (\Pi^\star\tilde{\mathcal{V}}^0, \tilde{\mathcal{W}}^0)$ where $\Pi^\star\tilde{\mathcal{V}}^0$ is of the form

$$\Pi^\star\tilde{\mathcal{V}}^0 = (\tilde{\mathtt{V}}^0_{\mathfrak{t}}, \tilde{\mathtt{V}}^0_d = 0, \tilde{\mathtt{P}}^0 = 0). \tag{47}$$

We recall that the operator $\mathcal{H}'$ is defined in (32). Write $\mathcal{U} := (\mathcal{V}, \mathcal{W})$ to find that

$$\mathcal{H}'(\mathcal{U}, \partial_{t,x,X})\tilde{\mathcal{U}} = (\mathcal{H}^{\star,'}(\mathcal{U}, \partial_{t,x,X})\tilde{\mathcal{V}}, \big((X\mathtt{v}^{\dagger,\flat}_d + \dot{\underline{\mathtt{V}}}_d + \tilde{\mathtt{V}}_d)\partial_X + \mathbf{X}_{\mathtt{v}^\dagger}\big)\tilde{\mathcal{W}}). \tag{48}$$

where

$$\begin{aligned} \mathcal{H}^{\star,'}(\mathcal{U}, \partial_{t,x,X})\tilde{\mathcal{V}} := {} & \big((X\mathtt{v}^{\dagger,\flat}_d + \dot{\underline{\mathtt{V}}}_d + \tilde{\mathtt{V}}_d) S^\star(v^\dagger, \mathcal{W})\partial_X + \mathfrak{L}^\star(v^\dagger, \mathcal{W}, \partial_{t,x})\big)\tilde{\mathcal{V}} \\ & + (S^\star(v^\dagger, \mathcal{W}) - S^\star(v^\dagger, \underline{\mathcal{W}}))\mathbf{X}_{\mathtt{v}^\dagger}\, \underline{\mathcal{V}} + K^0_I(\mathcal{V}, \tilde{\mathcal{W}}, X\tilde{\mathcal{W}}). \end{aligned} \tag{49}$$

Under the condition (44) the equation (45) is equivalent to the following set of two equations for $(t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+$: the scalar equation

$$\big((X\mathtt{v}^{\dagger,\flat}_d + \dot{\underline{\mathtt{V}}}^0_d + \tilde{\mathtt{V}}^0_d)\partial_X + \mathbf{X}_{\mathtt{v}^\dagger}\big)\tilde{\mathcal{W}}^0 = 0 \tag{50}$$

and the system

$$\Pi^\star \mathcal{H}^{\star,'}(\mathcal{U}^0,\partial_{t,x,X})\Pi^\star\tilde{\mathcal{V}}^0 = 0 \tag{51}$$

We will use that thanks to (44) and (43), we have

$$\tilde{\mathtt{V}}_d^0 = 0 \text{ and } \underline{\dot{\mathtt{V}}}_d^0 = 0. \tag{52}$$

(*i*) First thanks to (52) the scalar equation (50) reads $\Xi\tilde{\mathcal{W}}^0 = 0$, where $\Xi := \mathtt{v}_d^{\dagger,\flat} X\partial_X + \mathbf{X}_{\mathtt{v}^\dagger}$ is a linear transport operator tangent to $\{x_d = 0\}$ and to $\{X = 0\}$. Therefore there exists one (and only one) solution $\tilde{\mathcal{W}}^0$ in the boundary layer profile space $\mathcal{N}(T_\dagger)$ of $\Xi\tilde{\mathcal{W}}^0 = 0$ such that $\tilde{\mathcal{W}}^0|_{t=0} = \tilde{\mathcal{W}}^0_{\text{init}}$.

(*ii*) We now look for $\Pi^\star\,\tilde{\mathcal{V}}^0$ i.e. for $\tilde{\mathtt{V}}_{\mathtt{t}}^0$ (see (32)). The analysis of (51) takes three steps:
1. The system is first order symmetric hyperbolic (see (49) and Properties 4.1, (1)).
2. Thanks to (49), to section 4 and to (52) we see that the operator $\Pi^\star\,\mathcal{H}^{\star,'}(\mathcal{U},\partial_{t,x,X})\Pi^\star$ is tangent to $\{x_d = 0\}$ and to $\{X = 0\}$.
3. Remark that the system (51) is affine with respect to $\tilde{\mathtt{V}}_{\mathtt{t}}^0$.

Therefore there exists one (and only one) $\Pi^\star\,\tilde{\mathcal{V}}_0$ solution of the equation (51) on $(0,T_\dagger)\times\Omega\times\mathbb{R}_+$ with $\Pi^\star\,\tilde{\mathcal{V}}^0|_{t=0} = \tilde{\mathcal{V}}^0_{\text{init}}$.

The proof of Lemma 7.13 is therefore completed. □

Remark 7.14. *Referring to section* 4*, we get*

$$\Pi^\star\, S^\star(v^\dagger,\underline{\mathcal{W}})\Pi^\star = \rho(\mathtt{p}^\dagger,\mathcal{W}^0)Id_{d-1} \quad \text{and } \Pi^\star\,\boldsymbol{L}^\star\Pi^\star = 0, \tag{53}$$

We denote by $K^0_{I,\mathtt{t}}$ *the* $d-1$ *first components of* K^0_I*. Thanks to* (53)*, we deduce from the equation* (45) *the following equation for the tangential velocity* $\tilde{\mathtt{V}}_{\mathtt{t}}^0$*:*

$$\rho(\mathtt{p}^\dagger,\mathcal{W}^0)(\mathtt{v}_d^{\dagger,\flat}X\partial_X + \boldsymbol{X}_{\mathtt{v}^\dagger})\,\tilde{\mathtt{V}}_{\mathtt{t}}^0 + K^0_{I,\mathtt{t}}(\mathcal{V}^0,\tilde{\mathcal{W}}^0,X\tilde{\mathcal{W}}^0) = \tilde{f}^0_{\mathtt{t}},$$

where

$$\tilde{f}^0_{\mathtt{t}} := (\rho(\mathtt{p}^\dagger,\mathcal{W}^0) - \rho(\mathtt{p}^\dagger,\underline{\mathcal{W}}^0))\boldsymbol{X}_{\mathtt{v}^\dagger}\,\underline{\mathtt{V}}^0_{\mathtt{t}}$$

can be seen at this step as a source term. Referring to Proposition (6.2)*, we can compute*

$$K^0_{I,\mathtt{t}}(\mathcal{V}^0,\tilde{\mathcal{W}}^0,X\tilde{\mathcal{W}}^0) := (\rho(\mathtt{p}^\dagger,\mathcal{W}^0) - \rho(\mathtt{p}^\dagger,\underline{\mathcal{W}}^0))\frac{\boldsymbol{X}_{\mathtt{v}^\dagger}\,\mathtt{v}^\dagger_{\mathtt{t}}}{x_d}X$$
$$+(\rho(\mathtt{p}^\dagger,\mathcal{W}^0)\mathtt{v}^\dagger - \rho(\mathtt{p}^\dagger,\underline{\mathcal{W}}^0)\underline{\mathtt{V}}^0).\partial_x\,\mathtt{v}^\dagger_{\mathtt{t}}.$$

The first step is achieved since we have solved the problem $(\mathcal{S}^0(T_\dagger))$. Let us now turn to the following problems, the $(\mathcal{S}^j(T_\dagger))$ for $j \geqslant 1$.

Step 2.

For all $j \geqslant 1$, the problem $(\mathcal{S}^j(T_\dagger))$ splits into several problems. First, we solve:

$$(Id - \Pi)\tilde{\Phi}^{j-1} = 0 \quad \text{where}\,(t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+ \tag{54}$$

then

$$\Pi\,\tilde{\Phi}^j = 0 \quad \text{where}\,(t,x,X) \in (0,T_\dagger)\times\Omega\times\mathbb{R}_+ \tag{55}$$

and

$$\begin{cases} \underline{\Phi}_v^j = \underline{\Phi}_w^{j+1} = 0 & \text{where}\,(t,x) \in (0,T_\dagger) \times \Omega, \\ \underline{\mathtt{V}}_d^j = -\tilde{\mathtt{V}}_d^j|_{X=0} & \text{where}\,(t,x) \in (0,T_\dagger) \times \Gamma. \end{cases} \tag{56}$$

To do so, we will proceed in three steps.

(i) We solve (54) and we define $(Id - \Pi)\tilde{\mathcal{U}}^j$ as the unique solution in $\mathcal{N}(T)$ of

$$\mathbf{L}_d\,\partial_X (Id - \Pi)\tilde{\mathcal{U}}^j = -(Id - \Pi)(\mathcal{H}'(\mathcal{U}^0, \partial_{t,x,X})\tilde{\mathcal{U}}^{j-1} - \tilde{\mathcal{Q}}^{j-1}).$$

We stress that the operator ∂_X is an automorphism of $\mathcal{N}(T)$. Notice that we cannot prescribe arbitrarily $(Id - \Pi)\tilde{\mathcal{U}}^j$ at $t = 0$.

(ii) The problem (56) is equivalent to a linearized Euler problem as in Lemma 7.11 for $(\underline{\mathcal{V}}^j, \underline{\mathcal{W}}^{j+1})$. We obtain easily the existence of profiles $\underline{\mathcal{V}}^j$, $\underline{\mathcal{W}}^{j+1}$ in $H^\infty((0,T_\dagger) \times \Omega)$ solution of this problem with $\Pi^\star\, \underline{\mathcal{V}}^j|_{t=0} = \underline{\mathcal{V}}^j_{\text{init}}$ and $\underline{\mathcal{W}}^{j+1}|_{t=0} = \underline{\mathcal{W}}^{j+1}_{\text{init}}$.

(iii) We define $\Pi\tilde{\mathcal{U}}^j$ as the unique solution in $\mathcal{N}(T_\dagger)$ of the linear hyperbolic initial boundary value:

$$\Pi\,\mathcal{H}'(\mathcal{U}^0, \partial_{t,x,X})\Pi\tilde{\mathcal{U}}^j = \Pi(\tilde{\mathcal{Q}}^j - (Id - \Pi)\mathcal{H}'(\mathcal{U}^0, \partial_{t,x,X})\tilde{\mathcal{U}}^j)$$

with $\Pi\tilde{\mathcal{U}}^j|_{t=0} = \tilde{\mathcal{U}}^j_{\text{init}}$. Notice that unlike the case $j = 0$ it is un-useful to split $\Pi\tilde{\mathcal{U}}^j$. Thus (55) is proved.

The proof of Theorem 7.10 is therefore completed. □

8. Stability

In this section, we are interested in the existence and the propagation of exact solutions of (1) asymptotic to the formal solutions constructed in the previous section. Thus we assume that formal solutions $(u^\varepsilon_{\text{formal}})_\varepsilon$ of (1) on $(0,T_\dagger) \times \Omega$ of the form

$$u^\varepsilon_{\text{formal}} = (v^\dagger + \varepsilon V^\varepsilon_{\text{formal}}, W^\varepsilon_{\text{formal}}) \quad \text{where } U^\varepsilon_{\text{formal}} := (V^\varepsilon_{\text{formal}}, W^\varepsilon_{\text{formal}})$$

is an expansion

$$\sum_{n\geqslant 0} \varepsilon^n\, \mathcal{U}^n(t,x,\frac{x_d}{\varepsilon}) \text{ with } \mathcal{U}^n = (\mathcal{V}^n, \mathcal{W}^n) \in \mathcal{P}(T_\dagger),\ \underline{\mathcal{W}}^0 = w^\dagger$$

are given. We obtain approximate solutions $u^\varepsilon_a = (v^\varepsilon_a, W^\varepsilon_a)$ with

$$v^\varepsilon_a = v^\dagger + \varepsilon V^\varepsilon_a \tag{57}$$

of the system (1), choosing for $n \in \mathbb{N}$,

$$\begin{aligned} V^\varepsilon_a(t,x) &:= \sum_{k=0}^n \varepsilon^k\, \mathcal{V}^k(t,x,\frac{x_d}{\varepsilon}), \\ W^\varepsilon_a(t,x) &:= \sum_{k=0}^n \varepsilon^k\, \mathcal{W}^k(t,x,\frac{x_d}{\varepsilon}) + \varepsilon^{n+1}\,\underline{\mathcal{W}}^{n+1}(t,x). \end{aligned}$$

We split u^ε_a into $u^\varepsilon_a = (\mathtt{v}^\varepsilon_a, \mathtt{p}^\varepsilon_a, \mathtt{s}^\varepsilon_a)$ where $\mathtt{v}^\varepsilon_a, \mathtt{p}^\varepsilon_a, \mathtt{s}^\varepsilon_a$ have respectively the physical meaning of velocity, pressure and entropy. As a consequence, there holds $v^\varepsilon_a = (\mathtt{v}^\varepsilon_a, \mathtt{p}^\varepsilon_a)$. The velocity $\mathtt{v}^\varepsilon_a$ (resp. $\mathtt{p}^\varepsilon_a$)) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}$). We denote $U^\varepsilon_a = (V^\varepsilon_a, W^\varepsilon_a)$ and split V^ε_a into

$V_a^\varepsilon := (\mathtt{V}_a^\varepsilon, \mathtt{P}_a^\varepsilon)$, where the velocity $\mathtt{V}_a^\varepsilon$ (resp. $\mathtt{P}_a^\varepsilon$)) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}$). We split $\mathtt{V}_a^\varepsilon$ into $\mathtt{V}_a^\varepsilon = (\mathtt{V}_{a,1}^\varepsilon, ..., \mathtt{V}_{a,d}^\varepsilon)$. We will denote

$$W_a^{\varepsilon,\diamond}(t,x) := \sum_{k=1}^{n} \varepsilon^{k-1}\, \mathcal{W}^k(t,x,\frac{x_d}{\varepsilon}) + \varepsilon^n\, \underline{\mathcal{W}}^{n+1}(t,x).$$

As a consequence, there holds

$$W_a^\varepsilon(t,x) := \mathcal{W}^0(t,x,\frac{x_d}{\varepsilon}) + \varepsilon W_a^{\varepsilon,\diamond}(t,x). \tag{58}$$

We now present the results of this section.

8.1. **Results.** The results of this section are given through three theorems: Theorem 8.1 (subsection 8.1.1), Theorem 8.2 (subsection 8.1.2) and Theorem 8.3(subsection 8.1.3). Then we will prove Theorem 2.1 given in introduction thanks to Theorem 8.2 (subsection 8.1.4). Finally we will give a sketch of proof (subsection 8.1.5). We will use an original functional setting which involves ε-conormal regularity and anisotropic Sobolev spaces. Let us fix this. We denote $Z_0 := \partial_t$, $Z_i := \partial_i$ for $1 \leq i \leq d-1$ and $Z_d := h(x_d)\partial_d$, where h is a bounded and C^∞ function on $\mathbb{R}$ such that $h(x_d) \neq 0$ for $x_d \neq 0$, $h(x_d) = x_d$ when $0 \leqslant x_d \leqslant 1$ and $h(x_d) = 1$ when $x_d \geqslant 2$. The family $(Z_i)_{0 \leqslant i \leqslant d}$ generates the algebra of C^∞ tangent vector fields to the boundary $\Gamma := \{x_d = 0\}$. For all $l \in \mathbb{N}$, we denote by Z^l the collection of the operators of the form $Z_0^{\alpha_0}...Z_d^{\alpha_d}$ where $\alpha_0, ..., \alpha_d$ are in $\mathbb{N}$ and satisfy $|\alpha| := \alpha_0 + ... + \alpha_d = l$. For all $k, l \in \mathbb{N}$, we denote by $\mathfrak{D}_\varepsilon^{k,l}$ the collection of the operators $(\varepsilon\partial_d)^k\, Z^l$. To simplify, we introduce the notations

$$L^2(T) := L^2((0,T)\times\mathbb{R}^d_+) \quad \text{and} \quad L^\infty(T) := L^\infty((0,T)\times\mathbb{R}^d_+).$$

The family $(U_a^\varepsilon)_{\varepsilon\in]0,1]}$ verifies the following estimates:

$$\sup_{0<\varepsilon\leqslant 1} ||\mathfrak{D}_\varepsilon^{k,l}\, U_a^\varepsilon||_{L^\infty(T_\dagger)} < \infty. \tag{59}$$

For $m \in \mathbb{N}$ and $T > 0$, we denote by $\mathbf{E}^m(T)$ the set

$$\mathbf{E}^m(T) := \{(u^\varepsilon)_\varepsilon \in L^2(T) / \quad \sup_{0<\varepsilon\leqslant 1} \sum_{0\leqslant 2k+l\leqslant m} ||\mathfrak{D}_\varepsilon^{k,l}\, u||_{L^2(T)} < \infty\}.$$

To be clear about our notations, let us stress that in the previous inequality, the sum

$$\sum_{0\leqslant 2k+l\leqslant m} ||\mathfrak{D}_\varepsilon^{k,l}\, u||_{L^2(T)} \text{ stands for } \sum_{0\leqslant 2k+|\alpha|\leqslant m} ||(\varepsilon\partial_d)^k\, Z^\alpha\, u||_{L^2(T)}.$$

We look for solutions $(u^\varepsilon)_\varepsilon \in H^\infty$ of the problem (1) of the form

$$u^\varepsilon := (v^\varepsilon, W^\varepsilon) \text{ with } v^\varepsilon = v_a^\varepsilon + \varepsilon^{M+1}\, V_R^\varepsilon,\ W^\varepsilon = W_a^\varepsilon + \varepsilon^M\, W_R^\varepsilon. \tag{60}$$

We denote $U_R^\varepsilon := (V_R^\varepsilon, W_R^\varepsilon)$. We split V_R^ε into $V_R^\varepsilon := (\mathtt{V}_R^\varepsilon, \mathtt{P}_R^\varepsilon)$ and the velocity $\mathtt{V}_R^\varepsilon$ into $\mathtt{V}_R^\varepsilon := (\mathtt{V}_{R,1}^\varepsilon, ..., \mathtt{V}_{R,d}^\varepsilon)$. We denote the tangential velocity $\mathtt{v}_{R,\mathtt{t}}^\varepsilon := (\mathtt{V}_{R,1}^\varepsilon, ..., \mathtt{V}_{R,d-1}^\varepsilon)$. We introduce V^ε such that $v^\varepsilon = v^\dagger + \varepsilon V^\varepsilon$ as in the previous sections (cf. (17)). As a consequence of (17) and (60) there holds

$$V^\varepsilon = V_a^\varepsilon + \varepsilon^M V_R^\varepsilon. \tag{61}$$

This implies that in particular the following relation holds for the velocity:

$$\mathtt{v}^\varepsilon = \mathtt{v}^\varepsilon_a + \varepsilon^M \mathtt{v}^\varepsilon_R. \tag{62}$$

8.1.1. *Propagation.* The following theorem states a result of propagation.

