# Variables aléatoires continues

$f(x) = kx$ densité de $X$ si $\int_0^1 f(x)\,dx = 1 \iff k = 2$

$$\int_0^1 f(x)\,dx = \int_0^1 kx\,dx = \left[k\frac{x^2}{2}\right]_0^1 = \frac{k}{2}$$

donc $f(x) = 2x$.

$$\mathbb{E}(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot \cancel{k}\, 2x\,dx = \left[2\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\mathrm{Var}(X) = \int_0^1 x^2 f(x)\,dx - \mathbb{E}(X)^2 = \int_0^1 2x^3\,dx - \frac{4}{9} = \left[2\frac{x^4}{4}\right]_0^1 - \frac{4}{9} = \frac{2}{4} - \frac{4}{9}.$$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}.$$

Même chose pour $f(x) = k(1-|x|)$ sur $[-1,1]$

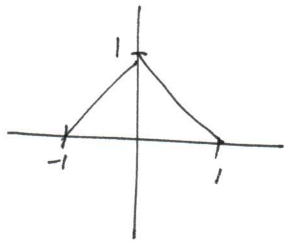
il faut $\int_{-1}^1 k(1-|x|)\,dx = 1$ or, $|x| = -x$ pour $x \in [-1,0]$ et $|x| = x$ pour $x \in [0,1]$

$$\int_{-1}^0 k(1+x)\,dx + \int_0^1 k(1-x)\,dx = \left[k\left(x+\frac{x^2}{2}\right)\right]_{-1}^0 + \left[k\left(x-\frac{x^2}{2}\right)\right]_0^1$$

$$= k(0) - k\left(-1+\frac{1}{2}\right) + k\left(1-\frac{1}{2}\right) - 0$$

$$= k\left(1 - \frac{1}{2} + 1 - \frac{1}{2}\right) = k$$

donc $k = 1$ et $f(x) = 1 - |x|$




$$\mathbb{E}(X) = \int_{-1}^1 x f(x)\,dx = \int_{-1}^1 x(1-|x|)\,dx =$$

$$= \underbrace{\int_{-1}^1 x\,dx}_{=0} - \underbrace{\int_{-1}^1 x|x|\,dx}_{=0} = 0$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \underbrace{\mathbb{E}(X)^2}_{=0} = \int_{-1}^1 x^2(1-|x|)\,dx = \int_{-1}^1 x^2 - \int_{-1}^1 x^2|x|\,dx =$$

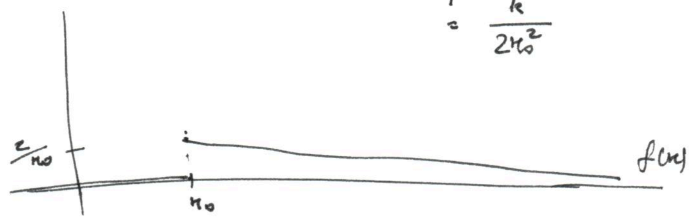
$$= \left[\frac{x^3}{3}\right]_{-1}^1 - 2\int_0^1 x^3\,dx = \frac{1}{3} - \left(-\frac{1}{3}\right) - 2\left[\frac{x^4}{4}\right]_0^1 = \frac{2}{3} - 2\cdot\frac{1}{4} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

**Ex. 41.** $\begin{cases} f(x) = \dfrac{k}{x^3} \text{ si } x \geq x_0 \quad x_0 > 0 \\ f(x) = 0 \text{ sinon} \end{cases}$

il faut $\int_{x_0}^{+\infty} f(x)\,dx = 1$ donc $1 = \int_{x_0}^{+\infty} \frac{k}{x^3}\,dx = \left[k\,\frac{x^{-2}}{-2}\right]_{x_0}^{+\infty} = 0 - \left(\frac{k \cdot x_0^{-2}}{-2}\right) = \frac{k}{2x_0^2}$

on a $\frac{k}{2x_0^2} = 1 \iff k = 2x_0^2$

$f(x) = \frac{2x_0^2}{x^3}$




Alors

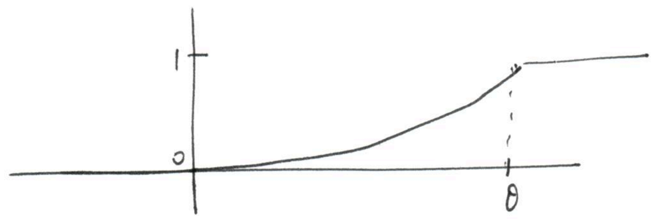
$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 \text{ si } x \leq x_0 \\ = \int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x} \frac{2x_0^2}{t^3} = \left[\cancel{2}x_0^2 \cdot \frac{t^{-2}}{-\cancel{2}}\right]_{x_0}^{x} \end{cases}$

