TD. 2. Applications linéaires Feuille 1

Ex. 1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x \\ 0 \\ x-y \end{pmatrix}$$

$$f(u+v) = f\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = f\begin{pmatrix} x+x' \\ y+y' \end{pmatrix} = \begin{pmatrix} 2(x+x') \\ 0 \\ (x+x')-(y+y') \end{pmatrix} = \begin{pmatrix} 2x+2x' \\ 0 \\ (x-y)+(x'-y') \end{pmatrix} = \begin{pmatrix} 2x \\ 0 \\ x-y \end{pmatrix} + \begin{pmatrix} 2x' \\ 0 \\ x'-y' \end{pmatrix} =$$

$$= f(u) + f(v) \checkmark$$

$$f(\lambda v) = f\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ 0 \\ \lambda x - \lambda y \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 0 \\ x-y \end{pmatrix} = \lambda f(v) \checkmark \Rightarrow f \text{ linéaire.}$$

$f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$$\begin{pmatrix} z \\ u \end{pmatrix} \longmapsto \begin{pmatrix} \bar z \\ \bar u \end{pmatrix}$$

- si $\mathbb{C}^2$ est considéré comme $\mathbb{R}$-esp. vect. alors :

$$f\begin{pmatrix} z+z' \\ u+u' \end{pmatrix} = \begin{pmatrix} \overline{z+z'} \\ \overline{u+u'} \end{pmatrix} = \begin{pmatrix} \bar z + \bar z' \\ \bar u + \bar u' \end{pmatrix} = \begin{pmatrix} \bar z \\ \bar u \end{pmatrix} + \begin{pmatrix} \bar z' \\ \bar u' \end{pmatrix} = f\begin{pmatrix} z \\ u \end{pmatrix} + f\begin{pmatrix} z' \\ u' \end{pmatrix} \checkmark$$

$$f\Big(\underset{\in \mathbb{R}}{\lambda}\begin{pmatrix} z \\ u \end{pmatrix}\Big) = f\begin{pmatrix} \lambda z \\ \lambda u \end{pmatrix} = \begin{pmatrix} \overline{\lambda z} \\ \overline{\lambda u} \end{pmatrix} = \begin{pmatrix} \bar\lambda \cdot \bar z \\ \bar\lambda \cdot \bar u \end{pmatrix} = \begin{pmatrix} \lambda \cdot \bar z \\ \lambda \cdot \bar u \end{pmatrix} = \lambda \begin{pmatrix} \bar z \\ \bar u \end{pmatrix} = \lambda f\begin{pmatrix} z \\ u \end{pmatrix} \checkmark$$

$\bar\lambda = \lambda$ car $\lambda \in \mathbb{R}$

$\Rightarrow$ f est $\mathbb{R}$-linéaire.

- Si $\mathbb{C}^2$ est $\mathbb{C}$-esp. vect. alors :

additivité : ok.

$$f\Big(\underset{\in \mathbb{C}}{\lambda}\begin{pmatrix} z \\ u \end{pmatrix}\Big) = \begin{pmatrix} \bar\lambda \cdot \bar z \\ \bar\lambda \cdot \bar u \end{pmatrix} = \bar\lambda \begin{pmatrix} \bar z \\ \bar u \end{pmatrix}$$ donc f n'est pas $\mathbb{C}$-linéaire.

en gén. $\bar\lambda \neq \lambda$ ex. $\lambda = i$

Ex. 2. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x-y \\ y-z \end{pmatrix}$$

noyau : ker f : $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker f \Leftrightarrow \begin{cases} x-y=0 \\ y-z=0 \end{cases} \quad \begin{cases} x=y \\ z=y \end{cases}$ donc $\begin{cases} x=t \\ y=t \\ z=t \end{cases}$ $t \in \mathbb{R}$,

c-à-d $\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

Image : $\dim \operatorname{Im} f = \dim \mathbb{R}^3 - \dim \ker f = 3-1 = 2 = \dim \mathbb{R}^2$

~~$\operatorname{Im} f = \langle f(e_1), f(e_2), f(e_3) \rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rangle$~~ $\operatorname{Im} f \subseteq \mathbb{R}^2$ et $\dim \operatorname{Im} f = \dim \mathbb{R}^2$

$\Rightarrow$ f surjective, $\operatorname{Im} f = \mathbb{R}^2$.

$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$

$P \longmapsto P' - P$ linéaire

ker f : $P(x) = a_0 + a_1 X + a_2 X^2$

$f(P) = a_1 + 2a_2 X - (a_0 + a_1 X + a_2 X^2) = (a_1 - a_0) + (2a_2 - a_1)X - a_2 X^2$

$P \in \ker f \iff \begin{cases} a_1 - a_0 = 0 \\ 2a_2 - a_1 = 0 \\ -a_2 = 0 \end{cases} \quad \begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases}$ ker $f = 0$ donc f injective, et puisque f endomorphisme, f surjective, $\mathrm{Im} f = \mathbb{R}_2[X]$.