Theorem 8.1. *Let $m > \frac{d}{2} + 5$, $M > \frac{1}{2}$ and $T \in]0, T_\dagger[$. We assume that we have a family of exact solutions $(u^\varepsilon)_\varepsilon \in H^\infty((0,T) \times \Omega)$ of the problem (1) of the form (60) where the family $(U^\varepsilon_R)_\varepsilon$ is in the set $\mathbf{E}^m(T)$. Then there is $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon \in]0, \varepsilon_0]$, the solution u^ε can be extended in a solution in $H^\infty((0,T_\dagger) \times \Omega)$ of the problem (1) and is of the form (60) where we have extended $(U^\varepsilon_R)_\varepsilon$ in a family of $\mathbf{E}^m(T_\dagger)$.*

The assumption $M > \frac{1}{2}$ means that the u^ε_a are close to u^ε. Theorem 8.1 claims that it is possible to extend the u^ε, for $\varepsilon \in]0, \varepsilon_0]$, till $T_\dagger$ i.e. till the ground state exists.

Let us explain why we use the set $\mathbf{E}^m(T)$. We begin with a brief review about smooth solutions of characteristic hyperbolic initial boundary value problem. We referred to the work of O. Guès [13].[7] First the boundary matrix of the system (15) is $A_d(u) := S(u)\mathtt{v}_d + \mathbf{L}_d$. When $x_d = 0$, $\mathtt{v}_d = 0$ and the rank of $A_d(u) = \mathbf{L}_d$ is constant. This suggests to use an extra-derivative namely $x_d\,\partial_d$ to the tangential derivatives $\partial_t, ..., \partial_{d-1}$. This yields the notion of conormal regularity. Then, to handle normal derivatives for characteristic problem needs carefulness. An example by [18] shows that in general there are not H^s estimates (unlike the noncharacteristic case). We can extirpate $A_d\,\partial_d u$ from the equation but because the matrix A_d is not invertible, this does not provide estimates for the components in the kernel of A_d. However these components satisfy a transport equation (cf. [28], [9], [13]) with a source term which contains two conormal derivatives of all the components. An iteration yields the idea to use one normal derivative for two conormal derivatives. Thus in [13], O. Guès uses the spaces

$$E^m(T) := \{u \in L^2(T) / \sum_{0 \leqslant 2k+l \leqslant m} ||\mathfrak{D}^{k,l}_\varepsilon\, u||_{L^2(T)} < \infty\}.$$

Here we face a singular perturbation problem. More precisely, we look at boundary layers which corresponds to variations in $\frac{x_d}{\varepsilon}$. That is why we introduce some more adapted sets $\mathbf{E}^m(T)$ with the derivatives $\varepsilon\partial_d$ instead of ∂_d. This idea of using some derivatives with ε in factor for some singular perturbations problems was used in [15], [14], [3],... Here, this idea is applied to anisotropic Sobolev spaces.

Let us also mention here one technical point. In [13], O. Guès uses a reduction of the system. For the Euler system, this corresponds to choose the thermodynamic variables p, $\mathtt{v}$, s.

8.1.2. *Existence.* We also give the following result of existence.

Theorem 8.2. *Let $m > \frac{d}{2} + 4$, $M > \frac{1}{2}$. There exists $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon \in]0, \varepsilon_0]$, there exists a solution $u^\varepsilon \in H^\infty((0,T_\dagger) \times \Omega)$ of (1) of the form (60) where $(U^\varepsilon_R)_\varepsilon$ is in $\mathbf{E}^m(T_\dagger)$.*

[7] An analysis in L^2 was achieved by J. Rauch [27].

It is important to understand that 8.2 (as Theorem 8.1) yields exact solutions till $T_\dagger$. Theorem 2.1 given in the introduction is a consequence of Theorem 8.2 as we will see in section 8.1.4.

8.1.3. *One only derivative.* Because the problem (1) comes from a system of conservation laws and $\lambda(u,\xi)$ is an eigenvalue with constant multiplicity (cf. section 2), it is possible to obtain L^∞ estimates, even for $d \geqslant 1$. We refer to papers [23], [20] of G. Métivier, paper [26] of J. Rauch and M. Reed and paper [13]. Therefore we can weaken the regularity of the solution and prove a propagation result for some solutions admitting only one normal derivative in L^2. We introduce the sets

$$\mathbf{A}^m(T) := \{(u^\varepsilon)_\varepsilon \in L^2(T)/ \quad \sup_{0<\varepsilon\leqslant 1} \sum_{0\leqslant l\leqslant m} ||Z^l\, u||_{L^2(T)} + \sum_{0\leqslant l\leqslant m-2} ||\mathfrak{D}^{k,l}_\varepsilon\, u||_{L^2(T)} < \infty\}.$$

We will also use some norms built on L^∞. We denote by Z^ε the collection of the derivatives Z_0,...,Z_d and $\varepsilon\partial_d$. Because the boundary is characteristic, we will need not only the Lipschitz norms but higher order L^∞ control, as O. Guès in [13] and G. Métivier in [20]. We denote by $L^\infty(T)$ the space $L^\infty(T) = L^\infty((0,T)\times\mathbb{R}^d_+)$. We introduce the norms

$$||u||_{m,T} \quad := \quad \sum_{0\leqslant k\leqslant m} ||Z^k u||_{L^\infty(T)}, \text{where } m\in\mathbb{N}, \quad \begin{cases} ||u||^*_{\varepsilon,T} & := \quad ||u||_{0,T} + ||Z^\varepsilon u||_{1,T}, \\ ||u||_{\varepsilon,Lip,T} & := \quad ||u||_{0,T} + ||Z^\varepsilon u||_{0,T}. \end{cases}$$

Remember that, by abuse of notation, we denote for example $||Z^\varepsilon u||_{0,T}$ for

$$\sum_{0\leqslant i\leqslant d} ||Z_i u||_{0,T} + ||\varepsilon\partial_d u||_{0,T}.$$

We introduce the sets

$$\mathbf{\Lambda}^m(T) := \{(u^\varepsilon)_\varepsilon \in L^\infty(T)/ \quad \sup_{0<\varepsilon\leqslant 1} ||u||^*_{\varepsilon,T} < \infty\}.$$

It is also possible to tackle the limit case $M = \frac{1}{2}$, but we can prove the propagation only till $T_1 \in]T, T_\dagger]$ (and not till $T_\dagger$). We incorporate this limit case in the following Theorem.

Theorem 8.3. *Let $m > \frac{d}{2}+5$, $M \geqslant \frac{1}{2}$, $T \in]0,T_\dagger[$. We assume that we have a family of exact solutions $(u^\varepsilon)_\varepsilon$ of the problem* (1) *on $(0,T)$ of the form* (60) *where $(U^\varepsilon_R)_\varepsilon$ is in $\mathbf{A}^m(T)\cap\mathbf{\Lambda}^m(T)$. Then there exists $T_1 \in]T,T_\dagger]$ and there is $\varepsilon_0 \in]0,1]$ such that for all $\varepsilon \in]0,\varepsilon_0]$, the solution u^ε can be extended in a solution on $(0,T_1)$ of the problem* (1) *and is of the form* (60) *where we have extended $(U^\varepsilon_R)_\varepsilon$ in a function of $\mathbf{A}^m(T_1)\cap\mathbf{\Lambda}^m(T_1)$. Moreover if $M > \frac{1}{2}$, we can take $T_1 = T_\dagger$.*

Remark 8.4. *It could be also possible to treat the limit case $M = \frac{1}{2}$ with the sets $\mathbf{E}^m(T_\dagger)$ and to incorporate a result of propagation till $T_1 \in]T,T_\dagger]$ in Theorem* 8.1. *We did not do so for sake of clarity.*

Remark 8.5. *In this paper, we consider a ground state $u^\dagger$ in $H^\infty((0,T_\dagger)\times\Omega)$ and formal solutions with H^∞ regularity. It could also be possible to extend to ground states of high but finite regularity.*

8.1.4. *Proof of Theorem* 2.1 *assuming Theorem* 8.2. We are going to prove Theorem 2.1 assuming Theorem 8.2. Thus we assume a solution $\tilde{\mathtt{S}}^0 \in \mathcal{N}(T_\dagger)$ of (4) is given. Remember that for $0 < \varepsilon \leqslant 1$, we denote by $\check{u}^\varepsilon$ the function $\check{u}^\varepsilon := (\mathtt{v}^\dagger, \mathtt{p}^\dagger, \check{\mathtt{s}}^\varepsilon)$ where $\check{\mathtt{s}}^\varepsilon$ is given by (3). We will denote $\check{v}^\varepsilon := (\mathtt{v}^\dagger, \mathtt{p}^\dagger)$ and $\check{W}^\varepsilon := \check{\mathtt{s}}^\varepsilon$. Thanks to section 7 we know that it is possible to construct some formal solutions $(u^\varepsilon_{\text{formal}})_\varepsilon$ of (1) on $(0, T_\dagger)$ which includes this profile $\tilde{\mathtt{S}}^0$. We apply Theorem 8.2 with $n = 1$ and $M > 1$. This provides a family of solutions $u^\varepsilon \in H^\infty((0, T_\dagger) \times \Omega)$ of (1) of the form (60) where $(U^\varepsilon_R)_\varepsilon$ is in $\mathbf{E}^m(T_\dagger)$. It remains to prove that $\check{u}^\varepsilon - u^\varepsilon$ tends to 0 in $H^1((0, T_\dagger) \times \Omega)$ when $\varepsilon \to 0^+$. To do this use the form (60) to see that $v^\varepsilon - \check{v}^\varepsilon = \varepsilon V^\varepsilon_a + \varepsilon^{M+1} V^\varepsilon_R$ and $W^\varepsilon - \check{W}^\varepsilon = \varepsilon \check{W}^{\varepsilon,\diamond}_a + \varepsilon^{M+1} \check{W}^\varepsilon_R$. Then notice that by definition the families $(\sqrt{\varepsilon} W^{\varepsilon,\diamond}_a)_\varepsilon$ and $(\sqrt{\varepsilon} V^\varepsilon_a)_\varepsilon$ are bounded in H^1 and that the family $(\varepsilon \mathcal{U}^\varepsilon_R)_\varepsilon$ is bounded in H^1.

8.1.5. *Sketch of proof.* The rest of this section is devoted to the proof of Theorem 8.1 and Theorem 8.2. Our method is based on the sets $\mathbf{E}^m$ and on some estimates uniform with respect to ε. The proof of Theorem 8.2 needs carefulness about the existence of compatible data. Subsection 8.4 is devoted to this question. In subsection 8.2 we reduce to the case $n + \frac{1}{2} > M$. In subsection 8.3 we perform a reduction in a problem for $(U^\varepsilon_R)_\varepsilon$ which are the real unknown. We obtain (cf. Prop. 8.6) that U^ε_R satisfies a quasi-linear symmetric hyperbolic boundary value problem. As for the originating Euler problem, the boundary is conservative and characteristic of constant multiplicity. At first look, this system is singular with respect to ε because of a factor ε^{-1} in the equation of V^ε_R. However a further analysis reveals that in fact the singular term contains $\frac{x_d}{\varepsilon} W^\varepsilon_R$ as factor. This will be a key point in order to surmount the apparent singularity. In order obtain existence of $(U^\varepsilon_R)_\varepsilon$ till $T_\dagger$, we will use a family of iterative schemes (subsection 8.10). Thus we will supply in subsection 8.5 linear estimates which are the core the proof. We will successively perform L^2 estimates (subsection 8.6), conormal estimates (subsection 8.8) and normal estimates. Several difficulties occur and are melt. First, the boundary is characteristic. As we have explained it above, to tackle this problem we get inspired by the paper of O. Guès [13]. We adapt the method of O. Guès substituting the derivative $\varepsilon \partial_d$ to the derivative ∂_d in order to obtain uniform estimates. Moreover, we use estimates of $\frac{x_d}{\varepsilon} W^\varepsilon_R$ in order to surmount the apparent singularity.

The proof of Theorem 8.3 is not detailed as it obtained from Theorem 3 of [13] in the same way Theorem 8.1 and Theorem 8.2 are inspired from Theorem 1 and 2 of [13]. The discussion about $M - \frac{1}{2}$ appears in the proof in subsection 8.10 when we use the Sobolev embedding Lemma 8.10. Some minor modifications at this step allow to tackle the limit case $M = \frac{1}{2}$.

8.2. **Reduction to the case** $n + \frac{1}{2} > M$**.** Assume that Theorem 8.1 is proved when $n + \frac{1}{2} > M$ and let us show how to extend to the general case $n \in \mathbb{N}$. To do so, consider $\hat{n} \geqslant M - \frac{1}{2}$ and approximate solutions of order $\hat{n}$:

$$\hat{u}^\varepsilon_a = (\hat{v}^\varepsilon_a, \hat{w}^\varepsilon_a) = (v^\dagger + \varepsilon \hat{V}^\varepsilon_a, \hat{W}^\varepsilon_a)$$

of the system (1), choosing

$$\hat{V}_a^\varepsilon(t,x) \;:=\; \sum_{k=0}^{\hat{n}} \varepsilon^k \, \mathcal{V}^k(t,x,\frac{x_d}{\varepsilon}),$$
$$\hat{W}_a^\varepsilon(t,x) \;:=\; \sum_{k=0}^{\hat{n}} \varepsilon^k \, \mathcal{V}^k(t,x,\frac{x_d}{\varepsilon}) + \varepsilon^{\hat{n}+1} \, \underline{\mathcal{W}}^{n+1}(t,x).$$

Thus the $\hat{u}_a^\varepsilon$ are more accurate approximate solutions than the u_a^ε. We define a family $(\hat{U}_R^\varepsilon)_\varepsilon$ of functions in $\mathbf{E}^m(T)$ by setting $\hat{U}_R^\varepsilon := U_R^\varepsilon + \varepsilon^{-M}\,(U^\varepsilon - \hat{U}_a^\varepsilon)$. On $(0,T)$, we get $U^\varepsilon = \hat{U}_a^\varepsilon + \varepsilon^M \hat{U}_R^\varepsilon$. Applying Theorem 8.1, we obtain some extensions of the $(\hat{U}_R^\varepsilon)_\varepsilon$ in a family of $\mathbf{E}^m(T_\dagger)$. Then we extend $(U_R^\varepsilon)_\varepsilon$ in a family of $\mathbf{E}^m(T_\dagger)$, setting

$$U_R^\varepsilon = \hat{U}_R^\varepsilon - \varepsilon^{-M}\,(U^\varepsilon - \hat{U}_a^\varepsilon).$$

8.3. **Reduction to a problem for the remainder U_R^ε.** Because the $(U_R^\varepsilon)_\varepsilon$ are the real unknown, we begin with a reduction. We will use some properties of the approximate solutions $(u_a^\varepsilon)_\varepsilon$. Looking at the table of subsection 7.2.2, we see that

$$\mathcal{H}^\varepsilon(U_a^\varepsilon, \partial_{t,x})U_a^\varepsilon = \varepsilon^{n+\frac{1}{2}} \begin{bmatrix} R_v^\varepsilon \\ \varepsilon R_W^\varepsilon \end{bmatrix} \quad \text{for all } (t,x) \in (0,T_\dagger) \times \Omega, \tag{63}$$

with

$$R_v^\varepsilon := (Id - \Pi^*)\tilde{\Phi}_v^n + \varepsilon \underline{\Phi}_v^{n+1} + ..., \quad R_W^\varepsilon := \tilde{\Phi}_w^n + \varepsilon \underline{\Phi}_w^{n+2} + ...$$

Notice that if $\tilde{\Phi} \in \mathcal{N}(T)$ then the family $(\Phi^\varepsilon)_\varepsilon$ defined by

$$\Phi^\varepsilon(t,x) := \varepsilon^{-\frac{1}{2}}\, \tilde{\Phi}(t,x,\frac{x_d}{\varepsilon}) \quad \text{for all } (t,x) \in (0,T) \times \Omega,$$

is in $\mathbf{E}^m(T)$, for all $m \in \mathbb{N}$. Thus the family $(\varepsilon^{-\frac{1}{2}}\, R^\varepsilon)_\varepsilon$ defined by

$$R^\varepsilon(t,x) := \begin{bmatrix} \varepsilon R_v^\varepsilon \\ R_W^\varepsilon \end{bmatrix} \quad \text{for all } (t,x) \in (0,T) \times \Omega,$$

is in $\in \mathbf{E}^m(T)$, for all $m \in \mathbb{N}$. The system of equations (63) is equivalent to the two equations, for all $(t,x) \in (0,T_\dagger) \times \Omega$,

$$\mathfrak{L}^\star(u_a^\varepsilon, \partial_{t,x})V_a^\varepsilon + \tfrac{1}{\varepsilon}K_v(v^\dagger, \partial v^\dagger, u_a^\varepsilon) = \varepsilon^{n+\frac{1}{2}} R_v^\varepsilon, \tag{64}$$
$$\mathbf{X}_{\mathrm{v}_a^\varepsilon} W_a^\varepsilon = \varepsilon^{n+\frac{3}{2}} R_W^\varepsilon. \tag{65}$$

Multiplying the equation (64) by ε, using that $u^\dagger$ satisfies the equation (15), we obtain that the family $(u_a^\varepsilon)_\varepsilon$ satisfies

$$\mathfrak{L}(u_a^\varepsilon, \partial_{t,x})u_a^\varepsilon = \varepsilon^{n+\frac{1}{2}} \begin{bmatrix} \varepsilon R_v^\varepsilon \\ \varepsilon R_W^\varepsilon \end{bmatrix} \text{ when } x_d > 0, \text{ and } \mathrm{v}_{a,d}^\varepsilon = 0 \text{ when } x_d = 0.$$

Proposition 8.6. *There are some $(d+1)\times(d+1)$ matrices J^α, some functions $J^{\beta,1}$, $J^{\beta,2}$ with values in $\mathbb{R}^{d+1}$, such that for all $\varepsilon\in]0,1]$, a function $u^\varepsilon=(v^\varepsilon,W^\varepsilon)$ of the form* (60) *verifies* (1) *if and only if $U_R^\varepsilon:=(V_R^\varepsilon,W_R^\varepsilon)$ verifies for $t\in(0,T)$*

$$(\mathfrak{L}^*(u^\varepsilon,\partial_{t,x})+J^\alpha).V_R^\varepsilon+W_R^\varepsilon.(J^{\beta,1}+\frac{x_d}{\varepsilon}.J^{\beta,2})=-\varepsilon^{n+\frac{1}{2}-M}R_v^\varepsilon, \tag{66}$$

$$\boldsymbol{X}_{\mathtt{v}^\varepsilon}\,W_R^\varepsilon+\varepsilon\mathtt{v}_R^\varepsilon.\partial_x\,W_a^\varepsilon=-\varepsilon^{n+\frac{3}{2}-M}R_W^\varepsilon \qquad \text{for } x\in\Omega, \tag{67}$$

$$\mathtt{v}_{R,d}^\varepsilon=0 \qquad \text{for } x\in\Gamma. \tag{68}$$

Moreover, the matrices J^α and the functions $J^{\beta,1}$, $J^{\beta,2}$ depend in a C^∞ way of

$$(\varepsilon,V_a^\varepsilon,Z_\varepsilon V_a^\varepsilon,W_a^{\varepsilon,\diamond},\varepsilon^M\,U_R^\varepsilon).$$

Remember that the notation $W_a^{\varepsilon,\diamond}$ is defined in (58).

