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Séminaire É. D. P. (2008-2009), Exposé n° XIII, 11 p.

<http://sedp.cedram.org/item?id=SEDP_2008-2009____A13_0>

Séminaire *Équations aux dérivées partielles*
Centre de mathématiques Laurent Schwartz, 2008-2009
Exposé n° XIII, 1-11

Viscous profiles of vortex patches

Franck Sueur*

Many systems of partial differential equations which propagate singularities are actually some singular limits of more realistic systems with an extra regularizing term of higher order. We refer here to ferromagnetism, to shell theory or to fluid mechanics, from which we borrow the terminology of inviscid/viscous systems.

Let us briefly remind, to motivate our analysis, some basic results on the propagation of the singularities, referring to the survey [13] by Gårding for a more detailed overview of the topic. For linear equations a fundamental result has been given by Hörmander: the wave front set of the solution at time t of a system of real principal type is the image of the wave front set of the Cauchy data by the canonical transformation generated by the Hamiltonian of the system, that is by the determinant of the principal symbol of the system.

For nonlinear systems, this is in general not true anymore: phenomena due to self spreading or crossing may occur [3]. However propagation can be obtain when one restrain the study to singularities more robust with respect to the nonlinearity. One may turn to conormal singularities. Loosely speaking, given a filtration $(E_s)_{s\in\mathbb{R}}$ of microlocalizable spaces (here we shall make use of the Besov spaces) and a closed set Σ, the distributions of index s conormal to Σ are the one in E_s which remain in the same space under (possibly repeated) action of the vector fields tangent to Σ. Such distributions have been introduced by Hörmander as a particular case of the Lagrangian distributions in his Fourier operator theory (see Hörmander [15] for a comprehensive expository), and the works of Bony [4], Alinhac [1], Chemin [6] have basically shown that when a conormal singularity is weak enough with respect to nonlinear effects it propagates nicely along the bicharacteristics, even if the evolution of the singular support itself is an unknown.

Our goal is precisely to explain in such a case how to construct some viscous profiles, that is to obtain some expansions for the solutions of the viscous perturbations of the inviscid system which describe as well as possible their behaviour with respect not only to the space-time variables but also to the viscosity coefficient, in the vanishing limit. Our main motivation here is the construction of viscous profiles of vortex patches. A full treatment is done in [20]. We also refer to the proceeding [21]. We take here this new opportunity to present our work starting with a digression about the conormal self-similarity of the viscous smoothing.

1 Conormal self-similarity of the viscous smoothing

There are many way to make appear viscous smoothing; the goal of this section is to show how viscosity creates self-similar fast scales in the singular directions. Here we shall simply consider the heat equation

$$\partial_t u^\nu = \nu \Delta_x u^\nu, \text{ for } t \geqslant 0 \text{ and } x \in \mathbb{R}^n, \tag{1}$$

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where n is in the set $\mathbb{N}^*$ of all natural numbers except zero, and ν is a nonnegative real. This has two obvious advantages: the equation is linear and for the inviscid system (i.e. when $\nu = 0$) the singularities are stationary.

We will also get rid of the difficulties linked to geometry by considering as initial data some distributions conormal to linear spaces. This allows to use some representation in straight coordinates through partial Fourier transform. More precisely we prescribe at $t = 0$ the oscillatory integral

