Small-time global exact controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions

Jean-Michel Coron Frédéric Marbach Franck Sueur

Abstract

In this work, we investigate the small-time global exact controllability of the Navier-Stokes equation, both towards the null equilibrium state and towards weak trajectories. We consider a viscous incompressible fluid evolving within a smooth bounded domain, either in 2D or in 3D. The controls are only located on a small part of the boundary, intersecting all its connected components. On the remaining parts of the boundary, the fluid obeys a Navier slip-with-friction boundary condition. Even though viscous boundary layers appear near these uncontrolled boundaries, we prove that small-time global exact controllability holds. Our analysis relies on the controllability of the Euler equation combined with asymptotic boundary layer expansions. Choosing the boundary controls with care enables us to guarantee good dissipation properties for the residual boundary layers, which can then be exactly canceled using local techniques.

Keywords. Controllability, Navier-Stokes, Boundary Layers

1 Introduction

1.1 Description of the fluid system

We consider a smooth bounded connected domain Ω in $\mathbb{R}^d$, with $d = 2$ or $d = 3$. Although some drawings will depict Ω as a very simple domain, we do not make any other topological assumption on Ω. Inside this domain, an incompressible viscous fluid evolves

J.-M. Coron: Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris cedex 05; e-mail: coron@ann.jussieu.fr

F. Marbach*: Corresponding author. Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris cedex 05; e-mail: frederic.marbach@upmc.fr

F. Sueur: Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351 cours de la libration, 33405 Talence; e-mail: franck.sueur@math.u-bordeaux1.fr

Mathematics Subject Classification (2010): Primary 93B05; Secondary 93C20

under the Navier-Stokes equations. We will name u its velocity field and p the associated pressure. We assume that we are able to act on the fluid flow only on a open part Γ of the full boundary $\partial\Omega$, where Γ intersects all connected components of $\partial\Omega$ (this geometrical hypothesis is used in the proofs of Lemma 2). On the remaining part of the boundary, $\partial\Omega \setminus \Gamma$, we assume that the fluid flow satisfies Navier slip-with-friction boundary conditions. Hence, (u, p) satisfies:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ N(u) = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{1.1}$$

Here and in the sequel, n denotes the outward pointing normal to the domain. For a vector field f, we introduce $[f]_{\text{tan}}$ its tangential part, $D(f)$ the rate of strain tensor (or shear stress) and $N(f)$ the tangential Navier boundary operator defined as:

$$[f]_{\text{tan}} := f - (f\cdot n)n, \tag{1.2}$$

$$D_{ij}(f) := \frac{1}{2}\left(\partial_i f_j + \partial_j f_i\right), \tag{1.3}$$

$$N(f) := [D(f)n + Mf]_{\text{tan}}. \tag{1.4}$$

Eventually, in (1.4), M is a smooth matrix valued function, describing the friction near the boundary. This is a generalization of the usual condition involving a single scalar parameter $\alpha \geq 0$ (i.e. $M = \alpha I_d$). For flat boundaries, such a scalar coefficient measures the amount of friction. When $\alpha = 0$ and the boundary is flat, the fluid slips along the boundary without friction. When $\alpha \to +\infty$, the friction is so intense that the fluid is almost at rest near the boundary and, as shown by Kelliher in [58], the Navier condition $[D(u)n + \alpha u]_{\text{tan}} = 0$ converges to the usual Dirichlet condition.

1.2 Controllability problem and main result

Let T be an allotted positive time (possibly very small) and u_* an initial data (possibly very large). The question of small-time global exact null controllability asks whether, for any T and any u_*, there exists a trajectory u (in some appropriate functional space) defined on $[0,T]\times\Omega$, which is a solution to (1.1), satisfying $u(0,\cdot) = u_*$ and $u(T,\cdot) = 0$. In this formulation, system (1.1) is seen as an underdetermined system. The controls used are the implicit boundary conditions on Γ and can be recovered from the constructed trajectory *a posteriori*.

We define the space $L^2_\gamma(\Omega)$ as the closure in $L^2(\Omega)$ of smooth divergence free vector fields which are tangent to $\partial\Omega \setminus \Gamma$. For $f \in L^2_\gamma(\Omega)$, we do not require that $f\cdot n = 0$ on the controlled boundary Γ. Of course, due to the Stokes theorem, such functions satisfy

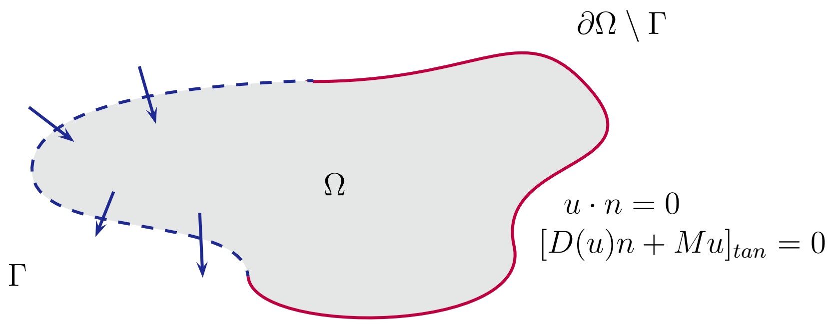

Figure 1: Setting of the main Navier-Stokes control problem.

$\int_\Gamma f \cdot n = 0$. The main result of this paper is the following small-time global exact null controllability theorem:

Theorem 1. *Let* $T > 0$ *and* $u_* \in L^2_\gamma(\Omega)$. *There exists a weak controlled trajectory* u *(see Definition 1) such that* $u \in \mathcal{C}^0_w([0,T]; L^2_\gamma(\Omega)) \cap L^2((0,T); H^1(\Omega))$ *of* (1.1) *satisfying* $u(0,\cdot) = u_*$ *and* $u(T,\cdot) = 0$.

Remark 1. Even though a unit dynamic viscosity is used in equation (1.1), Theorem 1 remains true for any fixed positive viscosity ν thanks to a straightforward scaling argument. Some works also consider the case when the friction matrix M depends on ν (see [73] or [89]). This does not impact our proofs in the sense that we could still prove that: for any $\nu > 0$, for any $T > 0$, for any smooth M_ν, for any initial data u_*, one can find boundary controls (depending on all these quantities) driving the initial data back to the null equilibrium state at time T.

Remark 2. Theorem 1 is stated as an existence result. The lack of uniqueness both comes from the fact that multiple controls can drive the initial state to zero and from the fact that it is not known whether weak solutions are unique for the Navier-Stokes equation in 3D (in 2D, it is known that weak solutions are unique). Always in the 3D case, if the initial data u_* is smooth enough, it would be interesting to know if we can build a strong solution to (1.1) driving u_* back to zero (in 2D, global existence of strong solutions is known). We conjecture that building strong controlled trajectories is possible. What we do prove here is that, if the initial data u_* is smooth enough, then our small-time global approximate null control strategy drives any weak solution starting from this initial state close to zero.

Although most of this paper is dedicated to the proof of Theorem 1 concerning the null controllability, we also explain in Section 5 how one can adapt our method to obtain small-time global exact controllability towards any weak trajectory (and not only the null equilibrium state).

1.3 A challenging open problem as a motivation

The small-time global exact null controllability problem for the Navier-Stokes equation was first suggested by Jacques-Louis Lions in the late 80's. It is mentioned in [61] in a setting where the control is a source term supported within a small subset of the domain (this situation is similar to controlling only part of the boundary). In Lions' original question, the boundary condition on the uncontrolled part of the boundary is the Dirichlet boundary condition. Using our notations and our boundary control setting, the system considered is:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{1.5}$$

Open Problem (OP) *For any $T > 0$ and u_* in $L^2(\Omega)$ which is divergence free and vanishes on $\partial\Omega \setminus \Gamma$, does there exist a trajectory of system* (1.5) *with $u(0,\cdot) = u_*$ such that $u(T,\cdot) = 0$?*

This is a very challenging open problem because the Dirichlet boundary condition gives rise to boundary layers that have a larger amplitude than Navier slip-with-friction boundary layers. However, we expect that the method we introduce here can inspire later works on the more difficult case of the Dirichlet boundary condition, at least for some favorable geometric and functional settings.

1.4 Known results and related previous works

1.4.1 Local results

A first approach to study the controllability of Navier-Stokes systems is to deal with the quadratic convective term as a perturbation term and obtain results using the diffusive term. Of course, this kind of approach is mostly efficient for local results, where the quadratic term is indeed small. Most local proofs rely on Carleman estimates for the adjoint system.

For the Dirichlet boundary condition, Imanuvilov proves in [57] small-time local controllability to the trajectories for 2D and 3D. This result has since been improved in [30] by Fernández-Cara, Guerrero, Imanuvilov and Puel. Their proof uses Carleman estimates and weakens the regularity assumed on the trajectories. In particular, their proof implies null controllability with small initial data in L^{2d-2}.

For Navier slip-with-friction boundary conditions, two papers were published in 2006. In [54], the authors prove a local controllability result to the trajectories in 2D domains, assuming the initial data is close to the trajectory in H^1. In [48], Guerrero, proves small-time local controllability to the trajectories for 2D and 3D domains, with general nonlinear Navier boundary conditions. His result implies null controllability when the initial data is small in H^3. It is likely that this result could be improved to lower this hypothesis to L^{2d-2} as in the Dirichlet case.

1.4.2 Global results

The second approach goes the other way around: see the viscous term as a perturbation of the inviscid dynamic and try to deduce the controllability of Navier-Stokes from the controllability of Euler. This approach is efficient to obtain small-time results, as inviscid effects prevail in this asymptotic. However, if one does not control the full boundary, boundary layers appear near the uncontrolled boundaries $\partial\Omega\backslash\Gamma$. Thus, most known results try to avoid this situation.

In [19], the first author and Fursikov prove a small-time global exact null controllability result when the domain is a manifold without border (in this setting, the control is a source term located in a small subset of the domain). Likewise, in [31], Fursikov and Imanuvilov prove small-time global exact null controllability when the control is supported on the whole boundary (i.e. $\Gamma = \partial\Omega$). In both cases, there is no boundary layer.

Another method to avoid the difficulties is to choose more gentle boundary conditions. In a simple geometry (a 2D rectangular domain), Chapouly proves in [13] small-time global exact null controllability for Navier-Stokes under the boundary condition $\nabla \times u = 0$ on uncontrolled boundaries. Let $[0, L] \times [0, 1]$ be the considered rectangle. Her control acts on both vertical boundaries at $x_1 = 0$ and $x_1 = L$. Uncontrolled boundaries are the horizontal ones at $x_2 = 0$ and $x_2 = 1$. She deduces the controllability of Navier-Stokes from the controllability of Euler by linearizing around an explicit reference trajectory $u^0(t, x) := (h(t), 0)$, where h is a smooth profile. Hence, the Euler trajectory already satisfies all boundary conditions and there is no boundary layer to be expected at leading order.

For Navier slip-with-friction boundary conditions in 2D, the first author proves in [17] a small-time global approximate null controllability result. He proves that exact controllability can be achieved in the interior of the domain. However, this is not the case near the boundaries. The approximate controllability is obtained in the space $W^{-1,\infty}$, which is not a strong enough space to be able to conclude to global exact null controllability using a local result. The residual boundary layers are too strong and have not been sufficiently handled during the control design strategy.

For Dirichlet boundary conditions, Guerrero, Imanuvilov and Puel prove in [49] (respectively in [50]) for a square (resp. a cube) where one side (resp. one face) is not controlled, a small time result which looks like global approximate null controllability. Their method consists in adding a new source term (a control supported on the whole domain Ω) to absorb the boundary layer. They prove that this additional control can be chosen small in $L^p((0,T);H^{-1}(\Omega))$, for $1 < p < p_0$ (with $p_0 = 8/7$ in 2D and $4/3$ in 3D). However, this norm is too weak to take a limit and obtain the result stated in Open Problem (OP) (without this fully supported additional control). Moreover, the $H^{-1}(\Omega)$ estimate seems to indicate that the role of the inner control is to act on the boundary layer directly where it is located, which is somehow in contrast with the goal of achieving controllability with controls supported on only part of the boundary.

All the examples detailed above tend to indicate that a new method is needed, which fully takes into account the boundary layer in the control design strategy.

1.4.3 The ”well-prepared dissipation” method

In [63], the second author proves small-time global exact null controllability for the Burgers equation on the line segment $[0,1]$ with a Dirichlet boundary condition at $x = 1$ (implying the presence of a boundary layer near the uncontrolled boundary $x = 1$). The proof relies on a method involving a *well-prepared dissipation* of the boundary layer. The sketch of the method is the following:

1. **Scaling argument.** Let $T > 0$ be the small time given for the control problem. Introduce $\varepsilon \ll 1$ a very small scale. Perform the usual small-time to small-viscosity fluid scaling $u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x)$, yielding a new unknown u^ε, defined on a large time scale $[0, T/\varepsilon]$, satisfying a vanishing viscosity equation. Split this large time interval in two parts: $[0,T]$ and $[T, T/\varepsilon]$.

2. **Inviscid stage.** During $[0,T]$, use (up to the first order) the same controls as if the system was inviscid. This leads to good interior controllability (far from the boundaries, the system already behaves like its inviscid limit) but creates a boundary layer residue near uncontrolled boundaries.

3. **Dissipation stage.** During the long segment $[T, T/\varepsilon]$, choose null controls and let the system dissipate the boundary layer by itself thanks to its smoothing term. As $\varepsilon \to 0$, the long time scale compensates exactly for the small viscosity. However, as $\varepsilon \to 0$, the boundary layer gets thinner and dissipates better.

The key point in this method is to separate steps 2 and 3. Trying to control both the inviscid dynamic and the boundary layer at the end of step 2 is too hard. Instead, one chooses the inviscid controls with care during step 2 in order to prepare the self-dissipation of the boundary layer during step 3. This method will be used in this paper and enhanced to prove our result. In order to apply this method, we will need a very precise description of the boundary layers involved.

1.5 Boundary conditions and boundary layers for Navier-Stokes

Physically, boundary layers are the fluid layers in the immediate vicinity of the boundaries of a domain, where viscous effects prevail. Mathematically, they appear when studying vanishing viscosity limits while maintaining strong boundary conditions. There is a huge literature about boundary conditions for partial differential equations and the associated boundary layers. In this paragraph, we give a short overview of some relevant references in our context for the Navier-Stokes equation.

1.5.1 Adherence boundary condition

The strongest and most commonly used boundary condition for Navier-Stokes is the full adherence (or no-slip) boundary condition $u = 0$. This condition is most often referred to as the Dirichlet condition although it was introduced by Stokes in [83]. Under this condition, fluid particles must remain at rest near the boundary. This generates large amplitude boundary layers.

In 1904, Prandtl proposed an equation describing the behavior of boundary layers for this adherence condition in [75]. Heuristically, these boundary layers are of amplitude $\mathcal{O}(1)$ and of thickness $\mathcal{O}(\sqrt{\nu})$ for a vanishing viscosity ν. Although his equation has been extensively studied, much is still to be learned.

Both physically and numerically, there exists situations where the boundary layer separates from the border: see [20], [53], [86], or [87]. Mathematically, it is known that solutions with singularities can be built [26] and that the linearized system is ill-posed in Sobolev spaces [33]. The equation has also been proved to be ill-posed in a non-linear context in [52]. Moreover, even around explicit shear flow solutions of the Prandtl equation, the equation for the remainder between Navier-Stokes and Euler+Prandtl is also ill-posed (see [46] and [47]).

Most positive known results fall into two families. First, when the initial data satisfies a monotonicity assumption, introduced by Oleinik in [71], [72]. See also [1], [45], [67] and [93] for different proof techniques in this context. Second, when the initial data are analytic, it is both proved that the Prandtl equations are well-posed [77] and that Navier-Stokes converges to an Euler+Prandtl expansion [78]. For historical reviews of known results, see [25] or [70]. We also refer to [62] for a comprehensive recent survey.

Physically, the main difficulty is the possibility that the boundary layer separates and penetrates into the interior of the domain (which is prevented by the Oleinik monotonicity assumption). Mathematically, Prandtl equations lack regularization in the tangential direction thus exhibiting a loss of derivative (which can be circumvented within an analytic setting).

1.5.2 Friction boundary conditions

Historically speaking, the adherence condition is posterior to another condition stated by Navier in [69] which involves friction. The fluid is allowed to slip along the boundary but undergoes friction near the impermeable walls. Originally, it was stated as:

$$u \cdot n = 0 \quad \text{and} \quad [D(u)n + \alpha u]_{\text{tan}} = 0, \tag{1.6}$$

where α is a scalar positive coefficient. Mathematically, α can depend (smoothly) on the position and be a matrix without changing much the nature of the estimates.

This condition has been justified from the boundary condition at the microscopic scale in [15] for the Boltzmann equation. See also [44] or [66] for other examples of such derivations.

Although the adherence condition is more popular in the mathematical community, the slip-with-friction condition is actually well suited for a large range of applications. For instance, it is an appropriate model for turbulence near rough walls [59] or in acoustics [35]. It is used by physicists for flat boundaries but also for curved domains (see [27], [51] or [74]). Physically, α is homogeneous to $1/b$ where b is a length, named *slip length*. Computing this parameter for different situations, both theoretically or experimentally is important for nanofluidics and polymer flows (see [2] or [12]).

Mathematically, the convergence of the Navier-Stokes equation under the Navier slip-with-friction condition to the Euler equation has been studied by many authors. For 2D, this subject is studied in [14] and [58]. For 3D, this subject is treated in [36] and [65]. To obtain more precise convergence results, it is necessary to introduce an asymptotic expansion of the solution u^ε to the vanishing viscosity Navier-Stokes equation involving a boundary layer term. In [56], Iftimie and the third author prove a boundary layer expansion. This expansion is easier to handle than the Prandtl model because the main equation for the boundary layer correction is both linear and well-posed in Sobolev spaces. Heuristically, these boundary layers are of amplitude $\mathcal{O}(\sqrt{\nu})$ and of thickness $\mathcal{O}(\sqrt{\nu})$ for a vanishing viscosity ν.

1.5.3 Slip boundary conditions

When the physical friction between the inner fluid and the solid boundary is very small, one may want to study an asymptotic model describing a situation where the fluid perfectly slips along the boundary. Sadly, the perfect slip situation is not yet fully understood in the mathematical literature.

2D. In the plane, the situation is easier. In 1969, Lions introduced in [60] the free boundary condition $\omega = 0$. This condition is actually a special case of (1.6) where α depends on the position and $\alpha(x) = 2\kappa(x)$, where $\kappa(x)$ is the curvature of the boundary at $x \in \partial\Omega$. With this condition, good convergence results can be obtained from Navier-Stokes to Euler for vanishing viscosities.

3D. In the space, for flat boundaries, slipping is easily modeled with the usual impermeability condition $u \cdot n = 0$ supplemented by any of the following equivalent conditions:

$$\partial_n [u]_{\text{tan}} = 0, \tag{1.7}$$

$$[D(u)n]_{\text{tan}} = 0, \tag{1.8}$$

$$[\nabla \times u]_{\text{tan}} = 0. \tag{1.9}$$

For general non-flat boundaries, these conditions cease to be equivalent. This situation gives rise to some confusion in the literature about which condition correctly describes a *true slip* condition.

Formally, condition (1.8) can be seen as the limit when $\alpha \to 0$ of the usual Navier slip-with-scalar-friction condition (1.6). As for condition (1.9) it can be seen as the natural extension in 3D of the 2D Lions free boundary condition. Let $x \in \partial\Omega$. We note T_x the tangent space to $\partial\Omega$ at x. The Weingarten map (or shape operator) $M_{\text{w}}(x)$ at x is defined as a linear map from T_x into itself such that $M_{\text{w}}(x)\tau := \nabla_\tau n$ for any τ in T_x. The image of $M_{\text{w}}(x)$ is contained in T_x. Indeed, since $|n|^2 = 1$ in a neighborhood of $\partial\Omega$, $0 = \nabla_\tau(n^2) = 2n \cdot \nabla_\tau n = 2n \cdot M_{\text{w}}\tau$ for any τ.

Lemma 1 ([5], [36])**.** *If Ω is smooth, the shape operator M_{w} is smooth. For any $x \in \partial\Omega$ it defines a self-adjoint operator with values in T_x. Moreover, for any divergence free vector field u satisfying $u \cdot n = 0$ on $\partial\Omega$, we have:*

$$[D(u)n + M_{\text{w}}u]_{tan} = \frac{1}{2}(\nabla \times u) \times n. \tag{1.10}$$

Even though it is a little unusual, it seems that condition (1.9) actually better describes the situation of a fluid slipping along the boundary. The convergence of the Navier-Stokes equation to the Euler equation under this condition has been extensively studied. In particular, let us mention the works by Beirao da Veiga, Crispo et al. (see [4], [5], [6], [7], [8], [9] and [21]), by Berselli et al. (see [10], [11]) and by Xiao, Xin et al. (see [88], [89], [90], [91] and [92]).

