Contrôle 3, corrigé

Ex. 1. 1.a) $\deg R < 3$ donc $\deg R = 0, 1$ ou 2.

b) on écrit $A(X) = (X-\alpha)^3 Q(X) + R(X)$ alors $A(\alpha) = 0 + R(\alpha) = R(\alpha)$.

c) $A'(X) = 3(X-\alpha)^2 Q(X) + (X-\alpha)^3 Q'(X) + R'(X)$ donc $A'(\alpha) = 0 + R'(\alpha) = R'(\alpha)$.

d) $A''(X) = 6(X-\alpha)Q(X) + 3(X-\alpha)^2 Q'(X) + 3(X-\alpha)^2 Q'(X) + (X-\alpha)^3 Q''(X) + R''(X)$

donc $A''(\alpha) = R''(\alpha)$.

2. a) $X^3 - 3X^2 + 3X + 1 = (X-1)^3$

b) $n=1$: $X = 0 \cdot (X-1)^3 + X \Rightarrow R_1(X) = X$

$n=2$: $X^2 = 0 \cdot (X-1)^3 + X^2 \Rightarrow R_2(X) = X^2$

$n \geq 3$: $X^n = (X-1)^3 Q(X) + R_n(X)$ et $R_n(X) = aX^2 + bX + c$ $\quad R_n'(X) = 2aX + b$

on utilise 1. avec $\alpha = 1$, $P(X) = (X-1)^3$, $A_n(X) = X^n$ $\quad R_n''(X) = 2a$

$A_n'(X) = nX^{n-1}$

$A_n''(X) = n(n-1)X^{n-2}$

$R_n(1) = a + b + c = A_n(1) = 1$

$R_n'(1) = 2a + b = A_n'(1) = n$

$R_n''(1) = 2a = A_n''(1) = n(n-1)$

$$\Rightarrow \begin{cases} a+b+c = 1 \\ 2a+b = n \\ 2a = n(n-1) \end{cases} \rightarrow \begin{cases} a = \dfrac{n(n-1)}{2} \\ b = n - n(n-1) = n(1-n+1) = n(2-n) \\ c = 1 - a - b = 1 - \dfrac{n(n-1)}{2} - n(2-n) \end{cases}$$

$$= \frac{2 - n^2 + n - 4n + 2n^2}{2} = \frac{n^2 - 3n + 2}{2} = \frac{(n-1)(n-2)}{2}$$

donc $R_n(X) = \dfrac{n(n-1)}{2} X^2 + n(2-n)X + \dfrac{(n-1)(n-2)}{2}$

cette formule vaut aussi pour $n=1$.

Ex. 2.

1. $\dim \mathbb{R}_n[X] = n+1, \quad \mathcal{B}_n = \{1, x, x^2, \dots, x^n\}$

2. a) $$\Phi(P+Q) = \begin{pmatrix} (P+Q)(2) \\ (P+Q)''(1) \\ (P+Q)(0) - 2(P+Q)(1) \end{pmatrix} = \begin{pmatrix} P(2) + Q(2) \\ P''(1) + Q''(1) \\ P(0) + Q(0) - 2P(1) - 2Q(1) \end{pmatrix} = \Phi(P) + \Phi(Q)$$

et $\Phi(\lambda P) = \lambda \Phi(P)$. donc Φ linéaire.

$\dim \mathbb{R}_3[X] = 4$, $\dim \mathbb{R}^3 = 3$, donc Φ n'est sûrement pas un isomorphisme.

b) $$1 \longmapsto \begin{pmatrix} 1 \\ 0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad x \longmapsto \begin{pmatrix} 2 \\ 0 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$x^2 \longmapsto \begin{pmatrix} 4 \\ 2 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}, \quad x^3 \longmapsto \begin{pmatrix} 8 \\ 6 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ -2 \end{pmatrix}$$

c) $$\begin{array}{ll} \mathcal{B}_3 : & \quad 1 \quad\;\; x \quad\;\; x^2 \quad\; x^3 \\ M = & \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 2 & 6 \\ -1 & -2 & -2 & -2 \end{pmatrix} \end{array}$$

~~d) $M \sim$ III (1 2 4 8) I−III (0 4 6 10) II (0 0 2 6)~~ ~~rk $M = 3$, donc~~

~~dim Im Φ = 3~~

~~dim ker Φ = 4 − 3 = 1 (formule du rang)~~

d) $$M \sim \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}+\mathrm{I} \end{array} \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rk $M = 2$ donc $\dim \operatorname{im} \Phi = 2$

et $\dim \ker \Phi = 4 - 2 = 2$ (formule du rang).

$$\begin{cases} a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \\ a_2 + 3a_3 = 0 \end{cases} \qquad \begin{cases} a_0 = -2a_1 - 8a_3 - 4a_2 \\ a_2 = -3a_3 \end{cases}$$

$$\begin{cases} a_0 = -2a_1 + 12a_3 - 8a_3 \\ a_2 = -3a_3 \end{cases} \qquad \begin{cases} a_0 = -2a_1 + 4a_3 \\ a_2 = -3a_3 \end{cases} \Rightarrow \cancel{\mathrm{Vect}}\ \ker \Phi = \mathrm{Vect}\left\{ \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

c-à-d $\ker \Phi = \mathrm{Vect}\{x^3 - 3x^2 + 4, \ x - 2\}$

et $\operatorname{im} \Phi = \mathrm{Vect}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \right\}$ (je choisis deux colonnes indépendantes de M).