Proof. We only prove the if part. The converse, easier, is left to the reader. Thus we assume that (u^ε) satisfies (1). We begin to show (66). This is the most difficult part of the proof. We will proceed in four steps.

Step 1.

According to Proposition 6.1, the function U^ε verifies (18). In particular we have

$$\mathfrak{L}^\star(u^\varepsilon,\partial_{t,x})V^\varepsilon+K_v(u^\varepsilon)=0.$$

Moreover because the family of functions $(u^\varepsilon)_\varepsilon$ is of the form (60), for each ϵ, the function W^ε is of the form (20) where $\mathcal{W}^0\in\mathcal{P}(T)$ is such that $\underline{\mathcal{W}}^0=w^\dagger$ and

$$W^{\varepsilon,\diamond}=W_a^{\varepsilon,\diamond}+\varepsilon^{M-1}\,W_R^\varepsilon. \tag{69}$$

Thus we can use Proposition 6.2 and we get that V^ε satisfies

$$\mathfrak{L}^\star(u^\varepsilon,\partial_{t,x})V^\varepsilon+K_v^\flat(V^\varepsilon,\varepsilon V^\varepsilon,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond})=0, \tag{70}$$

where we denote

$$K_v^\flat(V^\varepsilon,\varepsilon V^\varepsilon,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond}) \tag{71}$$

instead of

$$K_v^\flat(\varepsilon,x_d,\dot{v}^\dagger,v^{\dagger,\flat},V^\varepsilon,\varepsilon V^\varepsilon,\dot{w}^\dagger,w^{\dagger,\flat},\tilde{\mathcal{W}}^0,\frac{x_d}{\varepsilon}\tilde{\mathcal{W}}^0,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond})$$

because the arguments $(V^\varepsilon,\varepsilon V^\varepsilon,W^{\varepsilon,\diamond},\varepsilon W^{\varepsilon,\diamond})$ are the ones which are important at this step of the analysis.

Now we want to perform an expansion into Equation (70) thanks to (60). Since by definition (cf. section 4), $\mathfrak{L}^\star(u^\varepsilon,\partial_{t,x}):=S^\star(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon}\,V^\varepsilon+\mathbf{L}^\star(u^\varepsilon,\partial_x)$, there are two difficulties in this program. They concerns the terms $S^\star(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon}\,V^\varepsilon$ and (71). They are tackled in the two following steps.

Step 2.

We first look at the term $S^*(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon}\,V^\varepsilon$.

Lemma 8.7. *There are some $(d+1)\times(d+1)$ matrices $S^{*,\alpha}$ and some functions $S^{*,\beta}$ with values in $\mathbb{R}^{d+1}$ such that*

$$S^*(u^\varepsilon)\boldsymbol{X}_{\mathtt{v}^\varepsilon} V^\varepsilon = S^*(u^\varepsilon_a)\boldsymbol{X}_{\mathtt{v}^\varepsilon_a} V^\varepsilon_a + \varepsilon^M\, S^*(u^\varepsilon)\boldsymbol{X}_{\mathtt{v}^\varepsilon}\, V^\varepsilon_R + \varepsilon^M S^{*,\alpha}.V^\varepsilon_R + \varepsilon^M\, W^\varepsilon_R.S^{*,\beta}. \tag{72}$$

Moreover, the matrices $S^{*,\alpha}$ and $S^{*,\beta}$ depend in a C^∞ way of

$$(\varepsilon, V^\varepsilon_a, Z_\varepsilon V^\varepsilon_a, \varepsilon^M\, U^\varepsilon_R). \tag{73}$$

Proof. We proceed in three steps.

(1) Thanks to (61) we get

$$\mathbf{X}_{\mathtt{v}^\varepsilon} V^\varepsilon = \mathbf{X}_{\mathtt{v}^\varepsilon}\, V^\varepsilon_a + \varepsilon^M\, \mathbf{X}_{\mathtt{v}^\varepsilon}\, V^\varepsilon_R. \tag{74}$$

Thanks to the definition of the particle derivative and to (60) we get

$$\mathbf{X}_{\mathtt{v}^\varepsilon} = \mathbf{X}_{\mathtt{v}^\varepsilon_a} + \varepsilon^{M+1}\, \mathtt{V}^\varepsilon_R.\partial_x. \tag{75}$$

We plug the expansion (75) into the first term of the right side of (74). We obtain:

$$\mathbf{X}_{\mathtt{v}^\varepsilon} V^\varepsilon = \mathbf{X}_{\mathtt{v}^\varepsilon_a}\, V^\varepsilon_a + \varepsilon^M\, \mathbf{X}_{\mathtt{v}^\varepsilon}\, V^\varepsilon_R + \varepsilon^M\, \mathtt{V}^\varepsilon_R.\varepsilon\partial_x\, V^\varepsilon_a.$$

We deduce by applying the matrices $S^*(u^\varepsilon)$ that

$$S^*(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon} V^\varepsilon = S^*(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon_a} V^\varepsilon_a + \varepsilon^M\, S^*(u^\varepsilon)\mathbf{X}_{\mathtt{v}^\varepsilon} V^\varepsilon_R + \varepsilon^M\, S^*(u^\varepsilon)\mathtt{V}^\varepsilon_R.\varepsilon\partial_x\, V^\varepsilon_a. \tag{76}$$

(2) We apply (22) with $u_1 = u^\varepsilon_a$ and $u_2 = \varepsilon^M\,(\varepsilon V^\varepsilon_R, W^\varepsilon_R)$ and we use that the $(u^\varepsilon)_\varepsilon$ are of the form (60). We obtain

$$S^*(u^\varepsilon) = S^*(u^\varepsilon_a) + \varepsilon^{M+1}\, \mathtt{P}^\varepsilon_R.S^{*,\flat}_{1,R} + \varepsilon^M\, W^\varepsilon_R.S^{*,\flat}_{2,R}, \tag{77}$$

where

$$S^{*,\flat}_{1,R} := S^{*,\flat}_1(u^\varepsilon_a, \varepsilon^M\,(\varepsilon V^\varepsilon_R, W^\varepsilon_R)) \text{ and } S^{*,\flat}_{2,R} := S^{*,\flat}_2(u^\varepsilon_a, \varepsilon^M\,(\varepsilon V^\varepsilon_R, W^\varepsilon_R)).$$

We use (77) to expand the first term and the last term of the right hand side of (76). We introduce the $(d+1)$ square matrices $S^{*,\alpha}$ such that for all $V = (\mathtt{V}, \mathtt{P}) \in \mathbb{R}^{d+1}$,

$$S^{*,\alpha} V := S^*(u^\varepsilon)\mathtt{V}.\varepsilon\partial_x\, V^\varepsilon_a + \varepsilon\mathtt{P}.S^{*,\flat}_{1,R}.\mathbf{X}_{\mathtt{v}^\varepsilon_a}\, V^\varepsilon_a.$$

We also introduce the functions $S^{*,\beta} := S^{*,\flat}_{2,R}.\mathbf{X}_{\mathtt{v}^\varepsilon_a}\, V^\varepsilon_a$ which take values in $\mathbb{R}^{d+1}$. Thus we get (72).

(3) Thanks to (57) the following relation holds $\mathtt{v}^\varepsilon_a = \mathtt{v}^\dagger + \varepsilon\mathtt{V}^\varepsilon_a$. As a consequence by definition of the particle derivative we get $\mathbf{X}_{\mathtt{v}^\varepsilon_a} = \mathbf{X}_{\mathtt{v}^\dagger} + \mathtt{V}^\varepsilon_a.\varepsilon\partial_x$. Moreover since $\mathtt{v}^\dagger$ is tangent to the boundary $\{x_d = 0\}$ so is $\mathbf{X}_{\mathtt{v}^\dagger}$. As a consequence the term $\mathbf{X}_{\mathtt{v}^\varepsilon_a} V^\varepsilon_a$ can be expressed thanks to V^ε_a, $Z_\varepsilon V^\varepsilon_a$ and is not singular. Thus the matrices $S^{*,\alpha}$ and the the functions $S^{*,\beta}$ are C^∞ with respect to (73).

The proof of Lemma 8.7 is now completed. ∎

Step 3.

We now look at the term (71). Let us briefly introduce our strategy. We would like to write (71) as a perturbation of

$$K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond})$$

of order ε^M. But since we have only a factor ε^{M-1} into (69) a naive Taylor expansion fails to give the desired result. The trick lies in the way the term (71) depends of $W^{\varepsilon,\diamond}$ (cf. Proposition 6.2). More precisely thanks to Proposition 6.2, the function $K_v^\flat(V_1, V_2, W_1, W_2)$ is affine with respect to W_1 with $\mathbf{X}_{\mathbf{v}^\dagger} v^\dagger$ as a factor in the leading term. This allows to factorize the perturbation terms by ε^M and $\frac{x_d}{\varepsilon}$.

Lemma 8.8. *There are some $(d+1)\times(d+1)$ matrices $K_v^{\flat,\alpha}$ and some C^∞ functions $K_v^{\flat,\beta,1}$, $K_v^{\flat,\beta,2}$ with values in $\mathbb{R}^{d+1}$ such that*

$$\begin{aligned} K_v^\flat(V^\varepsilon, \varepsilon V^\varepsilon, W^{\varepsilon,\diamond}, \varepsilon W^{\varepsilon,\diamond}) &= K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond}) \\ &\quad + \varepsilon^M K_v^{\flat,\alpha}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon).V_R^\varepsilon \\ &\quad + \varepsilon^M W_R^\varepsilon.K_v^{\flat,\beta,1}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon) \\ &\quad + \varepsilon^M \frac{x_d}{\varepsilon} W_R^\varepsilon.K_v^{\flat,\beta,2}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}). \end{aligned} \tag{78}$$

Proof. We will proceed in three steps.

(1) We begin to expand the term (71) with respect to its arguments V^ε, $\varepsilon V^\varepsilon$ and $\varepsilon W^{\varepsilon,\diamond}$ by using that V^ε is of the form (61) and $W^{\varepsilon,\diamond}$ of the form (69). More precisely by a first order Taylor development, there exist some C^∞ $(d+1)\times(d+1)$ matrices $K_v^{\flat,\alpha}$ and some C^∞ functions $K_v^{\flat,\beta,1}$ with values in $\mathbb{R}^{d+1}$ such that

$$\begin{aligned} K_v^\flat(V^\varepsilon, \varepsilon V^\varepsilon, W^{\varepsilon,\diamond}, \varepsilon W^{\varepsilon,\diamond}) &= K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond}) \\ &\quad + \varepsilon^M K_v^{\flat,\alpha}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon).V_R^\varepsilon \\ &\quad + \varepsilon^M W_R^\varepsilon.K_v^{\flat,\beta,1}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon). \end{aligned}$$

(2) We now deal with the argument $W^{\varepsilon,\diamond}$ which needs more carefulness. According to Proposition 6.2, the function

$$K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond})$$

is affine with respect to $W^{\varepsilon,\diamond}$ with This means that there is a C^∞ function $K_v^{\flat,\eta}$ with values in $\mathbb{R}^{d+1}$ such that

$$\begin{aligned} K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond}) &= K_v^\flat(V_a^\varepsilon, \varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon W_a^{\varepsilon,\diamond}) \\ &\quad + \varepsilon^{M-1} \mathbf{X}_{\mathbf{v}^\dagger} v^\dagger.W_R^\varepsilon.K_v^{\flat,\eta}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}). \end{aligned}$$

(3) We define

$$K_v^{\flat,\beta,2}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}) := \frac{\mathbf{X}_{\mathbf{v}^\dagger\, v^\dagger}}{x_d}.K_v^{\flat,\eta}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}).$$

So the function $K_v^{\flat,\beta,2}$ is C^∞ and thanks to (1) and (2) we get (78).

The proof of Lemma 8.8 is now completed. □

Step 4.

We define the matrices $(d+1)\times(d+1)$ matrices J^α

$$\begin{aligned} J^\alpha(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon) &:= K_v^{\flat,\alpha}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon) \\ &\quad + S^{*,\alpha}(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, \varepsilon^M U_R^\varepsilon), \end{aligned}$$

and, for $i \in \{1,2\}$, the $\mathbb{R}^{d+1}$-valued function

$$\begin{aligned} J^{\beta,i}(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon) &:= K_v^{\flat,\beta,i}(\varepsilon, V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M U_R^\varepsilon) \\ &\quad + S^{*,\beta}(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, \varepsilon^M U_R^\varepsilon). \end{aligned}$$

Thanks to Lemma 8.8 and 8.7 and Equations (64) and (70), we show that V_R^ε satisfies (66). We have achieved the most difficult part of the proof. To complete the if part, it remains to prove (67) and (68).

Let us first deal with (66). Thanks to (60), we get

$$\mathbf{X}_{\mathtt{v}^\varepsilon} W^\varepsilon = \mathbf{X}_{\mathtt{v}^\varepsilon} W_a^\varepsilon + \varepsilon^M \mathbf{X}_{\mathtt{v}^\varepsilon} W_R^\varepsilon.$$

Use (75) to expand the first term of the right side above to get:

$$\mathbf{X}_{\mathtt{v}^\varepsilon} W^\varepsilon = \mathbf{X}_{\mathtt{v}_a^\varepsilon} W_a^\varepsilon + \varepsilon^M \mathbf{X}_{\mathtt{v}^\varepsilon} W_R^\varepsilon + \varepsilon^M \mathtt{V}_R^\varepsilon . (\varepsilon \partial_x W_a^\varepsilon),$$

Then use Equation (65) to get (67).

We turn finally to (68). Since u^ε is of the form (60) and since $\mathtt{v}^\dagger$ is tangent to the boundary $\{x_d = 0\}$ so is $\mathtt{V}^\varepsilon$. Since $\mathtt{V}_a^\varepsilon$ is tangent to the boundary (thanks to the previous section) and thanks to (62), we get (68) and therefore complete the proof of Proposition 8.6.

□

In the formulas (66) – (67) – (68), there is a singular factor ε^{-1} which appears in the term $\frac{x_d}{\varepsilon} W_R^\varepsilon J^{\beta,2}$. One idea would be to try to obtain estimates for W_R^ε ponderated by some ε. The difficulty lies in the fact that the equation (68) of W_R^ε involves V_R^ε in return by the term $\varepsilon \mathtt{V}_R^\varepsilon . \partial_x W_a^\varepsilon$. Fortunately, because of the special form of $\partial_x W_a^\varepsilon$, we will see that it is possible to find good estimates of $\frac{x_d}{\varepsilon} W_R^\varepsilon$. This allows to overcome the false singularity.

8.4. **Initial data.** In order to obtain smooth solutions $(U_R^\varepsilon)_{\varepsilon\in]0,1]}$ of (66) – (67) – (68), some compatibility conditions for the initial data $(U_{R,\text{init}}^\varepsilon)_{\varepsilon\in]0,1]}$ at $t=0$ are necessary. More precisely the initial data $U_{R,\text{init}}^\varepsilon$ must be compatible with the boundary condition (68), what implies a family of conditions on the corner $\{t = x_d = 0\}$. To introduce these compatibility conditions, we assume that some smooth solutions U_R^ε of (66) – (67) – (68) on $(0,T)$ with $T > 0$ are given. Because we will need to look carefully to the different components, we introduce some appropriated notations. We split $U_{R,\text{init}}^\varepsilon$ into $U_{R,\text{init}}^\varepsilon = (\mathtt{V}_{R,\text{init}}^\varepsilon, \mathtt{P}_{R,\text{init}}^\varepsilon, \mathtt{S}_{R,\text{init}}^\varepsilon)$ where the components are respectively the meaning of velocity, pressure and entropy. We split the velocity $\mathtt{V}_{R,\text{init}}^\varepsilon$ into $\mathtt{V}_{R,\text{init}}^\varepsilon = (\mathtt{V}_{R,\text{init},1}^\varepsilon, ..., \mathtt{V}_{R,\text{init},d}^\varepsilon)$. Following the notations of section 4, we denote the tangential velocity $\mathtt{V}_{R,\text{init},\mathtt{t}}^\varepsilon = (\mathtt{V}_{R,\text{init},1}^\varepsilon, ..., \mathtt{V}_{R,\text{init},d-1}^\varepsilon)$. Restricting the boundary condition (68) at $t=0$, we obtain the compatibility condition at the order 0

$$(\mathcal{R}^{0,\varepsilon}) : \ \mathtt{V}_{R,\text{init},d}^\varepsilon|_{x_d=0} = 0.$$

Let us now explain what are the compatibility conditions at order $j \geqslant 1$. We will split R_v^ε into $R_v^\varepsilon = (R_V^\varepsilon, R_p^\varepsilon)$, where $R_V^\varepsilon := (R_{V_1}^\varepsilon, ..., R_{V_d}^\varepsilon)$ (resp. R_p^ε) takes its values in $\mathbb{R}^d$ (resp.

$\mathbb{R}$). For all $l \in \mathbb{N}$, we denote by ∂_x^l the collection of the operators of the form $\partial_0^{\alpha_0} ... \partial_d^{\alpha_d}$ where $\alpha_0, ..., \alpha_d$ are in $\mathbb{N}$ and satisfy $|\alpha| := \alpha_0 + ... + \alpha_d = l$. From the equation (66), we can extirpate $\partial_t \mathtt{V}^\varepsilon_{R,d}$ in function of spatial derivatives and so by restriction its trace $\partial_t \mathtt{V}^\varepsilon_{R,d}|_{t=x_d=0}$ on the corner $\{t = x_d = 0\}$. More precisely, there exists a C^∞ function $\mathcal{H}_1$ such that

$$\partial_t \mathtt{V}^\varepsilon_{R,d}|_{t=x_d=0} = \mathcal{H}_1((\partial_x^l U^\varepsilon_{R,\text{init}}|_{x_d=0})_{l \leqslant 1}) - \partial_t R^\varepsilon_{\mathtt{V}_d}|_{t=x_d=0}.$$

In fact, the function $\mathcal{H}_1$ also depends in a C^∞ way of $t, x, v^\dagger$ and its derivatives, the profiles $\mathcal{U}^j$ and their ε-conormal derivatives. We purposely not specify these arguments for sake of clarity. By iteration, we can also express the time derivatives $\partial_t^j \mathtt{V}^\varepsilon_{R,d}$ for $j \geqslant 2$ by the equation (66). Therefore there exist some C^∞ functions $\mathcal{H}^j$ such that

$$\partial_t^j \mathtt{V}^\varepsilon_{R,d}|_{t=x_d=0} = \mathcal{H}_j((\partial_x^l U^\varepsilon_{R,\text{init}}|_{x_d=0})_{l \leqslant j}) - \partial_t^j R^\varepsilon_{\mathtt{V}_d}|_{t=x_d=0}.$$

On the other hand, we can apply ∂_t^j to the boundary condition (68) and restrict the resulting condition to the corner $\{t = x_d = 0\}$. Thus we get $\partial_t^j \mathtt{V}^\varepsilon_{R,d}|_{t=x_d=0} = 0$. As a consequence for $j \geqslant 1$, we define the jth order compatibility condition

$$(\mathcal{R}^{j,\varepsilon}): \ \mathcal{H}_j((\partial_x^l U^\varepsilon_{R,\text{init}}|_{x_d=0})_{l \leqslant j}) = \partial_t^j R^\varepsilon_{\mathtt{V}_d}|_{t=x_d=0}.$$

Thus for $\varepsilon \in]0,1]$, if for $T > 0$, $U^\varepsilon_R \in H^\infty((0,T) \times \Omega)$ satisfies the equation $(66)-(67)-(68)$ and $U^\varepsilon_R|_{t=0} = U^\varepsilon_{R,\text{init}}$, then U^ε_R satisfies the compatibility condition $(\mathcal{R}^{j,\varepsilon})_{j \in \mathbb{N}}$. Next proposition will show that there exist some families $(U^\varepsilon_{R,\text{init}})_{\varepsilon \in]0,1]}$ which satisfy the compatibility condition $(\mathcal{R}^{j,\varepsilon})_{j \in \mathbb{N}}$ and some estimates uniform with respect to ε. Moreover we can prescribe arbitrarily the components $(\mathtt{V}^\varepsilon_{R,\text{init},\mathtt{t}}, \mathtt{S}^\varepsilon_{R,\text{init}})_\varepsilon$ among the families which satisfy convenient uniform estimates.