The difficulty comes from the fact that the Euler equation (which models the behavior of a perfect fluid, not subject to friction) is only associated with the $u \cdot n = 0$ boundary condition for an impermeable wall. Any other supplementary condition will be violated for some initial data. Indeed, as shown in [8], even the persistence property is false for condition (1.9) for the Euler equation: choosing an initial data such that (1.9) is satisfied does not guarantee that it will be satisfied at time $t > 0$.

1.6 Plan of the paper

The paper is organized as follows:

- In Section 2, we consider the special case of the slip boundary condition (1.9). This case is easier to handle because no boundary layer appears. We prove Theorem 1 in this simpler setting in order to explain some elements of our method.
- In Section 3, we introduce the boundary layer expansion that we will be using to handle the general case and we prove that we can apply the well-prepared dissipation method to ensure that the residual boundary layer is small at the final time.
- In Section 4, we introduce technical terms in the asymptotic expansion of the solution and we use them to carry out energy estimates on the remainder. We prove Theorem 1 in the general case.
- In Section 5 we explain how the well-prepared dissipation method detailed in the case of null controllability can be adapted to prove small-time global exact controllability to the trajectories.

2 A special case with no boundary layer

In this section, we consider the special case where the friction coefficient M is the shape operator M_{w}. On the uncontrolled boundary, thanks to Lemma 1, the flow satisfies:

$$u \cdot n = 0 \quad \text{and} \quad [\nabla \times u]_{\text{tan}} = 0. \tag{2.1}$$

In this setting, we can build an Euler trajectory satisfying this overdetermined boundary condition. The Euler trajectory by itself is thus an excellent approximation of the Navier-Stokes trajectory, up to the boundary. This allows us to present some elements of our method in a simple setting before moving on to the general case which involves boundary layers.

As in [17], our strategy is to deduce the controllability of the Navier-Stokes equation in small time from the controllability of the Euler equation. In order to use this strategy, we are willing to trade small time against small viscosity using the usual fluid dynamics scaling. Even in this easier context, Theorem 1 is new for multiply connected 2D domains and for all 3D domains since [17] only concerns simply connected 2D domains. This condition was also studied in [13] in the particular setting of a rectangular domain.

2.1 Domain extension and weak controlled trajectories

We start by introducing a smooth extension $\mathcal{O}$ of our initial domain Ω. We choose this extended domain in such a way that $\Gamma \subset \mathcal{O}$ and $\partial\Omega \setminus \Gamma \subset \partial\mathcal{O}$ (see Figure 2.1 for a simple case). This extension procedure can be justified by standard arguments. Indeed, we already assumed that Ω is a smooth domain and, up to reducing the size of Γ, we can assume that its intersection with each connected component of $\partial\Omega$ is smooth. From now on, n will denote the outward pointing normal to the extended domain $\mathcal{O}$ (which coincides with the outward pointing normal to Ω on the uncontrolled boundary $\partial\Omega \setminus \Gamma$). We will also need to introduce a smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi = 0$ on $\partial\mathcal{O}$, $\varphi > 0$ in $\mathcal{O}$ and $\varphi < 0$ outside of $\bar{\mathcal{O}}$. Moreover, we assume that $|\varphi(x)| = \mathrm{dist}(x, \partial\mathcal{O})$ in a small neighborhood of $\partial\mathcal{O}$. Hence, the normal n can be computed as $-\nabla\varphi$ close to the boundary and extended smoothly within the full domain $\mathcal{O}$. In the sequel, we will refer to Ω as the *physical domain* where we try to build a controlled trajectory of (1.1). Things happening within $\mathcal{O} \setminus \Omega$ are technicalities corresponding to the choice of the controls and we advise the reader to focus on true physical phenomenons happening inside Ω.

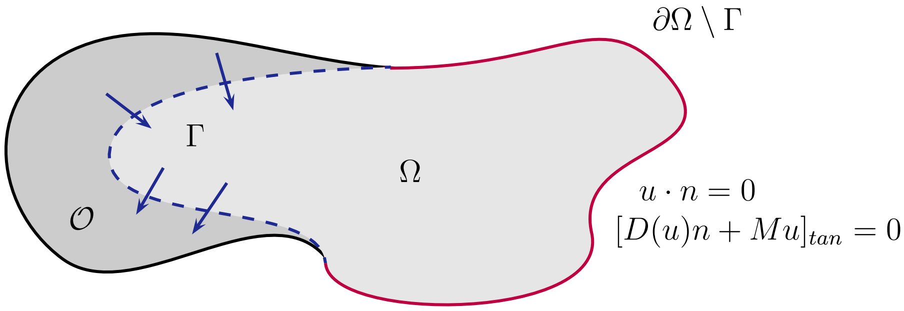

Figure 2: Extension of the physical domain $\Omega \subset \mathcal{O}$.

Definition 1. Let $T > 0$ and $u_* \in L^2_\gamma(\Omega)$. Let $u \in \mathcal{C}^0_w([0,T]; L^2_\gamma(\Omega)) \cap L^2((0,T); H^1(\Omega))$. We will say that u is a *weak controlled trajectory* of system (1.1) with initial condition u_* when u is the restriction to the physical domain Ω of a *weak Leray solution* in the space $\mathcal{C}^0_w([0,T]; L^2(\mathcal{O})) \cap L^2((0,T); H^1(\mathcal{O}))$ on the extended domain $\mathcal{O}$, which we still denote by u, to:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = \xi & \text{in } \mathcal{O}, \\ \operatorname{div} u = \sigma & \text{in } \mathcal{O}, \\ u \cdot n = 0 & \text{on } \partial\mathcal{O}, \\ N(u) = 0 & \text{on } \partial\mathcal{O}, \\ u(0,\cdot) = u_* & \text{in } \mathcal{O}, \end{cases} \tag{2.2}$$

where $\xi \in H^1((0,T), L^2(\mathcal{O})) \cap \mathcal{C}^0([0,T], H^1(\mathcal{O}))$ is a forcing term supported in $\bar{\mathcal{O}} \setminus \bar{\Omega}$, σ is a smooth non homogeneous divergence condition also supported in $\bar{\mathcal{O}} \setminus \bar{\Omega}$ and u_* has been extended to $\mathcal{O}$ such that the extension is tangent to $\partial\mathcal{O}$ and satisfies the compatibility condition $\operatorname{div} u_* = \sigma(0,\cdot)$.

Allowing a non vanishing divergence outside of the physical domain is necessary both for the control design process and because we did not restrict ourselves to controlling initial data satisfying $u_* \cdot n = 0$ on Γ. Defining weak Leray solutions to (2.2) is a difficult question when one tries to obtain optimal functional spaces for the non homogeneous source terms. For details on this subject, we refer the reader to [28], [29] or [76]. In our case, since the divergence source term is smooth, an efficient method is to start by solving a (stationary or evolution) Stokes problem in order to lift the non homogeneous divergence condition. We define u_σ as the solution to:

$$\begin{cases} \partial_t u_\sigma - \Delta u_\sigma + \nabla p_\sigma = 0 & \text{in } \mathcal{O}, \\ \operatorname{div} u_\sigma = \sigma & \text{in } \mathcal{O}, \\ u_\sigma \cdot n = 0 & \text{on } \partial\mathcal{O}, \\ N(u_\sigma) = 0 & \text{on } \partial\mathcal{O}, \\ u_\sigma(0, \cdot) = u_* & \text{in } \mathcal{O}. \end{cases} \tag{2.3}$$

Smoothness (in time and space) of σ immediately gives smoothness on u_σ. These are standard maximal regularity estimates for the Stokes problem in the case of the Dirichlet boundary condition. For Navier boundary conditions (sometimes referred to as Robin boundary conditions for the Stokes problem), we refer to [80], [81] or [82]. Decomposing $u = u_\sigma + u_h$, we obtain the following system for u_h:

$$\begin{cases} \partial_t u_h + ((u_\sigma + u_h) \cdot \nabla)(u_\sigma + u_h) - \Delta u_h + \nabla p_h = \xi & \text{in } \mathcal{O}, \\ \operatorname{div} u_h = 0 & \text{in } \mathcal{O}, \\ u_h \cdot n = 0 & \text{on } \partial\mathcal{O}, \\ N(u_h) = 0 & \text{on } \partial\mathcal{O}, \\ u_h(0, \cdot) = 0 & \text{in } \mathcal{O}. \end{cases} \tag{2.4}$$

Defining weak Leray solutions to (2.4) is a standard procedure. They are defined as measurable functions satisfying the variational formulation of (2.4) and some appropriate energy inequality. For in-depth insights on this topic, we refer the reader to the classical references by Temam [85] or Galdi [32]. In our case, let $L^2_{\text{div}}(\mathcal{O})$ denote the closure in $L^2(\mathcal{O})$ of the space of smooth divergence free vector fields tangent to $\partial\mathcal{O}$. We will say that $u_h \in \mathcal{C}^0_w([0,T]; L^2_{\text{div}}(\mathcal{O})) \cap L^2((0,T); H^1(\mathcal{O}))$ is a weak Leray solution to (2.4) if it satisfies the variational formulation:

$$\begin{aligned} &- \iint_{\mathcal{O}} u_h \partial_t \phi + \iint_{\mathcal{O}} ((u_\sigma \cdot \nabla) u_h + (u_h \cdot \nabla) u_\sigma + (u_h \cdot \nabla) u_h)\, \phi \\ &\quad + 2 \iint_{\mathcal{O}} D(u_h) : D(\phi) + 2 \iint_{\partial\mathcal{O}} [M u_h]_{\text{tan}}\, \phi = \iint_{\mathcal{O}} (\xi - (u_\sigma \cdot \nabla) u_\sigma)\, \phi, \end{aligned} \tag{2.5}$$

for any $\phi \in \mathcal{C}^\infty_c([0,T), \bar{\mathcal{O}})$ which is divergence free and tangent to $\partial\mathcal{O}$. We moreover require that they satisfy the so-called strong energy inequality for almost every $\tau < t$:

$$\begin{aligned} |u_h(t)|_{L^2}^2 + 4 \iint_{(\tau,t)\times\mathcal{O}} |D(u_h)|^2 \leq |u_h(\tau)|_{L^2}^2 - 4 \iint_{(\tau,t)\times\partial\mathcal{O}} [M u_h]_{\text{tan}} u_h \\ + \iint_{(\tau,t)\times\mathcal{O}} \sigma u_h^2 + 2(u_h \cdot \nabla) u_\sigma u_h + 2\left(\xi - (u_\sigma \cdot \nabla) u_\sigma\right) u_h. \end{aligned} \tag{2.6}$$

In (2.6), the boundary term is well defined. Indeed, from the Galerkin method, we can obtain strong convergence of Galerkin approximations u_h^n towards u_h in $L^2((0,T); L^2(\partial\mathcal{O}))$ (see [56, page 155]).

Although uniqueness of weak Leray solutions is still an open question, it is easy to adapt the classical Leray-Hopf theory proving global existence of weak solutions to the case of Navier boundary conditions (see [14] for 2D or [55] for 3D). Once forcing terms ξ and σ are fixed, there exists thus at least one weak Leray solution u to (2.2).

In the sequel, we will mostly work within the extended domain. Our goal will be to explain how we choose the external forcing terms ξ and σ in order to guarantee that the associated controlled trajectory vanishes within the physical domain at the final time.

2.2 Time scaling and small viscosity asymptotic expansion

The global controllability time T is small but fixed. Let us introduce a positive parameter $\varepsilon \ll 1$. We will be even more ambitious and try to control the system during the shorter time interval $[0, \varepsilon T]$. We perform the scaling: $u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x)$ and $p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x)$. Similarly, we set $\xi^\varepsilon(t,x) := \varepsilon^2 \xi(\varepsilon t, x)$ and $\sigma^\varepsilon(t,x) := \varepsilon\sigma(\varepsilon t, x)$. Now, $(u^\varepsilon, p^\varepsilon)$ is a solution to the following system for $t \in (0,T)$:

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon & \text{in } (0,T)\times\mathcal{O}, \\ \operatorname{div} u^\varepsilon = \sigma^\varepsilon & \text{in } (0,T)\times\mathcal{O}, \\ u^\varepsilon \cdot n = 0 & \text{on } (0,T)\times\partial\mathcal{O}, \\ [\nabla \times u^\varepsilon]_{\text{tan}} = 0 & \text{on } (0,T)\times\partial\mathcal{O}, \\ u^\varepsilon|_{t=0} = \varepsilon u_* & \text{in } \mathcal{O}. \end{cases} \tag{2.7}$$

Due to the scaling chosen, we plan to prove that we can obtain $|u^\varepsilon(T,\cdot)|_{L^2(\mathcal{O})} = o(\varepsilon)$ in order to conclude with a local result. Since ε is small, we expect u^ε to converge to the solution of the Euler equation. Hence, we introduce the following asymptotic expansion for:

$$u^\varepsilon = u^0 + \varepsilon u^1 + \varepsilon r^\varepsilon, \tag{2.8}$$

$$p^\varepsilon = p^0 + \varepsilon p^1 + \varepsilon \pi^\varepsilon, \tag{2.9}$$

$$\xi^\varepsilon = \xi^0 + \varepsilon \xi^1, \tag{2.10}$$

$$\sigma^\varepsilon = \sigma^0. \tag{2.11}$$

Let us provide some insight behind expansion (2.8)-(2.11). The first term $(u^0, p^0, \xi^0, \sigma^0)$ is the solution to a controlled Euler equation. It models a smooth reference trajectory around which we are linearizing the Navier-Stokes equation. This trajectory will be chosen in such a way that it flushes the initial data out of the domain in time T. The second term (u^1, p^1, ξ^1) takes into account the initial data u_*, which will be flushed out of the physical domain by the flow u^0. Eventually, $(r^\varepsilon, \pi^\varepsilon)$ contains higher order residues. We need to prove $|r^\varepsilon(T,\cdot)|_{L^2(\mathcal{O})} = o(1)$ in order to conclude the proof of Theorem 1.

2.3 A return method trajectory for the Euler equation

At order $\mathcal{O}(1)$, the first part (u^0, p^0) of our expansion is a solution to the Euler equation. Hence, the pair (u^0, p^0) is a *return-method-like* trajectory of the Euler equation on $(0,T)$:

$$\begin{cases} \partial_t u^0 + \left(u^0 \cdot \nabla\right) u^0 + \nabla p^0 = \xi^0 & \text{in } (0,T) \times \mathcal{O}, \\ \operatorname{div} u^0 = \sigma^0 & \text{in } (0,T) \times \mathcal{O}, \\ u^0 \cdot n = 0 & \text{on } (0,T) \times \partial\mathcal{O}, \\ u^0(0,\cdot) = 0 & \text{in } \mathcal{O}, \\ u^0(T,\cdot) = 0 & \text{in } \mathcal{O}, \end{cases} \tag{2.12}$$

where ξ^0 and σ^0 are smooth forcing terms supported in $\bar{\mathcal{O}} \setminus \bar{\Omega}$. We want to use this reference trajectory to flush any particle outside of the physical domain within the fixed time interval $[0,T]$. Let us introduce the flow Φ^0 associated with u^0:

$$\begin{cases} \Phi^0(t,t,x) = x, \\ \partial_s \Phi^0(t,s,x) = u^0(s, \Phi^0(t,s,x)). \end{cases} \tag{2.13}$$

Hence, we look for trajectories satisfying:

$$\forall x \in \bar{\mathcal{O}}, \exists t_x \in (0,T), \quad \Phi^0\left(0, t_x, x\right) \notin \bar{\Omega}. \tag{2.14}$$

We do not require that the time t_x be the same for all $x \in \mathcal{O}$. Indeed, it might not be possible to flush all of the points outside of the physical domain at the same time. Property (2.14) is obvious for points x already located in $\bar{\mathcal{O}} \setminus \bar{\Omega}$. For points lying within the physical domain, we use:

Lemma 2. *There exists a solution* $(u^0, p^0, \xi^0, \sigma^0) \in \mathcal{C}^\infty([0,T] \times \bar{\mathcal{O}}, \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$ *to system* (2.12) *such that the flow* Φ^0 *defined in* (2.13) *satisfies* (2.14)*. Moreover,* u^0 *can be chosen such that:*

$$\nabla \times u^0 = 0 \quad \textit{in} \quad [0,T] \times \bar{\mathcal{O}}. \tag{2.15}$$

Moreover, $(u^0, p^0, \xi^0, \sigma^0)$ *are compactly supported in* $(0,T)$*. In the sequel, when we need it, we will implicitly extend them by zero after* T*.*

This lemma is the key argument of multiple papers concerning the small-time global exact controllability of Euler equations. We refer to the following references for detailed statements and construction of these reference trajectories. First, the first author used it in [16] for 2D simply connected domains, then in [18] for general 2D domains when Γ intersects all connected components of $\partial\Omega$. Glass adapted the argument for 3D domains (when Γ intersects all connected components of the boundary), for simply connected domains in [37] then for general domains in [39]. He also used similar arguments to study the obstructions to approximate controllability in 2D when Γ does not intersect all connected components of the boundary for general 2D domains in [40]. Here, we use the assumption that our control domain Γ intersects all connected parts of the boundary $\partial\Omega$. The fact that condition (2.15) can be achieved is a direct consequence of the construction of the reference profile u^0 as a potential flow: $u^0(t,x) = \nabla\theta^0(t,x)$, where θ^0 is smooth.

2.4 Convective term and flushing of the initial data

We move on to order $\mathcal{O}(\varepsilon)$. Here, the initial data u_* comes into play. We build u^1 as the solution to:

$$\begin{cases} \partial_t u^1 + \left(u^0\cdot\nabla\right)u^1 + \left(u^1\cdot\nabla\right)u^0 + \nabla p^1 = \Delta u^0 + \xi^1 & \text{in } (0,T)\times\mathcal{O}, \\ \operatorname{div} u^1 = 0 & \text{in } (0,T)\times\mathcal{O}, \\ u^1\cdot n = 0 & \text{on } (0,T)\times\partial\mathcal{O}, \\ u^1(0,\cdot) = u_* & \text{in } \mathcal{O}, \end{cases} \tag{2.16}$$

where ξ^1 is a forcing term supported in $\bar{\mathcal{O}}\setminus\bar{\Omega}$. Formally, equation (2.16) also takes into account a residual term Δu^0. Thanks to (2.15), we have $\Delta u^0 = \nabla(\operatorname{div} u^0) = \nabla\sigma^0$. It is thus smooth, supported in $\bar{\mathcal{O}}\setminus\bar{\Omega}$ and can be canceled by incorporating it into ξ^1. The following lemma is natural thanks to the choice of a good flushing trajectory u^0:

Lemma 3. *Let $u_* \in H^3(\mathcal{O})\cap L^2_{\mathrm{div}}(\mathcal{O})$. There exists a force $\xi^1 \in \mathcal{C}^1([0,T],H^1(\mathcal{O}))\cap \mathcal{C}^0([0,T],H^2(\mathcal{O}))$ such that the solution u^1 to system* (2.16) *satisfies $u^1(T,\cdot)=0$. Moreover, u^1 is bounded in $L^\infty((0,T),H^3(\mathcal{O}))$. In the sequel, it is implicit that we extend (u^1,p^1,ξ^1) by zero after T.*

This lemma is mostly a consequence of the works on the Euler equation, already mentioned in the previous paragraph, due to the first author in 2D, then to Glass in 3D. However, in these original works, the regularity obtained for the constructed trajectory would not be sufficient in our context. Thus, we provide in Appendix A a constructive proof which enables us to obtain the regularity for ξ^1 and u^1 stated in Lemma 3. We only give here a short overview of the main idea of the proof. The interested reader can also start with the nice introduction given by Glass in [38].

