Feuille 10 : Fractions rationnelles

1) $$\frac{1}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x+2}$$

a : $$\frac{1}{(x+1)(x+2)} = a + \frac{b}{x+1}x + \frac{c}{x+2}x$$

$x=0$ · $$\frac{1}{2} = a + 0 \qquad a = \frac{1}{2}$$

b : $$\frac{1}{x(x+2)} = \frac{a}{x}(x+1) + b + \frac{c}{x+2}(x+1)$$

$x=-1$ $$\frac{1}{(-1)(1)} = b \qquad b = -1$$

c : $$\frac{1}{x(x+1)} = \frac{a}{x}(x+2) + \frac{b}{x+1}(x+2) + c$$

$x=-2$ $$\frac{1}{(-2)(-1)} = c \qquad c = \frac{1}{2}$$

~~etc~~

2) $$\frac{1}{x(x+i)^2(x-i)^2} = \frac{a}{x} + \frac{b}{(x+i)} + \frac{c}{(x+i)^2} + \frac{d}{(x-i)} + \frac{e}{(x-i)^2}$$

$$\frac{1}{(x+i)^2(x-i)^2} = a + x\left(\text{———}\right)$$

$x=0$: $$\frac{1}{(-1)(-1)} = a \qquad a = 1$$

$(x+i)^2$: $$\frac{1}{x(x-i)^2} = c + (x+i)\left(\text{machin}\right)$$

$x=-i$ $$\frac{1}{(-i)(-2i)^2} = c \qquad \cancel{c=1/4i} \qquad c = \frac{1}{4i} = -\frac{1}{4}i$$

$$(-i)(-4) = 4i$$

$(x-i)^2$: $$\frac{1}{x(x+i)^2} = e + (x-i)\left(\text{——}\right)$$

$x=i$ $$\frac{1}{i(2i)^2} = e \qquad e = \frac{1}{-4i} = \frac{1}{4}i$$

$$-4i$$

Maintenant il reste à dét. les coeffs b et d:

p. ex. j'évalue en $2i, -2i$

$$\begin{cases} \dfrac{1}{(2i)(3i)^2(i)^2} = \dfrac{1}{2i} + \dfrac{b}{3i} + \dfrac{-\frac{1}{4}i}{(3i)^2} + \dfrac{d}{i} + \dfrac{\frac{1}{4}i}{-1} \\[2ex] \dfrac{1}{(-2i)(-i)^2(-3i)^2} = \dfrac{1}{-2i} + \dfrac{b}{-i} + \dfrac{-\frac{1}{4}i}{(-i)^2} + \dfrac{d}{-3i} + \dfrac{\frac{1}{4}i}{(-3i)^2} \end{cases}$$

c.a.d

$$\begin{cases} \dfrac{1}{18i} = -\frac{1}{3}ib - id - \frac{1}{2}i + \frac{1}{36}i - \frac{1}{4}i \\[2ex] \dfrac{1}{-i18} = ib + \frac{1}{3}id + \frac{1}{2}i + \frac{1}{4}i - \frac{1}{36}i \end{cases}$$

$$\begin{cases} -\frac{1}{18} = -\frac{1}{3}b - d - \frac{1}{2} + \frac{1}{36} - \frac{1}{4} \\[1ex] \frac{1}{18} = b + \frac{1}{3}d + \frac{1}{2} + \frac{1}{4} - \frac{1}{36} \end{cases} \qquad \begin{cases} -\frac{1}{3}b - d = -\frac{1}{18} + \frac{1}{2} - \frac{1}{36} + \frac{1}{4} = \frac{-2+18-1+9}{36} = \cancel{\frac{24}{36}} = \frac{2}{3} \\[1ex] b + \frac{1}{3}d = \frac{1}{18} - \frac{1}{2} - \frac{1}{4} + \frac{1}{36} = -\frac{2}{3} \end{cases}$$

$$\begin{cases} -b - 3d = 2 \\ 3b + d = -2 \end{cases} \qquad b = \frac{\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix}} = \frac{-4}{8} = -\frac{1}{2}$$

$$d = \frac{\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}}{8} = \frac{-4}{8} = -\frac{1}{2}$$

ergo $F(x) = \dfrac{1}{x} - \dfrac{\frac{1}{2}}{(x+i)} - \dfrac{\frac{1}{4}i}{(x+i)^2} - \dfrac{\frac{1}{2}}{x-i} + \dfrac{\frac{1}{4}i}{(x-i)^2}$.

