Ex. sur les déterminants

Feuille 1.

Ex. 1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 0 & 2 & 4 \\ 1 & 1 & 2 \\ -1 & -1 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -2\,(5) + 4\,(0) = -10$$

$$\begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & k \end{vmatrix} = aehk.$$

Ex. 2. base $\Leftrightarrow$ $\det \underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 1 & m & m^2 \\ 1 & 3 & 9 \end{pmatrix}}_{\overset{\shortparallel}{A}} \neq 0.$

Or, $\det A = 9m + 2m^2 + 12 - 4m - 3m^2 - 18 = -m^2 + 5m - 6$

$m^2 - 5m + 6 = 0 \Leftrightarrow m = \dfrac{5 \pm \sqrt{25-24}}{2} = \dfrac{5\pm 1}{2} = 2,\ 3$

donc pour $m = 2, 3$ non base, pour $m \neq 2, 3$ base.

Ex. 3. $A_m = \begin{pmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{pmatrix}$ inversible si $\det A \neq 0$.

Or, $\det A = \det\begin{pmatrix} 1 & m & 1 \\ m & 1 & 1 \\ 1 & 1 & m \end{pmatrix}\cdot(-1) = \det\begin{pmatrix} 1 & m & 1 \\ 0 & -m^2+1 & -m+1 \\ 0 & -m+1 & -1+m \end{pmatrix}(-1) = (-1)(m-1)\det\begin{pmatrix} 1 & m & 1 \\ 0 & -m^2+1 & -m+1 \\ 0 & -1 & 1 \end{pmatrix}$

$= (-1)(m-1)(1-m)\det\begin{pmatrix} 1 & m & 1 \\ 0 & 1+m & 1 \\ 0 & -1 & 1 \end{pmatrix} = (m-1)^2 \cdot \begin{vmatrix} 1+m & 1 \\ -1 & 1 \end{vmatrix} = (m-1)^2(m+2)$ (*) les racines sont 1 et -2.

donc A_m inversible pour $m \neq 1, -2$.

$A_{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ $\qquad \det A_{-1} = (-1-1)^2(-1+2) = 4\,(1) = 4.$

$$\Rightarrow A_{-1}^{-1} = \frac{1}{\det A_{-1}}\cdot (A^c)^t = \frac{1}{4}\begin{pmatrix} +\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}^t =$$

$$= \frac{1}{4}\cdot\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}^t = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

vérification: $A_{-1}\cdot\dfrac{1}{\det A_{-1}}(A^c)^t = \dfrac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ok.

(Avec le pivot de Gauss: classique)

$P_{A_{-1}}(X) = \det(A_{-1} - X\,\mathbb{1}_3)$ (polynôme caractéristique) $= \det\begin{pmatrix} -1-X & 1 & 1 \\ 1 & -1-X & 1 \\ 1 & 1 & -1-X \end{pmatrix} =$ (fait en (*), pour $m = -1-X$)

$= (-1-X-1)^2(-1-X+2)$

$= (X+2)^2(-X+1)$

et l'on a :

$$P_{A_{-1}}(A_{-1}) = \left(A_{-1}^{2} + 2\,\mathbb{1}\right)^2\left(-A_{-1}+\mathbb{1}\right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

donc A_{-1} annule $(x^2+4x+4)(-x+1) = -x^3+x^2-4x^2\cancel{+4x}\ \cancel{-4x}+4$

$$= -x^3-3x^2+4$$

on en déduit, $-A^3-3A^2+4\,\mathbb{1}=0 \Rightarrow A\left(-A^2-3A\right) = -4\,\mathbb{1} \Rightarrow$

$$\Rightarrow A\left\{\frac{1}{4}\left(A^2+3A\right)\right\} = \mathbb{1}_3 \Rightarrow A^{-1} = \frac{1}{4}\left(A^2+3A\right)$$

$$\Rightarrow A_1^{-1} = \frac{1}{4}\left(\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + 3\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} + 3\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix} 0 & \cancel{4}\,2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

ok.

enfin $\begin{cases} -x+y+z=4 \\ x-y+z=2 \\ x+y-z=0 \end{cases} \Leftrightarrow A_{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_1^{-1}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

l'unique solution est $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Ex. 4. $D_n = \begin{vmatrix} 3 & 1 & 0 & \cdots & & 0 \\ 2^2 & 5 & 1 & \cdots & & 0 \\ 0 & 3^2 & 7 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & (n-1)^2 & (2n-1) & 1 \\ 0 & \cdots & & & n^2 & 2n+1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 & \cdots & & & 0 \\ 2^2 & 5 & 1 & \cdots & & & 0 \\ & 3^2 & 7 & 1 & \cdots & & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & (n-2)^2 & (2n-3) & 1 & 0 \\ & & & & (n-1)^2 & (2n-1) & 1 \\ & & & & & n^2 & 2n+1 \end{vmatrix}$

on développe selon la dernière ligne :

$$D_n = (2n+1)\begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 2^2 & 5 & 1 & \cdots & 0 \\ 0 & 3^2 & 7 & 1 \cdots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & & (n-1)^2 & 2n-1 \end{vmatrix} - n^2\begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 2^2 & 5 & 1 & \cdots & 0 \\ 0 & 3^2 & 7 & 1 \cdots & 0 \\ & & \ddots & \ddots & \\ & & (n-2)^2 & (2n-3) & 0 \\ & & & (n-1)^2 & 1 \end{vmatrix}$$

ici on dév. sur la dernière colonne

$$= (2n+1)\,D_{n-1} - n^2\begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 2^2 & 5 & 1 & \cdots & 0 \\ 0 & 3^2 & 7 & 1 \cdots & 0 \\ & & \ddots & \ddots & 1 \\ & & & (n-2)^2 & (2n-3) \end{vmatrix} = (2n+1)\,D_{n-1} - n^2\,D_{n-2}.$$

Feuille 2.

$$u_n = D_n - (n+1) D_{n-1}$$

$$= (2n+1) D_{n-1} - n^2 D_{n-2} - (n+1) D_{n-1}$$

$$= n D_{n-1} - n^2 D_{n-2}$$

$$= n \underbrace{\left(D_{n-1} - n D_{n-2}\right)}_{u_{n-1}} = n\, u_{n-1}$$

or a $D_2 = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = 11$ $\quad u_2 = D_2 - (2+1) D_1 = 11 - 3 \cdot 3 = 2$

$u_3 = 3 \cdot u_2 = 3 \cdot 2 \quad u_4 = 4 \cdot 3 \cdot 2$ etc. $\rightarrow \boxed{u_n = n!}$

$v_n = \dfrac{D_n}{(n+1)!}$ on a $\underset{\underset{n!}{\|}}{u_n} = v_n (n+1)! - (n+1)\, n!\, v_{n-1}$

d'où $\quad v_n - v_{n-1} = \dfrac{1}{n+1} \qquad v_n = v_{n-1} + \dfrac{1}{n+1}$

or, $\quad v_1 = \dfrac{3}{2} = 1 + \dfrac{1}{2} \qquad v_2 = v_1 + \dfrac{1}{3} = 1 + \dfrac{1}{2} + \dfrac{1}{3}$ etc.

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} = \sum_{k=1}^{n+1} \frac{1}{k}$$

et $\quad D_n = (n+1)! \displaystyle\sum_{k=1}^{n+1} \frac{1}{k} = \sum_{k=1}^{n+1} \frac{(n+1)!}{k}$