Ex. 3. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}13\\12\\6\end{pmatrix} \quad f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-8\\-7\\-4\end{pmatrix} \quad f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-12\\-12\\-5\end{pmatrix}$

$F = \{u \mid f(u) = u\} = \{u \mid f(u) - u = 0\} = \{u \mid (f - id)(u) = 0\} = \ker(f - id)$

$G = \{u \mid f(u) = -u\} = \ker(f + id)$ les noyaux sont toujours des sous-e.p. linéaires.

pour F :

$(f - id)\begin{pmatrix}x\\y\\z\end{pmatrix} = f\begin{pmatrix}x\\y\\z\end{pmatrix} - id\begin{pmatrix}x\\y\\z\end{pmatrix} = x f\begin{pmatrix}1\\0\\0\end{pmatrix} + y f\begin{pmatrix}0\\1\\0\end{pmatrix} + z f\begin{pmatrix}0\\0\\1\end{pmatrix} - id\begin{pmatrix}x\\y\\z\end{pmatrix}$

$= x\begin{pmatrix}13\\12\\6\end{pmatrix} + y\begin{pmatrix}-8\\-7\\-4\end{pmatrix} + z\begin{pmatrix}-12\\-12\\-5\end{pmatrix} - \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}13x - 8y - 12z - x\\ 12x - 7y - 12z - y\\ 6x - 4y - 5z - z\end{pmatrix}$

$\in \ker \iff \begin{cases} 12x - 8y - 12z = 0 \\ 12x - 8y - 12z = 0 \\ 6x - 4y - 6z = 0 \end{cases} \quad \begin{cases} 3x - 2y - 3z = 0 \\ 3x - 2y - 3z = 0 \\ 3x - 2y - 3z = 0 \end{cases} \iff 3x - 2y - 3z = 0$

1 équation $\Rightarrow \dim F = 3 - 1 = 2$. $\quad x = \frac{2}{3}y + z \quad F = \left\langle \begin{pmatrix}\frac{2}{3}\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$

même travail pour G.

Ex. 4. $f\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = x f(e_1) + y f(e_2) + z f(e_3) + t f(e_4) = x\begin{pmatrix}0\\-1\\1\\-1\end{pmatrix} + y\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} =$

$= \begin{pmatrix} y + z \\ -x - y + t \\ x + y - t \\ -x + z + t \end{pmatrix}$

$f^2\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = f(f(t)) = f\begin{pmatrix} y + z \\ -x - y + t \\ x + y - t \\ -x + z + t \end{pmatrix} = \begin{pmatrix} (-x - y + t) + (x + y - t) \\ -(y + z) - (-x - y + t) + (-x + z + t) \\ (y + z) + (-x - y + t) - (-x + z + t) \\ -(y + z) + (x + y - t) + (-x + z + t) \end{pmatrix} =$

$= \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ $\quad f \circ f = 0 \Rightarrow \forall u \in \mathbb{R}^4,\ f(f(u)) = 0 \Rightarrow f(u) \in \ker f \Rightarrow$

$\Rightarrow \mathrm{Im} f \subseteq \ker f$.

Feuille 2.

$u \in \ker f \Leftrightarrow$ $u = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$\begin{cases} y + z = 0 & \mathrm{III} \\ -x - y + t = 0 & \mathrm{II}+\mathrm{III} \\ x + y - t = 0 & \mathrm{IV}+\mathrm{III} \\ -x + z + t = 0 & \mathrm{I} \end{cases} \quad \begin{cases} x + y - t = 0 \\ 0 = 0 \\ y + z = 0 \\ y + z = 0 \end{cases} \quad \begin{cases} x + y - t = 0 \\ y + z = 0 \end{cases}$$

$$\begin{cases} x = -y + t = z + t \\ y = -z \end{cases} \qquad \ker f = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

on observe que $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = f(e_2)$

et $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = f(e_3)$

donc $\ker f \subseteq \operatorname{Im} f$. <u>Conclusion :</u> $\ker f = \operatorname{Im} f = \rangle$.

<u>Ex. 5.</u> $(E, +, \cdot)$ $f^2 = f$.

$g = id - f$ $\quad g^2(u) = g(g(u)) = g((id - f)u) = g(u - f(u)) = (id - f)(u - f(u)) =$

$= u - f(u) - f(u) + f^2(u) = u - f(u) - f(u) + f(u) = u - f(u) = (id - f)(u) = g(u) \Rightarrow g^2 = g.$

$\ker g = \{u \mid g(u) = 0\} = \{u \mid (id - f)u = 0\} = \{u \mid u = f(u)\}$ ok.

$\operatorname{Im} f = \{f(u) \mid u \in E\}$: si $v \in \operatorname{Im} f$, $v = f(u)$, $\exists u \in E$. Alors $f(v) = f^2(u) = f(u) = v$.

donc si $v \in \operatorname{Im} f$, $f(v) = v$. Donc $\operatorname{Im} f \subseteq \{u \mid f(u) = u\}$.

Vice-versa, si u est t. q. $f(u) = u$, alors $f(f(u)) = f(u) = u$, donc $u = f(f(u))$,

et $u \in \operatorname{Im} f$. Donc $\operatorname{Im} f \supseteq \{u \mid f(u) = u\}$, d'où l'égalité.

On a : $\ker f = \{u \mid f(u) = 0\}$,

$\operatorname{Im} f = \{u \mid f(u) = u\}$.

Donc $\ker f \cap \operatorname{Im} f = \{u \mid f(u) = 0$ et $f(u) = u\} = \{u \mid f(u) = 0 = u\} = 0$.

$\Rightarrow$ $\operatorname{Im} f$ et $\ker f$ en somme directe.

Si <u>E de dim. finie,</u> alors $\dim E = \dim \operatorname{Im} f + \dim \ker f \Rightarrow$

$\dim(\operatorname{Im} f \oplus \ker f) = E \Rightarrow \operatorname{Im} f \oplus \ker f = E$.

f est la projection sur $\operatorname{Im} f$, de direction $\ker f$.

<u>C'est vrai en général :</u> $v \in E$, $v = f(v) + (v - f(v))$

$f(v - f(v)) = f(v) - f(f(v)) = f(v) - f(v) = 0 \Rightarrow v - f(v) \in \ker f$.

et : $f(v) \in \operatorname{Im} f$,

donc $E = \operatorname{Im} f + \ker f$ et $\cap = 0$. $\Rightarrow \oplus$.