Sage: $\dfrac{1}{x} + \dfrac{-\frac{1}{2}x + \frac{3}{4}i}{(x-i)^2} + \dfrac{-\frac{1}{2}x - \frac{3}{4}i}{(x+i)^2}$

$$\frac{-\frac{1}{2}x - \frac{1}{2}i - \frac{1}{4}i}{(x+i)^2} = \frac{-\frac{1}{2}x - \frac{3}{4}i}{(x+i)^2}$$

Ok, les deux écritures s'équivalent

$$\frac{-\frac{1}{2}(x-i) + \frac{1}{4}i}{(x-i)^2} = \frac{-\frac{1}{2}x + \frac{3}{4}i}{(x-i)^2} \quad \text{ok.}$$

$$\frac{a}{x+\alpha} + \frac{b}{(x+\alpha)^2} = \frac{a(x+\alpha)+b}{(x+\alpha)^2} = \frac{aX + (a\alpha+b)}{(x+\alpha)^2}$$

$$\frac{aX+b}{(X+\alpha)^2} = \frac{A}{X+\alpha} + \frac{B}{(X+\alpha)^2} = \frac{AX + A\alpha + B}{(X+\alpha)^2}$$

$A = a \quad A\alpha + B = b \quad B = b - A\alpha$

$$x^4+x^2+1 = (x^2+ax+b)(x^2+cx+d)$$

$$\cancel{x^2 = \frac{-1\pm\sqrt{1-4}}{2}}$$

$$= x^4 + cx^3 + dx^2 + ax^3 + acx^2 + \cancel{bd}\, adx + bx^2 + bcx + bd$$

$$= x^4 + (c+a)x^3 + (d+ac+b)x^2 + (ad+bc)x + bd$$

$$\begin{cases} a+c=0 \\ d+ac+b=1 \\ ad+bc=0 \\ bd=1 \end{cases} \qquad \begin{cases} a=-c \\ b=\frac{1}{d} \\ d-c^2+\frac{1}{d}=1 \\ \cancel{ad} - dc + \frac{1}{d}c = 0 \end{cases}$$

$$c\left(-d+\frac{1}{d}\right)=0$$

$c=0$:

$$\begin{cases} c=0 \\ a=0 \\ d+\frac{1}{d}=1 \\ b=\frac{1}{d} \end{cases} \qquad \begin{cases} c=0 \\ a=0 \\ d^2+1=d \\ b=\frac{1}{d} \end{cases} \qquad \begin{cases} a=0 \\ c=0 \\ d^2-d+1=0 \\ b=\frac{1}{d} \end{cases} \quad \cancel{\mathbb{R}}$$

$d=\frac{1}{d}$ $\quad d=\pm 1$

$$\begin{cases} d=1 \\ b=1 \\ a=-c \\ -c^2=-1 \end{cases} \qquad \begin{cases} d=1 \\ b=1 \\ a=-c \\ c=\pm 1 \end{cases}$$

$$\begin{cases} d=-1 \\ b=-1 \\ -c^2=3 \end{cases} \quad \text{NOU}$$

$$\boxed{(x^2-x+1)(x^2+x+1)}$$

$$\frac{x^{\cancel{2}}}{x^4+x^2+1} = \frac{ax+b}{x^2-x+1} + \frac{cx+d}{x^2+x+1}$$

$$\cancel{\frac{x}{x^2+x+1} = ax+b+}\quad \left(\frac{x}{x^2+x+1} = ax+b + \frac{(cx+d)(x^2+x+1)}{x^2+x+1} \right)$$

$$= \frac{(ax+b)(x^2+x+1) + (cx+d)(x^2-x+1)}{x^4+x^2+1}$$

$$\begin{array}{l|l} x^3 & a+c=0 \\ x^2 & a+b-c+d=0 \\ x & a+b+c-d=1 \\ 1 & b+d=0 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \cancel{0} & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$\frac{\frac{1}{2}}{x^2-x+1} + \frac{-\frac{1}{2}}{x^2+x+1}$$

$$x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{3}\,i}{2} \qquad x = \frac{-1\pm\sqrt{3}\,i}{2}$$

$$\frac{\frac{1}{2}}{x^2-x+1} = \frac{A}{x-\frac{1-\sqrt{3}\,i}{2}} + \frac{B}{x-\frac{1+\sqrt{3}\,i}{2}} = \frac{\frac{1}{2\sqrt{3}}\,i}{x-\frac{1-\sqrt{3}\,i}{2}} + \frac{-\frac{1}{2\sqrt{3}}\,i}{x-\frac{1+\sqrt{3}\,i}{2}}$$