$$u_0(x) = \int_{\mathbb{R}^d} a_0(z, \zeta) e^{iy.\zeta} d\zeta, \tag{2}$$

where d is an integer such that $1 \leqslant d \leqslant n$, where $y = (x_1, ..., x_d)$, $z = (x_{d+1}, ..., x_n)$ is a splitting of the variable $x = (y, z)$ in $\mathbb{R}^n$; where $\xi = (\zeta, \eta)$ is the corresponding splitting of the dual variable, and where a_0 is in the space $S^p_{1,0}(\mathbb{R}^{n-d} \times \mathbb{R}^d)$ of the (uniform) symbols of order $p \in \mathbb{R}$ and of type $1, 0$, which means that for all $\alpha, \beta \in \mathbb{N}^n$ there exists $C_{\alpha,\beta} > 0$ such that for all $(z, \zeta) \in \mathbb{R}^{n-d} \times \mathbb{R}^d$, $|\partial^\alpha_\zeta \partial^\beta_z a| \leqslant C_{\alpha,\beta}(1 + |\zeta|)^{p-|\alpha|}$.[1] Such distributions u_0 have their wave front set included in N^*F, where F is the subspace of $\mathbb{R}^n$ given by the equation $y = 0$ and N^*F is its conormal bundle, without the section zero, given by the equation $y = 0, \eta = 0, \zeta \neq 0$; and when the symbol a_0 vanishes for z outside of a compact set, u_0 is characterized by the conditions $x^\alpha \partial^\beta u_0 \in B^{-p-d/2}_{2,\infty}(\mathbb{R}^n)$, for any $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha'| \geqslant |\beta'|$. Here α', β' are the first groups of variables in the splitting of α, β corresponding to the one of x; and $B^s_{p,q}(\mathbb{R}^n)$ denotes the (non-homogeneous) Besov space with s as regularity exponent, with p as integral-exponent and with q as sum-exponent. Note that the vector fields $x^\alpha \partial^\beta$ with $|\alpha'| \geqslant |\beta'|$ are tangential to the subspace F and generate all such (smooth) vector fields.

Some examples of conormal distributions are given by multiple distribution layers: let α denote a multi-index $\alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, with $|\alpha| = p$ and let $\mu(z)$ be in the Schwartz space $\mathcal{S}(\mathbb{R}^{n-d})$ of rapidly decreasing functions on $\mathbb{R}^{n-d}$. Then

$$a_0(z, \zeta) := \mu(z)(i\zeta)^\alpha := \mu(z)(i\zeta_1)^{\alpha_1} ... (i\zeta_r)^{\alpha_r}$$

is in $S^p_{1,0}(\mathbb{R}^{n-d} \times \mathbb{R}^d)$. The corresponding distributions $u_0 := \mu \otimes \delta^{(\alpha)}$ act on test functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ according to the formula:

$$< \mu \otimes \delta^{(\alpha)}, \varphi >_{\mathcal{D}', \mathcal{D}} = (-1)^{|\alpha|} \int_{\mathbb{R}^{n-d}} \mu(z) \partial^\alpha_y \varphi(0, z) dz.$$

Notice that any distribution whose support is included in a compact subset of F and whose wave front set is in the conormal bundle N^*F can be written as a linear combination of multiple layers.

For such an initial data Fourier transform yields for the solution u^ν of (1) the following representation formula

$$u^\nu(t, x) = \int_{\mathbb{R}^d} a^{\nu t}(z, \zeta) e^{iy.\zeta} d\zeta, \tag{3}$$

where

$$a^\lambda(z, \zeta) := e^{-\lambda \zeta^2} (i\zeta)^\alpha \cdot e^{-\lambda \Delta_z} \mu(z)$$

[1] To avoid any confusion let us precise that we use the following notations: when a letter γ denotes a multi-index $\gamma := (\gamma_1, ..., \gamma_r) \in \mathbb{N}^r$, then $|\gamma|$ denotes the length $\gamma_1 + ... + \gamma_r$, and when a variable $y = (y_1, ..., y_r)$ runs into $\mathbb{R}^r$, ∂^γ_y denotes the multi-derivative $\partial^\gamma_y := \partial^{\gamma_1}_1 ... \partial^{\gamma_r}_r$.

is in $S^{-\infty} := \cup_{p\in\mathbb{R}} S^p_{1,0}$ for $\lambda > 0$. Expanding the second exponential in the definition of a^λ by the Taylor formula we get that the solution u^ν may be expanded (at least formally) in powers of νt as follows:

$$u^\nu(t,x) \sim \sum_{q=0}^{\infty} (\nu t)^{q-\frac{p+d}{2}}\, \mathcal{U}^q((\nu t)^{-\frac{1}{2}} y, z), \tag{4}$$

where

$$\begin{aligned} \mathcal{U}^q(Y,z) &:= \int_{\mathbb{R}^d} \mathcal{A}^q(z,\zeta) e^{iY.\zeta} d\zeta, \text{ with } \mathcal{A}^q(z,\zeta) := e^{-\zeta^2}(i\zeta)^\alpha \cdot \frac{1}{q!}\Delta_z^q \mu(z), \\ &= \partial_Y^\alpha \varphi_0(Y)\cdot \frac{1}{q!}\Delta_z^q\mu(z) \text{ with } \varphi_0(Y) := \frac{1}{(4\pi)^{d/2}} e^{-Y^2/4}. \end{aligned} \tag{5}$$

Let us draw some conclusions on these examples: the solutions develop self-similar fast scales in the singular directions, with an amplitude which increases with the strength of the singularity. Moreover the solution may be described by a complete asymptotic expansion which involves internal layer profiles localized around the singular support of the initial data.

