<u>Ex. 81.</u> $\alpha = \sqrt{2} + \sqrt{3} \Rightarrow \alpha^2 = 2 + 2\sqrt{6} + 3$

$\Rightarrow \alpha^2 - 5 = 2\sqrt{6}$

$\Rightarrow (\alpha^2 - 5)^2 = 24$ c-à-d $\underbrace{\alpha^4 - 10\alpha^2 + 1}_{\in\,\mathbb{Q}[X]} = 0$

<u>Ex. 82.</u> On sait que

$\alpha^3 + 3\alpha^2 + 2\alpha - 4 = 0 \qquad \alpha = \frac{2}{\beta}$

$\Rightarrow \frac{8}{\beta^3} + 3 \cdot \frac{4}{\beta^2} + 2 \cdot \frac{2}{\beta} - 4 = 0 \Rightarrow \frac{8 + 12\beta + 4\beta^2 - 4\beta^3}{\beta^3} = 0$

β et $\alpha \neq 0 \Rightarrow -4\beta^3 + 4\beta^2 + 12\beta + 8 = 0$ c-à-d $\beta^3 - \beta^2 - 3\beta - 2 = 0$

<u>Ex. 83.</u> i) je fais la division euclidienne et j'écris :

$P(X) = (X-a)(X-b)\,Q(X) + R(X)$ (*) $\quad R(X) = \alpha X + \beta$

si $P(a) = P(b) = 0 \Rightarrow R(a) = R(b) = 0 \Rightarrow \dots \Rightarrow \alpha X + \beta = 0 \Rightarrow (X-a)(X-b) \mid P(X)$

Viceversa si $(X-a)(X-b) \mid P(X) \Rightarrow R(X) = 0$ et (*) implique $P(a) = P(b) = 0$.

ii) Application immédiate de i).

<u>Ex. 84.</u> $\alpha = 2 + \sqrt{5} \Rightarrow \alpha - 2 = \sqrt{5} \Rightarrow \alpha^2 - 4\alpha + 4 = 5 \Rightarrow \overbrace{\alpha^2 - 4\alpha - 1}^{P} = 0$

$\bar{\alpha}$ est aussi racine de P (c'est son conjugué)

Si $Q \in \mathbb{Q}[X]$ t.q. $Q(\alpha) = 0 \Rightarrow$ je fais la division par P :

$Q(X) = P(X)\,Q_1(X) + R(X) \qquad \deg R(X) < 2$ donc $R(X) = aX + b$

j'évalue en α : $0 = Q(\alpha) = R(\alpha) = a(2+\sqrt{5}) + b$

$a, b \in \mathbb{Q}$, et $2+\sqrt{5}$ est irrationel, donc $\Leftrightarrow a = b = 0 \Rightarrow R(X) = 0$

c-à-d $P(X) \mid Q(X)$. Donc $Q(\bar{\alpha}) = 0$ car $P(\bar{\alpha}) = 0$.

~~En général si α est racine de $a_nX^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0$ avec $a_i \in \mathbb{R}$, alors $\bar\alpha$ est racine aussi, car $a_n(\bar\alpha)^n + a_{n-1}(\bar\alpha)^{n-1} + \dots + a_1\bar\alpha + a_0 = \overline{a_n\alpha^n + \dots + a_1\alpha + a_0} = \bar{0} = 0$.~~

<u>Ex. 85.</u> Algorithme d'Euclide :

$$\begin{array}{r|l} X^4 + X^3 - 3X^2 - 4X - 1 & X^3 + X^2 - X - 1 \\ \hline -X^4 - X^3 + X^2 + X & X \\ \hline -2X^2 - 3X - 1 & \end{array}$$

$P(X) = X \cdot Q(X) + \overbrace{(-2X^2 - 3X - 1)}^{R_1}$

$Q(X) = \left(-\frac{1}{2}X + \frac{1}{4}\right) R_1(X) + \boxed{\overbrace{\left(-\frac{3}{4}X - \frac{3}{4}\right)}^{R_2}}$