$= \left[-\left(\frac{x_0}{t}\right)^2\right]_{x_0}^{x} = -\frac{x_0^2}{x^2} + \frac{x_0^2}{x_0^2} = 1 - \frac{x_0^2}{x^2}$ pour $x \geq x_0$.

**Ex. 42.** $f(x) = \frac{2x}{\theta^2}$ si $0 \leq x \leq \theta$, $f(x) = 0$ $\quad \theta > 0$

$\int_0^\theta \frac{2x}{\theta^2}\,dx = \frac{1}{\theta^2}\left[x^2\right]_0^\theta = \frac{1}{\theta^2}\,\theta^2 = 1$ ok ; densité.

Alors $F(x) = \mathbb{P}(X \leq x) = \begin{cases} = 0 \text{ si } x < 0 \\ = \int_0^x \frac{2t}{\theta^2}\,dt = \frac{1}{\theta^2}\left[t^2\right]_0^x = \frac{1}{\theta^2}(x^2) = \frac{x^2}{\theta^2} \text{ si } 0 \leq x \leq \theta \\ = 1 \text{ si } x > \theta \end{cases}$




$E(X) = \int_0^\theta x f(x)\,dx = \int_0^\theta x \cdot \frac{2x}{\theta^2}\,dx = \frac{1}{\theta^2}\int_0^\theta 2x^2\,dx =$

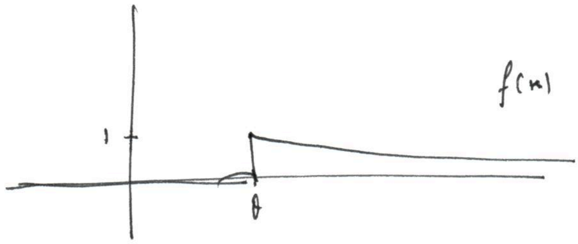
$= \frac{1}{\theta^2}\left[2\,\frac{x^3}{3}\right]_0^\theta = \frac{1}{\theta^2} \cdot 2 \cdot \frac{\theta^3}{3} = \frac{2}{3}\theta^2$.

$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \int_0^\theta x^2 f(x)\,dx - \frac{4}{9}\theta^4 = \int_0^\theta \frac{2x^3}{\theta^2}\,dx - \frac{4}{9}\theta^4 = \frac{2}{\theta^2}\left[\frac{x^4}{4}\right]_0^\theta - \frac{4}{9}\theta^4 =$

$= \frac{\cancel{2}}{\theta^2}\,\frac{\theta^4}{\cancel{4}_2} - \frac{4}{9}\theta^4 = \frac{\theta^2}{2} - \frac{4}{9}\theta^4$.

Ex. 43.

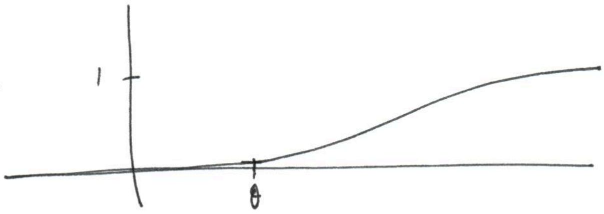
$f(x) = e^{-(x-\theta)} \quad x > \theta \qquad f(x) = 0$ sinon




$$F(x) = \begin{cases} 0 & \text{si } x < \theta \\ \int_\theta^x e^{-(t-\theta)}\,dt = \end{cases}$$

$$= \left[-e^{-(t-\theta)}\right]_\theta^x = -e^{-(x-\theta)} + e^0 = 1 - e^{-(x-\theta)}$$

si $x \geq \theta$




$X_1, \dots, X_n$ i.i.d.