Proposition 8.9. *Let $(\hat{\mathtt{V}}^\varepsilon_{R,init,\mathtt{t}}, \hat{\mathtt{S}}^\varepsilon_{R,init})_\varepsilon$ be a family bounded in $H^\infty(\Omega)$. Then there exists a family $(U^\varepsilon_{R,init})_\varepsilon$ bounded in $H^\infty(\Omega)$ such that for all $\varepsilon \in]0,1]$, $(\mathtt{V}^\varepsilon_{R,init,\mathtt{t}}, \mathtt{S}^\varepsilon_{R,init}) = (\hat{\mathtt{V}}^\varepsilon_{R,init,\mathtt{t}}, \hat{\mathtt{S}}^\varepsilon_{R,init})$ and that for all $j \in \mathbb{N}$, the compatibility condition $(\mathcal{R}^{\varepsilon,j})$ is verified.*

The underlying reason to the possibility to choose arbitrarily the tangential velocity and the entropy is that these components are the characteristic ones. Indeed we will see in the proof below the crucial role played by the normal derivatives. We need for uniform estimates in order to find a family of solutions $(U^\varepsilon_R)_\varepsilon$ in $E^\infty((0,T_\dagger) \times \Omega)$. We succeed in this goal obtaining in Proposition 8.9 a family $(U^\varepsilon_{R,\text{init}})_\varepsilon$ which even do not contain singularities in $\frac{x_d}{\varepsilon}$. We refer to [37], [36] for other examples of existence of initial data compatible at all order with uniform existence in a setting of boundary layer theory. We will use in the proof of Proposition 8.9, as in the references above, a Borel lemma.

Proof. We will proceed in three steps. Let us briefly sum up them. In the first step, we use a piece of algebra to analyse the way the normal derivatives are involved in the compatibility conditions $(\mathcal{R}^{\varepsilon,j})$. Thanks to this first step, then we define some families of functions which are expected to be the traces on $\{x_d = 0\}$ of the normal derivatives of an initial data which satisfy the compatibility conditions. The third step is devoted to that lifting.

Step 1.

We begin to analyze more accurately the compatibility conditions ($\mathcal{R}^{\varepsilon,j}$) and more particularly the way the functions $(\mathcal{H}_j)_{j\in\mathbb{N}^*}$ involve the normal derivatives of $\mathtt{v}^\varepsilon_{R,\text{init},d}$ and $\mathtt{p}^\varepsilon_{init}$. For sake of clarity let us explain our strategy on the following toy problem:

$$\partial_t\,\mathtt{v}_d + \rho^{-1}\,\partial_d\mathtt{p} = 0, \tag{79}$$

$$\partial_t\,\mathtt{p} + \boldsymbol{\alpha}^{-1}\,\partial_d\,\mathtt{v}_d = 0 \qquad \text{for}\, x_d > 0, \tag{80}$$

$$\mathtt{v}_d = 0 \qquad \text{for}\, x_d = 0. \tag{81}$$

This toy problem is a deliberately oversimplified version of (66) – (67) – (68) used to investigate the way to express the successive time derivatives $\partial_t^j\,\mathtt{v}_d$, for $j\in\mathbb{N}$. In particular notice that to get this toy problem from (66) – (67) – (68) we keep only the equation for $\mathtt{p}$ and $\mathtt{v}_d$ which are the characteristics components. Apply ∂_t to (79) to get

$$\partial_t^2\mathtt{v}_d = -\rho^{-1}\,\partial_t\partial_d\mathtt{p} - \big(\partial_t(\rho^{-1})\big)\partial_d\mathtt{p}. \tag{82}$$

To eliminate the first term of the right hand side above, use (80) and plug in (82). Thus we get

$$\partial_t^2\mathtt{v}_d = (\rho\boldsymbol{\alpha})^{-1}\,\partial_d^2\mathtt{v}_d - (\partial_t(\rho^{-1}))\partial_d\mathtt{p}. \tag{83}$$

We see now that the first term of the right hand side above involves a two times x_d derivative whereas the second one involves only a first x_d derivative. Apply one more ∂_t to (83) to get

$$\partial_t^3\mathtt{v}_d \;=\; (\rho\boldsymbol{\alpha})^{-1}\,\partial_d^2\partial_t\mathtt{v}_d + \partial_t\big((\rho\boldsymbol{\alpha})^{-1}\big).\partial_d^2\mathtt{v}_d - \partial_t^2(\rho^{-1}))\partial_d\mathtt{p} - (\partial_t(\rho^{-1}))\partial_d\partial_t\mathtt{p}. \tag{84}$$

Use (79) and (80) to eliminate the first time derivatives in the right hand side above to get

$$\partial_t^3\mathtt{v}_d = \rho^{-2}\,\boldsymbol{\alpha}^{-1}\,\partial_d^2\mathtt{p} + \text{l.o.t.} \tag{85}$$

where l.o.t. involves x_d derivatives of $\mathtt{p}$ and $\mathtt{v}_d$ of order lower than 2. By iteration, we find the following kind of relation for all $j\in\mathbb{N}^*$

$$\partial_t^{2j-1}\mathtt{v}_d = \rho^{-j}\boldsymbol{\alpha}^{-(j-1)}\,\partial_d^{2j-1}\mathtt{p} + \text{l.o.t.} \tag{86}$$

where l.o.t. involves x_d derivatives of $\mathtt{p}$ and $\mathtt{v}_d$ of order lower than $2j-1$, and

$$\partial_t^{2j}\mathtt{v}_d = (\rho\boldsymbol{\alpha})^{-j}\,\partial_d^{2j}\mathtt{v}_d + \text{l.o.t.} \tag{87}$$

where l.o.t. involves x_d derivatives of $\mathtt{p}$ and $\mathtt{v}_d$ of order lower than $2j$.

Such a strategy works also for the complete problem (66) – (67) – (68). In order to formulate what we get, we introduce some notations for the initial values of $\mathtt{p}^\varepsilon$, $\mathtt{v}^\varepsilon$, $\rho(\mathtt{p}^\varepsilon,\mathtt{v}^\varepsilon)$ and $\boldsymbol{\alpha}(\mathtt{p}^\varepsilon,\mathtt{v}^\varepsilon)$: we denote

$$\left\{\begin{array}{lcl}\mathtt{p}^\varepsilon_{\text{init}} & := & \mathtt{p}^\varepsilon_a|_{t=0} + \varepsilon^{M+1}\,\mathtt{p}^\varepsilon_{R,\text{init}},\\ \mathtt{s}^\varepsilon_{\text{init}} & := & \mathtt{s}^\varepsilon_a|_{t=0} + \varepsilon^{M}\,\mathtt{s}^\varepsilon_{R,\text{init}},\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{lcl}\rho^\varepsilon & := & \rho(\mathtt{p}^\varepsilon_{\text{init}},\mathtt{s}^\varepsilon_{\text{init}})\\ \boldsymbol{\alpha}^\varepsilon & := & \boldsymbol{\alpha}(\mathtt{p}^\varepsilon_{\text{init}},\mathtt{s}^\varepsilon_{\text{init}}).\end{array}\right.$$

Remember that, by assumption, ρ^ε and $\boldsymbol{\alpha}^\varepsilon$ are strictly positive. We introduce for all $j\in\mathbb{N}^*$, the set $I_j := \{\alpha := (\alpha_1,...,\alpha_d)/\ \alpha_1+...+\alpha_d \leqslant j \text{ and } \alpha_d < j\}$ and we denote respectively $\mho^\varepsilon_j$ and Ω^ε_j the families

$$\mho^\varepsilon_j := (\partial_x^\alpha U^\varepsilon_{R,\text{init}}|_{x_d=0})_{\alpha\in I_j} \text{ and } \Omega^\varepsilon_j := (\partial_x^l U^\varepsilon_{R,\text{init}}|_{x_d=0})_{l\leqslant j}.$$

We also introduce for all $j\in\mathbb{N}^*$, the functions

$$\gamma^\varepsilon_{2j-1} := (\boldsymbol{\alpha}^\varepsilon)^{-(j-1)}(\rho^\varepsilon)^{-j} \text{ and } \gamma^\varepsilon_{2j} := (\boldsymbol{\alpha}^\varepsilon)^{-j}(\rho^\varepsilon)^{-j}.$$

Indeed there are some C^∞ functions $(\tilde{\mathcal{H}}_j)_{j\in\mathbb{N}^*}$ such that for all $j\in\mathbb{N}^*$,

$$\mathcal{H}_{2j-1}(\Omega^\varepsilon_{2j-1}) = \tilde{\mathcal{H}}_{2j}(\mho^\varepsilon_{2j-1}) + \big(\gamma^\varepsilon_{2j-1}\partial_d^{2j-1}\mathtt{p}^\varepsilon_{R,\text{init}}\big)|_{x_d=0}, \tag{88}$$

$$\mathcal{H}_{2j}(\Omega^\varepsilon_{2j}) = \tilde{\mathcal{H}}_{2j}(\mho^\varepsilon_{2j}) + \big(\gamma^\varepsilon_{2j}\partial_d^{2j}\mathtt{v}^\varepsilon_{R,\text{init},d}\big)|_{x_d=0}. \tag{89}$$

Step 2.

This step is devoted to the construction of some families of functions $(\dot{U}^{\varepsilon,j}_{R,\text{init}})_\varepsilon\in H^\infty(\Gamma)$ which are expected to be the traces on $\{x_d=0\}$ of the normal derivatives $\partial_d^j U^\varepsilon_{R,\text{init}}$ of an initial data $U^\varepsilon_{R,\text{init}}$ which satisfy the compatibility conditions. The term $\dot{U}^{\varepsilon,j}_{R,\text{init}}$ will be split into

$$\dot{U}^{\varepsilon,j}_{R,\text{init}} := (\dot{\mathtt{V}}^{\varepsilon,j}_{R,\text{init}}, \dot{\mathtt{P}}^{\varepsilon,j}_{R,\text{init}}, \dot{S}^{\varepsilon,j}_{R,\text{init}}).$$

We will denote

$$\dot{\rho}^\varepsilon := \rho(\dot{\mathtt{p}}^\varepsilon_{\text{init}}, \dot{\mathtt{s}}^\varepsilon_{\text{init}}) \text{ and } \dot{\boldsymbol{\alpha}}^\varepsilon := \boldsymbol{\alpha}(\dot{\mathtt{p}}^\varepsilon_{\text{init}}, \dot{\mathtt{s}}^\varepsilon_{\text{init}})$$

where

$$\dot{\mathtt{p}}^\varepsilon_{\text{init}} := \mathtt{p}^\varepsilon_a|_{t=x_d=0} + \varepsilon^{M+1}\,\dot{\mathtt{P}}^{\varepsilon,0}_{R,\text{init}} \text{ and } \dot{\mathtt{s}}^\varepsilon_{\text{init}} := \mathtt{s}^\varepsilon_a|_{t=x_d=0} + \varepsilon^{M}\,\dot{\mathtt{S}}^{\varepsilon,0}_{R,\text{init}}.$$

Let us go on with a convention. When $\alpha := (\alpha_1,...,\alpha_d)\in\mathbb{N}^d$, we will denote $\alpha' := (\alpha_1,...,\alpha_{d-1})\in\mathbb{N}^{d-1}$. The notation $\partial_{\alpha'}$ will stand for $\partial_0^{\alpha_0}...\partial_{d-1}^{\alpha_{d-1}}$. By iteration, we see that there exists for all $j\in\mathbb{N}$, a family of functions $(\dot{U}^{\varepsilon,j}_{R,\text{init}})_\varepsilon$ bounded in $H^\infty(\Gamma)$ such that

$$(\dot{\mathtt{V}}^{\varepsilon,0}_{R,\text{init},\mathtt{t}}, \dot{\mathtt{S}}^{\varepsilon,0}_{R,\text{init}}) = (\hat{\mathtt{V}}^{\varepsilon}_{R,\text{init},\mathtt{t}}, \hat{\mathtt{S}}^{\varepsilon,0}_{R,\text{init}})|_{x_d=0},\ \dot{\mathtt{V}}^{\varepsilon}_{R,\text{init},d}|_{x_d=0}=0 \tag{90}$$

and for all $j\in\mathbb{N}^*$,

$$\tilde{\mathcal{H}}_{2j}(\dot{\mho}^\varepsilon_{2j-1}) + \dot{\gamma}^\varepsilon_{2j-1}\dot{\mathtt{p}}^{\varepsilon,2j-1}_{R,\text{init}} = \Upsilon^\varepsilon_{2j-1} \text{ and } \tilde{\mathcal{H}}_{2j}(\dot{\mho}^\varepsilon_{2j}) + \dot{\gamma}^\varepsilon_{2j}\dot{\mathtt{v}}^{\varepsilon,2j}_{R,\text{init},d} = \Upsilon^\varepsilon_{2j}. \tag{91}$$

Let us explain the notations used above. We denote

$$\dot{\mho}^\varepsilon_j := (\partial_x^{\alpha'}\dot{U}^{\varepsilon,\alpha_d}_{R,\text{init}})_{\alpha\in I_j} \text{ and } \Upsilon^\varepsilon_j := \partial_t^j R^\varepsilon_{\mathtt{V}_d}|_{t=x_d=0}.$$

and for all $j\in\mathbb{N}^*$ by:

$$\dot{\gamma}^\varepsilon_{2j-1} := (\dot{\boldsymbol{\alpha}}^\varepsilon)^{-(j-1)}(\dot{\rho}^\varepsilon)^{-j} \text{ and } \gamma^\varepsilon_{2j} := (\dot{\boldsymbol{\alpha}}^\varepsilon)^{-j}(\dot{\rho}^\varepsilon)^{-j}.$$

Step 3.

We use a Lemma by E. Borel (cf. [37]) to get a family $(U_{R,\text{init}})_\varepsilon$ bounded in $H^\infty(\Omega)$ such that for all $j\in\mathbb{N}$, for all ε,

$$(\mathtt{V}^\varepsilon_{R,\text{init},\mathtt{t}}, \mathtt{S}^\varepsilon_{R,\text{init}}) = (\hat{\mathtt{V}}^\varepsilon_{R,\text{init},\mathtt{t}}, \hat{\mathtt{S}}^\varepsilon_{R,\text{init}}) \text{ and } \partial_d^j U^\varepsilon_{R,\text{init}}|_{x_d=0} := \dot{U}^{\varepsilon,j}_{R,\text{init}}.$$

Use (91), (90), (88), (89) to verify that such a family $U_{R,\text{init}})_\varepsilon$ verifies the compatibility condition $(\mathcal{R}^{j,\varepsilon})$, for all $j\in\mathbb{N}$.

The proof of Lemma 8.9 is now completed. □

8.5. **Linear estimate.** We begin to look at the following linear boundary problem:

$$(\mathfrak{L}^\star(\underline{u}^\varepsilon, \partial_{t,x}) + \underline{J}^\alpha).V_R^\varepsilon + W_R^\varepsilon.(\underline{J}^{\beta,1} + \frac{x_d}{\varepsilon}\underline{J}^{\beta,2}) = -R_v^\varepsilon \quad \text{when } x_d > 0, \tag{92}$$

$$\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} W_R^\varepsilon + \varepsilon \mathtt{V}_R^\varepsilon.\partial_x W_a^\varepsilon = -\varepsilon R_W^\varepsilon \quad \text{when } x_d > 0, \tag{93}$$

$$\mathtt{V}_{R,d}^\varepsilon = 0 \quad \text{when } x_d = 0, \tag{94}$$

where $\underline{J}^\alpha$ denotes the $(d+1)\times(d+1)$ matrix

$$\underline{J}^\alpha \quad := \quad J^\alpha(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M \underline{U}_R^\varepsilon),$$

and for $i \in \{1,2\}$, $\underline{J}^{\beta,i}$ denotes the $\mathbb{R}^{d+1}$-valued function

$$\underline{J}^{\beta,i} \quad := \quad J^{\beta,i}(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M \underline{U}_R^\varepsilon).$$

The family $(\underline{u}^\varepsilon)_\varepsilon := (\underline{v}^\varepsilon, \underline{W}^\varepsilon)_\varepsilon$ is a given family of the form

$$\underline{v}^\varepsilon := v_a^\varepsilon + \varepsilon^{M+1}\underline{V}_R^\varepsilon, \quad \underline{W}^\varepsilon := W_a^\varepsilon + \varepsilon^M \underline{W}_R^\varepsilon \tag{95}$$

with $\underline{U}_R^\varepsilon := (\underline{V}_R^\varepsilon, \underline{W}_R^\varepsilon) \in \mathbf{E}^m(T_\dagger)$. We also define $\underline{V}^\varepsilon$ such that

$$\underline{v}^\varepsilon = v^\dagger + \varepsilon \underline{V}^\varepsilon. \tag{96}$$

We combine (57), (95) and (96) to get

$$\underline{V}^\varepsilon = V_a^\varepsilon + \varepsilon^M \underline{V}_R^\varepsilon. \tag{97}$$

We split $\underline{V}_R^\varepsilon$ and $\underline{V}^\varepsilon$ into

$$\underline{V}_R^\varepsilon := (\underline{\mathtt{V}}_R^\varepsilon, \underline{\mathtt{P}}_R^\varepsilon) \text{ and } \underline{V}^\varepsilon := (\underline{\mathtt{V}}^\varepsilon, \underline{\mathtt{P}}^\varepsilon)$$

with

$$\underline{\mathtt{V}}_R^\varepsilon := (\underline{\mathtt{V}}_{R,1}^\varepsilon, ..., \underline{\mathtt{V}}_{R,d}^\varepsilon) \text{ and } \underline{\mathtt{V}}^\varepsilon := (\underline{\mathtt{V}}_1^\varepsilon, ..., \underline{\mathtt{V}}_d^\varepsilon).$$

The relation (97) implies that in particular the following relation holds for the velocity:

$$\underline{\mathtt{V}}_d^\varepsilon \quad = \quad \mathtt{V}_{a,d}^\varepsilon + \varepsilon^M \underline{\mathtt{V}}_{R,d}^\varepsilon. \tag{98}$$

$$\underline{\mathtt{V}}_d^\varepsilon \quad = \quad \mathtt{v}^\dagger + \varepsilon \underline{\mathtt{V}}_d^\varepsilon. \tag{99}$$

We denote $\underline{\mathtt{V}}_{R,\mathfrak{t}}^\varepsilon := (\underline{\mathtt{V}}_{R,1}^\varepsilon, ..., \underline{\mathtt{V}}_{R,d-1}^\varepsilon)$ (resp. $\underline{\mathtt{V}}_\mathfrak{t}^\varepsilon : (\underline{\mathtt{V}}_1^\varepsilon, ..., \underline{\mathtt{V}}_{d-1}^\varepsilon)$) the tangential part of the velocity $\underline{\mathtt{V}}_R^\varepsilon$ (resp. $\underline{\mathtt{V}}^\varepsilon$).