$R_1(X) = \left(\frac{8}{3}X + \frac{4}{3}\right)\left(R_2(X)\right) + 0$

$$\begin{array}{r|l} X^3 + X^2 - X - 1 & -2X^2 - 3X - 1 \\ \hline -X^3 - \frac{3X^2}{2} - \frac{1}{2}X & -\frac{1}{2}X + \frac{1}{4} \\ \hline -\frac{1}{2}X^2 - \frac{3}{2}X - 1 & \\ \frac{1}{2}X^2 + \frac{3}{4}X + \frac{1}{4} & \\ \hline -\frac{3}{4}X - \frac{3}{4} & \end{array}$$

$$\begin{array}{r|l} -2X^2-3X+1 & -\frac{3}{4}X-\frac{2}{4} \\ \hline +2X^2+\frac{8}{4}X & +\frac{8}{3}X\ \cancel{\frac{11}{16}}+\frac{4}{3} \\ \hline -\frac{4}{4}X+1 & \\ +\frac{4}{4}X+\frac{2}{16} & \\ \hline -\frac{5}{16} & \end{array}$$

le dernier reste non nul est le polynôme $-\frac{3}{4}X-\frac{3}{4}$ donc PGCD $(P, Q) = X+1$ (le PGCD est unitaire)

Même chose pour les autres paires de polynômes.

Ex. 86. i). $P \in \mathbb{R}[X]$: On fait la division :

$P(X) = (X-a)^2 Q(X) + R(X)$

si $(X-a)^2 \mid P(X)$ alors $R(X) = 0$ donc $P(X) = (X-a)^2 Q(X)$ et $P(a) = 0$ (clair)

$P'(X) = 2(X-a) Q(X) + (X-a)^2 Q'(X) \rightarrow P'(a) = 0$.

Vice versa, on a $R(X) = \alpha X + \beta$ et $P(a) = P'(a) = 0 \Rightarrow R(a) = R'(a) = 0$

$\begin{array}{l|l} R(a) & \alpha a + \beta = 0 \\ R'(a) & \alpha = 0 \end{array}$ $\begin{cases} \beta = 0 \\ \alpha = 0 \end{cases}$ donc $R(X) = 0$ et $(X-a)^2 \mid P(X)$.

ii) Application de i), car $P(1) = n-(n+1)+1 = 0$ et $P'(X) = n(n+1)X^n - (n+1)nX^{n-1}$
$P'(1) = 0$

iii) $P(1) = 0$, $P'(X) = 4X^3 - 8X + 4$ $P'(1) = 0$ donc $(X-1)^2 \mid P(X)$,
donc $P(X) = (X-1)^2 Q(X)$ avec $\deg Q = 2$, que je peux décomposer.

Ex. 87. On se sert de 86 a) : $(X+1)^2 \mid P \iff P(-1) = P'(-1) = 0$

$P(-1) = a + b + 1$ $\quad P'(X) = a \cdot 2(n+1) X^{2(n+1)-1} + b \cdot 2n X^{2n-1}$

$P'(-1) = -2a(n+1) - 2bn$

$\begin{cases} a + b = -1 \\ -2(n+1)a - 2nb = 0 \end{cases}$ $\begin{cases} a = -b-1 \\ -2(n+1)(-b-1) - 2nb = 0 \end{cases}$ $\begin{cases} a = -b-1 \\ \Rightarrow b(+2(n+1) - 2n) + 2(n+1) = 0 \end{cases}$

