TD 1. Espaces Vectoriels Feuille 1.

Ex. 1. $2\begin{pmatrix}1\\-3\end{pmatrix} - 5\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}2-5\cdot 2\\-6+5\end{pmatrix} = \begin{pmatrix}-8\\-1\end{pmatrix}$

$-3(x-1)+2(x^2+x+1) = -3x+3+2x^2+2x+2 = 2x^2-x+5$

$-5f+7g : x \longmapsto -5f(x)+7g(x) = -5(x^2-1)+7(2x^2-3) = 9x^2-16$

Ex. 2. i) $F_1 = \{(x,y,z)\in\mathbb{R}^3 \mid y=1\} \subset \mathbb{R}^3 = E_1$

$F_1 \not\leq E_1$ car $0 \notin F_1$.

ii) $F_2 = \{(x,y,z)\in\mathbb{R}^3 \mid x+y-z=0\} \subset \mathbb{R}^3 = E_1$

~~des~~ si (x,y,z), $(x',y',z') \in F_2 \Rightarrow (x+x', y+y', z+z') \in F_2$ car

$(x+x')+(y+y')-(z+z') = 0$ de même pour $\lambda(x,y,z)$. $0 \in F_2 \Rightarrow F_2 \leq E$.

$\qquad = 0 \qquad = 0$

$x+y-z=0 \Leftrightarrow x=-y+z \Leftrightarrow \begin{cases} x=-t+p\\ y=t\\ z=p\end{cases} \Leftrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = t\begin{pmatrix}-1\\1\\0\end{pmatrix} + p\begin{pmatrix}1\\0\\1\end{pmatrix}$

plan engendré par $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ et $\begin{pmatrix}1\\0\\1\end{pmatrix}$. $F_2 = \left\langle \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$.

iii) $F_3 = \left\{ \begin{pmatrix}\alpha+\beta\\ \alpha-\beta\\ \alpha\end{pmatrix} \;\middle|\; \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{R}^2 \right\} \subset \mathbb{R}^3$

$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\\ \alpha\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\-1\\0\end{pmatrix} \Longrightarrow F_3 = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle \leq \mathbb{R}^3$.

iv) $F_4 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = \overline{z_1}\} \subset \mathbb{C}^2$

$(z_1, z_2), (z_1', z_2') \in F_4$ ~~alors~~ on a

$\overline{z_1+z_1'} = \overline{z_1} + \overline{z_1'} = z_2 + z_2'$ ok, mais si $(z_1,z_2) \in F_4$, $\lambda \in \mathbb{C}$, on a

$(\lambda z_1, \lambda z_2)$ et $\overline{\lambda z_1} = \overline{\lambda}\,\overline{z_1} = \overline{\lambda}\, z_2 \neq \lambda z_2$ en général :

ex. $(i,-i) \in F_4$ mais $\underset{\overset{\shortparallel}{(-1,1)}}{i(i,-i)} \notin F_4 \Rightarrow F_4$ pas sous-espace.

("la conjugaison n'est pas linéaire").

Ex. 3. $F = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle \leq \mathbb{R}^3$: ~~par~~ avec générateurs

Pour avec paramètres : $\begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\alpha\\ \alpha+\beta\\ \beta\end{pmatrix} \Rightarrow \begin{cases} x=\alpha\\ y=\alpha+\beta\\ z=\beta\end{cases}$

Avec les <u>équations</u> :

$$\begin{cases} \beta = z \\ y = \alpha + z \\ x = \alpha \end{cases} \qquad \begin{cases} \beta = z \\ \alpha = x \\ \boxed{y = x + z} \end{cases}$$

$$G = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x+y+z=0 \right\} \qquad \text{équation}$$

$$x = -y - z \qquad \text{soit} \quad \begin{cases} x = -\lambda - \mu \\ y = \lambda \\ z = \mu \end{cases} \quad (\lambda, \mu) \in \mathbb{R}^2 \quad \text{paramètres}$$

$$\Rightarrow \quad \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \text{générateurs.}$$

$F \cap G$: plusieurs ~~techniques~~ méthodes, p. ex.

$$\begin{cases} y = x + z & \text{pour } F \\ x+y+z = 0 & \text{pour } G \end{cases} \qquad \begin{cases} x - y + z = 0 \\ x + y + z = 0 \end{cases} \quad \begin{matrix} \mathrm{I} \\ \mathrm{II}-\mathrm{I} \end{matrix} \begin{cases} x - y + z = 0 \\ 2y = 0 \end{cases} \qquad \begin{cases} x = -z \\ y = 0 \end{cases} \quad \text{éq.}$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \text{gén.}$$

<u>Ex. 4.</u>

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y = z = 0 \right\} \quad \Rightarrow \quad F = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle. \quad \dim F = 1$$

$$G = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x+y = 0 \right\} \quad G = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle \quad \dim G = 2$$

$$\dim F+G = \dim F + \dim G - (\dim F \cap G)$$

$$F \cap G : \quad \begin{cases} y = 0 \\ z = 0 \\ x + y = 0 \end{cases} \quad \begin{cases} y = 0 \\ z = 0 \\ x = 0 \end{cases} \qquad F \cap G = 0 \quad \Rightarrow \quad F \oplus G = \mathbb{R}^3.$$

<u>Ex. 5.</u> i) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$: ~~li~~ indép. pas générateurs.