The intuition behind the possibility to control u^1 is to introduce $\omega^1 := \nabla \times u^1$ and to write (2.16) in vorticity form, within the physical domain Ω:

$$\begin{cases} \partial_t \omega^1 + \left(u^0 \cdot \nabla\right) \omega^1 - \left(\omega^1 \cdot \nabla\right) u^0 = 0 & \text{in } (0,T) \times \Omega, \\ \omega^1(0,\cdot) = \nabla \times u_* & \text{in } \Omega. \end{cases} \tag{2.17}$$

The term $(\omega^1 \cdot \nabla)\, u^0$ is specific to the 3D setting and does not appear in 2D (where the vorticity is merely transported). Nevertheless, even in 3D, the support of the vorticity is transported by u^0. Thus, thanks to hypothesis (2.14), ω^1 will vanish inside Ω at time T provided that we choose null boundary conditions for ω^1 on the controlled boundary Γ when the characteristics enter in the physical domain. Hence, we can build a trajectory such that $\omega^1(T,\cdot) = 0$ inside Ω. Combined with the divergence free condition and null boundary data, this yields that $u^1(T,\cdot) = 0$ inside Ω, at least for simple geometries.

2.5 Energy estimates for the remainder

In this paragraph, we study the remainder defined in expansion (2.8). We write the equation for the remainder in the extended domain $\mathcal{O}$:

$$\begin{cases} \partial_t r^\varepsilon + (u^\varepsilon \cdot \nabla)\, r^\varepsilon - \varepsilon \Delta r^\varepsilon + \nabla \pi^\varepsilon = f^\varepsilon - A^\varepsilon r^\varepsilon, & \text{in } (0,T) \times \mathcal{O}, \\ \operatorname{div} r^\varepsilon = 0 & \text{in } (0,T) \times \mathcal{O}, \\ [\nabla \times r^\varepsilon]_{\text{tan}} = - \left[\nabla \times u^1\right]_{\text{tan}} & \text{on } (0,T) \times \partial\mathcal{O}, \\ r^\varepsilon \cdot n = 0 & \text{on } (0,T) \times \partial\mathcal{O}, \\ r^\varepsilon(0,\cdot) = 0 & \text{in } \mathcal{O}, \end{cases} \tag{2.18}$$

where we used the notations:

$$A^\varepsilon r^\varepsilon := (r^\varepsilon \cdot \nabla) \left(u^0 + \varepsilon u^1\right), \tag{2.19}$$

$$f^\varepsilon := \varepsilon \Delta u^1 - \varepsilon (u^1 \cdot \nabla) u^1. \tag{2.20}$$

We want to establish a standard $L^\infty(L^2) \cap L^2(H^1)$ energy estimate for the remainder. As usual, formally, we multiply equation (2.18) by r^ε and integrate by parts. Since we are considering weak solutions, some integration by parts may not be justified because we do not have enough regularity to give them a meaning. However, the usual technique applies: one can recover the estimates obtained formally from the variational formulation of the problem, the energy equality for the first terms of the expansion and the energy inequality of the definition of weak solutions (see [56, page 168] for an example of such an argument). We proceed term by term:

$$\int_{\mathcal{O}} \partial_t r^\varepsilon \cdot r^\varepsilon = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} |r^\varepsilon|^2, \tag{2.21}$$

$$\int_{\mathcal{O}} (u^\varepsilon \cdot \nabla)\, r^\varepsilon \cdot r^\varepsilon = -\frac{1}{2}\int_{\mathcal{O}} (\operatorname{div} u^\varepsilon)\, |r^\varepsilon|^2, \tag{2.22}$$

$$-\varepsilon \int_{\mathcal{O}} \Delta r^\varepsilon \cdot r^\varepsilon = \varepsilon \int_{\mathcal{O}} |\nabla \times r^\varepsilon|^2 - \varepsilon \int_{\partial\mathcal{O}} (r^\varepsilon \times (\nabla \times r^\varepsilon)) \cdot n, \tag{2.23}$$

$$\int_{\mathcal{O}} \nabla \pi^\varepsilon \cdot r^\varepsilon = 0. \tag{2.24}$$

In (2.22), we will use the fact that $\operatorname{div} u^\varepsilon = \operatorname{div} u^0 = \sigma^0$ is known and bounded independently of r^ε. In (2.23), we use the boundary condition on r^ε to estimate the boundary term:

$$\begin{aligned} \left|\int_{\partial\mathcal{O}} r^\varepsilon \times (\nabla \times r^\varepsilon) \cdot n\right| &= \left|\int_{\partial\mathcal{O}} r^\varepsilon \times (\nabla \times u^1) \cdot n\right| \\ &= \left|\int_{\mathcal{O}} \operatorname{div}\left(r^\varepsilon \times \omega^1\right)\right| \\ &= \left|\int_{\mathcal{O}} (\nabla \times r^\varepsilon) \cdot \omega^1 - r^\varepsilon \cdot (\nabla \times \omega^1)\right| \\ &\leq \frac{1}{2}\int_{\mathcal{O}} |\nabla \times r^\varepsilon|^2 + \frac{1}{2}\int_{\mathcal{O}} \left|\omega^1\right|^2 \\ &\quad + \frac{1}{2}\int_{\mathcal{O}} |r^\varepsilon|^2 + \frac{1}{2}\int_{\mathcal{O}} \left|\nabla \times \omega^1\right|^2. \end{aligned} \tag{2.25}$$

We split the forcing term estimate as:

$$\left|\int_{\mathcal{O}} f^\varepsilon \cdot r^\varepsilon\right| \leq \frac{1}{2}\,|f^\varepsilon|_2 \left(1 + |r^\varepsilon|_2^2\right). \tag{2.26}$$

Combining estimates (2.21)-(2.24), (2.25) and (2.26) yields:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|r^\varepsilon|_2^2 + \varepsilon|\nabla \times r^\varepsilon|_2^2 \leq & \left(2\varepsilon \left|u^1\right|_{H^2}^2 + |f^\varepsilon|_2\right) \\ & + \left(\varepsilon + \left|\sigma^0\right|_\infty + 2\,|A^\varepsilon|_\infty + |f^\varepsilon|_2\right) |r^\varepsilon|_2^2. \end{aligned} \tag{2.27}$$

Applying Grönwall's inequality by integrating over $(0, T)$ and using the null initial condition gives:

$$|r^\varepsilon|_{L^\infty(L^2)}^2 + \varepsilon\,|\nabla \times r^\varepsilon|_{L^2(L^2)}^2 = \mathcal{O}(\varepsilon). \tag{2.28}$$

This paragraphs proves that, once the source terms ξ^ε and σ^ε are fixed as above, any weak Leray solution to (2.7) is small at the final time. Indeed, thanks to Lemma 2 and Lemma 3, $u^0(T) = u^1(T) = 0$. At the final time, (2.28) gives:

$$|u^\varepsilon(T, \cdot)|_{L^2(\mathcal{O})} \leq \varepsilon\,|r^\varepsilon(T, \cdot)|_{L^2(\mathcal{O})} = \mathcal{O}(\varepsilon^{3/2}). \tag{2.29}$$

2.6 Regularization and local arguments

In this paragraph, we explain how to chain our arguments in order to prove Theorem 1. We will need to use a local argument to finish bringing the velocity field exactly to the null equilibrium state (see paragraph 1.4.1 for references on null controllability of Navier-Stokes):

Lemma 4 ([48]). *Let $T > 0$. There exists $\delta_T > 0$ such that, for any $u_* \in H^3(\mathcal{O})$ which is divergence free, tangent to $\partial\mathcal{O}$, satisfies the compatibility assumption $N(u_*) = 0$ on $\partial\mathcal{O}$ and of size $|u_*|_{H^3(\mathcal{O})} \leq \delta_T$, there exists a control $\xi \in H^1((0,T), L^2(\mathcal{O})) \cap \mathcal{C}^0([0,T], H^1(\mathcal{O}))$ supported outside of $\bar{\Omega}$ such that the strong solution to* (2.2) *with $\sigma = 0$ satisfies $u(T,\cdot) = 0$.*

In this context of small initial data, the existence and uniqueness of a strong solution is proved in [48]. We also use the following smoothing lemma for our Navier-Stokes system:

Lemma 5. *Let $T > 0$. There exists a continuous function C_T with $C_T(0) = 0$, such that, if $u_* \in L^2_{\mathrm{div}}(\mathcal{O})$ and $u \in \mathcal{C}^0_w([0,T]; L^2_{\mathrm{div}}(\mathcal{O})) \cap L^2((0,T); H^1(\mathcal{O}))$ is a weak Leray solution to* (2.2)*, with $\xi = 0$ and $\sigma = 0$:*

$$\exists t_u \in [0,T], \quad |u(t_u,\cdot)|_{H^3(\mathcal{O})} \leq C_T\left(|u_*|_{L^2(\mathcal{O})}\right). \tag{2.30}$$

Proof. This result is proved by Temam in [84, Remark 3.2] in the harder case of Dirichlet boundary condition. His method can be adapted to the Navier boundary condition and one could track down the constants to explicit the shape of the function C_T. For the sake of completeness, we provide a standalone proof in a slightly more general context (see Lemma 9, Section 5). □

We can now explain how we combine these arguments to prove Theorem 1. Let $T > 0$ be the allowed control time and $u_* \in L^2_\gamma(\Omega)$ the (potentially large) initial data to be controlled. The proof of Theorem 1 follows the following steps:

- We start by extending Ω into $\mathcal{O}$ as explained in paragraph 2.1. We also extend the initial data u_* to all of $\mathcal{O}$, still denoting it by u_*. We choose an extension such that $u_* \cdot n = 0$ on $\partial\mathcal{O}$ and $\sigma_* := \operatorname{div} u_*$ is smooth (and supported in $\mathcal{O} \setminus \Omega$). We start with a short preparation phase where we let σ decrease from its initial value to zero, relying on the existence of a weak solution once a smooth σ profile is fixed, say $\sigma(t,x) := \beta(t)\sigma_*$, where β smoothly decreases from 1 to 0. Then, once the data is divergence free, we use Lemma 5 to deduce the existence of a time $T_1 \in (0, T/4)$ such that $u(T_1,\cdot) \in H^3(\mathcal{O})$. This is why we can assume that the new "initial" data has H^3 regularity and is divergence free. We can thus apply Lemma 3.

- Let $T_2 := T/2$. Starting from this new smoother initial data $u(T_1, \cdot)$, we proceed with the small-time global approximate controllability method explained above on a time interval of size $T_2 - T_1 \geq T/4$. For any $\delta > 0$, we know that we can build a trajectory starting from $u(T_1, \cdot)$ and such that $u(T_2, \cdot)$ is smaller than δ in $L^2(\mathcal{O})$. It particular, it can be made small enough such that $C_{\frac{T}{4}}(\delta) \leq \delta_{\frac{T}{4}}$, where $\delta_{\frac{T}{4}}$ comes from Lemma 4 and the function $C_{\frac{T}{4}}$ comes from Lemma 5.

- Repeating the regularization argument of Lemma 5, we deduce the existence of a time $T_3 \in \left(\frac{T}{2}, \frac{3T}{4}\right)$ such that $u(T_3, \cdot)$ is smaller than $\delta_{\frac{T}{4}}$ in $H^3(\mathcal{O})$.

- We use Lemma 4 on the time interval $\left[T_3, T_3 + \frac{T}{4}\right]$ to reach exactly zero. Once the system is at rest, it stays there until the final time T.

This concludes the proof of Theorem 1 in the case of the slip condition. For the general case, we will use the same proof skeleton, but we will need to control the boundary layers. In the following sections, we explain how we can obtain small-time global approximate null controllability in the general case.

3 Boundary layer expansion and dissipation

As in the previous section, the allotted physical control time T is fixed (and potentially small). We introduce an arbitrary mathematical time scale $\varepsilon \ll 1$ and we perform the usual scaling $u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x)$ and $p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x)$. In this harder setting involving a boundary layer expansion, we do not try to achieve approximate controllability towards zero in the smaller physical time interval $[0, \varepsilon T]$ like it was possible to do in the previous section. Instead, we will use the virtually long mathematical time interval to dissipate the boundary layer. Thus, we consider $(u^\varepsilon, p^\varepsilon)$ the solution to:

$$
\begin{cases}
\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon & \text{in } (0, T/\varepsilon) \times \mathcal{O}, \\
\operatorname{div} u^\varepsilon = \sigma^\varepsilon & \text{in } (0, T/\varepsilon) \times \mathcal{O}, \\
u^\varepsilon \cdot n = 0 & \text{on } (0, T/\varepsilon) \times \partial\mathcal{O}, \\
N(u^\varepsilon) = 0 & \text{on } (0, T/\varepsilon) \times \partial\mathcal{O}, \\
u^\varepsilon|_{t=0} = \varepsilon u_* & \text{in } \mathcal{O}.
\end{cases}
\tag{3.1}
$$

Here again, we do not expect to reach exactly zero with this part of the strategy. However, we would like to build a sequence of solutions such that $|u(T, \cdot)|_{L^2(\mathcal{O})} = o(1)$. As in Section 2, this will allow us to apply a local result with a small initial data, a fixed time and a fixed viscosity. Due to the scaling chosen, this conditions translates into proving that $\left|u^\varepsilon\left(\frac{T}{\varepsilon}, \cdot\right)\right|_{L^2(\mathcal{O})} = o(\varepsilon)$. Following and enhancing the original boundary layer expansion for Navier slip-with-friction boundary conditions proved by Iftimie and the third author in [56], we introduce the following expansion:

$$u^\varepsilon(t,x) = u^0(t,x) + \sqrt{\varepsilon} v\left(t, x, \frac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \varepsilon u^1(t,x) + \ldots + \varepsilon r^\varepsilon(t,x), \tag{3.2}$$

$$p^\varepsilon(t,x) = p^0(t,x) + \varepsilon p^1(t,x) + \ldots + \varepsilon \pi^\varepsilon(t,x). \tag{3.3}$$

The forcing terms are expanded as:

$$\xi^\varepsilon(t,x) = \xi^0(t,x) + \sqrt{\varepsilon}\xi^v\left(t, x, \frac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \varepsilon \xi^1(t,x), \tag{3.4}$$

$$\sigma^\varepsilon(t,x) = \sigma^0(t,x). \tag{3.5}$$

Compared with expansion (2.8), expansion (3.2) introduces a boundary correction v. Indeed, u^0 does not satisfy the Navier slip-with-friction boundary condition on $\partial\mathcal{O}$. The purpose of the second term v is to recover this boundary condition by introducing the tangential boundary layer generated by u^0. In equations (3.2) and (3.3), the missing terms are technical terms which will help us prove that the remainder is small. We give the details of this technical part in Section 4. We use the same profiles u^0 and u^1 as in the previous section (extended by zero after T). Hence, $u^\varepsilon \approx \sqrt{\varepsilon} v$ after T and we must understand the behavior of this boundary layer residue that remains after the short inviscid control strategy.

3.1 Boundary layer profile equations

Since the Euler system is a first-order system, we have only been able to impose a single scalar boundary condition in (2.12) (namely, $u^0 \cdot n = 0$ on $\partial\mathcal{O}$). Hence, the full Navier slip-with-friction boundary condition is not satisfied by u^0. Therefore, at order $\mathcal{O}(\sqrt{\varepsilon})$, we introduce a tangential boundary layer correction v. This profile is expressed in terms both of the slow space variable $x \in \mathcal{O}$ and a fast scalar variable $z = \varphi(x)/\sqrt{\varepsilon}$. As in [56], v is the solution to:

$$\begin{cases} \partial_t v + \left[(u^0 \cdot \nabla)v + (v \cdot \nabla)u^0\right]_{\text{tan}} + u^0_\flat z \partial_z v - \partial_{zz} v = \xi^v & \text{in } \mathbb{R}_+ \times \bar{\mathcal{O}} \times \mathbb{R}_+, \\ \partial_z v(t,x,0) = g^0 & \text{in } \mathbb{R}_+ \times \bar{\mathcal{O}}, \\ v(0,x,z) = 0 & \text{in } \bar{\mathcal{O}} \times \mathbb{R}_+, \end{cases} \tag{3.6}$$

where we introduce the following definitions:

$$u^0_\flat(t,x) := -\frac{u^0(t,x) \cdot n(x)}{\varphi(x)} \quad \text{in } \mathbb{R}_+ \times \mathcal{O}, \tag{3.7}$$

$$g^0(t,x) := 2\chi(x) N(u^0)(t,x) \quad \text{in } \mathbb{R}_+ \times \mathcal{O}. \tag{3.8}$$

Unlike in [56], we introduced an inhomogeneous source term ξ^v in (3.6). This corresponds to a smooth control term whose x-support is located within $\bar{\mathcal{O}} \setminus \bar{\Omega}$. Using the transport term, this outside control will enable us to modify the behavior of v inside the physical domain Ω. Let us state the following points about equations (3.6), (3.7) and (3.8):

- The boundary layer profile depends on $d+1$ spatial variables (d slow variables x and one fast variable z) and is thus not set in curvilinear coordinates. This approach used in [56] lightens the computations. It is implicit that n actually refers to the extension $-\nabla\varphi$ of the normal (as explained in paragraph 2.1) and that this extends formulas (1.2) defining the tangential part of a vector field and (1.4) defining the Navier operator inside $\mathcal{O}$.

- The boundary profile is tangential, even inside the domain. For any $x \in \bar{\mathcal{O}}$, $z \geq 0$ and $t \geq 0$, we have $v(t,x,z)\cdot n(x) = 0$. It is easy to check that, as soon as the source term $\xi^v \cdot n = 0$, the evolution equation (3.6) preserves the relation $v(0,x,z)\cdot n(x) = 0$ of the initial time. This orthogonality property is the reason why equation (3.6) is linear. Indeed, the quadratic term $(v\cdot n)\partial_z v$ should have been taken into account if it did not vanish. In the sequel, we will check that our construction satisfies the property $\xi^v \cdot n = 0$.

- In (3.8), we introduce a smooth cut-off function χ, satisfying $\chi = 1$ on $\partial\mathcal{O}$. This is intended to help us guarantee that v is compactly supported near $\partial\mathcal{O}$, while ensuring that v compensates the Navier slip-with-friction boundary trace of u^0. See paragraph 3.4 for the choice of χ.

- Even though φ vanishes on $\partial\mathcal{O}$, $u^0_\flat$ is not singular near the boundary because of the impermeability condition $u^0 \cdot n = 0$. Since u^0 is smooth, a Taylor expansion proves that $u^0_\flat$ is smooth in $\bar{\mathcal{O}}$.

3.2 Large time asymptotic decay of the boundary layer profile

In the previous paragraph, we defined the boundary layer profile through equation (3.6) for any $t \geq 0$. Indeed, we will need this expansion to hold on the large time interval $[0, T/\varepsilon]$. Thus, we prefer to define it directly for any $t \geq 0$ in order to stress out that this boundary layer profile does not depend in any way on ε. Is it implicit that, for $t \geq T$, the Euler reference flow u^0 is extended by 0. Hence, for $t \geq T$, system (3.6) reduces to a parametrized heat equation on the half line $z \geq 0$ (where the slow variables $x \in \mathcal{O}$ play the role of parameters):

$$\begin{cases} \partial_t v - \partial_{zz} v = 0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \quad \text{for } t \geq T, \\ \partial_z v(t,x,0) = 0 & \text{in } \{0\} \times \mathcal{O}, \quad \text{for } t \geq T. \end{cases} \tag{3.9}$$

The behavior of the solution to (3.9) depends on its "initial" data $\bar{v}(x,z) := v(T,x,z)$ at time T. Even without any assumption on $\bar{v}$, this heat system exhibits smoothing properties and dissipates towards the null equilibrium state. It can for example be proved that:

$$|v(t,x,\cdot)|_{L^2(\mathbb{R}_+)} \lesssim t^{-\frac{1}{4}} |\bar{v}(x,\cdot)|_{L^2(\mathbb{R}_+)}. \tag{3.10}$$

However, as the equation is set on the half-line $z \geq 0$, the energy decay obtained in (3.10) is rather slow. Moreover, without any additional assumption, this estimate cannot be improved. It is indeed standard to prove asymptotic estimates for the solution $v(t,x,\cdot)$ involving the corresponding Green function (see [3], [24], or [79]). Physically, this is due to the fact that the average of v is preserved under its evolution by equation (3.9). The energy contained by low frequency modes decays slowly. Applied at the final time $t = T/\varepsilon$, estimate (3.10) yields:

$$\left|\sqrt{\varepsilon} v\left(\frac{T}{\varepsilon}, \cdot, \frac{\varphi(\cdot)}{\sqrt{\varepsilon}}\right)\right|_{L^2(\mathcal{O})} = \mathcal{O}\left(\varepsilon^{\frac{1}{2}+\frac{1}{4}+\frac{1}{4}}\right), \tag{3.11}$$

where the last $\varepsilon^{\frac{1}{4}}$ factor comes from the Jacobian of the fast variable scaling (see [56, Lemma 3, page 150]). Hence, the natural decay $\mathcal{O}(\varepsilon)$ obtained in (3.11) is not sufficient to provide an asymptotically small boundary layer residue in the physical scaling. After division by ε, we only obtain a $\mathcal{O}(1)$ estimate. This motivates the fact that we need to design a control strategy to enhance the natural dissipation of the boundary layer residue after the main inviscid control step is finished.