We assume that the normal component $\underline{\mathtt{V}}_{R,d}^\varepsilon$ of the velocity remainder satisfies $\underline{\mathtt{V}}_{R,d}^\varepsilon = 0$ when $x_d = 0$. Since the normal components $\mathtt{v}_a^\varepsilon$ and $\mathtt{v}^\dagger$ of the velocity $\mathtt{v}_{a,d}^\varepsilon$ when $x_d = 0$, the normal velocity $\underline{\mathtt{v}}_d^\varepsilon$ is such that $\underline{\mathtt{v}}_d^\varepsilon = 0$ when $x_d = 0$.

We introduce the classic spaces of conormal distributions

$$H^{0,m}(T) := \{u \in L^2(T)/\ 0 \leqslant k \leqslant m,\ Z^k u \in L^2(T)\}.$$

These spaces will be endowed by the following weighted norms:

$$|u|_{m,\lambda,T} \quad := \quad \sum_{0\leqslant k\leqslant m} \lambda^{m-k}\, ||e^{-\lambda T} Z^k u||_{L^2(T)}. \tag{100}$$

Because we face a characteristic boundary problem with functions in $\frac{x_d}{\varepsilon}$, we will also use the following norms:

$$|u|^{\mathcal{N}}_{\varepsilon,m,\lambda,T} := |u|_{m,\lambda,T} + |(\varepsilon\partial_d)u|_{m,\lambda,T} \tag{101}$$

$$|u|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} := \sum_{0\leqslant 2k+l\leqslant m} \lambda^{m-2k-l}\, |\mathfrak{D}^{k,l}_{\varepsilon}\,|_{0,\lambda,T} \tag{102}$$

A link between the L^2-type norm: $|.|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}$ and the L^∞-type norm: $||.||^*_{\varepsilon,T}$ is given by the following Sobolev Embedding lemma:

Lemma 8.10 (Sobolev Embedding)**.** *Let $m > \frac{n}{2} + 5$. There is $c > 0$ such that for all $T \in [0, T_\dagger]$, for all $u \in C_0^\infty((0,T)\times\Omega)$,*

$$\sqrt{\varepsilon}||u||^*_{\varepsilon,T} \leqslant cTe^{\lambda T}\, |u|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}.$$

Proof. First we recall Lemma $II.1.1$ of [13]. To do this we introduce the norms:

$$||u||^*_T := ||u||_{0,T} + ||u||_{1,T} \text{ and } |u|^E_{m,\lambda,T} := \sum_{0\leqslant 2k+l\leqslant m} \lambda^{m-2k-l}\, |\partial_d^k\, Z^l\, u|_{0,\lambda,T}.$$

Lemma $II.1.1$ of [13] shows that there exists $c > 0$ such that for all $u \in E^m$,

$$||u||^*_T \leq cTe^{\lambda T}|u|^E_{m,\lambda,T}.$$

We now perform a rescaling. More precisely let us introduce the family of functions $(\tilde{u}^\varepsilon)_{\varepsilon\in]0,1]}$ by

$$\tilde{u}^\varepsilon(t,x) := u(t,y,\varepsilon x_d) \quad \text{for all } (\varepsilon,t,x) \in]0,1]\times[0,T]\times\Omega.$$

We get

$$||\tilde{u}^\varepsilon||^*_T := ||u||^*_{\varepsilon,T} \text{ and } ||\tilde{u}^\varepsilon||^E_{m,\lambda,T} := \varepsilon^{-\frac{1}{2}}\, ||u||^{\mathbf{E}}_{\varepsilon,m,\lambda,T}$$

and then complete the proof of Lemma 8.10. □

We will use the following version of the Gagliardo-Nirenberg estimates (cf. [13]).

Lemma 8.11 (Gagliardo-Nirenberg)**.** *There exists $C > 0$ such that for all $T \in]0, T_\dagger]$, for all $u \in L^\infty(T)\cap H^{0,m}(T)$, for all l and m such that $l \leqslant k \leqslant m$ and for all real $\lambda \geqslant 1$*

$$\lambda^{k-l}|e^{-\frac{2}{p}\lambda t}\, Z^l u|_{L^p((0,T)\times\Omega)} \leqslant C||u||_{0,T}^{1-\frac{k}{m}}|u|_{m,\lambda,T}^{\frac{k}{m}},$$

where $p := \frac{2m}{k}$.

We will also use

Lemma 8.12 (Gagliardo-Nirenberg)**.** *Let $m \in \mathbb{N}$ be even. There exists $C > 0$ such that for all $T \in]0, T_\dagger]$, for all $u \in L^\infty(T)\cap E(T)$, for all l and m such that $l \leqslant k \leqslant m$, for all $\varepsilon \in]0,1]$ and for all real $\lambda \geqslant 1$,*

$$\lambda^{k-l}|e^{-\frac{2}{p}\lambda t}(\varepsilon\partial_d)^k Z^l u|_{L^p((0,T)\times\Omega)} \leqslant C||u||_{0,T}^{1-\frac{k}{m}}(|u|^{\mathbf{E}}_{\varepsilon,m,\lambda,T})^{\frac{k}{m}},$$

where $p := \frac{2m}{k}$.

Proof. It is a special case of $(Ap - II - 3)$ given in [13], p.643. □

Lemma 8.11 and 8.12 imply the following Moser's type inequalities.

Lemma 8.13 (Moser Inequalities)**.** *Let $F \in C^\infty$ such that $F(0) = 0$, and $\mathfrak{R} > 0$. There exists a real l such that for all $T \in [0, T_\dagger]$, if*

$$g \in L^\infty(T) \cap H^{0,m}(T) \quad (\text{resp. } L^\infty(T) \cap E^m(T) \text{ and } m \text{ even}),$$

verifies $||g||_{0,T} \leqslant \mathfrak{R}$, we have, for all $\lambda \geqslant 1$, for all $\varepsilon \in]0,1]$, we have:

$$|F(g)|_{m,\lambda,T} \leqslant l|g|_{m,\lambda,T}, \quad (\text{resp. } |F(g)|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} \leqslant l|g|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}).$$

Lemma 8.14. *For all $\underline{m} \in \mathbb{N}$, there exists a real c such that for all T verifying $0 \leqslant T \leqslant T_\dagger$, and $\alpha, \beta \in L^\infty(T)$, we have:*

(1) *If $\alpha, \beta \in H^{0,m}(T)$ and if $j, k, l \in \mathbb{N}$ are such that $0 \leqslant j + l \leqslant k \leqslant \underline{m}$:*

$$\lambda^{\underline{m}-k}|Z^j\alpha\ Z^l\beta|_{\varepsilon,0,\lambda,T} \leqslant c(||\alpha||_{0,T}\ |\beta|_{\underline{m},\lambda,T} + |\alpha|_{\underline{m},\lambda,T}\ ||\beta||_{0,T})$$
$$|\alpha\ \beta|_{\underline{m},\lambda,T} \leqslant c(||\alpha||_{0,T}\ |\beta|_{\underline{m},\lambda,T} + |\alpha|_{\underline{m},\lambda,T}\ ||\beta||_{0,T})$$

(2) *If m is even, if α, $\beta \in \mathbf{E}^m(T)$, and $k, k', l, l', p \in \mathbb{N}$ are such that $2(k+k')+l+l' \leqslant p \leqslant \underline{m}$:*

$$\begin{aligned}\lambda^{\underline{m}-p}|(\mathfrak{D}^{k,l}_\varepsilon\,\alpha)(\mathfrak{D}^{k',l'}_\varepsilon\,\alpha)|_{0,\lambda,T} &\leqslant c(||\alpha||_{0,\lambda,T}\ |\beta|^{\mathbf{E}}_{\varepsilon,\underline{m},\lambda,T} + |\alpha|^{\mathbf{E}}_{\varepsilon,\underline{m},\lambda,T}\ ||\beta||_{0,\lambda,T}),\\ |\alpha\ \beta|^{\mathbf{E}}_{\varepsilon,\underline{m},\lambda,T} &\leqslant c(||\alpha||_{0,T}\ |\beta|^{\mathbf{E}}_{\varepsilon,\underline{m},\lambda,T} + |\alpha|^{\mathbf{E}}_{\varepsilon,\underline{m},\lambda,T}\ ||\beta||_{0,T}).\end{aligned}$$

We combine Lemma 8.13 and Lemma 8.14 to find the following corollary.

Corollary 8.15. *Let $F \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that $F(0) = 0$ and $\mathfrak{R} > 0$. There exists $C > 0$ such that, for all $T \in [0, T_\dagger]$, for all functions $g \in Lip((0,T) \times \Omega) \cap \mathcal{N}(T)$, $g : \Omega_T \to \mathbb{R}^k$ which verify $||g||_{Lip(T)} \leqslant \mathfrak{R}$ then $F(g) \in \mathcal{N}(\Omega_T)$ and we have:*

$$|F(g)|^{\mathcal{N}}_{\varepsilon,m,\lambda,T} \leqslant C|g|^{\mathcal{N}}_{\varepsilon,m,\lambda,T}.$$

Proof. Referring to the definition of $|.|^{\mathcal{N}}_{\varepsilon,m,\lambda,T}$ (see (101)), we get

$$|F(g)|^{\mathcal{N}}_{\varepsilon,m,\lambda,T} := |F(g)|_{m,\lambda,T} + |\varepsilon\partial_d F(g)|_{m-2,\lambda,T}.$$

We use Lemma 8.13 to bound $|F(g)|_{m,\lambda,T}$. Referring to the definition of $|.|_{m,\lambda,T}$ (see (100)), we get

$$\begin{aligned}|\varepsilon\partial_d F(g)|_{m-2,\lambda,T} &= \sum_{0\leqslant k\leqslant m-2} \lambda^{m-2-k}||e^{-\lambda T}\ Z^k\varepsilon\partial_d F(g)||_{L^2(T)}\\ &= \sum_{0\leqslant k=k_1+k_2\leqslant m-2} \lambda^{m-2-k}\mathfrak{N}_{k_1,k_2}\end{aligned}$$

where

$$\mathfrak{N}_{k_1,k_2} := ||e^{-\lambda T}\ Z^{k_1}D_gF(g).Z^{k_1}\varepsilon\partial_d g||_{L^2(T)}.$$

Then we use Lemma 8.14 (1). □

We will prove the following linear ε-uniform estimates which will be useful in subsection 8.10.

Theorem 8.16. *Let m be an even integer larger than 2. Let $\mathfrak{R}$ be a strictly positive real. Then there is $\lambda_0 > 1$ such that if $\varepsilon^M ||\underline{U}_R^\varepsilon||^*_{\varepsilon,T_\dagger} \leqslant \mathfrak{R}$ then for all $\lambda \geqslant \lambda_0$, for all $\varepsilon \in]0, \varepsilon_0]$,*

$$|U_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_\dagger} \leqslant \lambda^{-1}\lambda_0.(1 + \varepsilon^M\, ||U_R^\varepsilon||^*_{\varepsilon,T_\dagger}.|\underline{U}_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_\dagger}).$$

Theorem 8.16 will be proved in subsections 8.6 to 8.9. We will proceed in several steps. In subsection 8.6, we will perform L^2 estimates by proceeding in three steps. We will estimate W_R^ε then $\frac{x_d}{\varepsilon} W_R^\varepsilon$ and finally V_R^ε. Higher order estimates are more delicate to obtain. In subsection 8.7, we will give preliminary results with some commutators estimates (cf. Proposition 8.21 and with some estimates of $\mathtt{v}_d^\varepsilon.\partial_d U_R^\varepsilon$ (cf. Lemma 8.24). Then we will also begin with conormal estimates (subsection 8.8) then looking for normal estimates (subsection 8.9). As in the L^2 estimate, we will look for estimates about W_R^ε, $\frac{x_d}{\varepsilon} W_R^\varepsilon$ and finally V_R^ε. Furthermore, in the normal estimate, we will distinguish the estimates about tangential velocity $\mathtt{v}_{R,\mathtt{t}}^\varepsilon$ which corresponds to characteristic components and the estimates about normal velocity $\mathtt{v}_{R,d}^\varepsilon$ which corresponds to a noncharacteristic component.

Thanks to the boundary conditions, we can factorize by x_d and by some smooth functions the normal velocities:

$$\mathtt{v}_{a,d} = x_d\, \mathtt{v}_{a,d}^\flat,\ \mathtt{V}_{R,d}^\varepsilon = x_d\, \mathtt{V}_{R,d}^{\varepsilon,\flat},\ \underline{\mathtt{V}}_d^\varepsilon = x_d\, \underline{\mathtt{V}}_d^{\varepsilon,\flat},\ \underline{\mathtt{v}}_d^\varepsilon = x_d\, \underline{\mathtt{v}}_d^{\varepsilon,\flat}. \tag{103}$$

This will ply a crucial role.

8.6. L^2 **estimate.** We will proceed in three steps estimating W_R^ε then $\frac{x_d}{\varepsilon} W_R^\varepsilon$ and finally V_R^ε. According to section 5, $\partial_x W_a^\varepsilon$ is of the form $\beta_1(\varepsilon, t, x) + \varepsilon^{-1}\beta_2(\varepsilon, t, x, \frac{x_d}{\varepsilon})$ where β_1 and β_2 are C_b^∞ with respect to their arguments including ε up to 0. Moreover β_2 is rapidly decreasing with respect to its last argument.

8.6.1. *Estimate of W_R^ε.* The aim of this subsection is to prove

Proposition 8.17. *Let $\mathfrak{R} > 0$. There exists $\lambda_0 > 1$ such that if $\varepsilon^M\, ||\underline{V}_R^\varepsilon||^*_{\varepsilon,T_\dagger} \leqslant \mathfrak{R}$ then for all $\lambda \geqslant \lambda_0$,*

$$|W_R^\varepsilon|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda_0}{\lambda}(1 + |V_R^\varepsilon|_{0,\lambda,T_\dagger}).$$

Proof. We rewrite the equation (93) as

$$\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} W_R^\varepsilon = f_I^\varepsilon \tag{104}$$

where $f_I^\varepsilon := (\varepsilon\beta_1 + \beta_2)V_R^\varepsilon - \varepsilon R_W^\varepsilon$. By a classic L^2 estimate, we obtain that for $\mathfrak{R} > 0$, there exists $\lambda_0 \geqslant 1$ such that if $||\underline{\mathtt{v}}^\varepsilon||_{Lip} \leqslant \mathfrak{R}$ then for all $\lambda \geqslant \lambda_0$,

$$|W_R^\varepsilon|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda}{\lambda_0}|f_I^\varepsilon|_{0,\lambda,T_\dagger}.$$

Using (95) and estimating $|f_I^\varepsilon|_{0,\lambda,T_\dagger}$ we end the proof. ∎

8.6.2. *Estimate of* $\frac{x_d}{\varepsilon} W^\varepsilon_R$. We will exploit the special form of the equation (93) thanks to a second estimate which concerns $\frac{x_d}{\varepsilon} W^\varepsilon_R$.

Proposition 8.18. *Let* $\mathfrak{R} > 0$. *There exists* $\lambda_0 > 1$ *such that if* $\varepsilon^M ||\underline{V}^\varepsilon||^*_{\varepsilon,T_\dagger} \leqslant \mathfrak{R}$ *then for all* $\lambda \geqslant \lambda_0$, $|\frac{x_d}{\varepsilon} W^\varepsilon_R|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda_0}{\lambda}(1 + |V^\varepsilon_R|_{0,\lambda,T_\dagger})$.

Proof. We will proceed in two steps. First we will look for an equation for $\frac{x_d}{\varepsilon} W^\varepsilon_R$ and then we will use a L^2 estimate for this equation.

1. We begin to calculate

$$\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} (\frac{x_d}{\varepsilon} W^\varepsilon_R) = \frac{x_d}{\varepsilon} \mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} W^\varepsilon_R + \frac{\underline{\mathtt{v}}^\varepsilon_d}{\varepsilon} W^\varepsilon_R. \tag{105}$$

Using the equation (93) and the last equation in (103), we deduce from the equation (105) that $\frac{x_d}{\varepsilon} W^\varepsilon_R$ verifies the equation

$$\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} (\frac{x_d}{\varepsilon} W^\varepsilon_R) - \underline{\mathtt{v}}^{\varepsilon,\flat}_d (\frac{x_d}{\varepsilon} W^\varepsilon_R) = f^\varepsilon_{II}, \tag{106}$$

where

$$f^\varepsilon_{II} := \frac{x_d}{\varepsilon}(-\varepsilon \mathtt{V}^\varepsilon_R . \partial_x W^\varepsilon_a - \varepsilon R^\varepsilon_W), \tag{107}$$

$$:= -x_d \beta_1 V^\varepsilon_R - \frac{x_d}{\varepsilon} \beta_2 V^\varepsilon_R - x_d R^\varepsilon_W. \tag{108}$$

2. By a classic L^2 estimate, we obtain that for $\mathfrak{R} > 0$, there exists $\lambda_0 \geqslant 1$ such that if

$$||\underline{\mathtt{v}}^\varepsilon||_{Lip((0,T_\dagger)\times\Omega)} + ||\underline{\mathtt{v}}^{\varepsilon,\flat}_d||_{L^\infty(T_\dagger)} \leqslant R$$

then for all $\lambda \geqslant \lambda_0$,

$$|\frac{x_d}{\varepsilon} W^\varepsilon_R|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda_0}{\lambda} |f^\varepsilon_{II}|_{0,\lambda,T_\dagger}.$$

Because of the form of $\underline{\mathtt{v}}^\varepsilon$ (see (95)), the family $(||\underline{\mathtt{v}}^\varepsilon||_{Lip} + ||\underline{\mathtt{v}}^{\varepsilon,\flat}_d||_{L^\infty})_\varepsilon$ is bounded when the family $(\varepsilon^M ||\underline{V}^\varepsilon_R||^*_{\varepsilon,T_\dagger})_\varepsilon$ is bounded. Moreover because β_2 is rapidly decreasing with respect to its last argument, the family $(\frac{x_d}{\varepsilon}\beta_2(\varepsilon, t, x, \frac{x_d}{\varepsilon}))_\varepsilon$ is bounded in L^∞. This ends the proof.