There are also conormal distributions with intermediate orders, for which it is still possible to construct some viscous profiles but they are less localized. To show this let us consider the homogeneous function $a_0(z,\zeta) := \mu(z)|\zeta|^p$ with $p \in \mathbb{R}$, $\frac{p}{2} \notin \mathbb{N}$. Such a symbol is not in C^∞ so that the corresponding distribution u_0 in (2) is not strictly in the previous setting. However for $p > -d$, the function $a_0(z,\zeta)$ is in L^1_{loc} so that (2) makes sense. The solution u^ν of (1) with such distribution u_0 as initial data also admits an expansion of the form (4)-(5) but with

$$\mathcal{A}^q(z,\zeta) := e^{-\zeta^2}|\zeta|^p \cdot \frac{1}{q!}\Delta_z^q \mu(z).$$

The corresponding profiles $\mathcal{U}^q$ are therefore in C^∞ but they are not sufficiently decreasing to be in $\mathcal{S}$, only enough to be in $L^{\frac{d}{d+p}+\epsilon}$ for any $\epsilon > 0$.

We arrive at the case $p \leqslant -d$. We shall consider the case $d = 1$ so that F is the hypersurface given by the equation $x_1 = 0$. We shall denote $p - C^\infty$ the space of the piecewise smooth functions with respect to F. Locally, their restriction to each open half-space defined by F extends in a C^∞ function defined on the whole space. It is part of the mathematical folklore that the space $p - C^\infty$ is the set of the distributions u_0 smooth out of F and which can be written as in (2) with a symbol a_0 locally[2] in S^{-1}, satisfying[3] $a_0(z,\zeta) \sim \sum_{q=1}^\infty a_q(z)\zeta^{-q}$. In this case $a_q(z) = \frac{1}{2\pi i^{q-1}}[\partial_y^{q-1} u_0]$ where $[\cdot]$ designates the discontinuity jumps across the hypersurface F. These symbols satisfy the transmission property cf. for instance [14], what suggests to construct their viscous profiles by a transmission strategy. To explain this strategy let us get rid of the transversal directions by considering the heat equation (1) in the one dimensional case $n = 1$, with as initial data the Heaviside Step Function H. The solution may read

$$u^\nu(t,x) = H(x) + \mathcal{U}_\pm(\frac{x}{\sqrt{\nu t}}) \text{ when } \pm x > 0, \text{ where } \mathcal{U}_\pm(X) := \frac{1}{\sqrt{\pi}} \int_{-\frac{X}{2}}^{\mp\infty} e^{-y^2} dy \text{ when } \pm X > 0.$$

[2]which means that for all $\alpha, \beta \in \mathbb{N}^n$, for all compact $K \subset \mathbb{R}^{n-d}$ there exists $C_{\alpha,\beta,K} > 0$ such that for all $(z,\zeta) \in K \times \mathbb{R}^d$, $|\partial_\zeta^\alpha \partial_z^\beta a| \leqslant C_{\alpha,\beta,K}(1+|\zeta|)^{p-|\alpha|}$.

[3]in the sense that for any $N \in \mathbb{N}^*$, $a_0(z,\zeta) - \sum_{q=1}^{N-1} a_q(z)\chi(\zeta)\zeta^{-q}$ is of order $-N$, where $\chi(\zeta)$ is a smooth function vanishing for $|\zeta| \leqslant 1$ and equal to 1 for $|\zeta| \geqslant 1$.