$$\frac{\frac{1}{2}}{x+\frac{1+\sqrt{3}\,i}{2}} = A + B\left(\text{———}\right) \qquad A = \frac{\frac{1}{2}}{\frac{1-\sqrt{3}\,i}{2}-\frac{1+\sqrt{3}\,i}{2}} = \frac{1}{\cancel{1}-\sqrt{3}\,i\,\cancel{-1}-\sqrt{3}\,i} = \frac{1}{-2\sqrt{3}\,i} = \frac{i}{2\sqrt{3}}$$

$$\frac{\frac{1}{2}}{x+\frac{1+\sqrt{3}\,i}{2}} = A\left(\text{———}\right) + B \qquad B = \frac{\frac{1}{2}}{\frac{1+\sqrt{3}\,i}{2}-\frac{1-\sqrt{3}\,i}{2}} = \frac{1}{1+\sqrt{3}\,i-1+\sqrt{3}\,i} = \frac{1}{2\sqrt{3}\,i} = -\frac{i}{2\sqrt{3}}$$

etc.

<u>Ex. 114.</u> $$\frac{1}{(x^2-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$$\parallel$$

$$\frac{1}{(x-1)(x+1)^3}$$

A: $$\frac{1}{(x+1)^3} = A + \left(\text{———}\right)(x-1) \qquad A = \frac{1}{(1+1)^3} = \frac{1}{8}$$

D: $$\frac{1}{x-1} = D + \left(\text{———}\right)(x+1) \qquad D = \frac{1}{-2} = -\frac{1}{2}$$

~~Ex~~ B: $$\frac{1}{(x-1)(x+1)^2} = A\,\frac{x+1}{x-1} + B + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$\lim_{x\to+\infty}$: $0 = A + B + 0 \qquad B = -A = -\frac{1}{8}$

j'évalue en 0 : $$\frac{1}{(-1)(1)} = \frac{A}{-1} + B + C + D$$

$$-1 = -\frac{1}{8} - \frac{1}{8} + C - \frac{1}{2} \qquad C = -1 + \frac{1}{8} + \frac{1}{8} + \frac{1}{2} = -\frac{1}{4}$$

$D = -\frac{4}{8} = -\frac{1}{2}$

$= \frac{1}{8} - \frac{1}{4} + D = -\frac{7}{8} =$

$$\frac{-\frac{1}{8}x^2 - \frac{1}{2}x - \frac{7}{8}}{(x+1)^3} = \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

~~2B+C~~ ~~C=~~ D=

$2B + C = -\frac{1}{2} \qquad C = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

$$= \frac{B(x+1)^2 + C(x+1) + D}{(x+1)^3} = x^2(B) + x(2B+C) + B+C+D$$

ok.

$$\frac{x^7+1}{(x^2+1)(x^2+x+1)} =$$

$= x^4 + x^3+x^2+x^2+x+1$

$= x^4+x^3+2x^2+x+1$

$$x^7+0x^6+0x^5+0x^4+0x^3+0x^2+0x+1 \;\Big|\; x$$

$$
\begin{array}{l|l}
x^7 \qquad\qquad 1 & x^4+x^3+2x^2+x+1 \\
-x^7-x^6-2x^5-x^4-x^3 & \hline x^3-x^2-x+2 \\
\hline
/ \quad -x^6-2x^5-x^4-x^3+0x^2+0x+1 & \\
+x^6+x^5+2x^4+x^3+x^2 & \\
\hline
/ \quad -x^5+x^4+0x^3+x^2+0x+1 & \\
+x^5+x^4+2x^3+x^2+x & \\
\hline
/ \quad 2x^4+2x^3+2x^2+x+1 & \\
-2x^4-2x^3-4x^2-2x-2 & \\
\hline
/ \quad / \quad -2x^2-x-1 &
\end{array}
$$

$$(x^3-x^2-x+2) - \frac{2x^2+x+1}{(x^2+1)(x^2+x+1)} = \frac{-x-1}{x^2+1} + \frac{x}{x^2+x+1} + (x^3-x^2-x+2)$$

$$\frac{2x^2+x+1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1}$$

$$= \frac{(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1)}{(\qquad)}$$

$$\begin{cases} B+D=1 \\ A+C=0 \\ \frac{1}{2}A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{3}D=\frac{1}{2} \end{cases}$$