□

8.6.3. *Estimate of* V^ε_R. We will prove the following result:

Lemma 8.19. *Let* $\mathfrak{R} > 0$. *There exists* $\lambda_0 > 0$ *such that if* $\varepsilon^M ||\underline{U}^\varepsilon_R||^*_{\varepsilon,T_\dagger} \leqslant \mathfrak{R}$ *then for all* $\lambda \geqslant \lambda_0$, $|V^\varepsilon_R|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda_0}{\lambda}$.

Proof. For each $\varepsilon \in]0, 1]$, the function V^ε_R is solution of the following boundary value problem:

$$\begin{cases} (\mathfrak{L}^\star(\underline{u}^\varepsilon, \partial_{t,x}) + \underline{J}^\alpha). V^\varepsilon_R = f^\varepsilon_{III} & \text{when } x_d > 0, \\ \mathtt{V}^\varepsilon_{R,d} = 0 & \text{when } x_d = 0, \end{cases} \tag{109}$$

where

$$f^\varepsilon_{III} := -W^\varepsilon_R \underline{J}^{\beta,1} - \frac{x_d}{\varepsilon} W^\varepsilon_R \underline{J}^{\beta,2} - R^\varepsilon_W. \tag{110}$$

It is a symmetric hyperbolic system for which the boundary is conservative and characteristic of constant multiplicity (cf. Properties 4.1). Thanks to [27], the following L^2 estimate holds: for $\mathfrak{R} > 0$, there exists $\lambda > 0$, such that if

$$||\underline{\mathrm{v}}^\varepsilon||_{Lip} + ||\underline{J}^\alpha||_{L^\infty} + ||\underline{J}^\beta||_{L^\infty} \leqslant \mathfrak{R}$$

then for all $\lambda \geqslant \lambda_0$,

$$|V_R^\varepsilon|_{0,\lambda,T_\dagger} \leqslant \frac{\lambda}{\lambda_0} |f_{III}^\varepsilon|_{0,\lambda,T_\dagger}.$$

Using the form of $\underline{\mathrm{v}}^\varepsilon$, $\underline{J}^\alpha$, $\underline{J}^\beta$ and estimating $|f_{III}^\varepsilon|_{0,\lambda,T_\dagger}$ thanks to subsection 8.6.1 and 8.6.2 yields the conclusion. □

8.7. **Higher order estimates.** We look for estimates uniform with respect to $0 < \varepsilon \leqslant 1$ of

$$\lambda^{m-2k-l} |(\varepsilon \partial_d)^k Z^l U_R^\varepsilon|_{0,\lambda,T_\dagger} \quad \text{for } 2k + l \leqslant m.$$

In this subsection, we give crucial preliminary technical results. We group them in two kinds. In a first time, we look for commutators estimates. In a second time, we look for estimates which will be useful when estimating source terms.

8.7.1. *Commutators estimates.* We begin with the estimates of the commutators

$$[\underline{J}, (\varepsilon \partial_d)^k Z^l] \varphi.$$

Because we will proceed in two steps, estimating first conormal estimates then normal estimates, we give specific estimates of the commutators with the derivatives Z^l for $l \leqslant m$ (i.e. in the limit case $k = 0$).

Proposition 8.20. *Let m be an integer. There is an increasing function $C : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that for all $\varepsilon \in]0, 1]$, if $\varepsilon^M ||\underline{U}_R^\varepsilon||_{0,T} \leqslant \mathfrak{R}$ then for all $\varphi \in C_0^\infty((0, T) \times \overline{\Omega})$, for all k, l such that $2k + l \leqslant m$, for all $\lambda \geqslant \lambda_0$,*

a) if m is even,

$$\lambda^{m-2k-l} |[\underline{J}(\underline{u}^\varepsilon), \mathfrak{D}_\varepsilon^{k,l}]\varphi|_{0,\lambda,T} \leqslant C(\mathfrak{R})(|\varphi|_{\varepsilon,m,\lambda,T}^{\mathbf{E}} + \varepsilon^M ||\varphi||_{0,T} \cdot |\underline{U}_R^\varepsilon|_{\varepsilon,m,\lambda,T}^{\mathbf{E}}),$$

b) for any m,

$$\lambda^{m-l} |[\underline{J}(\underline{u}^\varepsilon), Z^l]\varphi|_{0,\lambda,T} \leqslant C(\mathfrak{R})(|\varphi|_{m,\lambda,T} + \varepsilon^M ||\varphi||_{0,T} \cdot |\underline{U}_R^\varepsilon|_{m,\lambda,T}),$$

where $\underline{J}$ stands for $\underline{J}^\alpha$, $\underline{J}^{\beta,1}$ or $\underline{J}^{\beta,2}$.

Proof of (b). We introduce a notation: if $l \in \mathbb{N}, l \neq 0$ and ϕ a vector-valued function which components are denoted ϕ_i, we denote $Z^{<l>}\phi$ the collection of the terms of the form $Z^{l_1}\phi_{i_1} ... Z^{l_r}\phi_{i_r}$ where $1 \leqslant r \leqslant l$, $l_1 + ... l_r = l$ and $l_i \geqslant 1$ and if $l = 0$, $Z^{<l>}\phi = 0$. The commutator $[\underline{J}, Z^l]\varphi$ is a linear combination of terms of the form

$$\Phi(\varepsilon, V_a^\varepsilon, Z_\varepsilon V_a^\varepsilon, W_a^{\varepsilon,\diamond}, \varepsilon^M \underline{U}_R^\varepsilon).\mathfrak{P}$$

with

$$\mathfrak{P} := Z^{<l_1>}(V_a^\varepsilon).Z^{<l_2>}(Z_\varepsilon V_a^\varepsilon).Z^{<l_3>}(W_a^{\varepsilon,\diamond}).Z^{<l_4>}(\varepsilon^M \underline{U}_R^\varepsilon).Z^{l_5}\phi,$$

and $l_1+l_2+l_3+l_4+l_5=l$. Thanks to estimates (59) and Lemma 8.13, we get

$$\lambda^{m-l}|[\underline{J}(\underline{u}^\varepsilon),Z^l]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})\lambda^{m-l}|Z^{<l_4>}(\varepsilon^M\,\underline{U}^\varepsilon_R).Z^{l_5}\phi|_{m,\lambda,T}.$$

We end the proof thanks to Lemma 8.14 (1). □

Proof of (a). We introduce a notation: if $l\in\mathbb{N},l\neq 0$ and ϕ a vector-valued function which components are denoted ϕ_i, we denote $(\varepsilon\partial_d)^{<k>}\phi$ the collection of the terms of the form $(\varepsilon\partial_d)^{k_1}\phi_{i_1}...(\varepsilon\partial_d)^{k_r}\phi_{i_r}$ where $1\leqslant r\leqslant k$, $k_1+...+k_r=k$ and $k_i\geqslant 1$ and if $k=0$, $(\varepsilon\partial_d)^{<k>}\phi=0$. We denote $\mathfrak{D}^{<k,l>}_\varepsilon:=(\varepsilon\partial_d)^{<k>}Z^l$.

The commutator $[\underline{J},\mathfrak{D}^{k,l}_\varepsilon]\varphi$ is a linear combination of terms of the form

$$\Phi(\varepsilon,V^\varepsilon_a,Z_\varepsilon V^\varepsilon_a,W^{\varepsilon,\diamond}_a,\varepsilon^M\,\underline{U}^\varepsilon_R).\tilde{\mathfrak{P}}$$

with

$$\tilde{\mathfrak{P}}:=\mathfrak{D}^{<k_1,l_1>}_\varepsilon(V^\varepsilon_a).\mathfrak{D}^{<k_2,l_2>}_\varepsilon(Z_\varepsilon V^\varepsilon_a).\mathfrak{D}^{<k_3,l_3>}_\varepsilon(W^{\varepsilon,\diamond}_a).\mathfrak{D}^{<k_4,l_4>}_\varepsilon(\varepsilon^M\,\underline{U}^\varepsilon_R).\mathfrak{D}^{<k_5,l_5>}_\varepsilon\,\underline{J}\phi,$$

with $l_1+...+l_5=l$ and $k_1+...+k_5=k$. Thanks to estimates (59) and Lemma 8.13, we get

$$\lambda^{m-2k-l}|[\underline{J}(\underline{u}^\varepsilon),Z^l]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})\lambda^{m-l}|Z^{<l_4>}(\varepsilon^M\,\underline{U}^\varepsilon_R).Z^{l_5}\,\underline{J}\phi|_{m,\lambda,T}.$$

We end the proof thanks to Lemma 8.14 (2). □

A main point would be to estimate some commutators involving $S(\underline{u}^\varepsilon)X_{\underline{\mathtt{v}}^\varepsilon}$. We will estimate separately $S(\underline{u}^\varepsilon)\underline{\mathtt{v}}^\varepsilon_d\partial_d$ and the one $S(\underline{u}^\varepsilon)X'_{\underline{\mathtt{v}}^\varepsilon}$ where

$$X'_{\mathtt{v}}:=X_{\mathtt{v}}-\mathtt{v}_d\partial_d=\partial_t+\sum_{i=1}^{d-1}\mathtt{v}_i\partial_i.$$

Moreover because we will proceed in two steps, estimating first conormal estimates then normal estimates, we give specific estimates of the commutators with the derivatives Z^l for $l\leqslant m$ (i.e. in the limit case $k=0$).

Proposition 8.21. *Let m be an integer. There is an increasing function $C:\mathbb{R}_+\to\mathbb{R}^*_+$ such that for all $\varepsilon\in]0,1]$, if $\varepsilon^M\,||\underline{U}^\varepsilon_R||^*_{\varepsilon,T}\leqslant\mathfrak{R}$ then for all $\varphi\in C^\infty_0((0,T)\times\overline{\Omega})$, for all k,l such that $2k+l\leqslant m$, for all $\lambda\geqslant\lambda_0$,*

a) if m is even,

$$\lambda^{m-2k-l}|[S(\underline{u}^\varepsilon).\underline{\mathtt{v}}^\varepsilon_d\partial_d,\mathfrak{D}^{k,l}_\varepsilon]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}+\varepsilon^M||\varphi||^*_{\varepsilon,T}.|\underline{U}^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}),$$

b) for any m,

$$\lambda^{m-l}|[S(\underline{u}^\varepsilon).\underline{\mathtt{v}}^\varepsilon_d\partial_d,Z^l]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|^{\mathcal{N}}_{\varepsilon,m,\lambda,T}+\varepsilon^M||\varphi||^*_{\varepsilon,T}.|\underline{U}^\varepsilon_R|^{\mathcal{N}}_{\varepsilon,m,\lambda,T}),$$

c) if m is even,

$$\lambda^{m-2k-l}|[S(\underline{u}^\varepsilon).\boldsymbol{X}'_{\underline{\mathtt{v}}^\varepsilon},\mathfrak{D}^{k,l}_\varepsilon]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}+\varepsilon^M||\varphi||_{1,T}.|\underline{U}^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}),$$

d) for any m,

$$\lambda^{m-l}|[S(\underline{u}^\varepsilon).\boldsymbol{X}'_{\underline{\mathtt{v}}^\varepsilon},Z^l]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|_{m,\lambda,T}+\varepsilon^M||\varphi||_{1,T}.|\underline{U}^\varepsilon_R|_{m,\lambda,T}).$$

We begin with the proof of $c)$ and $d)$ which are simpler. The estimates $c)$ and $d)$ are consequences of the following lemma.

Lemma 8.22. *Let m be an integer. Let F be a C^∞ function. There is an increasing function $C:\mathbb{R}_+\to\mathbb{R}_+^*$ such that for all $\varepsilon\in]0,1]$, if $\varepsilon^M\,||\underline{U}_R^\varepsilon||^*_{\varepsilon,T}\leqslant\mathfrak{R}$ then for all $\varphi\in C_0^\infty((0,T)\times\overline{\Omega})$, for all k,l such that $2k+l\leqslant m$, for all $\lambda\geqslant\lambda_0$,*

(i) *if m is even,*

$$\lambda^{m-2k-l}|[F(\underline{u}^\varepsilon)Z,\mathfrak{D}_\varepsilon^{k,l}]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}+\varepsilon^M||\varphi||_{1,T}.|\underline{U}_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}),$$

(ii) *for any m,*

$$\lambda^{m-l}|[F(\underline{u}^\varepsilon)Z,Z^l]\varphi|_{0,\lambda,T}\leqslant C(\mathfrak{R})(|\varphi|_{m,\lambda,T}+\varepsilon^M||\varphi||_{1,T}.|\underline{U}_R^\varepsilon|_{m,\lambda,T}).$$

Proof of (ii). The commutator $[F(\underline{u}^\varepsilon).Z,Z^l]\varphi$ is a linear combination of terms of the form

$$Z^{l_1}F(\underline{u}^\varepsilon).Z^{l_2}\varphi \quad \text{when } l_1+l_2=l+1,\ l_1\geqslant 1,\ l_2\geqslant 1. \tag{111}$$

We introduce $l_1':=l_1-1$, $l_2':=l_2'-1$. Thus the term (111) reads

$$Z^{l_1'}(ZF(\underline{u}^\varepsilon)).Z^{l_2'}(Z\varphi).$$

We apply Lemma 8.14 (1) with

$$\alpha:=ZF(\underline{u}^\varepsilon)\ ,\beta:=Z\varphi\ ,\underline{m}:=m-1$$

and obtain the estimate

$$\lambda^{m-l}|[F(\underline{u}^\varepsilon).Z,Z^l]\varphi|_{0,\lambda,T}\leqslant C(||F(\underline{u}^\varepsilon)||_{1,T}.|\varphi|_{m,\lambda,T}+|F(\underline{u}^\varepsilon)|_{m,\lambda,T}.||\varphi||_{1,T}).$$

We use that $(u^\varepsilon)_\varepsilon$ is of the form (95) and apply Lemma 8.13 to complete the proof. □

Proof of (i). We assume that $k\geqslant 1$. The case $k=0$ corresponds to (ii). The commutator $[F(\underline{u}^\varepsilon).Z,\mathfrak{D}_\varepsilon^{k,l}]\varphi$ is a linear combination of terms of the form

$$\mathfrak{D}_\varepsilon^{k,l_1}F(\underline{u}^\varepsilon).Z^{l_2}\varphi \qquad \text{with } l_1+l_2=l+1,\ l_1\geqslant 1,\ l_2\geqslant 1, \tag{112}$$

$$(\varepsilon\partial_d)^{k_1}F(\underline{u}^\varepsilon).\mathfrak{D}_\varepsilon^{k_2,l+1}\varphi \qquad \text{with } k_1+k_2=k,\ k_1\geqslant 1. \tag{113}$$

- We begin to deal with the term (112). We introduce $k':=k-1$, $l_2':=l_2'-1$. We commute $\varepsilon\partial_d$ with Z^{l_1}. Thus it suffices to deal with terms of the form

  $$\mathfrak{D}_\varepsilon^{k',l_1}(\varepsilon\partial_dF(\underline{u}^\varepsilon)).Z^{l_2'}(Z\varphi).$$

  We apply Lemma 8.14 (1) with

  $$\alpha:=\varepsilon\partial_dF(\underline{u}^\varepsilon)\ ,\beta:=Z\varphi\ ,\underline{m}:=m-2. \tag{114}$$

  We obtain the estimate

  $$\lambda^{m-l}|[F(\underline{u}^\varepsilon).Z,Z^l]\varphi|_{0,\lambda,T}\leqslant C(||F(\underline{u}^\varepsilon)||_{\varepsilon,Lip,T}.|\varphi|^{\mathbf{E}}_{m,\lambda,T}+|F(\underline{u}^\varepsilon)|^{\mathbf{E}}_{m,\lambda,T}.||\varphi||_{1,T}).$$

  We end the estimate as in the proof of (ii).
- To deal with the term (113) and complete the proof, we directly apply Lemma 8.14 (1) to (114) and proceed in the same way.

□

We tackle the Proof of b).

Proof of b). The commutator $[S(\underline{u}^\varepsilon).\underline{\mathtt{v}}_d^\varepsilon\partial_d, Z^l]\varphi$ is a linear combination of some terms of the form

$$Z^{l_1}S(\underline{u}^\varepsilon).Z^{l_2}\underline{\mathtt{v}}_d^\varepsilon.Z^{l_3}\partial_d\varphi, \quad \text{with } l_1+l_2+l_3=l,\ l_1+l_2\geqslant 1.$$

Because of (99) it suffices to deal with the terms

$$Z^{l_1}S(\underline{u}^\varepsilon).Z^{l_2}\mathtt{v}_d^\dagger.Z^{l_3}\partial_d\varphi, \tag{115}$$

$$Z^{l_1}S(\underline{u}^\varepsilon).Z^{l_2}\underline{\mathtt{V}}_d^\varepsilon.Z^{l_3}(\varepsilon\partial_d)\varphi. \tag{116}$$

Step 1. *We deal with the terms of the form* (115).

We use (12) and commute x_d with Z^{l_3}, we have to deal with the terms of the form

$$Z^{l_1}S(\underline{u}^\varepsilon).Z^{l_2}\mathtt{v}_d^\dagger.Z^{l_3'}Z\varphi, \quad \text{with } l_3'\leqslant l_3.$$

We use L^∞ estimate of $\mathtt{v}^\dagger$. Thus we have to control

$$\lambda^{m-l}|Z^{l_1}S(\underline{u}^\varepsilon).Z^{l_3'}Z\varphi|_{0,\lambda,T}. \tag{117}$$

We distinguish two cases:

- if $l_1\leqslant 1$, we control $Z^{l_1}S(\underline{u}^\varepsilon)$ in L^∞ norm by Lemma 8.13 and the term (117) is bounded by $c.|\varphi|_{m,\lambda,T}$.
- if $l_1\geqslant 2$, we apply Lemma 8.14 with

$$\alpha := Z^2S(\underline{u}^\varepsilon),\ \beta := Z\varphi,\ \underline{m} := m-2.$$

Step 2. *We deal with the terms of the form* (116).

We distinguish two cases:

- if $l_1\leqslant 1$, we control $Z^{l_1}S(\underline{u}^\varepsilon)$ in L^∞ norm and then we deal with

$$\lambda^{m-l}|Z^{l_2}\underline{\mathtt{V}}_d^\varepsilon.\mathfrak{D}_\varepsilon^{1,l_3}\,\varphi|_{0,\lambda,T}.$$

  with $l_2+l_3\leqslant l$. We distinguish two sub cases:

  (*a*) if $l_2\geqslant 2$, we introduce $l_2' := l_2-2$. Because of (98), we study the terms

$$\lambda^{m-l}|Z^{l_2'}(Z^2\mathtt{V}_{a,d}^\varepsilon).\mathfrak{D}_\varepsilon^{1,l_3}\,\varphi|_{0,\lambda,T}, \tag{118}$$

$$\varepsilon^M\lambda^{m-l}|Z^{l_2'}(Z^2\underline{\mathtt{V}}_{R,d}^\varepsilon).\mathfrak{D}_\varepsilon^{1,l_3}\,\varphi|_{0,\lambda,T}. \tag{119}$$

  We use the L^∞ estimates of $(Z^2\mathtt{V}_{a,d}^\varepsilon)$ to control the term (118). To control the term (119), we apply Lemma 8.14 with

$$\alpha := Z^2\underline{\mathtt{V}}_{R,d}^\varepsilon,\ \beta := \varepsilon\partial_d\varphi,\ \underline{m} := m-2.$$

  (*b*) if $l_2\leqslant 1$, we use the third equation in (103). We commute the factor x_d with the derivative Z^{l_3}. Thus, to control the term (119), we study terms of the form

$$\lambda^{m-l}|Z^{l_2}(\varepsilon\underline{\mathtt{V}}_d^{\varepsilon,\flat}).Z^{l_3'}(x_d\partial_d)\varphi|_{0,\lambda,T},$$

  with $l_3'\leqslant l_3$. We control $Z^{l_2}(\varepsilon\underline{\mathtt{V}}_d^{\varepsilon,\flat})$ in L^∞ norm by $||\underline{\mathtt{V}}_{R,d}^\varepsilon||_{\varepsilon,T}^*$.
- if $l_1\geqslant 2$, we apply twice the Moser inequality of Lemma 8.14 with

$$\alpha := Z^2S(\underline{u}^\varepsilon),\ \beta := \underline{\mathtt{V}}_d^\varepsilon,\ \gamma := \varepsilon\partial_d\varphi,\ \underline{m} := m-2.$$

□

We now give the proof of a).