One then see the "viscous" solutions u^ν as the sum of the "inviscid" solution u^0 plus a "double initial-(internal) boundary layer" $\mathcal{U}_\pm$ which satisfies an elliptic transmission problem. In effect they solve

$$\partial_X^2 \mathcal{U}_\pm + \frac{X}{2}\partial_X \mathcal{U}_\pm = 0 \quad \text{when} \quad \pm X > 0,$$

match the continuity conditions of u^ν and $\partial_x u^\nu$ at the internal boundary $x = X = 0$:

$$1 + \mathcal{U}_+|_{X=0^+} = \mathcal{U}_-|_{X=0^-} \text{ and } \partial_X \mathcal{U}_+|_{X=0^+} = \partial_X \mathcal{U}_-|_{X=0^-}$$

and vanish (i.e. $\mathcal{U}_\pm(X) \to 0$) when $X \to \pm\infty$. More generally for an initial data u_0 with $[\partial_y^{q-1} u_0] = 0$ for $1 \leqslant q \leqslant p$ and $[\partial_y^p u_0] \neq 0$ the corresponding solution u^ν develops an internal layer of amplitude $(\nu t)^{\frac{p}{2}}$. In addition one has to add to the expansion some regular terms since even for an analytic data u_0 the solution u^ν expands into

$$u^\nu(t,x) = \sum_{q=0}^{\infty} (\nu t)^q \, \frac{1}{q!} \partial_x^{2q} \, u_0(x).$$

2 Viscous profiles of vortex patches

Let us now consider the equations of incompressible fluid mechanics which read

$$\partial_t v^\nu + v^\nu \cdot \nabla v^\nu + \nabla p^\nu = \nu \Delta v^\nu \tag{6}$$

$$\text{div } v^\nu = 0, \tag{7}$$

where v^ν and p^ν respectively denote the velocity and the pressure of the fluid. When $\nu = 0$ the equations (6)-(7) are the **Euler** equations whereas $\nu > 0$ corresponds to the **Navier-Stokes** equations. We will consider the academic case where the spatial derivative x is in $\mathbb{R}^d$ for $d = 2$ or 3. A key quantity is the vorticity

$$\omega^\nu := \text{curl } v^\nu \tag{8}$$

which satisfies the equation

$$\partial_t \omega^\nu + v^\nu \cdot \nabla \omega^\nu = \omega^\nu \cdot \nabla v^\nu + \nu \Delta \omega^\nu. \tag{9}$$

We shall consider as initial data some **vortex patches** which are basically fluid configurations with the vorticity discontinuous across a hypersurface. The problem was initially considered for the Euler equations (when $\nu = 0$) in two dimensions. The vorticity is then scalar and a natural example of discontinuous vorticity is the characteristic function of a bounded domain. For such an initial data the existence and uniqueness of a solution was given by Yudovich in [22]. Actually Yudovich's theorem even deals with the more general case of an initial vorticity which is a bounded function with compact support. The corresponding velocity field is log-lipschitzian and admits a bicontinuous flow $\mathcal{X}^0$. In the two-dimensional case the stretching term (the first one in the r.h.s. of (9)) vanishes and the vorticity is simply transported by the flow, this implies that in the case of a vortex patch as initial data the vorticity ω^0 at time t remains a vortex patch relative to a domain which is homeomorphic to the initial domain. However Yudovich's approach does not allow to study precisely the evolution of the smoothness of the boundary of the patch. Numerical experiments of Zabusky in [23] suggested that singularities of the boundary of the patches would develop, presumably in finite time whereas the ones of Buttke [5] suggested a loss of smoothness. In [18] Majda announced local-in-time existence and conjectured that there are smooth initial curves such that the curve becomes nonrectifiable in finite time. However Chemin has shown that things go the other way by proving the persistence of the smoothness of the boundary in [8] (see also his recent survey [9]). Its proof uses the vorticity smoothness with respect to the vector fields tangential to the boundary of the patch. These vector fields move with the fluid and their own smoothness is therefore linked to the smoothness of the fluid velocity. There were numerous works after Chemin's result[4]. In particular it was extended to the three dimensional case by Gamblin and Saint-Raymond [12], and then by Zhang and Qiu [24]. Because of the stretching term it is only a short time result.