$M_n = \min(X_1, \dots, X_n)$

On détermine $F_{M_n}(x) = \mathbb{P}(M_n \leq x) = \mathbb{P}\left(\min(X_1, \dots, X_n) \leq x\right) = 1 - \mathbb{P}\left(\min(X_1, \dots, X_n) > x\right) =$

$= 1 - \mathbb{P}\left(\{X_1 > x\} \cap \dots \cap \{X_n > x\}\right) = 1 - \mathbb{P}(X_1 > x) \cdots \mathbb{P}(X_n > x) =$

$$= 1 - \Big(1 - \underbrace{\mathbb{P}(X_1 \leq x)}_{F_{X_1}(x)}\Big) \cdots \Big(1 - \underbrace{\mathbb{P}(X_n \leq x)}_{F_{X_n}(x)}\Big) = \begin{cases} 0 & \text{si } x < \theta \\ 1 - \left(1 - 1 + e^{-(x-\theta)}\right) \cdots \left(1 - 1 + e^{-(x-\theta)}\right) = \end{cases}$$

$$= 1 - \left(e^{-(x-\theta)}\right)^n = 1 - e^{-n(x-\theta)}$$

donc $F_{M_n}(x) = \begin{cases} 0 & \text{si } x < \theta \\ 1 - e^{-n(x-\theta)} & \text{si } x \geq \theta \end{cases}$

pour trouver la densité, il suffit de dériver $F_{M_n}(x)$ : $f_{M_n}(x) = \begin{cases} 0, & x < \theta \\ n\,e^{-n(x-\theta)}, & x \geq \theta. \end{cases}$

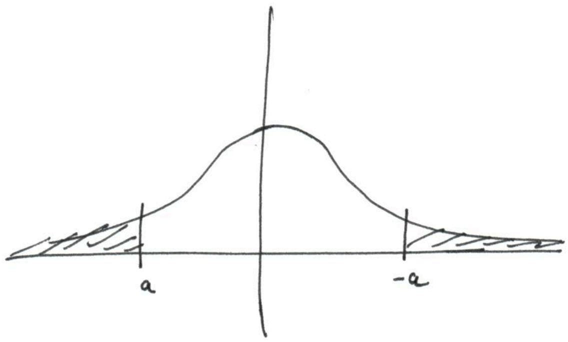
Ex. 44. $X \sim \mathcal{N}(0,1)$ On fait ces calculs avec les tables

a) $\mathbb{P}(0{,}53 < X < 2{,}73) = \mathbb{P}(X < 2{,}73) - \mathbb{P}(X < 0{,}53) = 0{,}9968 - 0{,}7019 = 0{,}2949$

$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X \leq a)$

b) ~~$\mathbb{P}($~~ si $a \leq 0$ :

$\mathbb{P}(X \leq a) =$ ~~$1-\mathbb{P}(X$~~ $\mathbb{P}(X > -a) = 1 - \mathbb{P}(X \leq -a)$




donc $\mathbb{P}(X \leq -0{,}53) = 1 - \mathbb{P}(X \leq 0{,}53)$

$= 1 -$ ~~$\mathbb{P}$~~ $0{,}7019$

et donc $\mathbb{P}(-0{,}53 < X < 2{,}73) =$ ~~$\mathbb{P}($~~ $0{,}9968 - 1 + 0{,}7019 = 0{,}6987$

c) $\mathbb{P}(|X| \geq a) = 2\,\mathbb{P}(X > a) = 2\left(1 - \mathbb{P}(X < a)\right)$ par symétrie

$(a > 0)$

donc $\mathbb{P}(|X| > 2{,}27) = 2\left(1 - 0{,}9884\right) = 0{,}0232$

d) $\mathbb{P}(|X| < a) = 1 - \mathbb{P}(|X| > a)$
$= 1 - 2\left(1 - \mathbb{P}(X<a)\right)$
$= 2\,\mathbb{P}(X<a) - 1$

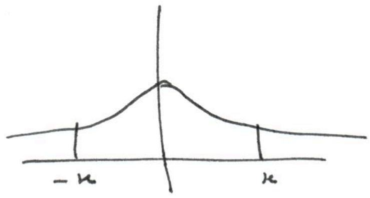
donc $\mathbb{P}(|X| < 1,45) = 2 \cdot 0,9265 - 1 \approx 0,853$

e) $\mathbb{P}(|X| > x) = 0,78$ (table 2)