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{cases} \alpha = 0 \\ \alpha + \beta = 0 \\ \beta = 0 \end{cases} \qquad \alpha = \beta = 0 \quad \Rightarrow \text{indép.}$$

Mais $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$.

ii) $\begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix}$ indép. générateurs.

$$\alpha\begin{pmatrix}1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\end{pmatrix} + \gamma\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{cases}\alpha+\beta=0\\ \alpha+\gamma=0\\ \beta+\gamma=0\end{cases} \qquad \begin{cases}\alpha+\beta=0\\ -\beta+\gamma=0\\ \beta+\gamma=0\end{cases} \qquad \begin{cases}\alpha=0\\ \beta=0\\ \gamma=0\end{cases}$$

dim $= 3 \Rightarrow$ générateurs.

iii) $\begin{pmatrix}-6\\2\end{pmatrix} \begin{pmatrix}9\\-3\end{pmatrix}$. On observe que $-\frac{3}{2}\begin{pmatrix}-6\\2\end{pmatrix} = \begin{pmatrix}9\\-3\end{pmatrix} \Rightarrow$ dépendants.

Non générateurs car $\begin{pmatrix}0\\1\end{pmatrix} \notin \left\langle \begin{pmatrix}-6\\2\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}3\\1\end{pmatrix} \right\rangle$. $\Rightarrow$ Pas base de $\mathbb{R}^2$.

iv) $$\alpha\begin{pmatrix}-1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\-2\\2\end{pmatrix} + \gamma\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases}-\alpha + a\gamma = 0\\ \alpha - 2\beta + b\gamma = 0\\ 2\beta + c\gamma = 0\end{cases} \ (\mathrm{II}+\mathrm{I}) \begin{cases}-\alpha + a\gamma = 0\\ -2\beta + (a+b)\gamma = 0\\ 2\beta + c\gamma = 0 \ (\mathrm{III}+\mathrm{II})\end{cases} \begin{cases}-\alpha + a\gamma = 0\\ -2\beta + (a+b)\gamma = 0\\ (a+b+c)\gamma = 0\end{cases}$$

si $a+b+c \neq 0 \Rightarrow \begin{cases}\alpha=0\\ \beta=0\\ \gamma=0\end{cases} \Rightarrow$ indép.

si $a+b+c = 0 \Rightarrow \begin{cases}-\alpha = -a\gamma\\ -2\beta = -(a+b)\gamma\end{cases} \quad \begin{cases}\alpha = a\gamma\\ \beta = \frac{1}{2}(a+b)\gamma\end{cases} \Rightarrow$ dépendants.

$\Rightarrow$ pas générateurs de $\mathbb{R}^3$.

Ex. 6. base canonique de $\mathbb{R}_2(x)$: $1, x, x^2$ $\quad \dim_{\mathbb{R}} \mathbb{R}_2[x] = 3$.

$F = \{ P \in \mathbb{R}_2(x) \mid xP' = P \}$

a) $F \leq \mathbb{R}_2(x)$: $0 \in F$ ✓, $x(P+Q)' = x(P'+Q') = xP' + xQ' = P+Q$ ✓

$x(\lambda P)' = x\lambda P' = \lambda x P' = \lambda P$ ✓

b) $1 \notin F$ car $x\cdot(1)' = x \cdot 0 = 0 \Rightarrow F \subsetneq \mathbb{R}_2[x]$.

c) Soit $P(X) = a_0 + a_1X + a_2X^2$ $\qquad a_1X + 2a_2X^2 = a_0 + a_1X + a_2X^2 \Leftrightarrow$

$\cancel{a_1 + 2a_2X = a_0 + a_1X + a_2X^2}$

$\Rightarrow XP'(X) = X(a_1 + 2a_2X) = P(X) \Leftrightarrow$

$\cancel{\begin{cases}a_0 = a_1\\ 2a_2 = a_1\\ a_2 = 0\end{cases}} \quad \cancel{\begin{cases}a_0 = a_1\\ a_2 = \frac{1}{2}a_1\\ a_2 = 0\end{cases}} \quad \cancel{\begin{cases}a_2 = 0\\ a_1 = 0\\ a_0 = 0\end{cases}} \Leftrightarrow \quad \cancel{P(X) = a_1 +}$

$$\begin{cases}0 = a_0\\ a_1 = a_1\\ 2a_2 = a_2\end{cases} \qquad \begin{cases}a_0 = 0\\ a_1 = a_1\\ a_2 = 0\end{cases} \Rightarrow P(X) = a_1X \quad a_1 \in \mathbb{R}.$$

$\Rightarrow F = \langle X \rangle$. $\dim F = 1$.

Ex. 7.

i) $\alpha u + \beta v + \gamma w = 0 \iff \alpha u(x) + \beta v(x) + \gamma w(x) = 0 \quad \forall x \in \mathbb{R}$

$\iff \alpha \cdot 1 + \beta \sin 4x + \gamma \cos 4x = 0 \quad \forall x \in \mathbb{R}$

pour $x = 0$, $x = \frac{\pi}{4}$, $x = \frac{\pi}{8}$

$$\left\{\begin{array}{ll} \alpha + 0 + \gamma = 0 & \\ \alpha + 0 - \gamma = 0 & (II - I) \\ \alpha + \beta + 0 = 0 & (III - I) \end{array}\right. \quad \left.\begin{array}{l} \alpha + \gamma = 0 \\ -2\gamma = 0 \\ \beta - \gamma = 0 \end{array}\right. \Rightarrow \gamma = \beta = \alpha = 0$$

ii) Puisque $\boxed{\sin 2x = 2 \sin x \cos x} \Rightarrow$

$\Rightarrow \quad \sin 4x = 2 \sin 2x \cos 2x$ d'où $\frac{1}{2} \sin 4x = \sin(2x) \cos(2x) \Rightarrow \frac{1}{2} v = g$.

et puisque $\sin^2(2x) = \dfrac{1 - \cos 4x}{2} = \frac{1}{2} - \frac{1}{2} \cos 4x$ on a $f = \frac{1}{2} u - \frac{1}{2} w$.