In three dimensions a vortex patch is defined as follows. Let be given a compact connected hypersurface Γ_0 in the Hölder class $C^{s+1,r}$ where s is in $\mathbb{N}$ and $0 < r < 1$. This means that there exists a function[5] $\varphi_0 \in C^{s+1,r}(\mathbb{R}^3, \mathbb{R})$ such that an equation of Γ_0 is given by $\Gamma_0 = \{\varphi_0 = 0\}$, with $\nabla\varphi_0 \neq 0$ in a neighborhood of Γ_0. According to Jordan's theorem $\mathbb{R}^3 \backslash \Gamma_0$ has two distinct connected components. One of them is bounded (the "interior"), we shall denote it $\mathcal{O}_{0,+}$, and the other one (the "exterior") is unbounded, we shall denote it $\mathcal{O}_{0,-}$. We assume that $\mathcal{O}_{0,\pm} = \{\pm\varphi_0 > 0\}$. We consider a divergence free initial velocity v_0 in $L^2(\mathbb{R}^3)$ whose vorticity $\omega_0 :=$ curl v_0 is in the Hölder space $C^{s,r}_c(\mathcal{O}_{0,\pm})$, that is a compactly supported vorticity which is $C^{s,r}$ on each side of Γ_0.

As already said the persistence of the patch smoothness is only linked to the iterated action of the vector fields tangential to the boundary. The persistence of the initial piecewise smoothness needs a little bit more of work (but this is needed for our construction of viscous profiles). The two-dimensional case was proved by Depauw in the case $s = 0$ in [10] and by Huang [16] in the general case s in $\mathbb{N}$. In three dimensions Huang in [17] proved the case $s = 0$ in [17] (see also Dutrifoy's paper [11] section 3.1). The proof of the persistence of higher order piecewise smoothness is given in [20].

Theorem 2.1. *There exists $T > 0$ and a unique solution*

$$v^0 \in L^\infty(0, T; Lip(\mathbb{R}^3)) \cap Lip(0, T; L^2(\mathbb{R}^3))$$

to the **Euler** *equations:*

$$\partial_t v^0 + v^0 \cdot \nabla v^0 + \nabla p^0 = 0, \tag{10}$$

$$div\ v^0 = 0, \tag{11}$$

with v_0 as initial velocity. To say more let us denote

(i). D the material derivative

$$D := \partial_t + v^0 \cdot \nabla,$$

(ii). φ^0 the solution of

$$D\varphi^0 = 0, \quad with\ \varphi^0|_{t=0} = \varphi_0, \tag{12}$$

[4] We refer to [20] for a detailed survey.

[5] For an open subset $\mathcal{O}$ of $\mathbb{R}^3$, the Hölder space $C^{s,r}(\mathcal{O})$ is the set of the functions of class $C^s(\mathcal{O})$ such that

$$\|u\|_{C^{s,r}(\mathcal{O})} := \sup_{|\alpha|\leqslant s}\big(\|\partial^\alpha u\|_{L^\infty(\mathcal{O})} + \sup_{x\neq y\in\mathcal{O}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^r}\big) < +\infty.$$

(iii). $\mathcal{X}^0$ *the flow map defined by* $\partial_t \mathcal{X}^0(t,x) = v^0(t, \mathcal{X}^0(t,x))$ *with initial data* $\mathcal{X}^0(0,x) = x$*,*

(iv). $\Gamma(t)$ *the compact connected hypersurface* $\Gamma(t) := \mathcal{X}^0(t, \Gamma_0) = \{\varphi^0(t,.) = 0\}$ *which separates* $\mathcal{O}_\pm(t) := \mathcal{X}^0(t, \mathcal{O}_{0,\pm}) = \{\pm\varphi^0(t,.) > 0\}$.

Then for each $t \in (0,T)$ *the boundary* $\Gamma(t)$ *is* $C^{s+1,r}$*, the vorticity*

$$\omega^0(t) := \mathit{curl}\ v^0(t) \tag{13}$$

is $C_c^{s,r}(\mathcal{O}_\pm(t))$*, and*

$$\varphi^0 \in L^\infty(0,T; C^{1,r}(\mathbb{R}^d)) \cap L^\infty(0,T; C^{s+1,r}(\mathcal{O}_\pm(t))).$$

Moreover there exists $\eta > 0$ *such that for* $0 \leqslant t \leqslant T$*, and* x *such that* $|\varphi^0(t,x)| < \eta$ *the vector* $n(t,x) := \nabla_x \varphi^0(t,x)$ *satisfies* $n(t,x) \neq 0$*. For each* $t \in (0,T)$ *the function* $(\omega^0.n)(t,.)$ *is* $C^{0,r}$ *on* $\{|\varphi^0(t,.)| < \eta\}$*. Finally the internal boundary* $\Gamma(t)$ *is analytic with respect to time and the restrictions on each side of the boundary of the flow* $\mathcal{X}^0$ *are also analytic with respect to time with values in* $C^{s+1,r}$*.*

The final statement in Theorem 2.1 about the smoothness of the boundary with respect to time originates in Chemin's pioneering work [7] (note that the vector field D is also tangential to the boundary) and was extended into analyticity with respect to time by Serfati [19] (see also its doctoral thesis). This shall be useful for proving smoothness properties of the viscous profiles that we shall now construct.