$x = 0,279$

f) $\mathbb{P}(|X| < x) = 1 - \mathbb{P}(|X| > x) = 0,22$ alors $\mathbb{P}(|X| > x) = 1 - 0,22 = 0,78$

donc $x = 0,279$




g) $\mathbb{P}(|X| < x) = 0,93$ alors

$\mathbb{P}(X > x) = 1 - 0,93$

$\mathbb{P}(|X| > x) = 2(1 - 0,93) = 0,14 \Rightarrow x = 1,476$

h) $\mathbb{P}(X < x) = 0,22 \Rightarrow \left(\mathbb{P}(|X| > x) = 2(1 - 0,22) = 1,56\right.$ ~~> 1~~ impossible donc$\left.\right)$

$x$ doit être négatif $x < 0$

alors $\mathbb{P}(|X| > x) = 2 \cdot 0,22 = 0,44$ et $x = 0,772$.

Ex. 45. $X \sim \mathcal{N}(\mu, \sigma)$

$\mathbb{P}(X < 18) = \frac{25}{100} = 0,25$

$\mathbb{P}(18 < X < 25) = 0,4$

$\mathbb{P}(X > 25) = 0,35$

Soit $Y = \frac{X - \mu}{\sigma}$, alors $Y$ est normale centrée réduite. On sait

$\mathbb{P}\left(Y < \frac{18-\mu}{\sigma}\right) = 0,25$ ~~< 0~~ donc $\frac{18-\mu}{\sigma} < 0 \Rightarrow$

$\mathbb{P}\left(|Y| > \frac{\mu - 18}{\sigma}\right) = 2 \cdot 0,25 = 0,5 \xRightarrow{\text{(table)}} \frac{\mu - 18}{\sigma} = 0,674$

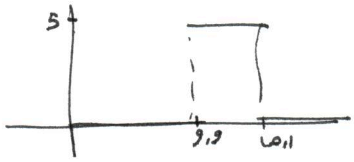
de même $\mathbb{P}\left(Y \geq \frac{25-\mu}{\sigma}\right) = 0,35$ donc $\frac{25-\mu}{\sigma} > 0$ et

$\mathbb{P}\left(|Y| \geq \frac{25-\mu}{\sigma}\right) = 2 \cdot 0,35 = 0,7 \xRightarrow{\text{(table)}} \frac{25-\mu}{\sigma} = 0,385$

$\Rightarrow \begin{cases} \mu - 18 = \sigma \cdot 0,674 \\ 25 - \mu = \sigma \cdot 0,385 \end{cases} \Rightarrow -18 + 25 = \sigma \cdot (0,674 + 0,385)$

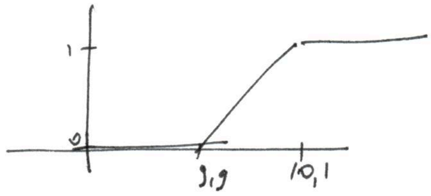
$\mu = 18 + \sigma \cdot 0,674 = 18 + \frac{-18 + 25}{0,674 + 0,385} \cdot 0,674$
$= 22,46$

Ex. 46. $X \sim U(9{,}9,\ 10.1)$




$$f(x) = \begin{cases} \frac{1}{0{,}2} = 5 \text{ pour } & 9{,}9 \leq x \leq 10{,}1 \\ 0 \text{ sinon} & \end{cases}$$

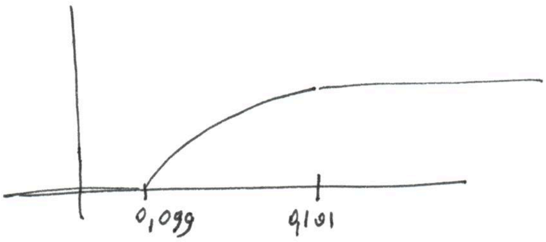
$$F(x) = \begin{cases} \int_{9{,}9}^{x} 5\, dt = 5(x - 9{,}9) & 9{,}9 \leq x \leq 10{,}1 \\ 0 & x \leq 9{,}9 \\ 1 & x > 10{,}1 \end{cases}$$




On considère $\frac{1}{X}$, qui varie entre $\frac{1}{10{,}1} = 0{,}099$ et $\frac{1}{9{,}9} = 0{,}101$