Our strategy will be to guarantee that $\bar{v}$ satisfies a finite number of vanishing moment conditions for $k \in \mathbb{N}$ of the form:

$$\forall x \in \mathcal{O}, \quad \int_{\mathbb{R}_+} z^k \bar{v}(x,z)\mathrm{d}z = 0. \tag{3.12}$$

These conditions also correspond to vanishing derivatives at zero for the Fourier transform in z of $\bar{v}$ (or its even extension to $\mathbb{R}$). If we succeed to kill enough moments in the boundary layer at the end of the inviscid phase, we can obtain arbitrarily good polynomial decay properties. For $s, n \in \mathbb{N}$, let us introduce the following weighted Sobolev spaces:

$$H^{s,n}(\mathbb{R}) := \left\{ f \in H^s(\mathbb{R}), \quad \sum_{\alpha=0}^{s} \int_{\mathbb{R}} (1+z^2)^n |\partial_z^\alpha f(z)|^2 \mathrm{d}z < +\infty \right\}, \tag{3.13}$$

which we endow with their natural norm. We prove in the following lemma that vanishing moment conditions yield polynomial decays in these weighted spaces for a heat equation set on the real line.

Lemma 6. *Let $s, n \in \mathbb{N}$ and $f_0 \in H^{s,n+1}(\mathbb{R})$ satisfying, for $0 \leq k < n$,*

$$\int_{\mathbb{R}} z^k f_0(z)\mathrm{d}z = 0. \tag{3.14}$$

Let f be the solution to the heat equation on $\mathbb{R}$ with initial data f_0:

$$\begin{cases} \partial_t f - \partial_{zz} f = 0 & \text{in } \mathbb{R}, \quad \text{for } t \geq 0, \\ f(0, \cdot) = f_0 & \text{in } \mathbb{R}, \quad \text{for } t = 0. \end{cases} \tag{3.15}$$

There exists a constant $C_{s,n}$ independent on f_0 such that, for $0 \leq m \leq n$,

$$|f(t, \cdot)|_{H^{s,m}} \leq C_{s,n} |f_0|_{H^{s,n+1}} \left| \frac{\ln(2+t)}{2+t} \right|^{\frac{1}{4} + \frac{n}{2} - \frac{m}{2}}. \tag{3.16}$$

Proof. For small times (say $t \leq 2$), the t function in the right-hand side of (3.16) is bounded below by a positive constant. Thus, inequality (3.16) holds because the considered energy decays under the heat equation. Let us move on to large times, e.g. assuming $t \geq 2$. Using Fourier transform in $z \mapsto \zeta$, we compute:

$$\hat{f}(t, \zeta) = e^{-t\zeta^2} \hat{f}_0(\zeta). \tag{3.17}$$

Moreover, from Plancherel's equality, we have the following estimate:

$$|f(t, \cdot)|^2_{H^{s,m}} \lesssim \sum_{j=0}^{m} \int_{\mathbb{R}} (1 + \zeta^2)^s \left| \partial_\zeta^j \hat{f}(t, \zeta) \right|^2 \mathrm{d}\zeta. \tag{3.18}$$

We use (3.17) to compute the derivatives of the Fourier transform:

$$\partial_\zeta^j \hat{f}(t, \zeta) = \sum_{i=0}^{j} \zeta^{i-j} P_{i,j} \left(t\zeta^2 \right) e^{-t\zeta^2} \partial_\zeta^i \hat{f}_0(\zeta), \tag{3.19}$$

where $P_{i,j}$ are polynomials with constant numerical coefficients. The energy contained at high frequencies decays very fast. For low frequencies, we will need to use assumptions (3.14). Writing a Taylor expansion of $\hat{f}_0$ near $\zeta = 0$ and taking into account these assumptions yields the estimates:

$$\left| \partial_\zeta^i \hat{f}_0(\zeta) \right| \lesssim |\zeta|^{n-i} \left| \partial_\zeta^n \hat{f}_0 \right|_{L^\infty} \lesssim |\zeta|^{n-i} |z^n f_0(z)|_{L^1} \lesssim |\zeta|^{n-i} |f_0|_{H^{0,n+1}}. \tag{3.20}$$

We introduce $\rho > 0$ and we split the energy integral at a cutting threshold:

$$\zeta^*(t) := \left| \frac{\rho \ln(2+t)}{2+t} \right|^{1/2}. \tag{3.21}$$

High frequencies. We start with high frequencies $|\zeta| \geq \zeta^*(t)$. For large times, this range actually almost includes the whole spectrum. Using (3.18) and (3.19) we compute the high energy terms:

$$\begin{aligned} \mathcal{W}^\sharp_{j,i,i'}(t) := \int_{|\zeta| \geq \zeta^*(t)} & \left(1 + \zeta^2\right)^s e^{-2t\zeta^2} |\zeta|^{i-j} |\zeta|^{i'-j} \\ & \cdot \left| P_{i,j}\left(t\zeta^2\right) P_{i',j}\left(t\zeta^2\right) \right| \left| \partial_\zeta^i \hat{f}_0 \right| \left| \partial_\zeta^{i'} \hat{f}_0 \right| \mathrm{d}\zeta. \end{aligned} \tag{3.22}$$

Plugging estimate (3.20) into (3.22) yields:

$$\begin{aligned}\mathcal{W}^{\sharp}_{j,i,i'}(t) \leq |f_0|^2_{H^{0,n+1}} \cdot \frac{e^{-t(\zeta^*(t))^2}}{|t|^{n-j+\frac{1}{2}}} \\ \cdot \int_{\mathbb{R}} \left(1+\zeta^2\right)^s e^{-t\zeta^2} \left|t\zeta^2\right|^{n-j} \left|P_{i,j}\left(t\zeta^2\right) P_{i',j}\left(t\zeta^2\right)\right| t^{\frac{1}{2}} \mathrm{d}\zeta.\end{aligned} \tag{3.23}$$

The integral in (3.23) is bounded from above for $t \geq 2$ through an easy change of variable. Moreover,

$$e^{-t(\zeta^*(t))^2} = e^{-\frac{\rho t}{2+t}\ln(2+t)} = (2+t)^{-\frac{\rho t}{2+t}} \leq (2+t)^{-\frac{\rho}{2}}. \tag{3.24}$$

Hence, for $t \geq 2$, combining (3.23) and (3.24) yields:

$$\mathcal{W}^{\sharp}_{j,i,i'}(t) \lesssim (2+t)^{-\frac{\rho}{2}} |f_0|^2_{H^{0,n+1}}. \tag{3.25}$$

In (3.21), we can choose any $\rho > 0$. Hence, the decay obtained in (3.25) can be arbitrarily good. This is not the case for the low frequencies estimates which are capped by the number of vanishing moments assumed on the initial data f_0.

Low frequencies. We move on to low frequencies $|\zeta| \leq \zeta^*(t)$. For large times, this range concentrates near zero. Using (3.18) and (3.19) we compute the low energy terms:

$$\begin{aligned}\mathcal{W}^{\flat}_{j,i,i'}(t) := \int_{|\zeta|\leq\zeta^*(t)} \left(1+\zeta^2\right)^s e^{-2t\zeta^2} |\zeta|^{i-j} |\zeta|^{i'-j} \\ \cdot \left|P_{i,j}\left(t\zeta^2\right) P_{i',j}\left(t\zeta^2\right)\right| \left|\partial^i_\zeta \hat{f}_0\right| \left|\partial^{i'}_\zeta \hat{f}_0\right| \mathrm{d}\zeta.\end{aligned} \tag{3.26}$$

Plugging estimate (3.20) into (3.26) yields:

$$\begin{aligned}\mathcal{W}^{\flat}_{j,i,i'}(t) \leq |f_0|^2_{H^{0,n+1}} \int_{|\zeta|\leq\zeta^*(t)} \left(1+\zeta^2\right)^s |\zeta|^{2n-2j} \\ \cdot \left|P_{i,j}\left(t\zeta^2\right) P_{i',j}\left(t\zeta^2\right)\right| e^{-2t\zeta^2} \mathrm{d}\zeta.\end{aligned} \tag{3.27}$$

The function $\tau \mapsto |P_{i,j}(\tau)P_{i',j}(\tau)|\, e^{-2\tau}$ is bounded on $[0,+\infty)$ thanks to the growth comparison theorem. Moreover, $(1+\zeta^2)^s$ can be bounded by $(1+\rho)^s$ for $|\zeta| \leq |\zeta^*(t)|$. Hence, plugging the definition (3.21) into (3.27) yields:

$$\mathcal{W}^{\flat}_{j,i,i'}(t) \lesssim |f_0|^2_{H^{0,n+1}} \left|\frac{\rho\ln(2+t)}{2+t}\right|^{\frac{1}{2}+n-j}. \tag{3.28}$$

Hence, choosing $\rho = 1 + 2n - 2m$ in equation (3.21) and summing estimates (3.25) with (3.28) for all indexes $0 \leq i, i' \leq j \leq m$ concludes the proof of (3.16). □

We will use the conclusion of Lemma 6 for two different purposes. First, it states that the boundary layer residue is small at the final time. Second, estimate (3.16) can also be used to prove that the source terms generated by the boundary layer in the equation of the remainder are integrable in large time. Indeed, for $n \geq 2$, f_0 and f satisfying the assumptions of Lemma 6, we have:

$$\|f\|_{L^1(H^{2,n-2})} \lesssim |f_0|_{H^{2,n+1}}. \tag{3.29}$$

3.3 Preparation of vanishing moments for the boundary layer profile

In this paragraph, we explain how we intend to prepare vanishing moments for the boundary layer profile at time T using the control term ξ^v of equation (3.6). In order to perform computations within the Fourier space in the fast variable, we want to get rid of the Neumann boundary condition at $z = 0$. This can be done by lifting the inhomogeneous boundary condition g^0 to turn it into a source term. We choose the simple lifting $-g^0(t,x)e^{-z}$. The homogeneous boundary condition will be preserved via an even extension of the source term. Let us introduce $V(t,x,z) \in \mathbb{R}^d$ defined for $t \geq 0$, $x \in \bar{\mathcal{O}}$ and $z \in \mathbb{R}$ by:

$$V(t,x,z) := v(t,x,|z|) + g^0(t,x)e^{-|z|}. \tag{3.30}$$

We also extend implicitly ξ^v by parity. Hence, V is the solution to the following evolution equation in $\mathbb{R}_+ \times \bar{\mathcal{O}} \times \mathbb{R}_+$:

$$\partial_t V + (u^0 \cdot \nabla)V + BV + u^0_\flat z \partial_z V - \partial_{zz} V = G^0 e^{-|z|} + \tilde{G}^0 |z| e^{-|z|} + \xi^v, \tag{3.31}$$

where we introduce:

$$B_{i,j} := \partial_j u^0_i - \left(n \cdot \partial_j u^0\right) n_i + (u^0 \cdot \nabla n_j) n_i \quad \text{for } 1 \leq i,j \leq d, \tag{3.32}$$

$$G^0 := \partial_t g^0 - g^0 + (u^0 \cdot \nabla) g^0 + B g^0, \tag{3.33}$$

$$\tilde{G}^0 := -u^0_\flat g^0. \tag{3.34}$$

Moreover, the initial condition is:

$$V(0,x,z) = 0 \quad \text{in } \bar{\mathcal{O}} \times \mathbb{R}_+. \tag{3.35}$$

Indeed $u^0(0,\cdot) = 0$ and hence $g^0(0,\cdot) = 0$. Similarly, we have $g^0(t,\cdot) = 0$ for $t \geq T$ since we extended u^0 by zero after T. As remarked for equation (3.6), equation (3.31) also preserves orthogonality with n. Indeed, the particular structure of the zeroth-order operator B is such that $[(u^0 \cdot \nabla)V + BV] \cdot n = 0$ for any function V such that $V \cdot n = 0$. We compute the partial Fourier transform $\hat{V}(t,x,\zeta) := \int_{\mathbb{R}} V(t,x,z) e^{-i\zeta z} \mathrm{d}z$. We obtain:

$$\partial_t \hat{V} + (u^0 \cdot \nabla)\hat{V} + \left(B + \zeta^2 - u^0_\flat\right)\hat{V} - u^0_\flat \zeta \partial_\zeta \hat{V} = \frac{2G^0}{1+\zeta^2} + \frac{2\tilde{G}^0(1-\zeta^2)}{(1+\zeta^2)^2} + \hat{\xi}^v. \tag{3.36}$$

To obtain the decay we are seeking, we will need to consider a finite number of derivatives of $\hat{V}$ at $\zeta = 0$. Thus, we introduce:

$$Q_k(t,x) := \partial^k_\zeta \hat{V}(t,x,\zeta = 0). \tag{3.37}$$

Let us compute the evolution equations satisfied by these quantities. Indeed, differentiating equation (3.36) k times with respect to ζ yields:

$$\begin{aligned}\partial_t \partial^k_\zeta \hat{V} + (u^0 \cdot \nabla)\partial^k_\zeta \hat{V} + \left(B + \zeta^2 - u^0_\flat\right)\partial^k_\zeta \hat{V} + 2k\zeta \partial^{k-1}_\zeta \hat{V} + k(k-1)\partial^{k-2}_\zeta \hat{V} \\ -u^0_\flat(\zeta\partial_\zeta + k)\partial^k_\zeta \hat{V} = \partial^k_\zeta \left[\frac{2G^0}{1+\zeta^2} + \frac{2\tilde{G}^0(1-\zeta^2)}{(1+\zeta^2)^2} + \hat{\xi}^v\right].\end{aligned} \tag{3.38}$$

Now we can evaluate at $\zeta = 0$ and obtain:

$$\partial_t Q_k + (u^0 \cdot \nabla) Q_k + BQ_k - u_\flat^0 (k+1) Q_k = \partial_\zeta^k \left[\frac{2G^0}{1+\zeta^2} + \frac{2\tilde{G}^0(1-\zeta^2)}{(1+\zeta^2)^2} + \hat{\xi}^v \right] \Bigg|_{\zeta=0} - k(k-1)Q_{k-2}. \tag{3.39}$$

In particular:

$$\partial_t Q_0 + (u^0 \cdot \nabla) Q_0 + BQ_0 - u_\flat^0 Q_0 = 2G^0 + 2\tilde{G}^0 + \left[\hat{\xi}^v \right]_{\zeta=0} \tag{3.40}$$

$$\partial_t Q_2 + (u^0 \cdot \nabla) Q_2 + BQ_2 - 3u_\flat^0 Q_2 = -2Q_0 - 4G^0 - 12\tilde{G}^0 + \left[\partial_\zeta^2 \hat{\xi}^v \right]_{\zeta=0}. \tag{3.41}$$

These equations can be brought back to ODEs using the characteristics method, by following the flow Φ^0. Moreover, thanks to their cascade structure, it is easy to build a source term ξ^v which prepares vanishing moments. We have the following result:

Lemma 7. *Let $n \geq 1$ and $u^0 \in \mathcal{C}^\infty([0,T] \times \bar{\mathcal{O}})$ be a fixed reference flow as defined in paragraph 2.3. There exists $\xi^v \in \mathcal{C}^\infty(\mathbb{R}_+ \times \bar{\mathcal{O}} \times \mathbb{R}_+)$ with $\xi^v \cdot n = 0$, whose x support is in $\bar{\mathcal{O}} \setminus \bar{\Omega}$, whose time support is compact in $(0,T)$, such that:*

$$\forall 0 \leq k < n, \forall x \in \bar{\mathcal{O}}, \quad Q_k(T,x) = 0. \tag{3.42}$$

Moreover, for any $s,p \in \mathbb{N}$, for any $0 \leq m \leq n$, the associated boundary layer profile satisfies:

$$|v(t,\cdot,\cdot)|_{H^p_x(H^{s,m}_z)} \lesssim \left| \frac{\ln(2+t)}{2+t} \right|^{\frac{1}{4}+\frac{n}{2}-\frac{m}{2}}, \tag{3.43}$$

where the hidden constant depends on the functional space and on u^0 but not on time.

Proof. **Reduction to independent control of n ODEs.** Once n is fixed, let $n' = \lfloor (n-1)/2 \rfloor$. We start by choosing smooth even functions of z, ϕ_j for $0 \leq j \leq n'$, such that $\partial_\zeta^{2k} \hat{\phi}_j(0) = \delta_{jk}$. We then compute iteratively the moments Q_{2j} (odd moments automatically vanish by parity) using $\xi_j^v := \xi_j^v(t,x)\phi_j(z)$ to control Q_{2j} without interfering with previously constructed controls. When computing the control at order j, all lower order moments $0 \leq i < j$ are known and their contribution as the one of Q_0 in (3.41) can be seen as a known source term.

Reduction to a null controllability problem. Let us explain why (3.40) is controllable. First, by linearity and since the source terms G^0 and $\tilde{G}^0$ are already known, fixed and tangential, it suffices to prove that, starting from zero and without these source terms, we could reach any smooth tangential state. Moreover, since the flow flushing property (2.14) is invariant through time reversal, it is also sufficient to prove that, in the absence of source term, we can drive any smooth tangential initial state to zero. These arguments can also be formalized using a Duhamel formula following the flow for equation (3.40).

Null controllability for a toy system. We are thus left with proving a null controllability property for the following toy system:

$$\begin{cases} \partial_t Q + (u^0 \cdot \nabla)Q + BQ + \lambda Q = \xi & \text{in } (0,T) \times \bar{\mathcal{O}}, \\ Q(0,\cdot) = Q_* & \text{in } \bar{\mathcal{O}}, \end{cases} \tag{3.44}$$

where $B(t,x)$ is defined in (3.32) and $\lambda(t,x)$ is a smooth scalar-valued amplification term. Thanks to the flushing property (2.14) and to the fact that $\bar{\mathcal{O}}$ is bounded, we can choose a finite partition of unity described by functions η_l for $1 \le l \le L$ with $0 \le \eta_l(x) \le 1$ and $\sum_l \eta_l \equiv 1$ on $\bar{\mathcal{O}}$, where the support of η_l is a small ball B_l centered at some $x_l \in \bar{\mathcal{O}}$. Moreover, we extract our partition such that: for any $1 \le l \le L$, there exists a time $t_l \in (\epsilon, T-\epsilon)$ such that $\mathrm{dist}(\Phi^0(0,t,B_l), \bar{\Omega}) \ge \delta/2$ for $|t-t_l| \le \epsilon$ where $\epsilon > 0$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\beta = 1$ on $(-\infty, -\epsilon)$ and $\beta = 0$ on $(\epsilon, +\infty)$. Let Q^l be the solution to (3.44) with initial data $Q^l_* := \eta_l Q_*$ and null source term ξ. We define:

$$Q(t,x) := \sum_{l=1}^{L} \beta(t-t_l) Q^l(t,x), \tag{3.45}$$

$$\xi(t,x) := \sum_{l=1}^{L} \beta'(t-t_l) Q^l(t,x). \tag{3.46}$$

Thanks to the construction, formulas (3.45) and (3.46) define a solution to (3.44) with a smooth control term ξ supported in $\bar{\mathcal{O}} \setminus \bar{\Omega}$, satisfying $\xi \cdot n = 0$ and such that $Q(T,\cdot) = 0$.

Decay estimate. For small times $t \in (0,T)$, when $\xi^v \neq 0$, estimate (3.43) can be seen as a uniform in time estimate and can be obtained similarly as the well-posedness results proved in [56]. For large times, $t \ge T$, the boundary layer profile equation boils down to the parametrized heat equation (3.9) and we use the conclusion of Lemma 6 to deduce (3.43) from (3.16). ☐

3.4 Staying in a small neighborhood of the boundary

The boundary layer correction defined in (3.6) is supported within a small x-neighborhood of $\partial\mathcal{O}$. This is legitimate because Navier boundary layers don't exhibit separation behaviors. Within this x-neighborhood, this correction lifts the tangential boundary layer residue created by the Euler flow but generates a non vanishing divergence at order $\sqrt{\varepsilon}$. In the sequel, we will need to find a lifting profile for this residual divergence (see (4.23)). This will be possible as long as the extension $n(x) := -\nabla\varphi(x)$ of the exterior normal to $\partial\mathcal{O}$ does not vanish on the x-support of v. However, there exists at least one point in $\mathcal{O}$ where $\nabla\varphi = 0$ because φ is a non identically vanishing smooth function with $\varphi = 0$ on $\partial\mathcal{O}$. Hence, we must make sure that, despite the transport term present in equation (3.6), the x-support of v will not encounter points where $\nabla\varphi$ vanishes.