Here we want to show that the solutions of the Navier-Stokes equations benefit from a conormal smoothing of the initial vorticity discontinuity into a layer of width $\sqrt{\nu t}$ around the hypersurface $\{\varphi^0(t,.) = 0\}$ where the discontinuity has been transported at the time t by the flow of the Euler equations. Therefore we expect that the solutions ω^ν of the Navier-Stokes equations with vortex patches as initial data can be described by an expansion of the form

$$\omega^\nu(t,x) \sim \omega^0(t,x) + \tilde{\omega}^\nu(t,x) \tag{14}$$

where $\tilde{\omega}^\nu$ denotes a perturbation mainly local and conormally self-similar, that is depending on the extra inner scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$, so that

$$\tilde{\omega}^\nu(t,x) := \tilde{\Omega}(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}), \tag{15}$$

with

$$\lim_{X\to\pm\infty} \tilde{\Omega}(t,x,X) = 0. \tag{16}$$

The velocity v^ν may be recovered by applying a pseudo-local operator of order -1 (with Fourier symbol $-\frac{\xi}{|\xi|^2}\wedge$) to the vorticity ω^ν so that we expect that the velocity v^ν given by the Navier-Stokes equations can be described by an asymptotic expansion of the form:

$$v^\nu(t,x) \sim v^0(t,x) + \sqrt{\nu t}\ \tilde{V}(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}), \tag{17}$$

where the profile $\tilde{V}(t,x,X)$ is also expected to satisfy

$$\lim_{X\to\pm\infty} \tilde{V}(t,x,X) = 0. \tag{18}$$

Plugging (14), (15) and (17) into the relations (8), taking into account (13) and equalling the leading order terms leads to

$$n \wedge \partial_X \tilde{V} = \tilde{\Omega}. \tag{19}$$

Hence the vorticity profile $\tilde{\Omega}$ has to satisfy the orthogonality condition:

$$\tilde{\Omega} \cdot n = 0. \tag{20}$$

This condition is not a surprise: since w^0 is divergence free $w^0 \cdot n$ is continuous so that no (large amplitude) layer is expected on the normal component of the vorticity.

Now the pressure p^ν can be recovered from the velocity v^ν by applying the operator divergence to the equation (6) which yields the Laplace problem:

$$\triangle_x \, p^\nu = - \sum_{1 \leqslant i,j \leqslant 3} (\partial_i v_j^\nu)(\partial_j v_i^\nu). \tag{21}$$

If the velocity v^ν satisfies the expansion (17), the r.h.s. of (21) should admit an expansion of the form:

$$\triangle_x \, p^\nu \quad \sim \quad \triangle_x \, p^0 + \tilde{F}(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}), \tag{22}$$

where the function $\tilde{F}$ vanishes for $X \to \pm\infty$. Since the Laplacian is of order -2 we are lead to consider a perturbation of order νt on the pressure:

$$p^\nu(t,x) \quad \sim \quad p^0(t,x) + \nu t \tilde{P}(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}), \tag{23}$$

where -once again- the fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ is expected to be a local inner scale (since the Laplacian operator is pseudo-local) so that

$$\lim_{X \to \pm\infty} \tilde{P}(t, x, X) \quad = \quad 0. \tag{24}$$

We plug the ansatz (17) and (23) into the equation (6). The leading order terms are of order $\sqrt{\nu t}^0$ and provide the equation

$$Dv^0 + \nabla p^0 + D\varphi^0 \partial_X \tilde{V} = 0, \tag{25}$$

which is actually satisfied since the velocity v^0 satisfies the Euler equations (10)-(11) and φ^0 satisfies the eikonal equation (12). At the following order $\sqrt{\nu t}$ we get the equality

$$D\tilde{V} + \tilde{V} \cdot n \, \partial_X \tilde{V} + \tilde{V} \cdot \nabla_x v^0 + \partial_X \tilde{P} \, n = \frac{1}{t}(|n|^2 \partial_X^2 \tilde{V} + \frac{X}{2} \partial_X \tilde{V} - \frac{1}{2} \tilde{V}). \tag{26}$$

