Corrigé du contrôle 2

Ex. 1. $A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$

1. pol. caractéristique : $\det \begin{pmatrix} 5-x & 6 \\ -3 & -4-x \end{pmatrix} = -20 - 5x + 4x + x^2 + 18 = x^2 - x + 2$

2. $\lambda_{1,2} = \dfrac{1 \pm \sqrt{1+8}}{2} = \dfrac{1 \pm 3}{2} = \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 2 \end{cases}$

3. A est diagonalisable, car λ_1, λ_2 réels et $\lambda_1 \neq \lambda_2$. $\Delta = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 6 & 6 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 6x + 6y = 0 \\ -3x - 3y = 0 \end{cases} \iff x + y = 0 \qquad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 6 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 3x + 6y = 0 \\ -3x - 6y = 0 \end{cases} \iff x + 2y = 0 \qquad v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

6. $P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

7. $P^{-1} = \dfrac{1}{-1+2} \begin{pmatrix} -1 & -2 \\ 1 & +1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & +1 \end{pmatrix}$

on vérifie que

$$P^{-1}AP = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \Delta \text{ ok.}$$

alors $A = P \Delta P^{-1}$

$$\Rightarrow A^n = P \Delta^n P^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^n & 2 \cdot 2^n \\ -(-1)^n & -2^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -(-1)^n + 2 \cdot 2^n & -2(-1)^n + 2 \cdot 2^n \\ (-1)^n - 2^n & 2(-1)^n - 2^n \end{pmatrix}$$

Ex. 2.

$$g'(x) = \frac{\left(\frac{1}{\cos^2 x} + 10x\right)\cos x - \left(\tan x + 5x^2\right)(-\sin x)}{\cos^2 x}$$

$$h'(x) = \frac{1}{1+\left(e^x\right)^2} \cdot e^x$$

Ex. 3.

$$\int_0^{\frac{\pi}{2}} x\cos x \, dx = \left[x \cdot \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \cdot 1 \, dx$$

$$= \frac{\pi}{2} \cdot 1 - 0 - \left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + (0 - 1) = \frac{\pi}{2} - 1$$

$$\int_{\frac{1}{2}}^1 \ln 2x \, dx = \left[x \cdot \ln 2x\right]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 x \cdot \frac{1}{\cancel{2}x} \cdot \cancel{2} \, dx$$

$$= \ln 2 - \underbrace{\frac{1}{2}\ln 1}_{=0} - \int_{\frac{1}{2}}^1 1 \cdot dx$$

$$= \ln 2 - \left[x\right]_{\frac{1}{2}}^1 = \ln 2 - \frac{1}{2}$$

Ex. 4.

$$\int_0^1 \frac{dx}{e^x + e^{-x}} \cancel{dx} =$$

$e^x = t$

$x = \ln t$

$dx = \frac{1}{t} dt$

$$= \int_{t=1}^{t=e} \frac{\frac{1}{t}\,dt}{t + \frac{1}{t}} \cancel{dt} = \int_{t=1}^{t=e} \frac{1}{t^2+1} \, dt = \left[\arctan t\right]_{t=1}^{t=e} =$$

$$= \arctan e - \arctan 1 = \arctan e - \frac{\pi}{4}$$

$$\int_0^{\frac{1}{2}} \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

$\arcsin x = t$

$x = \sin t$

$dx = \cos t \, dt$

$$= \int_{t=0}^{t=\frac{\pi}{6}} t \cdot \frac{1}{\underbrace{\sqrt{1-\sin^2 t}}_{\sqrt{\cos^2 t} = \cos t}} \cdot \cos t \, dt = \int_{t=0}^{t=\frac{\pi}{6}} t \cdot \frac{1}{\cancel{\cos t}} \cancel{\cos t} \, dt =$$

$$= \int_{t=0}^{t=\frac{\pi}{6}} t \, dt = \left[\frac{1}{2}t^2\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 .$$