en 0 : $\frac{1}{1 \cdot 1} = \frac{B}{1} + \frac{D}{1}$

en $\frac{x(2x^2+x+1)}{(x^2+1)(x^2+x+1)} = \frac{Ax^2+Bx}{x^2+1} + \frac{Cx^2+Dx}{x^2+x+1}$

en $+\infty$: $0 = A+C$

$$\frac{2x^2+x+1}{x^2+x+1} = Ax+B + (\text{——})(x^2+1)$$

~~en i : $\frac{-2+i+1}{-1+i+1} = Ai+B$~~

$$
\begin{array}{r}
x^3 \\ x^2 \\ x \\ 1
\end{array}
\begin{cases}
A+C=0 \\
A+B+\cancel{C}+D=2 \\
A+B+C=1 \\
B+D=1
\end{cases}
$$

en 1 : $\frac{2+1+1}{2 \cdot 3} = \frac{A+B}{2} + \frac{C+D}{3}$

$$
\left(\begin{array}{cccc|c}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & -\phi & 1 & 2 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1
\end{array}\right)
\sim
\left(\begin{array}{cccc|c}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{array}\right)
\quad
\begin{cases}
A=1 \\ B=1 \\ C=-1 \\ D=0
\end{cases}
$$

$$\frac{x^3+x}{(x^2+x+1)^2} = \frac{ax+b}{(x^2+x+1)} + \frac{cx+d}{(x^2+x+1)^2}$$

$$= \frac{(ax+b)(x^2+x+1)+cx+d}{(x^2+x+1)^2}$$

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ 1 \end{array} \left\{ \begin{array}{l} a=1 \\ a+b=0 \\ a+b+c=1 \\ b+d=0 \end{array} \right. \quad \left. \begin{array}{l} a=1 \\ b=-1 \\ c=1 \\ d=1 \end{array} \right.$$

final:

$$\frac{x^7+4x^6+10x^5+16x^4+18x^3+14x^2+8x+2}{(x+1)(x^3+2x^2+2x+1)^2}$$

$$(x+1)^3(x^2+x+1)^2$$

$$\cancel{(x+1)^2(x^2+x+1)}$$

$$x^4+x^2+1+2x^3+2x^2 = x^4+2x^3+3x^2+1$$

$$\begin{array}{r|rrr|r} & 1 & 2 & 2 & 1 \\ -1 & & -1 & -1 & -1 \\ \hline & 1 & 1 & 1 & / \end{array}$$

$$x^4+2x^3+2x^2+x+x^3+2x^2+2x+1 =$$

$$= x^4+3x^3+4x^2+3x+1$$

$$\begin{array}{rrrrrrrr} 1 & 4 & 10 & 16 & 18 & 14 & 8 & 2 \\ -1 & -3 & -4 & -3 & -1 & & & \\ \hline / & 1 & 6 & 13 & 17 & 14 & 8 & 2 \\ & -1 & -3 & -4 & -3 & -1 & & \\ \hline & / & 3 & 9 & 14 & 13 & 8 & 2 \\ & & -3 & -9 & -12 & -9 & -3 & \\ \hline & & / & / & \cancel{12x^3+4x^2+5x+2} & & & \end{array}$$

$$\begin{array}{|rrrrr} \cancel{1} & \cancel{3} & \cancel{4} & \cancel{3} & \cancel{1} \\ \hline \cancel{1} & \cancel{1} & \cancel{3} & \cancel{0} & \end{array}$$

$$\cancel{x^3+x^2+3x}$$

$$P = Q\cdot 1 + R$$

$$\frac{P}{Q} = 1 + \frac{\;}{---}$$

$$\frac{R}{Q} = - - \qquad \frac{P-Q}{Q} = \frac{\;}{--}$$

$$\cancel{= a + \frac{bx+c}{(x+1)} + \frac{dx+e}{(x+1)^2} + \frac{fx+g}{(x+1)^3} + \frac{hx+l}{(x^2+x+1)} + \frac{mx+n}{(x^2+x+1)^2}}$$

$$= a + \frac{b}{x+1} + \frac{c}{(x+1)^2} + \frac{d}{(x+1)^3} + \frac{ex+f}{x^2+x+1} + \frac{gx+h}{(x^2+x+1)^2}$$

$$\lim_{x\to+\infty} \quad a=1$$

$$F(x)-1 = \frac{-x^6-2x^5-2x^4+2x^2+3x+1}{(x+1)^3(x^2+x+1)^2} \qquad \text{etc.}$$

Sage: $\left[1,\ \frac{x}{(x+1)^3},\ \frac{-x^3-x+1}{x^4+2x^3+3x^2+2x+1}\right]$