Matrices, Exercices

Ex. 1. $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -1 & 3 \\ -1 & 5 & -3 \\ 3 & -3 & 3 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

si $\exists\, B \neq 0$ t.q. $AB = 0$, A non inversible, car si $\exists\, C$ inverse de A,

on aurait $\underset{\underset{\underset{B}{\|}}{\underset{\mathbb{1}\cdot B}{\|}}}{CAB} = C \cdot AB = C \cdot 0 = 0$ donc $B = 0$, absurde.

Rem. Un $B \neq 0$ t.q. $AB = 0$ est un diviseur (droit) de 0.

Si A a diviseurs de 0, A non inversible.

$$AB = A(A^2 - 5A + 6\,\mathbb{1}_3) = A^3 - 5A^2 + 6A = 0.$$

on sait donc que

Donc $A^3 = 5A^2 - 6A$.

ainsi $A^4 = A \cdot A^3 = A(5A^2 - 6A) = 5A^3 - 6A^2 = 5(5A^2 - 6A) - 6A^2 = 19A^2 - 6A$.

$A^5 = A(A^4) = \dots$ etc. A^n peut s'exprimer en fonction de $\underset{\mathbb{1}}{A^0}, A^1, A^2$.

plus en gén: A^6:

$$\begin{array}{l|l} X^6 + 0X^5 + 0X^4 + 0X^3 + 0X^2 + 0X + 0 & X^3 - 5X^2 + 6X \\ \hline -X^6 + 5X^5 - 6X^4 & X^3 + 5X^2 + 19X + 65 \\ \quad 5X^5 - 6X^4 & \\ \quad -5X^5 + 25X^4 - 30X^3 & \\ \quad\quad 19X^4 - 30X^3 & \\ \quad\quad -19X^4 + 95X^3 - 114X^2 & \\ \quad\quad\quad 65X^3 - 114X^2 & \\ \quad\quad\quad -65X^3 + 325X^2 - 390X & \\ \quad\quad\quad\quad 211X^2 - 390X & \end{array}$$

$$X^6 = (X^3 - 5X^2 + 6X)(X^3 + 5X^2 + 19X + 65) + (211X^2 - 390X)$$

$\Rightarrow A^6 = 0 + 211A^2 - 390A$.

(il faut trouver le reste de la division par A).

Exercice. si $P(X) = \underset{\neq 0}{a_0} + a_1X + \dots + a_mX^m$ et $P(A) = a_0\mathbb{1} + a_1A + \dots + a_mA^m = 0$ alors

alors $\mathbb{1} = -\frac{a_1}{a_0}A - \dots - \frac{a_m}{a_0}A^m = A\underbrace{\left(-\frac{a_1}{a_0}\mathbb{1} - \dots - \frac{a_m}{a_0}A^{m-1}\right)}_{A^{-1}}$.

ex. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$P(X) = -X^3 + 2X^2 + X - 2$.

Ex. 2.

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & 2a & 2b \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \qquad DA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$AD = DA \iff \begin{cases} 2a = a \\ 2b = b \end{cases} \iff a = b = 0.$$

$$N = A - D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

matrice nilpotente.

$N = A - D \Rightarrow A = D + N$

on a $DN = ND$: $\longrightarrow$ matrice diagonale et matrice nilpotente commutent. (on la vérifie directement).

donc on peut appliquer la formule du binôme : $A^n = (N+D)^n = \sum_{k=0}^{n} \binom{n}{k} N^k D^{n-k} =$

$$= \binom{n}{0} \mathbb{1} D^n + \binom{n}{1} N D^{n-1} + \underbrace{\underbrace{\binom{n}{2} N^2 D^{n-2}}_{=0} + \cancel{\dots}}_{=0}$$

$$= D^n + n N D^{n-1}$$

$$= \begin{pmatrix} 1 & & \\ & 2^n & \\ & & 2^n \end{pmatrix} + n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 2^{n-1} & \\ & & 2^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 2^n & \\ & & 2^n \end{pmatrix} + n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2^{n-1} \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}$$

je pose $u_n = 1 \quad \forall n$:

$$\begin{cases} u_{n+1} = 1 = u_n \\ v_{n+1} = 2v_n + w_n \\ w_{n+1} = 2w_n \end{cases} \quad \text{se réécrit} \quad \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = A^n \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$$

$$\text{donc} \begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ 2^n v_0 + n2^{n-1} w_0 \\ 2^n w_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2^n + n2^{n-1} \\ 2^n \end{pmatrix}$$

$u_0 = v_0 = w_0 = 1$