We consider the extended domain $\mathcal{O}$. Its boundary is the set $\{x \in \mathbb{R}^d;\ \varphi(x) = 0\}$. For any $\delta \geq 0$, we define $\mathcal{V}_\delta := \{x \in \mathbb{R}^d;\ 0 \leq \varphi(x) \leq \delta\}$. Hence, $\mathcal{V}_\delta$ is a neighborhood of $\partial\mathcal{O}$ in $\bar{\mathcal{O}}$. For δ large enough, $\mathcal{V}_\delta = \bar{\mathcal{O}}$. As mentioned in paragraph 3.1, φ was chosen such that $|\nabla\varphi| = 1$ and $|\varphi(x)| = \mathrm{dist}(x, \partial\mathcal{O})$ in a neighborhood of $\partial\mathcal{O}$. Let us introduce $\eta > 0$ such that this is true on $\mathcal{V}_\eta$. Hence, within this neighborhood of $\partial\mathcal{O}$, the extension $n(x) = -\nabla\varphi(x)$ of the outwards normal to $\partial\mathcal{O}$ is well defined (and of unit norm). We want to guarantee that v vanishes outside of $\mathcal{V}_\eta$.

Considering the evolution equation (3.36), we see it as an equation defined on the whole of $\mathcal{O}$. Thanks to its structure, we see that the support of $\hat{V}$ is transported by the flow of u^0. Moreover, $\hat{V}$ can be triggered either by fixed polluting right-hand side source term or by the control forcing term. We want to determine the supports of these sources such that $\hat{V}$ vanishes outside of $\mathcal{V}_\eta$.

Thanks to definitions (3.8), (3.33) and (3.34), the unwanted right-hand side source term of (3.36) is supported within the support of χ. We introduce η_χ such that $\mathrm{supp}(\chi) \subset \mathcal{V}_{\eta_\chi}$. For $\delta \geq 0$, we define:

$$S(\delta) := \sup\left\{\varphi\left(\Phi^0(t, t', x)\right);\ \ t, t' \in [0, T],\ \ x \in \mathcal{V}_\delta\right\} \geq \delta. \tag{3.47}$$

With this notation, η_χ includes the zone where pollution might be emitted. Hence $S(\eta_\chi)$ includes the zone that might be reached by some pollution. Iterating once more, $S(S(\eta_\chi))$ includes the zone where we might want to act using ξ^v to prepare vanishing moments. Eventually, $S(S(S(\eta_\chi)))$ corresponds to the maximum localization of non vanishing values for v.

First, since u^0 is smooth, Φ^0 is smooth. Moreover, φ is smooth. Hence, (3.47) defines a smooth function of δ. Second, due to the condition $u^0 \cdot n = 0$, the characteristics cannot leave or enter the domain and thus follow the boundaries. Hence, $S(0) = 0$. Therefore, by continuity of S, there exists $\eta_\chi > 0$ small enough such that $S(S(S(\eta_\chi))) \leq \eta$. We assume χ is fixed from now on.

3.5 Controlling the boundary layer exactly to zero

In view of what has been proved in the previous paragraphs, a natural question is whether we could have controlled the boundary layer exactly to zero (instead of controlling only a finite number of modes and relying on self-dissipation of the higher order ones). This was indeed our initial approach but it turned out to be impossible. The boundary layer equation (3.6) is not exactly null controllable at time T. In fact, it is not even exactly null controllable in any finite time greater than T. Indeed, since $u^0(t, \cdot) = 0$ for $t \geq T$, v is the solution to (3.9) for $t \geq T$. Hence, reaching exactly zero at time T is equivalent to reaching exactly zero at any later time.

Let us present a reduced toy model to explain the difficulty. We consider a rectangular domain and a scalar-valued unknown function v solution to the following system:

$$\begin{cases} \partial_t v + \partial_x v - \partial_{zz} v = 0 & [0,T] \times [0,1] \times [0,1], \\ v(t,x,0) = g(t,x) & [0,T] \times [0,1], \\ v(t,x,1) = 0 & [0,T] \times [0,1], \\ v(t,0,z) = q(t,z) & [0,T] \times [0,1], \\ v(0,x,z) = 0 & [0,1] \times [0,1]. \end{cases} \tag{3.48}$$

System (3.48) involves both a known tangential transport term and a normal diffusive term. At the bottom boundary, $g(t,x)$ is a smooth fixed pollution source term (which models the action of $N(u^0)$, the boundary layer residue created by our reference Euler flow). At the left inlet vertical boundary $x = 0$, we can choose a Dirichlet boundary value control $q(t,z)$. Hence, applying the same strategy as described above, we can control any finite number of vertical modes provided that $T \geq 1$.

However, let us check that it would not be reasonable to try to control the system exactly to zero at any given time $T \geq 1$. Let us consider a vertical slice located at $x_\star \in (0,1)$ of the domain at the final time and follow the flow backwards by defining:

$$v_\star(t,z) := v(t, x_\star + (t-T), z). \tag{3.49}$$

Hence, letting $T_\star := T - x_\star \geq 0$ and using (3.49), $v_\star$ is the solution to a one dimensional heat system:

$$\begin{cases} \partial_t v_\star - \partial_{zz} v_\star = 0 & [T_\star, T] \times [0,1], \\ v_\star(t,0) = g_\star(t) & [T_\star, T], \\ v_\star(t,1) = 0 & [T_\star, T], \\ v_\star(0,z) = q_\star(z) & [0,1], \end{cases} \tag{3.50}$$

where $g_\star(t) := g(t, x_\star + (t-T))$ is smooth but fixed and $q_\star(z) := q(T_\star, z)$ is an initial data that we can choose as if it was a control. Actually, let us change a little the definition of $v_\star$ to lift the inhomogeneous boundary condition at $z = 0$. We set:

$$v_\star(t,z) := v(t, x_\star + (t-T), z) - (1-z) g_\star(t). \tag{3.51}$$

Hence, system (3.50) reduces to:

$$\begin{cases} \partial_t v_\star - \partial_{zz} v_\star = -(1-z) g_\star'(t) & [T_\star, T] \times [0,1], \\ v_\star(t,0) = 0 & [T_\star, T], \\ v_\star(t,1) = 0 & [T_\star, T], \\ v_\star(0,z) = q_\star(z) & [0,1], \end{cases} \tag{3.52}$$

where we change the definition of $q_\star(z) := q(T_\star, z) - (1-z)g_\star(T_\star)$. Introducing the Fourier basis adapted to system (3.52), $e_n(z) := \sin(n\pi z)$, we can solve explicitly for the evolution of $v_\star$:

$$v_\star^n(T) = e^{-n^2\pi^2 T} v_\star^n(0) - \int_{T_\star}^{T} e^{-n^2\pi^2(T-t)} \langle 1-z, e_n \rangle g_\star'(t) \mathrm{d}t. \tag{3.53}$$

If we assume that the pollution term g vanishes at the final time, equation (3.53) and exact null controllability would impose the choice of the initial control data:

$$q_\star^n = \langle 1-z, e_n \rangle \int_{T_\star}^{T} e^{n^2\pi^2 t} g_\star'(t) \mathrm{d}t. \tag{3.54}$$

Even if the pollution term g is very smooth, there is nothing good to be expected from relation (3.54). Hoping for cancellations or vanishing moments is not reasonable because we would have to guarantee this relation for all Fourier modes n and all $x_\star \in [0,1]$. Thus, the boundary data control that we must choose has exponentially growing Fourier modes. Heuristically, it belongs to the dual of a Gevrey space.

The intuition behind relation (3.54) is that the control data emitted from the left inlet boundary undergoes a heat regularization process as they move towards their final position. In the meantime, the fixed polluting boundary data is injected directly at positions within the domain and undergoes less smoothing. This prevents any hope from proving exact null controllability for system (3.48) within reasonable functional spaces and explains why we had to resort to a low-modes control process.

Theorem 1 is an exact null controllability result. To conclude our proof, we use a local argument stated as Lemma 4 in paragraph 2.6 which uses diffusion in all directions. Boundary layer systems like (3.48) exhibit no diffusion in the tangential direction and are thus harder to handle. The conclusion of our proof uses the initial formulation of the Navier-Stokes equation with a fixed $\mathcal{O}(1)$ viscosity.

4 Estimation of the remainder and technical profiles

In the previous sections, we presented the construction of the Euler reference flushing trajectory u^0, the transported flow involving the initial data u^1 and the leading order boundary layer correction v. In this section, we follow on with the expansion and introduce technical profiles, which do not have a clear physical interpretation. The purpose of the technical decomposition we propose is to help us prove that the remainder we obtain is indeed small. We will use the following expansion:

$$u^\varepsilon = u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon \nabla\theta^\varepsilon + \varepsilon\{w\} + \varepsilon r^\varepsilon, \tag{4.1}$$

$$p^\varepsilon = p^0 + \varepsilon\{q\} + \varepsilon p^1 + \varepsilon\mu^\varepsilon + \varepsilon\pi^\varepsilon, \tag{4.2}$$

where v, w and q are profiles depending on t, x and z. For such a function $f(t,x,z)$, we use the notation $\{f\}$ to denote its evaluation at $z = \varphi(x)/\sqrt{\varepsilon}$. In the sequel, operators ∇, Δ, D and div only act on x variables. We will use the following straightforward commutation formulas:

$$\operatorname{div}\{f\} = \{\operatorname{div} f\} - n\cdot\{\partial_z f\}/\sqrt{\varepsilon} \tag{4.3}$$

$$\nabla\{f\} = \{\nabla f\} - n\{\partial_z f\}/\sqrt{\varepsilon}, \tag{4.4}$$

$$N(\{f\}) = \{N(f)\} - \frac{1}{2}\{[\partial_z f]_{\text{tan}}\}/\sqrt{\varepsilon}, \tag{4.5}$$

$$\varepsilon\Delta\{f\} = \varepsilon\{\Delta f\} + \sqrt{\varepsilon}\Delta\varphi\{\partial_z f\} - 2\sqrt{\varepsilon}\{(n\cdot\nabla)\partial_z f\} + |n|^2\{\partial_{zz} f\}. \tag{4.6}$$

Within the x-support of boundary layer terms, $|n|^2 = 1$.

4.1 Formal expansions of constraints

In this paragraph, we are interested in the formulation of the boundary conditions and the incompressibility condition for the full expansion. We plug expansion (4.1) into these conditions and identify the successive orders of power of $\sqrt{\varepsilon}$.

4.1.1 Impermeability boundary condition

The impermeability boundary condition $u^\varepsilon\cdot n = 0$ on $\partial\mathcal{O}$ yields:

$$u^0\cdot n = 0, \tag{4.7}$$

$$v(\cdot,\cdot,0)\cdot n = 0, \tag{4.8}$$

$$u^1\cdot n + \partial_n\theta^\varepsilon + w(\cdot,\cdot,0)\cdot n + r^\varepsilon\cdot n = 0. \tag{4.9}$$

By construction of the Euler trajectory u^0, equation (4.7) is satisfied. Since the boundary profile v is tangential, equation (4.8) is also satisfied. By construction, we also already have $u^1\cdot n = 0$. In order to be able to carry out integrations by part for the estimates of the remainder, we also would like to impose $r^\varepsilon\cdot n = 0$. Thus, we read (4.9) as a definition of $\partial_n\theta^\varepsilon$ once w is known:

$$\forall t\geq 0, \forall x\in\partial\mathcal{O},\quad \partial_n\theta^\varepsilon(t,x) = -w(t,x,0)\cdot n. \tag{4.10}$$

4.1.2 Incompressibility condition

The (almost) incompressibility condition $\operatorname{div} u^\varepsilon = \sigma^0$ in $\mathcal{O}$ (σ^0 is smooth forcing terms supported outside of the physical domain Ω) yields:

$$\operatorname{div} u^0 - n\cdot\{\partial_z v\} = \sigma^0, \tag{4.11}$$

$$\{\operatorname{div} v\} - n\cdot\{\partial_z w\} = 0, \tag{4.12}$$

$$\operatorname{div} u^1 + \operatorname{div}\nabla\theta^\varepsilon + \{\operatorname{div} w\} + \operatorname{div} r^\varepsilon = 0. \tag{4.13}$$

In (4.12) and (4.13), we used formula (4.3) to isolate the contributions to the divergence coming from the slow derivatives with the one coming from the fast derivative ∂_z. By construction $\operatorname{div} u^0 = \sigma^0$, $\operatorname{div} u^1 = 0$, $n \cdot \partial_z v = 0$ and we would like to work with $\operatorname{div} r^\varepsilon = 0$. Hence, we read (4.12) and (4.13) as:

$$n \cdot \{\partial_z w\} = \{\operatorname{div} v\}, \tag{4.14}$$

$$-\Delta\theta^\varepsilon = \{\operatorname{div} w\}. \tag{4.15}$$

4.1.3 Navier boundary condition

Last, we turn to the slip-with-friction boundary condition. Proceeding as above yields by identification:

$$N(u^0) - \frac{1}{2}\left[\partial_z v\right]_{\tan}\big|_{z=0} = 0, \tag{4.16}$$

$$N(v)\big|_{z=0} - \frac{1}{2}\left[\partial_z w\right]_{\tan}\big|_{z=0} = 0, \tag{4.17}$$

$$N(u^1) + N(\nabla\theta^\varepsilon) + N(w)\big|_{z=0} + N(r^\varepsilon) = 0. \tag{4.18}$$

By construction, (4.16) is satisfied. We will choose a basic lifting to guarantee (4.17). Last, we read (4.18) as an inhomogeneous boundary condition for the remainder:

$$N(r^\varepsilon) = g^\varepsilon := -N(u^1) - N(\nabla\theta^\varepsilon) - N(w)\big|_{z=0}. \tag{4.19}$$

4.2 Definitions of technical profiles

At this stage, the three main terms u^0, v and u^1 are defined. In this paragraph, we explain step by step how we build the following technical profiles of the expansion. For any $t \geq 0$, the profiles are built sequentially from the values of $v(t,\cdot,\cdot)$. Hence, they will inherit from the boundary layer profile its smoothness with respect to the slow variables x and its time decay estimates obtained from Lemma 6.

4.2.1 Boundary layer pressure

Equation (3.6) only involves the tangential part of the symmetrical convective product between u^0 and v. Hence, to compensate its normal part, we introduce as in [56] the pressure q which is defined as the unique solution vanishing as $z \to +\infty$ to:

$$\left[(u^0 \cdot \nabla)v + (v \cdot \nabla)u^0\right] \cdot n = \partial_z q. \tag{4.20}$$

Hence, we can now write:

$$\partial_t v + (u^0 \cdot \nabla)v + (v \cdot \nabla)u^0 + u^0_\flat z \partial_z v - \partial_{zz} v - n\partial_z q = 0. \tag{4.21}$$

This pressure profile vanishes as soon as u^0 vanishes, hence in particular for $t \geq T$. For any $p, s, n \in \mathbb{N}$, the following estimate is straightforward:

$$|q(t,\cdot,\cdot)|_{H^1_x(H^{0,0}_z)} \lesssim |v(t,\cdot,\cdot)|_{H^2_x(H^{0,2}_z)}. \tag{4.22}$$

4.2.2 Second boundary corrector

The first boundary condition v generates a non vanishing slow divergence and a non vanishing tangential boundary flux. The role of the profile w is to lift two unwanted terms that would be too hard to handle directly in the equation of the remainder. We define w as:

$$w(t,x,z) := -2e^{-z}N(v)(t,x,0) - n(x)\int_z^{+\infty} \operatorname{div} v(t,x,z')\mathrm{d}z' \tag{4.23}$$

Definition (4.23) allows to guarantee condition (4.17). Moreover, under the assumption $|n(x)|^2 = 1$ for any x in the x-support of the boundary layer, this definition also fulfills condition (4.12). In equation (4.23) it is essential that $n(x)$ does not vanish on the x-support of v. This is why we dedicated paragraph 3.4 to proving we could maintain a small enough support for the boundary layer. For any $p, s, n \in \mathbb{N}$, the following estimates are straightforward:

$$|[w(t,\cdot,\cdot)]_{\mathrm{tan}}|_{H^p_x(H^{s,n}_z)} \lesssim |v(t,\cdot,\cdot)|_{H^{p+1}_x(H^{1,1}_z)}, \tag{4.24}$$

$$|w(t,\cdot,\cdot)\cdot n|_{H^p_x(H^{0,n}_z)} \lesssim |v(t,\cdot,\cdot)|_{H^{p+1}_x(H^{0,n+2}_z)}, \tag{4.25}$$

$$|w(t,\cdot,\cdot)\cdot n|_{H^p_x(H^{s+1,n}_z)} \lesssim |v(t,\cdot,\cdot)|_{H^{p+1}_x(H^{s,n}_z)}. \tag{4.26}$$

Estimates (4.24), (4.25) and (4.26) can be grossly summarized sub-optimally by:

$$|w(t,\cdot,\cdot)|_{H^p_x(H^{s,n}_z)} \lesssim |v(t,\cdot,\cdot)|_{H^{p+1}_x(H^{s+1,n+2}_z)}. \tag{4.27}$$

4.2.3 Inner domain corrector

Once w is defined by (4.23), the collateral damage is that this generates a non vanishing boundary flux $w \cdot n$ on $\partial\mathcal{O}$ and a slow divergence. For a fixed time $t \geq 0$, we define θ^ε as the solution to:

$$\begin{cases} \Delta\theta^\varepsilon = -\{\operatorname{div} w\} & \text{in } \mathcal{O}, \\ \partial_n\theta^\varepsilon = -w(t,\cdot,0)\cdot n & \text{on } \partial\mathcal{O}. \end{cases} \tag{4.28}$$

System (4.28) is well-posed as soon as the usual compatibility condition between the source terms is satisfied. Using Stokes formula, equations (4.3) and (4.12), we compute:

$$\begin{aligned}\int_{\partial\mathcal{O}} w(t,\cdot,0)\cdot n &= \int_{\partial\mathcal{O}} \{w\}\cdot n = \int_{\mathcal{O}} \operatorname{div}\{w\} \\ &= \int_{\mathcal{O}} \{\operatorname{div} w\} - \varepsilon^{-\frac{1}{2}} n\cdot\{\partial_z w\} \\ &= \int_{\mathcal{O}} \{\operatorname{div} w\} - \varepsilon^{-\frac{1}{2}}\{\operatorname{div} v\} \\ &= \int_{\mathcal{O}} \{\operatorname{div} w\} - \varepsilon^{-\frac{1}{2}} \operatorname{div}\{v\} + \varepsilon^{-1} n\cdot\{\partial_z v\} \\ &= \int_{\mathcal{O}} \{\operatorname{div} w\} - \varepsilon^{-\frac{1}{2}} \int_{\partial\mathcal{O}} \{v\}\cdot n = \int_{\mathcal{O}} \{\operatorname{div} w\},\end{aligned} \tag{4.29}$$

where we used twice the fact that v is tangential. Thus, the compatibility condition is satisfied and system (4.28) has a unique solution. The associated potential flow $\nabla\theta^\varepsilon$ solves:

$$\begin{cases} \partial_t \nabla\theta^\varepsilon + \left(u^0\cdot\nabla\right)\nabla\theta^\varepsilon + \left(\nabla\theta^\varepsilon\cdot\nabla\right)u^0 = -\nabla\mu^\varepsilon, & \text{in } \mathcal{O} \quad \text{for } t\geq 0, \\ \operatorname{div}\nabla\theta^\varepsilon = -\{\operatorname{div} w\} & \text{in } \mathcal{O} \quad \text{for } t\geq 0, \\ \nabla\theta^\varepsilon\cdot n = -w_{|z=0}\cdot n & \text{on } \partial\mathcal{O} \quad \text{for } t\geq 0, \end{cases} \tag{4.30}$$

where the pressure term $\mu^\varepsilon := -\partial_t\theta^\varepsilon - u^0\cdot\nabla\theta^\varepsilon$ absorbs all other terms in the evolution equation (see (2.15)). Estimating roughly θ^ε using standard regularity estimates for the Laplace equation yields:

$$\begin{aligned}|\theta^\varepsilon(t,\cdot)|_{H^4_x} &\lesssim |\{\operatorname{div} w\}(t,\cdot)|_{H^2_x} + |w(t,\cdot,0)\cdot n|_{H^3_x} \\ &\lesssim \varepsilon^{\frac{1}{4}} |w(t)|_{H^4_x(H^{0,0}_z)} + \varepsilon^{-\frac{1}{4}} |w(t)|_{H^3_x(H^{1,0}_z)} \\ &\qquad + \varepsilon^{-\frac{3}{4}} |w(t)|_{H^2_x(H^{2,0}_z)} + |v(t)|_{H^3_x(H^{0,1}_z)} \\ &\lesssim \varepsilon^{-\frac{3}{4}} |w(t)|_{H^4_x(H^{2,0}_z)} + |v(t)|_{H^3_x(H^{0,1}_z)},\end{aligned} \tag{4.31}$$

where we used [55, Lemma 3, page 150] to benefit from the fast variable scaling. Similarly,

$$|\theta^\varepsilon(t,\cdot)|_{H^3_x} \lesssim \varepsilon^{-\frac{1}{4}} |w(t)|_{H^3_x(H^{1,0}_z)} + |v(t)|_{H^2_x(H^{0,1}_z)}, \tag{4.32}$$

$$|\theta^\varepsilon(t,\cdot)|_{H^2_x} \lesssim \varepsilon^{\frac{1}{4}} |w(t)|_{H^2_x(H^{0,0}_z)} + |v(t)|_{H^1_x(H^{0,1}_z)}. \tag{4.33}$$