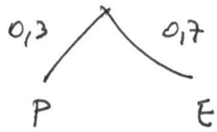
$$F(x) = P\left(\frac{1}{X} \leq x\right) = \begin{cases} 0 & \text{si } x < 0{,}099 \\ P\left(X \geq \frac{1}{x}\right) = 1 - F\left(\frac{1}{x}\right) & \text{si } 0{,}099 \leq x \leq 0{,}101 \\ 1 & x \geq 0{,}101 \end{cases}$$

$$= 1 - 5\left(\frac{1}{x} - 9{,}9\right)$$




donc $f_{\frac{1}{X}}$ s'obtient par dérivation

$$f(x) = \begin{cases} -5\left(-\frac{1}{x^2}\right) = \frac{5}{x^2} & \text{si } 0{,}099 \leq x \leq 0{,}101 \\ 0 \text{ sinon} & \end{cases}$$




Ex. 47. $v_p \sim \mathcal{N}(7.8,\ 0.8)$

$v_e \sim \mathcal{N}(8.2,\ 0.8)$

prob. totales

$$P(V \leq 7.4) = P(\{V \leq 7.4\} \mid P)\, P(P) + P(\{V \leq 7.4\} \mid E)\, P(E)$$

$$= P(v_p \leq 7.4)\, P(P) + P(v_e \leq 7.4)\, P(E)$$

$$= P\left(\frac{v_p - 7.8}{0.8} \leq \underbrace{\frac{7.4 - 7.8}{0.8}}_{= -0.5}\right) P(P) + P\left(\frac{v_e - 8.2}{0.8} \leq \underbrace{\frac{7.4 - 8.2}{0.8}}_{= -1}\right) P(E)$$

$$= \underbrace{(1 - 0.6915)}_{= 1 - P(\cdots \leq 0.5)} \cdot 0{,}3 + \underbrace{(1 - 0.8413)}_{= 1 - P(\cdots \leq 1)} \cdot 0.7 = 0.2$$

Ex. 47 (suite)

2. Bayes

$$\mathbb{P}(P \mid \{V \leq 7.4\}) = \frac{\mathbb{P}(\{V \leq 7.4\} \mid P)\ \mathbb{P}(P)}{\mathbb{P}(\{V \leq 7.4\})} = \frac{\mathbb{P}(\{V \leq 7.4\} \mid P)\ \mathbb{P}(P)}{\mathbb{P}(\{V \leq 7.4\} \mid P)\ \mathbb{P}(P) + \mathbb{P}(\{V \leq 7.4\} \mid E)\ \mathbb{P}(E)}$$

$$= \frac{(1-0.6915)\cdot 0.3}{0.2} \approx 0.46$$

3. ~~P~~ On a vu que

$$\underbrace{\mathbb{P}(V \leq x)}_{= F_V(x)} = \mathbb{P}(\{V \leq x\} \mid P)\ \cancel{0.3}\ \mathbb{P}(P) + \mathbb{P}(\{V \leq x\} \mid E)\ \mathbb{P}(E)$$

$$= F_{V_P}(x)\ \mathbb{P}(P) + F_{V_E}(x)\ \mathbb{P}(E)$$

en dérivant, $f_V(x) = f_{V_P}(x)\ \mathbb{P}(P) + f_{V_E}(x)\ \mathbb{P}(E)$ <u>mélange de deux lois normales</u>

~~on en déduit que $\forall$ $\mathbb{P}(P)\, v_P + \mathbb{P}(E)\, v_E$~~

$$\left( \begin{array}{l} \mathbb{E}[X] = \mu = \sum_{i=1}^{n} w_i \mu_i \\ \mathbb{E}[(X-\mu)^2] = \sigma^2 = \sum_{i=1}^{n} w_i \left( (\mu_i - \mu)^2 + \sigma_i^2 \right) \end{array} \right)$$

<u>expliquer</u>

Dans notre cas : $\mu = 0.3 \cdot 7.8 + 0.7 \cdot 8.2 = 8.08$

$$\sigma^2 = 0.3\left( (7.8 - 8.08)^2 + 0.8^2 \right) + 0.7\left( (8.2 - 8.08)^2 + 0.8^2 \right)$$