Proof of a). To explain in a clearer way our method, we will deal with the commutator

$$[\underline{\mathtt{v}}^{\varepsilon}_d\partial_d, \mathfrak{D}^{k,l}_{\varepsilon}]\varphi \tag{120}$$

instead of

$$[S(\underline{u}^{\varepsilon}).\underline{\mathtt{v}}^{\varepsilon}_d\partial_d, \mathfrak{D}^{k,l}_{\varepsilon}]\varphi.$$

This avoids heavy notations and does not change the mathematical analysis. The commutator (120) is a linear combination of some terms of the form

$$\mathfrak{D}^{k,l_1}_{\varepsilon}\, \underline{\mathtt{v}}^{\varepsilon}_d.Z^{l_2}\partial_d\varphi, \qquad \text{with } \mathrm{l}_1 \neq 0\ , l_1+l_2=l, \tag{121}$$

$$(\varepsilon\partial_d)^{k_1}\underline{\mathtt{v}}^{\varepsilon}_d.\mathfrak{D}^{k_2,l}_{\varepsilon}\,\partial_d\varphi, \qquad \text{with } \mathrm{k}_1 \neq 0\ , k_1+k_2=k. \tag{122}$$

We look at the terms of the form (121). Because of (99) it suffices to deal with the terms

$$\mathfrak{D}^{k,l_1}_{\varepsilon}\, \mathtt{v}^{\dagger}_d.Z^{l_2}\partial_d\varphi, \qquad \text{with } \mathrm{l}_1 \neq 0\ , l_1+l_2=l, \tag{123}$$

$$\mathfrak{D}^{k,l_1}_{\varepsilon}\, \underline{\mathtt{V}}^{\varepsilon}_d.Z^{l_2}(\varepsilon\partial_d)\varphi \qquad \text{with } \mathrm{l}_1 \neq 0\ , l_1+l_2=l. \tag{124}$$

We begin with the terms of the form (123). We use (12) and commute x_d with Z^{l_2}, we have to deal with the terms of the form

$$\mathfrak{D}^{k,l_1}_{\varepsilon}\, \mathtt{v}^{\dagger,\flat}_d.Z^{l'_2}(\varepsilon\partial_d)\varphi \quad \text{with } l'_2 \leqslant l_2.$$

Using L^{∞} estimate of $\mathtt{v}^{\dagger}$ yields the result. We now have a look for the terms of the form (124). We distinguish three cases:

- if $k \neq 0$ then we write $k = k'+1$ and we commute $\varepsilon\partial_d$ with Z^{l_1}. This yields some terms of the form

$$\mathfrak{D}^{k',l'}_{\varepsilon}(\varepsilon\partial_d\underline{\mathtt{V}}^{\varepsilon}_d)Z^{l_2}\varepsilon\partial_d\varphi$$

  with $l' \leqslant l_1$. We use (98) and study the terms

$$\lambda^{m-2k-l}|\mathfrak{D}^{k',l'}_{\varepsilon}(\varepsilon\partial_d\mathtt{V}^{\varepsilon}_{a,d})Z^{l_2}\varepsilon\partial_d\varphi|_{0,\lambda,T}, \tag{125}$$

$$\lambda^{m-2k-l}|\varepsilon^M\,\mathfrak{D}^{k',l'}_{\varepsilon}(\varepsilon\partial_d\underline{\mathtt{V}}^{\varepsilon}_{R,d})Z^{l_2}\varepsilon\partial_d\varphi|_{0,\lambda,T}. \tag{126}$$

  Using the L^{∞} estimates (59), we get (125) $\leqslant C|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}$. We apply Lemma 8.14 (2) to

$$\alpha := \varepsilon\partial_d\underline{\mathtt{V}}^{\varepsilon}_{R,d},\ \beta := \varepsilon\partial_d\varphi,\ \underline{m} = m-2$$

  to bound (126) by

$$C\varepsilon^M(|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + ||\varphi||_{\varepsilon,Lip,T}.|\underline{U}^{\varepsilon}_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}). \tag{127}$$

- if $k=0$ and $l_1 \geqslant 2$ then with $l_1 := l'_1+2$, the term (124) can be rewritten

$$\mathfrak{D}^{k,l'_1}_{\varepsilon}(Z^2\underline{\mathtt{V}}^{\varepsilon}_d).Z^{l_2}(\varepsilon\partial_d)\varphi.$$

  We use use (98) and study the terms

$$\lambda^{m-2k-l}|\mathfrak{D}^{k,l'_1}_{\varepsilon}(Z^2\mathtt{V}^{\varepsilon}_{a,d}).Z^{l_2}(\varepsilon\partial_d)\varphi|_{0,\lambda,T}, \tag{128}$$

$$\lambda^{m-2k-l}|\varepsilon^M\mathfrak{D}^{k,l'_1}_{\varepsilon}(Z^2\underline{\mathtt{V}}^{\varepsilon}_{R,d}).Z^{l_2}(\varepsilon\partial_d)\varphi|_{0,\lambda,T}. \tag{129}$$

We use the L^∞ estimates (59) to bound the term (128). We apply Lemma 8.14 with

$$\alpha := Z^2 \underline{V}^\varepsilon_{R,d}, \ \beta := \varepsilon \partial_d \varphi, \ \underline{m} = m-2$$

to bound (129) by (127).

- if $k=0$ and $l_1 = 1$, we use the third equation in (103) and commute x_d with Z^{l_2} yield a sum of terms of the form

$$Z^{l_1} \varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_d . Z^{l_2}(x_d \partial_d \varphi).$$

We bound it by $||\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_d||_{1,T}.|\varphi|_{m,\lambda,T}$ and conclude with $||\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_d||_{1,T} \leqslant ||\underline{\mathtt{V}}^{\varepsilon}_d||^*_{\varepsilon,T}$. We can deal with the terms (122) with the same methods.

□

8.7.2. *Source term estimates.* We now give two lemmas which will be useful when estimating source terms.

Lemma 8.23. *Let m be an integer. There is an increasing function $C : \mathbb{R}_+ \to \mathbb{R}^*_+$ such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \, ||\underline{U}^\varepsilon_R||_{0,T} \leqslant \mathfrak{R}$ then for all $\varphi \in C^\infty_0((0,T)\times\overline{\Omega})$, for all $\lambda \geqslant \lambda_0$,*
a) for any m,

$$|\underline{J}.\varphi|_{m,\lambda,T} \leqslant C(\mathfrak{R})(|\varphi|_{m,\lambda,T} + \varepsilon^M ||\varphi||_{0,T}.|\underline{U}^\varepsilon_R|_{m,\lambda,T}),$$

b) for m even,

$$|\underline{J}.\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} \leqslant C(\mathfrak{R})(|\varphi|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + \varepsilon^M ||\varphi||_{0,T}.|\underline{U}^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}).$$

where $\underline{J}$ stands for $\underline{J}^\alpha$, $\underline{J}^{\beta,1}$ or $\underline{J}^{\beta,2}$.

The proof of Lemma 8.23 is straightforward and mainly lies on Lemma 8.14. It is left to the reader.

Lemma 8.24. *Let m be an integer. There is an increasing function $C : \mathbb{R}_+ \to \mathbb{R}^*_+$ such that for all $\varepsilon \in]0,1]$, if $\varepsilon^M \, ||\underline{U}^\varepsilon_R||^*_{\varepsilon,T} \leqslant \mathfrak{R}$ then for all $\lambda \geqslant \lambda_0$,*
a) for any m,

$$|\underline{\mathtt{v}}^\varepsilon_d \partial_d U^\varepsilon_R|_{m-1,\lambda,T} \leqslant C(\mathfrak{R})(|U^\varepsilon_R|^{\mathcal{N}}_{\varepsilon,m,\lambda,T} + \varepsilon^M ||U^\varepsilon_R||^*_{\varepsilon,T}.|\underline{\mathtt{v}}^\varepsilon_R|^{\mathcal{N}}_{\varepsilon,m,\lambda,T}),$$

b) for m even,

$$|\underline{\mathtt{v}}^\varepsilon_d \partial_d U^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T} \leqslant C(\mathfrak{R})(|U^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + \varepsilon^M ||U^\varepsilon_R||_{\varepsilon,Lip}.|\underline{\mathtt{v}}^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}).$$

Proof of a). We will proceed in four steps.

(i) Because of (99) and (12), we get

$$|\underline{\mathtt{V}}^\varepsilon_d.\partial_d U^\varepsilon_R|_{m-1,\lambda,T} \leqslant C|U^\varepsilon_R|_{m,\lambda,T} + |\underline{\mathtt{V}}^\varepsilon_d.(\varepsilon\partial_d)U^\varepsilon_R|_{m-1,\lambda,T}.$$

(ii) In order to estimate $|\underline{\mathtt{V}}^\varepsilon_d.(\varepsilon\partial_d)U^\varepsilon_R|_{m-1,\lambda,T}$, we will control the terms of the form

$$\lambda^{m-1-l}|(Z^{l_1}\underline{\mathtt{V}}^\varepsilon_d).Z^{l_2}(\varepsilon\partial_d)U^\varepsilon_R|_{0,\lambda,T}, \tag{130}$$

where $l_1, l_2 \in \mathbb{N},\ l_1 + l_2 = l \leqslant m-1$. Because of (97) it suffices to deal with the terms of the form

$$\lambda^{m-1-l} |Z^{l_1} \mathtt{V}^{\varepsilon}_{a,d} . Z^{l_2} (\varepsilon\partial_d) U^{\varepsilon}_R|_{0,\lambda,T}, \tag{131}$$

$$\lambda^{m-1-l} |Z^{l_1} \underline{\mathtt{V}}^{\varepsilon}_{R,d} . Z^{l_2} (\varepsilon\partial_d) U^{\varepsilon}_R|_{0,\lambda,T}. \tag{132}$$

(*iii*) We begin with the term (131). We distinguish two cases:

- If $l_1 > 0$, then $l_2 \leqslant m$ we use the L^∞ estimates (59) and bound the term (131) by
$$c.|(\varepsilon\partial_d) U^{\varepsilon}_R|_{m-2,\lambda,T} \leqslant c.|U^{\varepsilon}_R|_{m,\lambda,T}.$$
- If $l_1 = 0$, then we use (103) and commute x_d with Z^{l_2} to find that (131) is a sum of terms of the form
$$\lambda^{m-1-l} |\varepsilon \mathtt{V}^{\varepsilon,\flat}_{a,d} Z^{l'_2} U^{\varepsilon}_R|_{0,\lambda,T}. \tag{133}$$
with $l'_2 \leqslant l_2 + 1$. Thus
$$(133) \leqslant C||\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_{a,d}||_\infty . |U^{\varepsilon}_R|_{m,\lambda,T} \leqslant C||\mathtt{V}^{\varepsilon}_{a,d}||^{*}_{\varepsilon,T} . |U^{\varepsilon}_R|_{m,\lambda,T}.$$

We now look at the term (132). We distinguish three cases:

(*iv*)
- If $l_1 \neq 0$, we apply Lemma 8.14 (1) to
$$\alpha := Z\underline{\mathtt{V}}^{\varepsilon}_{R,d},\ \beta := (\varepsilon\partial_d) U^{\varepsilon}_R,\ \underline{m} := m-2.$$
The term (132) is bounded by
$$c.\{||\underline{\mathtt{V}}^{\varepsilon}_{R,d}||_{1,T} . |U^{\varepsilon}_R|^{\mathcal{N}}_{\varepsilon,m,\lambda,T} + |\underline{\mathtt{V}}^{\varepsilon}_{R,d}|_{m,\lambda,T} . ||U^{\varepsilon}_R||_{\varepsilon,Lip}\}.$$
- If $l_1 = 0$, we use (103) and commute x_d with Z^{l_2} to find that (130) is a sum of terms of the form
$$\lambda^{m-1-l} |\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_{R,d} Z^{l'_2} U^{\varepsilon}_R|_{0,\lambda,T}. \tag{134}$$
with $l'_2 \leqslant l_2 + 1$. Thus the term (134) is bounded by $||\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_{R,d}||_\infty . |U^{\varepsilon}_R|_{m,\lambda,T}$.

□

The proof of *b*) follows the same path but is slightly more complicated.

Proof of b). We will proceed in four steps.

(*i*) Because of (99) and (12) we get
$$|\underline{\mathtt{v}}^{\varepsilon}_d . \partial_d U^{\varepsilon}_R|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T} \leqslant C|U^{\varepsilon}_R|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + |\underline{\mathtt{V}}^{\varepsilon}_d . (\varepsilon\partial_d) U^{\varepsilon}_R|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T}.$$

(*ii*) In order to estimate the last term of the right side above we will control the terms of the form
$$\lambda^{m-1-2k-l} |\mathfrak{D}^{k_1,l_1}_{\varepsilon} \underline{\mathtt{V}}^{\varepsilon}_d . \mathfrak{D}^{k_2,l_2}_{\varepsilon} (\varepsilon\partial_d) U^{\varepsilon}_R|_{0,\lambda,T}, \tag{135}$$
where
$$k_i, l_i \in \mathbb{N},\ k_1 + k_2 = k,\ l_1 + l_2 = l,\ 2k + l \leqslant m-1.$$

Because (97), it suffices to deal with the terms of the form

$$\lambda^{m-1-2k-l}\, |\mathfrak{D}_{\varepsilon}^{k_1,l_1}\, \mathtt{V}^{\varepsilon}_{a,d}.\mathfrak{D}_{\varepsilon}^{k_2,l_2}(\varepsilon\partial_d)U^{\varepsilon}_{R}|_{0,\lambda,T}, \tag{136}$$

$$\lambda^{m-1-2k-l}\, |\mathfrak{D}_{\varepsilon}^{k_1,l_1}\, \underline{\mathtt{V}}^{\varepsilon}_{R,d}.\mathfrak{D}_{\varepsilon}^{k_2,l_2}(\varepsilon\partial_d)U^{\varepsilon}_{R}|_{0,\lambda,T}. \tag{137}$$

(iii) We begin with the term (136). We distinguish two cases:

- If $l_1 + k_1 > 0$, then $2k_2 + l_2 \leqslant m$ we use the L^{∞} estimates (59) and bound the term (136) by
$$c.|(\varepsilon\partial_d)U^{\varepsilon}_{R}|^{\mathbf{E}}_{m-2,\lambda,T} \leqslant c.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{m,\lambda,T}.$$
- If $l_1 = k_1 = 0$, then we use (103) and commute x_d with $\mathfrak{D}_{\varepsilon}^{k_2,l_2}$ to find that (136) is a sum of terms of the form
$$\lambda^{m-1-2k-l}\, |\varepsilon \mathtt{V}^{\varepsilon,\flat}_{a,d}\, \mathfrak{D}_{\varepsilon}^{k_2',l_2'}\, U^{\varepsilon}_{R}|_{0,\lambda,T}. \tag{138}$$
with $k_2' \leqslant k_2$, $l_2' \leqslant l_2 + 1$. Thus
$$(138) \leqslant C||\varepsilon \mathtt{V}^{\varepsilon,\flat}_{a,d}||_{\infty}.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{m,\lambda,T} \leqslant C||\mathtt{V}^{\varepsilon}_{a,d}||^{*}_{\varepsilon,T}.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{m,\lambda,T}.$$

(iv) We now look at the term (137). We distinguish three cases:

- If $l_1 \neq 0$, we apply Lemma 8.14 (2) to
$$\alpha := Z\underline{\mathtt{V}}^{\varepsilon}_{R,d},\ \beta := (\varepsilon\partial_d)U^{\varepsilon}_{R},\ \underline{m} := m-2.$$
The term (137) is bounded by
$$c.\{||\underline{\mathtt{V}}^{\varepsilon}_{R,d}||_{1,T}.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + |\underline{\mathtt{V}}^{\varepsilon}_{R,d}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}.||U^{\varepsilon}_{R}||_{\varepsilon,Lip}\}.$$
- If $l_1 = 0$ and $k_1 \neq 0$, we apply Lemma 8.14 (2) to
$$\alpha := \varepsilon\partial_d\underline{\mathtt{V}}^{\varepsilon}_{R,d},\ \beta := \varepsilon\partial_d U^{\varepsilon}_{R},\ \underline{m} := m-2.$$
The term (137) is bounded by
$$\lambda^{-1}c.\{||\underline{\mathtt{V}}^{\varepsilon}_{R,d}||_{\varepsilon,Lip,T}.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T} + |\underline{\mathtt{V}}^{\varepsilon}_{R,d}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T}.||U^{\varepsilon}_{R}||_{\varepsilon,Lip}\}.$$
- If $l_1 = k_1 = 0$, we we use (103) and commute x_d with $\mathfrak{D}_{\varepsilon}^{k_2,l_2}$ to find that (135) is a sum of terms of the form
$$\lambda^{m-1-2k-l}|\varepsilon \underline{\mathtt{V}}^{\varepsilon,\flat}_{R,d}\, \mathfrak{D}_{\varepsilon}^{k_2',l_2'}\, U^{\varepsilon}_{R}|_{0,\lambda,T}. \tag{139}$$
with $k_2' \leqslant k_2$, $l_2' \leqslant l_2 + 1$. Thus the term (139) is bounded by $||\varepsilon\underline{\mathtt{V}}^{\varepsilon,\flat}_{R,d}||_{\infty}.|U^{\varepsilon}_{R}|^{\mathbf{E}}_{m,\lambda,T}$.

□

We now attack the conormal estimates of U^{ε}_{R}.