4.3 Equation for the remainder

In the extended domain $\mathcal{O}$, the remainder is a solution to:

$$\begin{cases} \partial_t r^\varepsilon - \varepsilon\Delta r^\varepsilon + (u^\varepsilon\cdot\nabla) r^\varepsilon + \nabla\pi^\varepsilon = \{f^\varepsilon\} - \{A^\varepsilon r^\varepsilon\} & \text{in } \mathcal{O} \quad \text{for } t\geq 0, \\ \operatorname{div} r^\varepsilon = 0 & \text{in } \mathcal{O} \quad \text{for } t\geq 0, \\ N(r^\varepsilon) = g^\varepsilon & \text{on } \partial\mathcal{O} \quad \text{for } t\geq 0, \\ r^\varepsilon\cdot n = 0 & \text{on } \partial\mathcal{O} \quad \text{for } t\geq 0, \\ r^\varepsilon(0,\cdot) = 0 & \text{in } \mathcal{O} \quad \text{at } t = 0. \end{cases} \tag{4.34}$$

Recall that g^ε is defined in (4.19). We introduce the amplification operator:

$$A^\varepsilon r^\varepsilon := (r^\varepsilon \cdot \nabla)\left(u^0 + \sqrt{\varepsilon} v + \varepsilon u^1 + \varepsilon \nabla \theta^\varepsilon + \varepsilon w\right) - (r^\varepsilon \cdot n)\left(\partial_z v + \sqrt{\varepsilon}\partial_z w\right) \quad (4.35)$$

and the forcing term:

$$f^\varepsilon := f^\varepsilon_\Delta + f^\varepsilon_\nabla + \tilde{f}^\varepsilon, \quad (4.36)$$

where:

$$\begin{aligned} f^\varepsilon_\Delta &:= (\Delta\varphi\partial_z v - 2(n\cdot\nabla)\partial_z v + \partial_{zz} w) + \sqrt{\varepsilon}(\Delta v + \Delta\varphi\partial_z w - 2(n\cdot\nabla)\partial_z w) \\ &\quad + \varepsilon(\Delta w + \Delta u^1 + \Delta\nabla\theta^\varepsilon) \quad (4.37) \\ f^\varepsilon_\nabla &:= -\left((v + \sqrt{\varepsilon}(w + u^1 + \nabla\theta^\varepsilon))\cdot\nabla\right)(v + \sqrt{\varepsilon}(w + u^1 + \nabla\theta^\varepsilon)) - (u^0\cdot\nabla)w \\ &\quad - (w\cdot\nabla)u^0 + (w + u^1 + \nabla\theta^\varepsilon)\cdot n\partial_z\left(v + \sqrt{\varepsilon}w\right) \quad (4.38) \\ \tilde{f}^\varepsilon &:= -\nabla q - \partial_t w. \quad (4.39) \end{aligned}$$

In equations (4.35) to (4.39), many functions depend on t, x and z. The differential operators ∇ and Δ only act on the slow variables x and the evaluation at $z = \varphi(x)/\sqrt{\varepsilon}$ is done *a posteriori* in (4.34). The derivatives in the fast variable direction are explicitly marked with the ∂_z operator. Moreover, most terms are independent of ε, except where explicitly stated in θ^ε and r^ε.

Expansion (4.1) contains 4 slowly varying profiles and 2 boundary layer profiles. Thus, computing $\varepsilon\Delta u^\varepsilon$ using formula (4.6) produces $4 + 2\times 4 = 12$ terms. Terms Δu^0 and $\{\partial_{zz} v\}$ have already been taken into account respectively in (2.16) and (3.6). Term Δr^ε is written in (4.34). The remaining 9 terms are gathered in the the forcing term (4.37).

Computing the non-linear term $(u^\varepsilon\cdot\nabla)u^\varepsilon$ using formula (4.4) produces $6\times 4 + 6\times 2\times 2 = 48$ terms. First, 8 have already been taken into account in (2.12), (2.16), (3.6) and (4.30). Moreover, 6 are written in (4.34) as $(u^\varepsilon\cdot\nabla)r^\varepsilon$, 7 more as the amplification term (4.35) and 25 in (4.38). The two missing terms $\{(v\cdot n)\partial_z v\}$ and $\{(v\cdot n)\partial_z w\}$ vanish because $v\cdot n = 0$.

4.4 Size of the remainder

We need to prove that equation (4.34) satisfies an energy estimate on the long time interval $[0, T/\varepsilon]$. Moreover, we need to estimate the size of the remainder at the final time and check that it is small. The key point is that the size of the source term $\{f^\varepsilon\}$ is small in $L^2(\mathcal{O})$. Indeed, for terms appearing at order $\mathcal{O}(1)$, the fast scaling makes us win a $\varepsilon^{\frac{1}{4}}$ factor (see for example [56, Lemma 3, page 150]). We proceed as we have done in the case of the shape operator in paragraph 2.5.

The only difference is the estimation of the boundary term (2.25). We have to take into account the inhomogeneous boundary condition g^ε and the fact that, in the general case, the boundary condition matrix M is different from the shape operator M_{w}. Using (1.10) allows us to write, on $\partial\mathcal{O}$:

$$(r^\varepsilon \times (\nabla \times r^\varepsilon)) \cdot n = ((\nabla \times r^\varepsilon) \times n) \cdot r^\varepsilon = 2\left(N(r^\varepsilon) + [(M - M_{\mathrm{w}})r^\varepsilon]_{\tan}\right) \cdot r^\varepsilon. \quad (4.40)$$

Introducing smooth extensions of M and M_{w} to the whole domain $\mathcal{O}$ also allows to extend the Navier operator N defined in (1.4), since the extension of the normal n extends the definition of the tangential part (1.2). Using (4.40), we transform the boundary term into an inner term:

$$\begin{aligned} \left| \int_{\partial\mathcal{O}} (r^\varepsilon \times (\nabla \times r^\varepsilon)) \cdot n \right| &= 2 \left| \int_{\partial\mathcal{O}} g^\varepsilon \cdot r^\varepsilon + ((M - M_{\mathrm{w}})r^\varepsilon) \cdot r^\varepsilon \right| \\ &= 2 \left| \int_{\mathcal{O}} \operatorname{div} \left[(g^\varepsilon \cdot r^\varepsilon) n + (((M - M_{\mathrm{w}})r^\varepsilon) \cdot r^\varepsilon) n \right] \right| \\ &\le \lambda \left| \nabla r^\varepsilon \right|_2^2 + C_\lambda \left(|r^\varepsilon|_2^2 + + |g^\varepsilon|_2^2 + |\nabla g^\varepsilon|_2^2 \right), \end{aligned} \quad (4.41)$$

for any $\lambda > 0$ to be chosen and where C_λ is a positive constant depending on λ. We intend to absorb the $|\nabla r^\varepsilon|_2^2$ term of (4.41) using the dissipative term. However, the dissipative term only provides the norm of the symmetric part of the gradient. We recover the full gradient using the Korn inequality. Indeed, since $\operatorname{div} r^\varepsilon = 0$ in $\mathcal{O}$ and $r^\varepsilon \cdot n = 0$ on $\partial\mathcal{O}$, the following estimate holds (see [23, Corollary 1, Chapter IX, page 212]):

$$|r^\varepsilon|^2_{H^1(\mathcal{O})} \le C_K \, |r^\varepsilon|^2_{L^2(\mathcal{O})} + C_K \, |\nabla \times r^\varepsilon|^2_{L^2(\mathcal{O})}. \quad (4.42)$$

We choose $\lambda = 1/(2C_K)$ in (4.41). Combined with (4.42) and a Grönwall inequality as in paragraph 2.5 yields an energy estimate for $t \in [0, T/\varepsilon]$:

$$|r^\varepsilon|^2_{L^\infty(L^2)} + \varepsilon \, |r^\varepsilon|^2_{L^2(H^1)} = \mathcal{O}(\varepsilon^{\frac{1}{4}}), \quad (4.43)$$

as long as we can check that the following estimates hold:

$$\|A^\varepsilon\|_{L^1(L^\infty)} = \mathcal{O}(1), \quad (4.44)$$

$$\varepsilon \, \|g^\varepsilon\|^2_{L^2(H^1)} = \mathcal{O}(\varepsilon^{\frac{1}{4}}), \quad (4.45)$$

$$\|f^\varepsilon\|_{L^1(L^2)} = \mathcal{O}(\varepsilon^{\frac{1}{4}}). \quad (4.46)$$

In particular, the remainder at time T/ε is small and we can conclude the proof of Theorem 1 with the same arguments as in paragraph 2.6. Therefore, it only remains to be checked that estimates (4.44), (4.46) and (4.45) hold on the time interval $[0, T/\varepsilon]$. In fact, they even hold on the whole time interval $[0, +\infty)$.

Estimates for A^ε. The two terms involving u^0 and u^1 vanish for $t \ge T$. Thus, they satisfy estimate (4.44). For $t \geq 0$, we estimate the other terms in A^ε in the following way:

$$\sqrt{\varepsilon}\,|\nabla v(t)|_{L^\infty} \lesssim \sqrt{\varepsilon}\,|v(t)|_{H^3_x(H^{1,0}_z)}\,, \tag{4.47}$$

$$\varepsilon\,|\nabla w(t)|_{L^\infty} \lesssim \varepsilon\,|w(t)|_{H^3_x(H^{1,0}_z)}\,, \tag{4.48}$$

$$|\partial_z v(t)|_{L^\infty} \lesssim |v(t)|_{H^2_x(H^{2,0}_z)}\,, \tag{4.49}$$

$$\sqrt{\varepsilon}\,|\partial_z w(t)|_{L^\infty} \lesssim \sqrt{\varepsilon}\,|w(t)|_{H^2_x(H^{2,0}_z)}\,, \tag{4.50}$$

$$\varepsilon\,\left|\nabla^2\theta^\varepsilon(t)\right|_{L^\infty} \lesssim \varepsilon\,|\theta^\varepsilon(t)|_{H^4}\,. \tag{4.51}$$

Combining these estimates with (4.31) and (4.27) yields:

$$\|A^\varepsilon\|_{L^1(L^\infty)} \lesssim \|u^0\|_{L^1_{[0,T]}(H^3)} + \varepsilon\|u^1\|_{L^1_{[0,T]}(H^3)} + \|v\|_{L^1(H^5_x(H^{3,2}_z))}\,. \tag{4.52}$$

Applying Lemma 7 with $p = 5$, $n = 4$ and $m = 2$ concludes the proof of (4.44).

Estimates for g^ε. For $t \geq 0$, using the definition of g^ε in (4.19), we estimate:

$$\varepsilon\left|N(u^1)(t)\right|^2_{H^1} \lesssim \varepsilon\left|u^1(t)\right|^2_{H^2}\,, \tag{4.53}$$

$$\varepsilon\,|N(\nabla\theta^\varepsilon)(t)|^2_{H^1} \lesssim \varepsilon\,|\theta^\varepsilon(t)|^2_{H^3}\,, \tag{4.54}$$

$$\varepsilon\left|N(w)_{|z=0}(t)\right|^2_{H^1} \lesssim \varepsilon\,|w(t)|^2_{H^2_x(H^{1,1}_z)}\,. \tag{4.55}$$

Combining these estimates with (4.32) and (4.27) yields:

$$\varepsilon\,\|g^\varepsilon\|^2_{L^2(H^1)} \lesssim \varepsilon\|u^1\|^2_{L^2_{[0,T]}(H^2)} + \varepsilon^{\frac{3}{4}}\,\|v\|^2_{L^2(H^4_x(H^{2,3}_z))}\,. \tag{4.56}$$

Applying Lemma 7 with $p = 4$, $n = 4$ and $m = 3$ concludes the proof of (4.45).

Estimates for f^ε. For $t \geq 0$, we estimate the 36 terms involved in the definition of f^ε in (4.36). The conclusion is that (4.46) holds as soon as v is bounded in $L^1(H^4_x(H^{3,4}_z))$. This can be obtained from Lemma 7 with $p = 4$, $n = 6$ and $m = 4$. Let us give a few examples of some of the terms requiring the most regularity. The key point is that all terms of (4.36) appearing at order $\mathcal{O}(1)$ involve a boundary layer term and thus benefit from the fast variable scaling gain of $\varepsilon^{\frac{1}{4}}$ in L^2 of [56, Lemma 3, page 150]. For example, with (4.27):

$$|\{\partial_{zz}w\}(t)|_{L^2} \lesssim \varepsilon^{\frac{1}{4}}\,|w(t)|_{H^1_x(H^{2,0}_z)} \lesssim \varepsilon^{\frac{1}{4}}\,|v(t)|_{H^2_x(H^{3,2}_z)}\,. \tag{4.57}$$

Using (4.32) and (4.27), we obtain:

$$\varepsilon\,|\Delta\nabla\theta^\varepsilon(t)|_{L^2} \lesssim \varepsilon^{\frac{3}{4}}\,|w(t)|_{H^3_x(H^{1,0}_z)} + |v(t)|_{H^2_x(H^{0,1}_z)} \lesssim \varepsilon^{\frac{3}{4}}\,|v(t)|_{H^4_x(H^{2,2}_z)}\,. \tag{4.58}$$

The time derivative $\{\partial_t w\}$ can be estimated easily because the time derivative commutes with the definition of w through formula (4.23). Moreover, $\partial_t v$ can be recovered from its evolution equation (3.6):

$$\begin{aligned} |\{\partial_t w\}(t)|_{L^2} &\lesssim \varepsilon^{\frac{1}{4}}\,|\partial_t w(t)|_{H^1_x(H^{0,0}_z)} \lesssim \varepsilon^{\frac{1}{4}}\,|\partial_t v(t)|_{H^2_x(H^{1,2}_z)} \\ &\lesssim \varepsilon^{\frac{1}{4}}\left(|v(t)|_{H^3_x(H^{2,4}_z)} + |\xi^v(t)|_{H^3_x(H^{2,4}_z)}\right). \end{aligned} \tag{4.59}$$

The forcing term ξ^v is smooth and supported in $[0,T]$. As a last example, consider the term $(\nabla\theta^\varepsilon \cdot n)\partial_z v$. We use the injection $H^1 \hookrightarrow L^4$ which is valid in 2D and in 3D and estimate (4.33):

$$\begin{aligned} |(\nabla\theta^\varepsilon \cdot n)\{\partial_z v\}(t)|_{L^2} &\lesssim |\nabla\theta^\varepsilon(t)|_{H^1} |\{\partial_z v\}(t)|_{H^1} \\ &\lesssim \varepsilon^{\frac{1}{4}} |v(t)|_{H^3_x(H^{1,2}_z)} |v(t)|_{H^2_x(H^{1,0}_z)} . \end{aligned} \tag{4.60}$$

As (3.43) both yields L^∞ and L^1 estimates in time, this estimation is enough to conclude. All remaining nonlinear convective terms can be handled in the same way or even more easily. The pressure term is estimated using (4.22).

These estimates conclude the proof of small-time global approximate null controllability in the general case. Indeed, both the boundary layer profile (thanks to Lemma 7) and the remainder are small at the final time. Thus, as announced in Remark 2, we have not only proved that there exists a weak trajectory going approximately to zero, but that any weak trajectory corresponding to our source terms ξ^ε and σ^ε goes approximately to zero. We combine this result with the local and regularization arguments explained in paragraph 2.6 to conclude the proof of Theorem 1 in the general case.

5 Global controllability to the trajectories

In this section, we explain how our method can be adapted to prove small-time global exact controllability to other states than the null equilibrium state. Since the Navier-Stokes equation exhibits smoothing properties, all possible target states must be smooth enough. Generally speaking, the exact description of the set of reachable states for a given controlled system is a difficult question. Already for the heat equation on a line segment, the complete description of this set is still open (see [22] and [64] for recent developments on this topic). The usual circumvention is to study the notion of global exact controllability *to the trajectories*. That is, we are interested in whether all known final states of the system are reachable from any other arbitrary initial state using a control:

Theorem 2. *Let $T > 0$. Assume that the intersection of Γ with each connected component of $\partial\Omega$ is smooth. Let $\bar{u} \in \mathcal{C}^0_w([0,T]; L^2_\gamma(\Omega)) \cap L^2((0,T); H^1(\Omega))$ be a fixed weak trajectory of* (1.1) *with smooth ξ. Let $u_* \in L^2_\gamma(\Omega)$ be another initial data unrelated with $\bar{u}$. Then there exists $u \in \mathcal{C}^0_w([0,T]; L^2_\gamma(\Omega)) \cap L^2((0,T); H^1(\Omega))$ a weak trajectory of* (1.1) *with $u(0,\cdot) = u_*$ satisfying $u(T,\cdot) = \bar{u}(T,\cdot)$.*

The strategy is very similar to the one described in the previous sections to prove the global null controllability. We start with the following lemma, asserting small-time global approximate controllability to smooth trajectories in the extended domain.

