Propriétés des coeff. binomiaux

Feuille 7

Ex. 3.17. ~~f(x)+coeff~~ $f(x) = (1+x)^n$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}.$$

$$\Rightarrow \quad \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1 \cdot 1 = (1+1)^n = 2^n$$

$$\sum_{k=0}^{n} (-1)^n \binom{n}{k} = (1-1)^n = 0.$$

$\displaystyle\sum_{0 \le 2k \le n} \binom{n}{2k}$: on calcule $\displaystyle\sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 2^n + 0$

$$= 2 \sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k}$$

on en déduit $\displaystyle\sum_{\substack{k=0 \\ k \text{ pair}}}^{n} \binom{n}{k} = \frac{1}{2} \cdot 2^n = 2^{n-1}.$

Ex. 3.18. $n \ge p \ge 1$

$$p \binom{n}{p} = p \cdot \frac{n!}{p!\,(n-p)!} = \frac{n!}{(p-1)!\,(n-p)!} = n \cdot \frac{(n-1)!}{(p-1)!\,((n-1)-(p-1))!} = n \binom{n-1}{p-1}$$

$$\Rightarrow \sum_{k=0}^{n} k \binom{n}{k} = 0 + \sum_{k=1}^{n} k \binom{n}{k} = \sum_{k=1}^{n} n \binom{n-1}{k-1} = n \sum_{k=1}^{n} \binom{n-1}{k-1}$$

$$= n \sum_{k-1=0}^{n-1} \binom{n-1}{k-1} = n \sum_{p=0}^{n-1} \binom{n-1}{p} = n\, 2^{n-1}$$

$k-1 = p$

de l'autre côté, $f'(x) = n(1+x)^{n-1}$ donc

$$f'(1) = n \cdot 2^{n-1} = n \sum_{p=0}^{n-1} \binom{n-1}{p}.$$

Ex. 3.19. Formules de Vandermonde

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}$$

: j'ai un ens. A de $n\cancel{+m}$ elmts, B de m elmts. choisir k elmts ~~p~~ en $A \cup B$ c'est fixer i, $0 \le i \le k$, choisir i elmts en A et $k-i$ en B. etc.

si $n = m$, $k = n$

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{k-i} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}^2.$$