8.8. **Conormal estimates.** We look for estimates of $\lambda^{m-l}|Z^l U^{\varepsilon}_{R}|_{0,\lambda,T_{\dagger}}$. As for L^2 estimates, we will proceed in three steps estimating

$$\lambda^{m-l}|Z^l W^{\varepsilon}_{R}|_{0,\lambda,T_{\dagger}},\ \lambda^{m-l}|Z^l(\frac{x_d}{\varepsilon}W^{\varepsilon}_{R})|_{0,\lambda,T_{\dagger}} \text{ and } \lambda^{m-l}|Z^l V^{\varepsilon}_{R}|_{0,\lambda,T_{\dagger}}.$$

8.8.1. *Estimate of* W_R^ε. We apply the derivative Z^l to the equation (104). We get

$$\mathbf{X}_{\underline{v}^\varepsilon}(Z^l W_R^\varepsilon) = \tilde{f}_I^\varepsilon,$$

where $\tilde{f}_I^\varepsilon := [\mathbf{X}_{\underline{\mathrm{v}}^\varepsilon}, Z^l] W_R^\varepsilon + Z^l f_I^\varepsilon$. Then we estimate $\tilde{f}_I^\varepsilon$ thanks to Lemma 8.14 and use a classic L^2 estimate.

8.8.2. *Estimate of* $\frac{x_d}{\varepsilon} W_R^\varepsilon$. We will proceed in three steps.

(1) If we apply naively the derivative Z^l to the equation (106), a difficulty occurs. It is linked to the control of the commutator $[\underline{\mathrm{v}}_d^{\varepsilon,\flat} \frac{x_d}{\varepsilon}, Z^l] W_R^\varepsilon$. To surmount the difficulty, we combine the relations (12) and (103) to find

$$\frac{x_d}{\varepsilon} \underline{\mathrm{v}}_d^{\varepsilon,\flat} = \frac{x_d}{\varepsilon} \mathrm{v}^{\dagger,\flat} + \underline{\mathrm{V}}_d^\varepsilon.$$

Thus by plugging this into Equ. (106) we get that $\frac{x_d}{\varepsilon} W_R^\varepsilon$ verifies the equation

$$\mathbf{X}_{\underline{v}^\varepsilon}(\frac{x_d}{\varepsilon} W_R^\varepsilon) = \underline{\mathrm{v}}_d^{\dagger,\flat} \frac{x_d}{\varepsilon} W_R^\varepsilon + \underline{\mathrm{V}}_d^\varepsilon . W_R^\varepsilon + f_{II}^\varepsilon. \tag{140}$$

(2) We apply the derivative Z^l to this equation and find for $Z^l(\frac{x_d}{\varepsilon} W_R^\varepsilon)$ the equation

$$(\mathbf{X}_{\underline{\mathrm{v}}^\varepsilon} - \underline{\mathrm{v}}_d^{\dagger,\flat}) Z^l(\frac{x_d}{\varepsilon} W_R^\varepsilon) = \tilde{f}_{II}^\varepsilon,$$

where

$$\tilde{f}_{II}^\varepsilon := [\mathbf{X}_{\underline{\mathrm{v}}^\varepsilon} - \underline{\mathrm{v}}_d^{\dagger,\flat}, Z^l] \frac{x_d}{\varepsilon} W_R^\varepsilon + Z^l(\underline{\mathrm{V}}_d^\varepsilon . W_R^\varepsilon) + Z^l f_{II}^\varepsilon. \tag{141}$$

(3) We conclude thanks to a classic L^2 estimate, by estimating the first term in the right side of (141) thanks to Lemma 8.14 and the second one thanks to Lemma 8.14 and to the previous subsection.

8.8.3. *Estimate of* V_R^ε. We will proceed in two steps.

(1) We apply the derivative Z^l to the boundary value problem (109) and find that $Z^l V_R^\varepsilon$ is solution of the following boundary value problem:

$$\begin{cases} (\mathfrak{L}^\star(\underline{u}^\varepsilon, \partial_{t,x}) + \underline{J}^\alpha).(Z^l V_R^\varepsilon) = \tilde{f}_{III}^\varepsilon & \text{when } x_d > 0, \\ Z^l V_{R,d}^\varepsilon = 0 & \text{when } x_d = 0, \end{cases}$$

where

$$\tilde{f}_{III}^\varepsilon := [S^\star(\underline{u}^\varepsilon)\mathbf{X}_{\underline{\mathrm{v}}^\varepsilon}, Z^l] V_R^\varepsilon + [\mathbf{L}^\star(\partial_x), Z^l] V_R^\varepsilon + [\underline{J}^\alpha, Z^l] V_R^\varepsilon + Z^l f_{III}^\varepsilon. \tag{142}$$

(2) To conclude as in subsection 8.6.3 with a L^2 estimate, we need some estimates of the source term $\tilde{f}_{III}^\varepsilon$. We are going to estimate each of the four terms in the right side of 142. We estimate the first one thanks to Proposition 8.21 and the third one thanks to lemma 8.14. We estimate the last one thanks to the subsection 8.8.1 and 8.8.2. It remains to explain how to estimate the second one. This term reads $[\mathbf{L}_d^\star \partial_d, Z^l] V_R^\varepsilon$ and it is a sum of terms of the form $Z^{l'} \mathbf{L}_d^\star \partial_d V_R^\varepsilon$ with $l' \leqslant l$. Then we extirpate $\mathbf{L}_d^\star \partial_d V_R^\varepsilon$ by the first equation of (109). We can apply the previous methods for the resulting terms. We use Lemma 8.24 to handle the most delicate point i.e. the estimate of $|\underline{\mathrm{v}}_d^\varepsilon . \partial_d U_R^\varepsilon|_{m-1,\lambda,T_\dagger}$.

8.9. **Normal estimates.** This is the last step in the proof of Theorem 8.16. We look for estimates of $\lambda^{m-2k-l}|\mathfrak{D}_\varepsilon^{k,l} U_R^\varepsilon|_{0,\lambda,T}$, with $k > 0$ and $2k + l \leqslant m$. Indeed the case $k = 0$ corresponds to the conormal estimates performed above. We proceed in four steps. We begin to estimate the noncharacteristic part: $\mathbf{L}_d^\star V_R^\varepsilon$ (cf. section 8.9.1). This corresponds to the pressure $\mathtt{P}_R^\varepsilon$ and the normal velocity $\mathtt{V}_{R,d}^\varepsilon$. The key point is to extirpate $\mathbf{L}_d^\star \partial_d V_R^\varepsilon$ from the equation (92) (as in section 8.8.3). Then we will estimate $|W_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_\dagger}$ (cf. section 8.9.2), $\frac{x_d}{\varepsilon}|W_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_\dagger}$ (cf. section 8.9.3) and finally $|\mathtt{V}_{R,\mathrm{t}}^\varepsilon|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_\dagger}$ (cf. section 8.9.4). To do this we will use some P.D.E. whose the principal part would be the transport operator $\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon}$. Since this operator is tangent to the boundary $\{x_d = 0\}$ there will be no need for boundary conditions. The analysis will involve some commutators which will be estimated thanks to Proposition 8.21.

8.9.1. *Normal estimates of $\mathbf{L}_d^\star V_R^\varepsilon$.* We notice that

$$|\mathbf{L}_d^\star \varepsilon\, \partial_d\, V_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m-2,\lambda,T} \leqslant \lambda^{-1}|\mathbf{L}_d^\star \varepsilon\, \partial_d\, V_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T}$$

and extirpate $\mathbf{L}_d^\star \varepsilon\partial_d V_R^\varepsilon$ from the equation (92). Among the resulting terms, the most delicate one to estimate is $|\underline{\mathtt{v}}_d^\varepsilon.\partial_d U_R^\varepsilon|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T_\dagger}$ which is given by Lemma 8.24.

8.9.2. *Normal estimates of W_R^ε* . We apply the derivative $\mathfrak{D}_\varepsilon^{k,l}$ to the equation (104). We find that $\mathfrak{D}_\varepsilon^{k,l} W_R^\varepsilon$ verifies the following equation:

$$\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon}(\varepsilon\partial_d)^k Z^l W_R^\varepsilon = \hat{f}_I^\varepsilon$$

where

$$\hat{f}_I^\varepsilon := [\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon}, (\varepsilon\partial_d)^k Z^l] W_R^\varepsilon + (\varepsilon\partial_d)^k Z^l f_I^\varepsilon.$$

We estimate the commutator $[\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon}, \mathfrak{D}_\varepsilon^{k,l}] W_R^\varepsilon$ thanks to Proposition 8.21 *a*) and *c*) and use a L^2 estimate.

8.9.3. *Normal estimates of $\frac{x_d}{\varepsilon} W_R^\varepsilon$* . We will proceed in three steps.

(1) As in section 8.8, in order to tackle the commutators which will appears we start with the equation (140). More precisely we apply the derivative $\mathfrak{D}_\varepsilon^{k,l}$ to the equation (140) to find for $\hat{W}_R^\varepsilon := \mathfrak{D}_\varepsilon^{k,l}(\frac{x_d}{\varepsilon} W_R^\varepsilon)$ the equation $(\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} - \mathtt{v}_d^{\dagger,\flat})\hat{W}_R^\varepsilon = \hat{f}_{III}^\varepsilon$ where the right side is

$$\hat{f}_{III}^\varepsilon := [\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} - \mathtt{v}_d^{\dagger,\flat}, \mathfrak{D}_\varepsilon^{k,l}]\hat{W}_R^\varepsilon + \mathfrak{D}_\varepsilon^{k,l}(\underline{\mathtt{v}}_d^\varepsilon.W_R^\varepsilon) + \mathfrak{D}_\varepsilon^{k,l} f_{III}^\varepsilon.$$

(2) We estimate the first term of the right side above thanks to Proposition 8.21, the second term of the right side thanks to Lemma 8.14 and the previous subsection. To estimate the third one we refer to the definition of f_{III}^ε in (110) and we use that because β_2 is rapidly decreasing with respect to its last argument, all the derivatives of $\frac{x_d}{\varepsilon}\beta_2(\varepsilon, t, x, \frac{x_d}{\varepsilon})$ are bounded in L^∞ uniformly with respect to ε. As a consequence $(\hat{f}_{III}^\varepsilon) \in \mathbf{E}^m$.
(3) We end with a L^2 estimate.

8.9.4. *Normal estimates of* $\mathtt{V}^\varepsilon_{R,\mathtt{t}}$. We will proceed in four steps.

(1) Looking at the first $d-1$ equations of (92), we find for $\mathtt{v}^\varepsilon_{R,\mathtt{t}}$ an equation of the form

$$(\underline{\rho}^\varepsilon.\mathbf{X}_{\underline{\mathtt{v}}^\varepsilon} + \underline{J}_1^\alpha).\mathtt{V}^\varepsilon_{R,\mathtt{t}} = f^\varepsilon_{IV} \tag{143}$$

where

$$\underline{\rho}^\varepsilon := \rho(\underline{\mathtt{p}}^\varepsilon, \underline{\mathtt{s}}^\varepsilon), \quad f^\varepsilon_{IV} := -\underline{J}_2^\alpha.\begin{bmatrix}\mathtt{V}^\varepsilon_{R,d}\\ \mathtt{P}^\varepsilon_R\end{bmatrix} - Z\mathtt{P}^\varepsilon_R - W^\varepsilon_R(\underline{J}^{\beta,1} + \frac{x_d}{\varepsilon}\underline{J}^{\beta,2}) - R^\varepsilon_{\mathtt{t}}. \tag{144}$$

The term $R^\varepsilon_{\mathtt{t}}$ corresponds to the first $d-1$ components of R^ε_v and $\underline{J}_1^\alpha$, $\underline{J}_2^\alpha$ are some matrices extracted from $\underline{J}^\alpha$.

(2) We apply the derivative $\mathfrak{D}^{k,l}_\varepsilon$ to Equation(143) and find for $\hat{\mathtt{V}}^\varepsilon_{R,\mathtt{t}} := \mathfrak{D}^{k,l}_\varepsilon \mathtt{V}^\varepsilon_{R,\mathtt{t}}$ the equation

$$(\underline{\rho}^\varepsilon X_{\underline{\mathtt{v}}^\varepsilon} + \underline{J}_1^\alpha).\hat{\mathtt{V}}^\varepsilon_{R,\mathtt{t}} = \hat{f}^\varepsilon_{IV} \tag{145}$$

where

$$\hat{f}^\varepsilon_{IV} := [\underline{\rho}^\varepsilon \mathbf{X}_{\underline{\mathtt{v}}^\varepsilon}, \mathfrak{D}^{k,l}_\varepsilon]\mathtt{V}^\varepsilon_{R,\mathtt{t}} + [\underline{J}_1^\alpha, \mathfrak{D}^{k,l}_\varepsilon]\mathtt{V}^\varepsilon_{R,\mathtt{t}} + \mathfrak{D}^{k,l}_\varepsilon f^\varepsilon_{IV}.$$

(3) We estimate the term $\hat{f}^\varepsilon_{IV}$. To do this we estimate the first commutator thanks to Proposition 8.21 and the second one thanks to Lemma 8.14. It remains to estimate the term $\mathfrak{D}^{k,l}_\varepsilon f^\varepsilon_{IV}$. We look at each of the four terms in (144). The contributions of the first term and of the last term are easy to estimate. We estimate the contribution of the third term thanks to the previous subsections (8.9.2 and 8.9.3) and to Lemma 8.14. It remains to explain how to estimate the second one. We have to control

$$\lambda^{m-2k-l}\, |\mathfrak{D}^{k,l+1}_\varepsilon\, \mathtt{P}^\varepsilon_R|_{0,\lambda,T_\dagger}.$$

Commuting $\varepsilon\partial_d$ with Z^{l+1}, we are lead to estimate the terms

$$\lambda^{m-2k-l}\, |\mathfrak{D}^{k',l'}_\varepsilon\, \varepsilon\partial_d \mathtt{P}^\varepsilon_R|_{0,\lambda,T_\dagger},$$

with $k' \leqslant k-1$, $l' \leqslant l+1$. Since $2k'+l' \leqslant 2k-1 \leqslant m-1$, all these terms are bounded by $|\varepsilon\partial_d \mathtt{P}^\varepsilon_R|^{\mathbf{E}}_{\varepsilon,m-1,\lambda,T_\dagger}$ which is estimated in subsection 8.9.2.

(4) We end by using a L^2 estimate for the equation (145).

Thus we get some normal estimates and the proof of Theorem 8.16 is therefore completed.

8.10. **An iterative scheme.** We define an iterative scheme $(U^{\varepsilon,\nu}_R)_{\varepsilon,\nu}$ by

$$(\mathfrak{L}^\star(u^{\varepsilon,\nu}, \partial_{t,x}) + J^{\alpha,\nu})V^{\varepsilon,\nu+1}_R + W^\varepsilon_R(\underline{J}^{\beta,1,\nu} + \frac{x_d}{\varepsilon}.\underline{J}^{\beta,2,\nu}) = R^\varepsilon_v \quad \text{when } x_d > 0,$$
$$\mathbf{X}_{\mathtt{v}^{\varepsilon,\nu}} W^{\varepsilon,\nu+1}_R + \varepsilon V^{\varepsilon,\nu+1}_R.\partial_x W^\varepsilon_a = -\varepsilon R^\varepsilon_W \quad \text{when } x_d > 0,$$
$$\mathtt{V}^{\varepsilon,\nu+1}_{R,d} = 0 \quad \text{when } x_d = 0,$$

when $J^{\alpha,\nu}$ denotes the $(d+1)\times(d+1)$ matrix

$$J^{\alpha,\nu} := J^\alpha(\varepsilon, V^\varepsilon_a, Z_\varepsilon V^\varepsilon_a, W^{\varepsilon,\flat}_a, \varepsilon^M U^{\varepsilon,\nu}_R),$$

for $i \in \{1,2\}$, $J^{\beta,i,\nu}$ denotes the $\mathbb{R}^{d+1}$-valued functions

$$J^{\beta,i,\nu} := J^{\beta,i}(\varepsilon, V^\varepsilon_a, Z_\varepsilon V^\varepsilon_a, W^{\varepsilon,\flat}_a, \varepsilon^M U^{\varepsilon,\nu}_R)$$

and

$$u^{\varepsilon,\nu} := u^{\dagger} + \varepsilon U^{\varepsilon,\nu}, \quad U^{\varepsilon,\nu} := U_a^{\varepsilon} + \varepsilon^M U_R^{\varepsilon,\nu}.$$

Because this step is now very classic in BKW theory (cf. [15], [14], [16], [37], [36], [38]...), we only sketch as a preview how to deduce from the Sobolev embedding lemma 8.10 and from the linear estimates of Theorem 8.16 the uniform boundedness of $(U_R^{\varepsilon,\nu})_{\varepsilon,\nu}$. In order to do so, we fix two strictly positive real h and R , λ and ε_0 such that

$$\lambda^{-1}\,\lambda_0 \leqslant \min(\frac{h}{2}, \frac{1}{2}), \tag{146}$$

$$\varepsilon_0^{M-\frac{1}{2}}\, cT_{\dagger} e^{\lambda T_{\dagger}} \leqslant \frac{1}{2h}. \tag{147}$$

Proposition 8.25. *If*

$$\varepsilon^M ||U_R^{\varepsilon,\nu}||^*_{\varepsilon,T_{\dagger}} \leqslant \mathfrak{R} \text{ and } ||U_R^{\varepsilon,\nu}||^{\mathbf{E}}_{\varepsilon,m,\lambda,T_{\dagger}} \leqslant h,$$

then

$$\varepsilon^M ||U_R^{\varepsilon,\nu+1}||^*_{\varepsilon,T_{\dagger}} \leqslant \mathfrak{R} \text{ and } ||U_R^{\varepsilon,\nu+1}||^{\mathbf{E}}_{\varepsilon,m,\lambda,T_{\dagger}} \leqslant h.$$

Proof. We will proceed in three steps.

(1) We begin applying the Sobolev embedding lemma 8.10:

$$\sqrt{\varepsilon}||U_R^{\varepsilon,\nu+1}||^*_{\varepsilon,T_{\dagger}} \leqslant cT_{\dagger} e^{\lambda T_{\dagger}}\,|U_R^{\varepsilon,\nu+1}|^E_{\varepsilon,m,\lambda,T_{\dagger}}.$$

Thanks to (147), we obtain

$$\varepsilon^M\;||U_R^{\varepsilon,\nu+1}||^*_{\varepsilon,T_{\dagger}} \leqslant \frac{1}{2h}|U_R^{\varepsilon,\nu+1}|^E_{\varepsilon,m,\lambda,T_{\dagger}}.$$

(2) We apply Theorem 8.16 and find

$$|U_R^{\varepsilon,\nu+1}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_{\dagger}} \leqslant \lambda^{-1}\lambda_0.(1+\varepsilon^M||U_R^{\varepsilon,\nu+1}||^*_{\varepsilon,T_{\dagger}}.|U_R^{\varepsilon,\nu}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_{\dagger}}).$$

Thanks to (146), we obtain

$$|U_R^{\varepsilon,\nu+1}|^{\mathbf{E}}_{\varepsilon,m,\lambda,T_{\dagger}} \leqslant h.$$

(3) Thanks to the Sobolev embedding lemma 8.10, we obtain $\varepsilon^M||U_R^{\varepsilon,\nu+1}||^*_{\varepsilon,T_{\dagger}} \leqslant \mathfrak{R}$.

□

Then we deduce the existence of functions $(U_R^{\varepsilon})_{\varepsilon}$ in $\mathbf{E}^m$ which satisfy (66), (67) and (68) by passing to the limit $\nu \to +\infty$. Thus we end the proof of Theorem 8.1 and 8.2 thanks to Proposition 8.9 and Proposition 8.6.

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