Lemma 8. *Let $T > 0$. Let $(\bar{u}, \bar{\xi}, \bar{\sigma}) \in \mathcal{C}^\infty([0,T] \times \bar{\mathcal{O}})$ be a fixed smooth trajectory of* (2.2)*. Let $u_* \in L^2_{\mathrm{div}}(\mathcal{O})$ be another initial data unrelated with $\bar{u}$. For any $\delta > 0$, there exists $u \in \mathcal{C}^0_w([0,T]; L^2_{\mathrm{div}}(\mathcal{O})) \cap L^2((0,T); H^1(\mathcal{O}))$ a weak Leray solution of* (2.2) *with $u(0,\cdot) = u_*$ satisfying $|u(T) - \bar{u}(T)|_{L^2(\mathcal{O})} \leq \delta$.*

Proof. We build a sequence $u^{(\varepsilon)}$ to achieve global approximate controllability to the trajectories. Still using the same scaling, we define it as:

$$u^{(\varepsilon)}(t,x) := \frac{1}{\varepsilon} u^\varepsilon \left(\frac{t}{\varepsilon}, x \right), \tag{5.1}$$

where u^ε solves the vanishing viscosity Navier-Stokes equation (3.1) with initial data εu_* on the time interval $[0, T/\varepsilon]$. As previously, this time interval will be used in two different stages. First, a short stage of fixed length T to achieve controllability of the Euler system by means of a return-method strategy. Then, a long stage $[T, T/\varepsilon]$, during which the boundary layer dissipates thanks to the careful choice of the boundary controls during the first stage. During the first stage, we use the expansion:

$$u^\varepsilon = u^0 + \sqrt{\varepsilon}\,\{v\} + \varepsilon u^{1,\varepsilon} + \ldots, \tag{5.2}$$

where $u^{1,\varepsilon}$ is built such that $u^{1,\varepsilon}(0,\cdot) = u_*$ and $u^{1,\varepsilon}(T,\cdot) = \bar{u}(\varepsilon T, \cdot)$. This is the main difference with respect to the null controllability strategy. Here, we need to aim for a non zero state at the first order. Of course, this is also possible because the state $u^{1,\varepsilon}$ is mostly transported by u^0 (which is such that the linearized Euler system is controllable). The profile $u^{1,\varepsilon}$ now depends on ε. However, since the reference trajectory belongs to $\mathcal{C}^\infty$, all required estimates can be made independent on ε. During this first stage, $u^{1,\varepsilon}$ solves the usual first-order system (2.16). For large times $t \geq T$, we change our expansion into:

$$u^\varepsilon = \sqrt{\varepsilon}\,\{v\} + \varepsilon \bar{u}(\varepsilon t, \cdot) + \ldots, \tag{5.3}$$

where the boundary layer profile solves the homogeneous heat system (3.9) and $\bar{u}$ is the reference trajectory solving the true Navier-Stokes equation. As we have done in the case of null controllability, we can derive the equations satisfied by the remainders in the previous equations and carry on both well-posedness and smallness estimates using the same arguments. Changing expansion (5.2) into (5.3) allows to get rid of some unwanted terms in the equation satisfied by the remainder. Indeed, terms such as $\varepsilon \Delta u^1$ or $\varepsilon(u^1 \nabla) u^1$ don't appear anymore because they are already taken into account by $\bar{u}$. One important remark is that it is necessary to aim for $\bar{u}(\varepsilon T) \approx \bar{u}(0)$ at the linear order and not towards the desired end state $\bar{u}(T)$. Indeed, the inviscid stage is very short and the state will continue evolving while the boundary layer dissipates. This explains our choice of pivot state. We obtain:

$$\left| u^{(\varepsilon)}(T) - \bar{u}(T) \right|_{L^2(\mathcal{O})} = \mathcal{O}\left(\varepsilon^{\frac{1}{8}} \right), \tag{5.4}$$

which concludes the proof of approximate controllability. ☐

We will also need the following regularization lemma:

Lemma 9. *Let $T > 0$. Let $\bar{u} \in \mathcal{C}^\infty([0,T] \times \bar{\mathcal{O}})$ be a fixed smooth function with $\bar{u} \cdot n = 0$ on $\partial\mathcal{O}$. There exists a smooth function C, with $C(0) = 0$, such that, for any $r_* \in L^2_{\mathrm{div}}(\mathcal{O})$ and any $r \in \mathcal{C}^0_w([0,T]; L^2_{\mathrm{div}}(\mathcal{O})) \cap L^2((0,T); H^1(\mathcal{O}))$, weak Leray solution to:*

$$\begin{cases} \partial_t r - \Delta r + (\bar{u}\cdot\nabla) r + (r\cdot\nabla)\bar{u} + (r\cdot\nabla) r + \nabla\pi = 0 & \text{in } [0,T] \times \mathcal{O}, \\ \mathrm{div}\, r = 0 & \text{in } [0,T] \times \mathcal{O}, \\ r \cdot n = 0 & \text{on } [0,T] \times \partial\mathcal{O}, \\ N(r) = 0 & \text{on } [0,T] \times \partial\mathcal{O}, \\ r(0,\cdot) = r_* & \text{in } \mathcal{O}, \end{cases} \tag{5.5}$$

the following property holds true:

$$\exists t_r \in [0,T], \quad |r(t_r,\cdot)|_{H^3(\mathcal{O})} \leq C\left(|r_*|_{L^2(\mathcal{O})}\right). \tag{5.6}$$

Proof. This regularization lemma is easy in our context because we assumed a lot of smoothness on the reference trajectory $\bar{u}$ and we are not demanding anything on the time t_r at which the solution becomes smoother. We only sketch out the steps that we go through. We repeatedly use the Korn inequality from [68, Theorem 10.2, page 299] to derive estimates from the symmetrical part of gradients. Let $\mathbb{P}$ denote the usual orthogonal Leray projector on divergence-free vectors fields tangent to the boundaries. We will use the fact $|\Delta r|_{L^2} \lesssim |\mathbb{P}\Delta r|_{L^2}$ which follows from maximal regularity result for the Stokes problem with $\mathrm{div}\, r = 0$ in $\mathcal{O}$, $r \cdot n = 0$ and $N(r) = 0$ on $\partial\mathcal{O}$. Our scheme is inspired from [32].

Weak solution energy estimate. We start with the usual weak solution energy estimate (which is included in the definition of a weak Leray solution to (5.5)), formally multiplying (5.5) by r and integrating by parts. We obtain:

$$\exists C_1, \text{for a.e. } t \in [0,T], \quad |r(t)|^2_{L^2(\mathcal{O})} + \int_0^t |r(t')|^2_{H^1(\mathcal{O})}\,\mathrm{d}t' \leq C_1 |r_*|^2_{L^2(\mathcal{O})}. \tag{5.7}$$

In particular (5.7) yields the existence of $0 \leq t_1 \leq T/3$ such that:

$$|r(t_1)|_{H^1(\mathcal{O})} \leq \sqrt{\frac{3C_1}{T}}\,|r_*|_{L^2(\mathcal{O})}. \tag{5.8}$$

Strong solution energy estimate. We move on to the usual strong solution energy estimate, multiplying (5.5) by $\mathbb{P}\Delta r$ and integrating by parts. We obtain:

$$\exists C_2, \forall t \in [t_1, t_1+\tau_1], \quad |r(t)|^2_{H^1(\mathcal{O})} + \int_{t_1}^t |r(t')|^2_{H^2(\mathcal{O})}\,\mathrm{d}t' \leq C_2 |r(t_1)|^2_{H^1(\mathcal{O})}, \tag{5.9}$$

where $\tau_1 \leq T/3$ is a short existence time coming from the estimation of the nonlinear term and bounded below as a function of $|r(t_1)|_{H^1(\mathcal{O})}$. See [32, Theorem 6.1] for a detailed proof. Our situation introduces an unwanted boundary term during the integration by parts of $\langle \partial_t r, \mathbb{P}\Delta r \rangle$:

$$\int_{t_1}^{t} \int_{\partial\mathcal{O}} [D(r)n]_{\text{tan}} [\partial_t r]_{\text{tan}} = - \int_{t_1}^{t} \int_{\partial\mathcal{O}} (Mr) \cdot \partial_t r. \tag{5.10}$$

Luckily, the Navier boundary conditions helps us win one space derivative. When M is a scalar (or a symmetric matrix), this term can be seen as a time derivative. In the general case, we have to conduct a parallel estimate for $\partial_t r \in L^2$ by multiplying equation (5.5) by $\partial_t r$, which allows us to maintain the conclusion (5.9). In particular, this yields the existence of $0 \leq t_2 \leq 2T/3$ such that:

$$|r(t_2)|_{H^2(\mathcal{O})} \leq \sqrt{\frac{C_2}{\tau_1}} \, |r(t_1)|_{H^1(\mathcal{O})} \, . \tag{5.11}$$

Third energy estimate. We iterate once more. We differentiate (5.5) with respect to time to obtain an evolution equation on $\partial_t r$ which we multiply by $\partial_t r$ and integrate by parts. We obtain:

$$\exists C_3, \forall t \in [t_2, t_2+\tau_2], \quad |\partial_t r(t)|^2_{L^2(\mathcal{O})} + \int_{t_2}^{t} |\partial_t r(t')|^2_{H^1(\mathcal{O})} \, \mathrm{d}t' \leq C_3 \, |\partial_t r(t_2)|^2_{L^2(\mathcal{O})} \, , \tag{5.12}$$

where τ_2 is a short existence time bounded from below as a function of $|\partial_t r(t_2)|_{L^2(\mathcal{O})}$, which is bounded at time t_2 since we can compute it from equation (5.5). Using (5.12), we deduce an $L^\infty(H^2)$ bound on r seeing (5.5) as a Stokes problem for r. Using the same argument as above, we find a time t_3 such that $r \in H^3$ with a quantitative estimate. □

Now we can prove Theorem 2. Even though $\bar{u}$ is only a weak trajectory on $[0, T]$, there exists $0 \leq T_1 < T_2 \leq T$ such that $\bar{u}$ is smooth on $[T_1, T_2]$. This is a classical statement (see [84, Remark 3.2] for the case of Dirichlet boundary conditions). We will start our control strategy by doing nothing on $[0, T_1]$. Thus, the weak trajectory u will move from u_* to some state $u(T_1)$ which we will use as a new initial data. Then, we use our control to drive $u(T_1)$ to $\bar{u}(T_2)$ at time T_2. After T_2, we choose null controls. The trajectory u follows $\bar{u}$. Hence, without loss of generality, we can assume that $T_1 = 0$ and $T_2 = T$. This allows to work with a smooth reference trajectory.

To finish the control strategy, we use the local result from [48]. According to this result, there exists $\delta_{T/3} > 0$ such that, if we succeed to prove that there exists $0 < \tau < 2T/3$ such that $|u(\tau) - \bar{u}(\tau)|_{H^3(\mathcal{O})} \leq \delta_{T/3}$, then there exist controls driving u to $\bar{u}(T)$ at time T. If we choose null controls $r := u - \bar{u}$ satisfies the hypothesis of Lemma 9. Hence, there exists $\delta > 0$ such that $C(\delta) \leq \delta_{T/3}$ and we only need to build a trajectory such that $|u(T/3) - \bar{u}(T/3)|_{L^2(\mathcal{O})} \leq \delta$, which is precisely what has been proved in Lemma 8. This concludes the proof of Theorem 2.

Perspectives

The results obtained in this work can probably be extended in following directions:

- As stated in Remark 2, for the 3D case, it would be interesting to prove that the constructed trajectory is a strong solution of the Navier-Stokes system (provided that the initial data is smooth enough). Since the first order profiles are smooth, the key point is whether we can obtain strong energy estimates for the remainder despite the presence of a boundary layer. In the uncontrolled setting, an alternative approach to the asymptotic expansion of [56] consists in introducing *conormal Sobolev spaces* to perform energy estimates (see [65]).

- As proposed in [41], [42] then [43], respectively for the case of perfect fluids (Euler equation) then very viscous fluids (stationary Stokes equation), the notion of *Lagrangian controllability* is interesting for applications. It is likely that the proofs of these references can be adapted to the case of the Navier-Stokes equation with Navier boundary conditions thanks to our method, since the boundary layers are located in a small neighborhood of the boundaries of the domain which can be kept separated from the Lagrangian trajectories of the considered movements. This adaptation might involve stronger estimates on the remainder.

- As stated after Lemma 2, the hypothesis that the control domain Γ intersects all connected components of the boundary $\partial\Omega$ of the domain is necessary to obtain controllability of the Euler equation. However, since we are dealing with the Navier-Stokes equation, it might be possible to release this assumption, obtain partial results in its absence, or prove that it remains necessary. This question is also linked to the possibility of controlling a fluid-structure system where one tries to control the position of a small solid immersed in a fluid domain by a control on a part of the external border only. Existence of weak solutions for such a system is studied in [34].

- At least for simple geometric settings of Open Problem (OP), our method might be adapted to the challenging Dirichlet boundary condition. In this case, the amplitude of the boundary layer is $\mathcal{O}(1)$ instead of $\mathcal{O}(\sqrt{\varepsilon})$ here for the Navier condition. This scaling deeply changes the equations satisfied by the boundary layer profile. Moreover, the new evolution equation satisfied by the remainder involves a difficult term $\frac{1}{\sqrt{\varepsilon}}(r^\varepsilon \cdot n)\partial_z v$. Well-posedness and smallness estimates for the remainder are much harder and might involve analytic techniques. We refer to paragraph 1.5.1 for a short overview of some of the difficulties to be expected.

More generally speaking, we expect that the *well-prepared dissipation* method can be applied to other fluid mechanics systems to obtain small-time global controllability results, as soon as asymptotic expansions for the boundary layers are known.

A Smooth controls for the linearized Euler equation

In this appendix, we provide a constructive proof of Lemma 3. The main idea is to construct a force term ξ^1 such that $\nabla \times u^1(T,\cdot) = 0$ in $\mathcal{O}$. Hence, the final time profile $U := u^1(T,\cdot)$ satisfies:

$$\begin{cases} \nabla \cdot U = 0 & \text{in } \mathcal{O}, \\ \nabla \times U = 0 & \text{in } \mathcal{O}, \\ U \cdot n = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{A.1}$$

For simply connected domains, this implies that $U = 0$ in $\mathcal{O}$. For multiply connected domains, the situation is more complex. Roughly speaking, a finite number of non vanishing solutions to (A.1) must be ruled out by sending in appropriate vorticity circulations. For more details on this specific topic, we refer to the original references: [18] for 2D, then [39] for 3D. Here, we give an explicit construction of a regular force term such that $\nabla \times u^1(T,\cdot) = 0$. The proof is slightly different in the 2D and 3D settings, because the vorticity formulation of (2.16) is not exactly the same. In both cases, we need to build an appropriate partition of unity.

A.1 Construction of an appropriate partition of unity

First, thanks to hypothesis (2.14), the continuity of the flow Φ^0 and the compactness of $\bar{\mathcal{O}}$, there exists $\delta > 0$ such that:

$$\forall x \in \bar{\mathcal{O}}, \exists t_x \in (0,T), \quad \operatorname{dist}\left(\Phi^0(0,t_x,x), \bar{\Omega}\right) \geq \delta. \tag{A.2}$$

Hence, there exists a smooth closed control region $K \subset \bar{\mathcal{O}}$ such that $K \cap \bar{\Omega} = \emptyset$ and:

$$\forall x \in \bar{\mathcal{O}}, \exists t_x \in (0,T), \quad \Phi^0(0,t_x,x) \in K. \tag{A.3}$$

Next, for each point $x \in \partial\mathcal{O} \cap K$, we build a small square (resp. a cube) centered at x, included in $\mathbb{R}^d \setminus \bar{\Omega}$, such that one side (resp. one face) is included in $\mathcal{O}$ and one side (resp. one face) is included in $\mathbb{R}^d \setminus \bar{\mathcal{O}}$. For each point $x \in K \setminus \partial\mathcal{O}$, we build a small square (resp. a cube) centered at x and included in $\mathcal{O} \setminus \bar{\Omega}$. Since these two sets of open squares (resp. cubes) cover K, which is compact, we can extract a finite number of squares (resp. cubes), labeled C_m for $1 \leq m \leq M$, that cover K and never intersect $\bar{\Omega}$. Thanks to (A.3) and to the continuity of the flow Φ^0:

$$\begin{gathered} \forall x \in \bar{\mathcal{O}}, \exists \epsilon_x > 0, \exists t_x \in (\epsilon_x, T-\epsilon_x), \exists m_x \in \{1,\dots M\}, \forall t' \in (0,T), \forall x' \in \bar{\mathcal{O}}, \\ |t'-t_x| < \epsilon_x \text{ and } |x-x'| < \epsilon_x \Rightarrow \Phi^0\left(0,t',x'\right) \in C_{m_x}. \end{gathered} \tag{A.4}$$

By compactness of $\bar{\mathcal{O}}$, we can find $\epsilon > 0$ and a finite number of small balls B_l for $1 \leq l \leq L$, covering $\bar{\mathcal{O}}$, such that, for any $1 \leq l \leq L$,

$$\exists t_l \in (\epsilon, T-\epsilon), \exists m_l \in \{1,\dots M\}, \forall t \in (t_l-\epsilon, t_l+\epsilon), \quad \Phi^0\left(0,t,B_l\right) \in C_{m_l}. \tag{A.5}$$

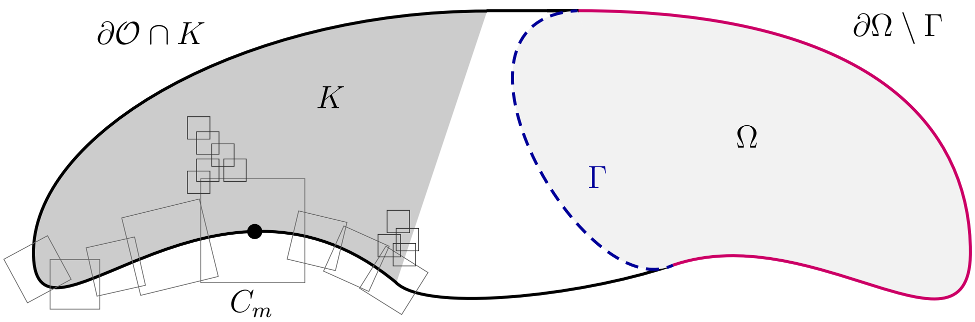

Figure 3: Paving the control region K with appropriate squares.

Hence, each ball spends a positive amount of time within a given square (resp. cube) where we can use a local control to act on the u^1 profile. This square (resp. cube) can be of one of two types as constructed above: either of *inner* type, or of *boundary* type. We also introduce a smooth partition of unity η_l for $1 \leq l \leq L$, such that $0 \leq \eta_l(x) \leq 1$, $\sum \eta_l \equiv 1$ and each η_l is compactly supported in B_l. Last, we introduce a smooth function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta \equiv 1$ on $(-\infty, \epsilon)$ and $\beta \equiv 0$ on $(\epsilon, +\infty)$.

A.2 Planar case

We consider the initial data $u_* \in H^3(\mathcal{O}) \cap L^2_{\text{div}}(\mathcal{O})$ and we split it using the constructed partition of unity. Writing (2.16) in vorticity form, $\omega^1 := \nabla \times u^1$ can be computed as $\sum \omega_l$ where ω_l is the solution to:

$$\begin{cases} \partial_t \omega_l + (\operatorname{div} u^0)\omega_l + \left(u^0 \cdot \nabla\right) \omega_l = \nabla \times \xi_l & \text{in } (0,T) \times \bar{\mathcal{O}}, \\ \omega_l(0, \cdot) = \nabla \times (\eta_l u_*) & \text{in } \bar{\mathcal{O}}. \end{cases} \tag{A.6}$$

We consider $\bar{\omega}_l$, the solution to (A.6) with $\xi_l = 0$. Setting $\omega_l := \beta(t - t_l)\bar{\omega}_l$ defines a solution to (A.6), vanishing at time T, provided that we can find ξ_l such that $\nabla \times \xi_l = \dot{\beta}\bar{\omega}_l$. The main difficulty is that we need ξ_l to be supported in $\bar{\mathcal{O}} \setminus \Omega$. Since $\dot{\beta} \equiv 0$ outside of $(-\epsilon, \epsilon)$, $\dot{\beta}\bar{\omega}_l$ is supported in C_{m_l} thanks to (A.5) because the support of ω_l is transported by (A.6). We distinguish two cases.

Inner balls. Assume that C_{m_l} is an inner square. Then B_l does not intersect $\partial\mathcal{O}$. Indeed, the streamlines of u^0 follow the boundary $\partial\mathcal{O}$. If there existed $x \in B_l \cap \partial\mathcal{O}$, then $\Phi^0(0, t_l, x) \in \partial\mathcal{O}$ could not belong to C_{m_l}, which would violate (A.5). Hence, B_l must be an inner ball. Then, thanks to Stokes' theorem, the average of $\omega_l(0, \cdot)$ on B_l is null (since the circulation of $\eta_l u_*$ along its border is null). Moreover, this average is preserved under the evolution by (A.6) with $\xi_l = 0$. Thus, the average of $\bar{\omega}_l$ is identically null. It remains to be checked that, if w is a zero-average scalar function supported in an inner square, we can find functions (ξ_1, ξ_2) supported in the same square such that $\partial_1\xi_2 - \partial_2\xi_1 = w$. Up to translation, rescaling and rotation, we can assume that the inner square is $C = [0,1]^2$. We define:

$$a(x_2) := \int_0^1 w(x_1, x_2) \mathrm{d}x_1, \tag{A.7}$$

$$b(x_2) := \int_0^{x_2} a(x) \mathrm{d}x, \tag{A.8}$$

$$\xi_1(x_1, x_2) := -c'(x_1) b(x_2), \tag{A.9}$$

$$\xi_2(x_1, x_2) := -c(x_1) a(x_2) + \int_0^{x_1} w(x, x_2) \mathrm{d}x, \tag{A.10}$$

where $c : \mathbb{R} \to [0,1]$ is a smooth function with $c \equiv 0$ on $(-\infty, 1/4)$ and $c \equiv 1$ on $(3/4, +\infty)$. Thanks to (A.7), a vanishes for $x_2 \notin [0,1]$. Thanks to (A.8), b vanishes for $x_2 \leq 0$ (because $a(x_2) = 0$ when $x_2 \leq 0$) and for $x_2 \geq 1$ (because the $b(x_2) = \int_C w = 0$ for $x_2 \geq 1$). Thanks to (A.9) and (A.10), (ξ_1, ξ_2) vanish outside of C and $\partial_1 \xi_2 - \partial_2 \xi_1 = w$. Thus, we can build ξ_l, supported in C_{m_l} such that $\nabla \times \xi_l = \dot{\beta} \bar{\omega}_l$.

Moreover, thanks to this explicit construction, the spatial regularity of ξ_l is at least as good as that of $\bar{\omega}_l$, which is the same as that of $\nabla \times (\eta_l u_*)$. If $u_* \in H^3(\mathcal{O})$, then $\xi_l \in \mathcal{C}^1([0,T], H^1(\mathcal{O})) \cap \mathcal{C}^0([0,T], H^2(\mathcal{O}))$. This remains true after summation with respect to $1 \leq l \leq L$ and for the following constructions exposed below. If the initial data u_* was smoother, we could also build smoother controls.