We now pay attention to the divergence free condition. Plugging the ansatz (17) into the equation (7), retaining the terms at order $\sqrt{\nu t}^0$ and taking into account that the velocity v^0 given by Euler is divergence free leads to the orthogonality equation: $n \cdot \partial_X \tilde{V} = 0$, which by integration, with the condition (18) leads to the condition:

$$n \cdot \tilde{V} = 0. \tag{27}$$

An important consequence of the condition (27) is to kill the second term in (26) which is the only nonlinear one.

The equation (26) involves both $\tilde{V}$ and $\tilde{P}$. However the pressure in the NS equations is not truly an unknown but can be recovered from the velocity (as recalled in (21)) so that we expect that the same holds for the profiles. One way to proceed is to project normally the equation (26), to take into account that the (non-unit) normal vector $n(t,x)$ satisfies the equation:

$$Dn = -{}^t(\nabla v^0) \cdot n. \tag{28}$$

and to use the condition (27) to get that

$$\partial_X \tilde{P} := -2\frac{(\tilde{V} \cdot \nabla_x v^0) \cdot n}{|n|^2}. \tag{29}$$

We now use the equation (29) to get rid of the pressure profile into the equation (26). Inverting the two sides and dividing by t, we have:

$$|n|^2 \partial_X^2 \tilde{V} + \frac{X}{2}\partial_X \tilde{V} - \frac{1}{2}\tilde{V} = t(D\tilde{V} + \tilde{V} \cdot \nabla_x v^0 - 2\frac{(\tilde{V} \cdot \nabla_x v^0)n}{|n|^2}n), \tag{30}$$

The vector field n may vanish, away the boundary of the patch, hence so may do the coefficient in front of the leading order in the equation (30). To remedy to this we consider a function a in the space

$$\mathcal{B} := L^\infty\Big([0,T], C^{0,r}(\mathbb{R}^d)\Big) \cap L^\infty\Big(0,T; C^{s,r}\big(\mathcal{O}_\pm(t)\big)\Big) \tag{31}$$

satisfying the condition

$$\inf_{[0,T]\times\mathbb{R}^d} a =: c > 0 \tag{32}$$

and such that $a = |n|^2$ when $|\varphi^0| < \eta$, and we consider for the profile $\tilde{V}(t,x,X)$ the *linear* partial differential equation:

$$L\tilde{V} = 0 \tag{33}$$

where the differential operator L is given by

$$L := E - t(D + A)$$

where E and A are some operators of respective order 2 and 0 acting formally on functions $V(t,x,X)$ as follows:

$$EV := a\partial_X^2 V + \frac{X}{2}\partial_X V - \frac{1}{2}V \text{ and } AV := V \cdot \nabla_x v^0 - 2\frac{(V \cdot \nabla_x v^0)n}{a}n.$$

The substitution of a instead of $|n|^2$ is almost harmless since their values are different only for $|\varphi^0| \geqslant \eta$, so that the corresponding values of the (expected, so far) solutions $V(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}})$ and $\tilde{V}(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}})$ respectively given by the equations (30) and (33) both tend to 0 as $\sqrt{\nu t}$ tends to 0, because of the condition (18).

The condition (32) yields some ellipticity with respect to X for the operator E. Roughly speaking the equation (33) is therefore hyperbolic in t,x and parabolic in t,X, but degenerates for $t=0$ into an elliptic equation in X. This degeneracy leads to the existence of parasite solutions.

In effect if we look for solutions V not depending on X and neglecting[6] the term involving A the equation (33) simplifies into the Fuschian differential equation $t\partial_t V = -V/2$, which admits an infinity of solutions i.e. $V(t) = C/\sqrt{t}$, for $C \in \mathbb{R}$. However only one is in $L^2(0,T)$, corresponding to $C = 0$; and of course we expect that our scaling is relevant enough to have a solution with $L^2(0,T)$ smoothness (at least), even in the case of the full equation (33). Except this $L^2(0,T)$ condition none initial condition at $t = 0$ has to be prescribe.