Boundary balls. Assume that C_{m_l} is a boundary square. Then, B_l can either be an inner ball or a boundary ball and we can no longer assume that the average of $\bar{\omega}_l$ is identically null. However, the same construction also works. Up to translation, rescaling and rotation, we can assume that the boundary square is $C = [0,1]^2$, with the side $x_2 = 0$ inside $\mathcal{O}$ and the side $x_2 = 1$ in $\mathbb{R}^2 \setminus \mathcal{O}$:

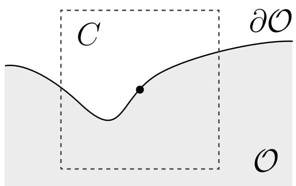

Figure 4: A boundary square

We start by extending w from $C \cap \bar{\mathcal{O}}$ to C, choosing a regular extension operator. Then, we use the same formulas (A.7), (A.8), (A.9) and (A.10). One checks that this defines a force which vanishes for $x_1 \leq 0$, for $x_1 \geq 1$ and for $x_2 \leq 0$.

A.3 Spatial case

In 3D, each vorticity patch ω_l satisfies:

$$\begin{cases} \partial_t \omega_l + \nabla \times (\omega_l \times u^0) = \nabla \times \xi_l & \text{in } (0,T) \times \bar{\mathcal{O}}, \\ \omega_l(0,\cdot) = \nabla \times (\eta_l u_*) & \text{in } \bar{\mathcal{O}}. \end{cases} \tag{A.11}$$

Equation (A.11) preserves the divergence-free condition of its initial data. Hence, proceeding as above, the only thing that we need to check is that, given a vector field $w = (w_1, w_2, w_3) : \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$\operatorname{support}(w) \subset (0,1)^3, \tag{A.12}$$

$$\operatorname{div}(w) = 0, \tag{A.13}$$

we can find a vector field $\xi = (\xi_1, \xi_2, \xi_3) : \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$\partial_2 \xi_3 - \partial_3 \xi_2 = w_1, \tag{A.14}$$

$$\partial_3 \xi_1 - \partial_1 \xi_3 = w_2, \tag{A.15}$$

$$\partial_1 \xi_2 - \partial_2 \xi_1 = w_3, \tag{A.16}$$

$$\text{support}(\xi) \subset (0,1)^3. \tag{A.17}$$

As in the planar case, we distinguish the case of inner and boundary cubes.

Inner cubes. Let $a \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that:

$$\int_0^1 a(x) \mathrm{d}x = 1, \tag{A.18}$$

$$\text{support}(a) \subset (0,1). \tag{A.19}$$

We define:

$$\xi_1(x_1, x_2, x_3) := a(x_1) h(x_2, x_3) \tag{A.20}$$

$$\xi_2(x_1, x_2, x_3) := \int_0^{x_1} (\partial_2 \xi_1 + w_3)(x, x_2, x_3) \mathrm{d}x, \tag{A.21}$$

$$\xi_3(x_1, x_2, x_3) := \int_0^{x_1} (\partial_3 \xi_1 - w_2)(x, x_2, x_3) \mathrm{d}x, \tag{A.22}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ will specified later on. From (A.21), one has (A.16). From (A.22), one has (A.15). From (A.12), (A.13), (A.21) (A.21) and (A.22), one has (A.14). Using (A.12), (A.20), (A.21) and (A.22) one checks that (A.17) holds if h satisfies

$$\text{support}(h) \subset (0,1)^2, \tag{A.23}$$

$$\partial_2 h(x_2, x_3) = W_2(x_2, x_3), \tag{A.24}$$

$$\partial_3 h(x_2, x_3) = W_3(x_2, x_3), \tag{A.25}$$

where

$$W_2(x_2,x_3) := -\int_0^1 w_3(x,x_2,x_3)\mathrm{d}x, \tag{A.26}$$

$$W_3(x_2,x_3) := \int_0^1 w_2(x,x_2,x_3)\mathrm{d}x. \tag{A.27}$$

From (A.12), (A.13), (A.26) and (A.27), one has:

$$\text{support}(W_2) \subset (0,1)^2, \ \text{support}(W_3) \subset (0,1)^2, \tag{A.28}$$

$$\partial_2 W_3 - \partial_3 W_2 = 0. \tag{A.29}$$

We define h by

$$h(x_2,x_3) := \int_0^{x_2} W_2(x,x_3)\mathrm{d}x, \tag{A.30}$$

so that (A.24) holds. From (A.28), (A.29) and (A.30), one gets (A.25). Finally, from (A.26), (A.28) and (A.30) one sees that (A.23) holds if and only if:

$$k(x_3) = 0, \tag{A.31}$$

where

$$k(x_3) := \int_0^1 \int_0^1 w_3(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2. \tag{A.32}$$

Using (A.12), (A.13) and (A.32), one sees that $k' \equiv 0$ and support$(k) \subset (0,1)$, which implies (A.31).

Boundary cubes. Now we consider a boundary cube. Up to translation, scaling and rotation, we assume that we are considering the cube $C = [0,1]^3$ with the face $x_1 = 0$ lying inside $\mathcal{O}$ and the face $x_1 = 1$ lying in $\mathbb{R}^3 \setminus \mathcal{O}$. Similarly as in the planar case, we choose a regular extension of w to C. We set $\xi_1 = 0$ and we define ξ_2 by (A.21) and ξ_3 by (A.22). One has (A.14), (A.15), (A.16) in $C \cap \bar{\mathcal{O}}$ with support$(\xi) \cap \bar{\mathcal{O}} \subset C$.

Acknowledgments. This research was supported by ERC Advanced Grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

References

[1] Alexandre, R., Wang, Y. G., Xu, C.-J., and Yang, T. Well-posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.*, 28(3):745–784, 2015.

[2] Barrat, J.-L. and Bocquet, L. Large slip effect at a nonwetting fluid-solid interface. *Physical review letters*, 82(23):4671, 1999.

[3] Bartier, J.-P., Blanchet, A., Dolbeault, J., and Escobedo, M. Improved intermediate asymptotics for the heat equation. *Appl. Math. Lett.*, 24(1):76–81, 2011.

[4] Beirão da Veiga, H. On the sharp vanishing viscosity limit of viscous incompressible fluid flows. In *New directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., pages 113–122. Birkhäuser Verlag, Basel, 2010.

[5] Beirão da Veiga, H. and Crispo, F. Sharp inviscid limit results under Navier type boundary conditions. An L^p theory. *J. Math. Fluid Mech.*, 12(3):397–411, 2010.

[6] Beirão da Veiga, H. and Crispo, F. Concerning the $W^{k,p}$-inviscid limit for 3-D flows under a slip boundary condition. *J. Math. Fluid Mech.*, 13(1):117–135, 2011.

[7] Beirão da Veiga, H. and Crispo, F. The 3-D inviscid limit result under slip boundary conditions. A negative answer. *J. Math. Fluid Mech.*, 14(1):55–59, 2012.

[8] Beirão da Veiga, H. and Crispo, F. A missed persistence property for the Euler equations and its effect on inviscid limits. *Nonlinearity*, 25(6):1661–1669, 2012.

[9] Beirão da Veiga, H., Crispo, F., and Grisanti, C.-R. Reducing slip boundary value problems from the half to the whole space. Applications to inviscid limits and to non-Newtonian fluids. *J. Math. Anal. Appl.*, 377(1):216–227, 2011.

[10] Berselli, L. C. Some results on the Navier-Stokes equations with Navier boundary conditions. *Riv. Math. Univ. Parma (N.S.)*, 1(1):1–75, 2010.

[11] Berselli, L. C. and Spirito, S. On the vanishing viscosity limit of 3D Navier-Stokes equations under slip boundary conditions in general domains. *Comm. Math. Phys.*, 316(1):171–198, 2012.

[12] Bocquet, L. and Barrat, J.-L. Flow boundary conditions from nano-to micro-scales. *Soft matter*, 3(6):685–693, 2007.

[13] Chapouly, M. On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions. *J. Differential Equations*, 247(7):2094–2123, 2009.

[14] Clopeau, T., Mikelić, A., and Robert, R. On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions. *Nonlinearity*, 11(6):1625–1636, 1998.

[15] Coron, F. Derivation of slip boundary conditions for the Navier-Stokes system from the Boltzmann equation. *J. Statist. Phys.*, 54(3-4):829–857, 1989.

[16] Coron, J.-M. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(3):271–276, 1993.

[17] Coron, J.-M. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM Contrôle Optim. Calc. Var.*, 1:35–75 (electronic), 1995/96.

[18] Coron, J.-M. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9)*, 75(2):155–188, 1996.

[19] Coron, J.-M. and Fursikov, A. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4):429–448, 1996.

[20] Cowley, S. Computer extension and analytic continuation of Blasius' expansion for impulsive flow past a circular cylinder. *J. Fluid Mech.*, 135:389–405, 1983.

[21] Crispo, F. On the zero-viscosity limit for 3D Navier-Stokes equations under slip boundary conditions. *Riv. Math. Univ. Parma (N.S.)*, 1(1):205–217, 2010.

[22] Dardé, J. and Ervedoza, S. On the reachable set for the one-dimensional heat equation. *ArXiv e-prints*, September 2016.

[23] Dautray, R. and Lions, J.-L. *Mathematical analysis and numerical methods for science and technology. Vol. 3*. Springer-Verlag, Berlin, 1990. Spectral theory and applications, With the collaboration of Michel Artola and Michel Cessenat, Translated from the French by John C. Amson.

[24] Duoandikoetxea, J. and Zuazua, E. Moments, masses de Dirac et décomposition de fonctions. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(6):693–698, 1992.

[25] E, W. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)*, 16(2):207–218, 2000.

[26] E, W. and Engquist, B. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.*, 50(12):1287–1293, 1997.

[27] Einzel, D., Panzer, P., and Liu, M. Boundary condition for fluid flow: curved or rough surfaces. *Physical review letters*, 64(19):2269, 1990.

[28] Farwig, R., Galdi, G., and Sohr, H. A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data. *J. Math. Fluid Mech.*, 8(3):423–444, 2006.

[29] Farwig, R., Kozono, H., and Sohr, H. Global weak solutions of the Navier-Stokes equations with nonhomogeneous boundary data and divergence. *Rend. Semin. Mat. Univ. Padova*, 125:51–70, 2011.

[30] Fernández-Cara, E., Guerrero, S., Imanuvilov, O., and Puel, J.-P. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.

[31] Fursikov, A. and Imanuilov, O. Exact controllability of the Navier-Stokes and Boussinesq equations. *Uspekhi Mat. Nauk*, 54(3(327)):93–146, 1999.

[32] Galdi, G. An introduction to the Navier-Stokes initial-boundary value problem. In *Fundamental directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., pages 1–70. Birkhäuser, Basel, 2000.

[33] Gérard-Varet, D. and Dormy, E. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.*, 23(2):591–609, 2010.

[34] Gérard-Varet, D. and Hillairet, M. Existence of weak solutions up to collision for viscous fluid-solid systems with slip. *Comm. Pure Appl. Math.*, 67(12):2022–2075, 2014.

[35] Geymonat, G. and Sánchez-Palencia, E. On the vanishing viscosity limit for acoustic phenomena in a bounded region. *Arch. Rational Mech. Anal.*, 75(3):257–268, 1980/81.

[36] Gie, G.-M. and Kelliher, J. Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions. *J. Differential Equations*, 253(6):1862–1892, 2012.

[37] Glass, O. Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles en dimension 3. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(9):987–992, 1997.

[38] Glass, O. Contrôlabilité de l'équation d'Euler tridimensionnelle pour les fluides parfaits incompressibles. In *Séminaire sur les Équations aux Dérivées Partielles, 1997–1998*, pages Exp. No. XV, 11. École Polytech., Palaiseau, 1998.

[39] Glass, O. Exact boundary controllability of 3-D Euler equation. *ESAIM Control Optim. Calc. Var.*, 5:1–44 (electronic), 2000.

[40] Glass, O. An addendum to a J. M. Coron theorem concerning the controllability of the Euler system for 2D incompressible inviscid fluids. "On the controllability of

2-D incompressible perfect fluids" [J. Math. Pures Appl. (9) **75** (1996), no. 2, 155–188; MR1380673 (97b:93010)]. *J. Math. Pures Appl. (9)*, 80(8):845–877, 2001.

[41] Glass, O. and Horsin, T. Approximate Lagrangian controllability for the 2-D Euler equation. Application to the control of the shape of vortex patches. *J. Math. Pures Appl. (9)*, 93(1):61–90, 2010.

[42] Glass, O. and Horsin, T. Prescribing the motion of a set of particles in a three-dimensional perfect fluid. *SIAM J. Control Optim.*, 50(5):2726–2742, 2012.

[43] Glass, O. and Horsin, T. Lagrangian controllability at low Reynolds number. *ArXiv e-prints*, February 2016.

[44] Golse, F. From the Boltzmann equation to the Euler equations in the presence of boundaries. *Comput. Math. Appl.*, 65(6):815–830, 2013.

[45] Grard-Varet, D., Maekawa, Y., and Masmoudi, N. Gevrey Stability of Prandtl Expansions for 2D Navier-Stokes. *ArXiv e-prints*, July 2016.

[46] Grenier, E. Boundary layers. In *Handbook of mathematical fluid dynamics. Vol. III*, pages 245–309. North-Holland, Amsterdam, 2004.

[47] Grenier, E., Guo, Y., and Nguyen, T. Spectral stability of Prandtl boundary layers: an overview. *Analysis (Berlin)*, 35(4):343–355, 2015.

[48] Guerrero, S. Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions. *ESAIM Control Optim. Calc. Var.*, 12(3):484–544 (electronic), 2006.

[49] Guerrero, S., Imanuvilov, O., and Puel, J.-P. Remarks on global approximate controllability for the 2-D Navier-Stokes system with Dirichlet boundary conditions. *C. R. Math. Acad. Sci. Paris*, 343(9):573–577, 2006.

[50] Guerrero, S., Imanuvilov, O., and Puel, J.-P. A result concerning the global approximate controllability of the Navier–Stokes system in dimension 3. *J. Math. Pures Appl. (9)*, 98(6):689–709, 2012.

[51] Guo, L., Chen, S., and Robbins, M. Slip boundary conditions over curved surfaces. *Physical Review E*, 93(1):013105, 2016.

[52] Guo, Y. and Nguyen, T. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.*, 64(10):1416–1438, 2011.

[53] Guyon, E., Hulin, J.-P., Petit, L., and de Gennes, P. G. *Hydrodynamique physique*. EDP sciences Les Ulis, 2001.

[54] Havârneanu, T., Popa, C., and Sritharan, S. Exact internal controllability for the two-dimensional Navier-Stokes equations with the Navier slip boundary conditions. *Systems Control Lett.*, 55(12):1022–1028, 2006.

[55] Iftimie, D. and Planas, G. Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions. *Nonlinearity*, 19(4):899–918, 2006.

[56] Iftimie, D. and Sueur, F. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Anal.*, 199(1):145–175, 2011.

[57] Imanuvilov, O. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72 (electronic), 2001.

[58] Kelliher, J. Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane. *SIAM J. Math. Anal.*, 38(1):210–232 (electronic), 2006.

[59] Launder, B. E. and Spalding, D. B. Lectures in mathematical models of turbulence. 1972.

[60] Lions, J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.

[61] Lions, J.-L. Exact controllability for distributed systems. Some trends and some problems. In *Applied and industrial mathematics (Venice, 1989)*, volume 56 of *Math. Appl.*, pages 59–84. Kluwer Acad. Publ., Dordrecht, 1991.

[62] Maekawa, Y. and Mazzucato, A. The Inviscid Limit and Boundary Layers for Navier-Stokes Flows. *ArXiv e-prints*, October 2016.

[63] Marbach, F. Small time global null controllability for a viscous Burgers' equation despite the presence of a boundary layer. *J. Math. Pures Appl. (9)*, 102(2):364–384, 2014.

[64] Martin, P., Rosier, L., and Rouchon, P. On the reachable states for the boundary control of the heat equation. *Appl. Math. Res. Express. AMRX*, (2):181–216, 2016.

[65] Masmoudi, N. and Rousset, F. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.*, 203(2):529–575, 2012.

[66] Masmoudi, N. and Saint-Raymond, L. From the Boltzmann equation to the Stokes-Fourier system in a bounded domain. *Comm. Pure Appl. Math.*, 56(9):1263–1293, 2003.

[67] Masmoudi, N. and Wong, T. K. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.*, 68(10):1683–1741, 2015.

[68] McLean, W. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[69] Navier, C.-L. Mémoire sur les lois du mouvement des fluides. *Mémoires de lAcadémie Royale des Sciences de lInstitut de France*, 6:389–440, 1823.

[70] Nickel, K. Prandtl's boundary-layer theory from the viewpoint of a mathematician. *Annual Review of Fluid Mechanics*, 5(1):405–428, 1973.

[71] Oleĭnik, O. On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *J. Appl. Math. Mech.*, 30:951–974 (1967), 1966.

[72] Oleĭnik, O. and Samokhin, V. *Mathematical models in boundary layer theory*, volume 15 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[73] Paddick, M. Stability and instability of Navier boundary layers. *Differential Integral Equations*, 27(9-10):893–930, 2014.

[74] Panzer, P., Liu, M., and Einzel, D. The effects of boundary curvature on hydrodynamic fluid flow: calculation of slip lengths. *International Journal of Modern Physics B*, 6(20):3251–3278, 1992.

[75] Prandtl, L. Uber flussigkeits bewegung bei sehr kleiner reibung. *Verhaldlg III Int. Math. Kong*, pages 484–491, 1904.

[76] Raymond, J.-P. Stokes and Navier-Stokes equations with a nonhomogeneous divergence condition. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1537–1564, 2010.

[77] Sammartino, M. and Caflisch, R. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433–461, 1998.

[78] Sammartino, M. and Caflisch, R. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(2):463–491, 1998.

[79] Schonbek, M. E. Decay of solutions to parabolic conservation laws. *Comm. Partial Differential Equations*, 5(5):449–473, 1980.

[80] Shibata, Y. and Shimada, R. On a generalized resolvent estimate for the Stokes system with Robin boundary condition. *J. Math. Soc. Japan*, 59(2):469–519, 2007.

[81] Shibata, Y. and Shimada, R. On the Stokes equation with Robin boundary condition. In *Asymptotic analysis and singularities—hyperbolic and dispersive PDEs and fluid mechanics*, volume 47 of *Adv. Stud. Pure Math.*, pages 341–348. Math. Soc. Japan, Tokyo, 2007.

[82] Shimada, R. On the L_p-L_q maximal regularity for Stokes equations with Robin boundary condition in a bounded domain. *Math. Methods Appl. Sci.*, 30(3):257–289, 2007.

[83] Stokes, G. G. *On the effect of the internal friction of fluids on the motion of pendulums*, volume 9. Pitt Press, 1851.

[84] Temam, R. Behaviour at time $t = 0$ of the solutions of semilinear evolution equations. *J. Differential Equations*, 43(1):73–92, 1982.

[85] Temam, R. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[86] van Dommelen, L. and Cowley, S. On the Lagrangian description of unsteady boundary-layer separation. I. General theory. *J. Fluid Mech.*, 210:593–626, 1990.

[87] van Dommelen, L. and Shen, S.-F. The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comput. Phys.*, 38(2):125–140, 1980.

[88] Wang, L., Xin, Z., and Zang, A. Vanishing viscous limits for 3D Navier-Stokes equations with a Navier-slip boundary condition. *J. Math. Fluid Mech.*, 14(4):791–825, 2012.

[89] Wang, X.-P., Wang, Y.-G., and Xin, Z. Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit. *Commun. Math. Sci.*, 8(4):965–998, 2010.

[90] Xiao, Y. and Xin, Z. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. *Comm. Pure Appl. Math.*, 60(7):1027–1055, 2007.

[91] Xiao, Y. and Xin, Z. Remarks on vanishing viscosity limits for the 3D Navier-Stokes equations with a slip boundary condition. *Chin. Ann. Math. Ser. B*, 32(3):321–332, 2011.

[92] Xiao, Y. and Xin, Z. On the inviscid limit of the 3D Navier-Stokes equations with generalized Navier-slip boundary conditions. *Commun. Math. Stat.*, 1(3):259–279, 2013.

[93] Xin, Z. and Zhang, L. On the global existence of solutions to the Prandtl's system. *Adv. Math.*, 181(1):88–133, 2004.