Now because of the parabolic nature of the Navier Stokes equations, we expect that v^ν and ω^ν are continuous including through $\varphi^0 = 0$ (these are the Rankine-Hugoniot conditions associated to the problem), which lead to the transmission conditions: $\tilde{V}$ and $\omega^0 + \tilde{\Omega}$ should be continuous, which (taking into account the equalities (19), (20) and (27)) is equivalent to the transmission conditions: $\tilde{V}$ and $n \wedge \omega^0 - |n|^2 \partial_X \tilde{V}$ should be continuous. More precisely this means a priori that

$$\tilde{V}|_{X=0^+,\varphi^0=0^+} - \tilde{V}|_{X=0^-,\varphi^0=0^-} = 0, \tag{34}$$

$$|n|^2\partial_X\tilde{V}|_{X=0^+,\varphi^0=0^+} - |n|^2\partial_X\tilde{V}|_{X=0^-,\varphi^0=0^-} = -(n\wedge\omega^0|_{\varphi^0=0^+} - n\wedge\omega^0|_{\varphi^0=0^-}). \tag{35}$$

Since X is the placeholder for $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ the function $\tilde{V}(t,x,X)$ needs to be defined only when X and $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ share the same sign. However it is useful to look for a profile $V(t,x,X)$ defined for (t,x,X) in the whole domain

$$\mathcal{D} := (0,T) \times \mathbb{R}^d \times \mathbb{R}.$$

As a consequence we will actually look at the following transmission conditions: for any $(t,x) \in (0,T) \times \mathbb{R}^d$,

$$[V] = 0 \;\text{ and } [\partial_X V] = -\frac{n\wedge(\omega^0_+ - \omega^0_-)}{a}, \tag{36}$$

where the brackets denote the jump $[V] = V|_{X=0^+} - V|_{X=0^-}$ across $\{X = 0\}$ and where $\omega^0_\pm$ are two functions in $L^\infty\Big((0,T), C^{s,r}\big(\mathbb{R}^d\big)\Big)$ such that $\omega^0_\pm|_{\mathcal{O}_\pm(t)} = \omega^0$.

The equation (33), with the transmission conditions (36) on the interface $\{X = 0\}$ and the conditions (18) for X at infinities are well-posed in appropriate spaces. Moreover the function $V(t,x,X)\cdot n(t,x)$ vanishes identically, what is self-consistent with the condition (27). The solution inherits the smoothness with respect to the usual variables t, x from the coefficients and is piecewise smooth and rapidly decreasing with respect to the fast variable X.

Theorem 2.2. *There exists exactly one function*

$$V(t,x,X) \in L^\infty\Big((0,T), C^{0,r}\big(\mathbb{R}^d, p-\mathcal{S}(\mathbb{R})\big)\Big) \cap L^\infty\Big(0,T; C^{s,r}\big(\mathcal{O}_\pm(t), p-\mathcal{S}(\mathbb{R})\big)\Big)$$

satisfying the equation (33) for $\pm X > 0$ and the transmission conditions (36).

We denote $p - \mathcal{S}(\mathbb{R})$ the space of the functions $f(X)$ whose restrictions to the half-lines $\mathbb{R}_\pm$ are in the Schwartz space of rapidly decreasing functions. Here the smoothness with respect to the vectorfield D is a key point. Loosely speaking when applying D to the equations and commuting, the coercivity in X is improved. Things go on the other way if one try to derivate in X first.

If piecewise smoothness of the initial data is sufficient it is possible to continue the expansion with respect to νt of the solutions of the Navier-Stokes equations. At the extreme limit if the initial data is piecewise smooth on each side of the interface $\{\varphi^0 = 0\}$ -that is if $s = +\infty$- then it

[6]Cf. Baouendi and Goulaouic's paper [2].

is possible to write a complete formal asymptotic expansion of the Navier-Stokes velocities of the form:

$$v^\nu(t,x) = v^0(t,x) + \sum_{j\geqslant 1} \sqrt{\nu t}^j \; V^j(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}) + O(\sqrt{\nu t}^\infty).$$

The construction of the V^j for $j \geqslant 2$ is more involved, they are the sum of a regular part and of a layer part. These expansions are stable: their lifetime is the one of the solution of the Euler equation ("the ground state") which traps the main part of the nonlinearity of the problem. We refer to [20] for a more detailed treatment.

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