$\begin{cases} 2b + 2(n+1) = 0 \\ a = -b-1 \end{cases}$ $\begin{cases} b = -(n+1) \\ a = +n+1-1 \end{cases}$ $\begin{cases} a = n \\ b = -(n+1) \end{cases}$

on a donc $P(X) = nX^{2(n+1)} - (n+1)X^{2n} + 1$ on peut faire le calcul directe
Pour connaître le quotient $Q(X)$ de $P(X) = Q(X)(X+1)^2$
qui donne :

$$
\begin{array}{l|l}
n X^{2(n+1)} + 0 - (n+1)X^{2n} + 0 \quad \cdots \quad +1 & X^2+2X+1 \\
\hline
-nX^{2n+2} \quad -2nX^{2n+1} - nX^{2n} & nX^{2n} - 2nX^{2n-1} + \\
\quad -2nX^{2n+1} - (2n+1)X^{2n} \quad \cdots \quad 1 & +(2n-1)X^{2n-2} - \\
\quad +2nX^{2n+1} + 4nX^{2n} + 2nX^{2n-1} & -(2n-2)X^{2n-3} + (2n-3)X^{2n-4} - \\
\quad\quad (2n-1)X^{2n} + 2nX^{2n-1} \quad\quad 1 & - \cdots - 2X + 1 \\
\quad\quad -(2n-1)X^{2n} - 2(2n-1)X^{2n-1} - (2n-1)X^{2n-2} & \\
\quad\quad\quad -(2n-2)X^{2n-1} \text{ etc.} &
\end{array}
$$

donc $Q(X) =$ ~~...~~ $nX^{2n} - 2nX^{2n-1} + (2n-1)X^{2n-2} - (2n-2)X^{2n-3} + (2n-3)X^{2n-4} - \cdots - 2X + 1$

On a aussi cette stratégie :

je cherche $Q(X) = a_0 + a_1X + a_2X^2 + \cdots + a_{2n-2}X^{2n-2} + a_{2n-1}X^{2n-1} + a_{2n}X^{2n}$

t.q. $(X^2+2X+1)\,Q(X) = nX^{2n+2} - (n+1)X^{2n} + 1$

en comparant les coefficients de X^j, $j=0 \dots 2n+2$, j'ai le système :

$$
\begin{array}{l}
2n+2 \\ 2n+1 \\ 2n \\ \vdots \\ 1 \\ 0
\end{array}
\left\{
\begin{array}{l}
a_{2n} = n \\
a_{2n-1} + 2a_{2n} = 0 \\
a_{2n-2} + 2a_{2n-1} + a_{2n} = -(n+1) \\
\cdots \\
a_0 + 2a_1 + \cancel{a_2} = 0 \\
a_0 = 1
\end{array}
\right.
$$

qui donne les mêmes coefficients.

Ex. 88. $nP = XP'$ $\qquad P(X) = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$

$n(a_0 + a_1X + \cdots + a_nX^n) = X(a_1 + 2a_2X + 3a_3X^2 + \cdots + (n-1)a_{n-1}X^{n-2} + na_nX^{n-1})$

$na_0 + na_1X + \cdots + na_nX^n = a_1X + 2a_2X^2 + 3a_3X^3 + \cdots + (n-1)a_{n-1}X^{n-1} + na_nX^n$

j'égale les coefficients :

$$
\left\{
\begin{array}{l}
na_0 = 0 \\
na_1 = a_1 \\
na_2 = 2a_2 \\
na_3 = 3a_3 \\
\vdots \\
na_{n-1} = (n-1)a_{n-1} \\
\cancel{na_n = na_n}
\end{array}
\right.
\qquad
\left\{
\begin{array}{l}
na_0 = 0 \\
(n-1)a_1 = 0 \\
(n-2)a_2 = 0 \\
\vdots \\
2a_{n-2} = 0 \\
a_{n-1} = 0
\end{array}
\right.
\Rightarrow
\left\{
\begin{array}{l}
a_0 = 0 \\
a_1 = 0 \\
\vdots \\
a_{n-1} = 0
\end{array}
\right.
$$

donc $P(X) = a_nX^n$.

Si $P' \mid P$, on a $P = (\alpha X + \beta)P'$ $\quad \exists\, \alpha, \beta \in \mathbb{R}$

en comparant les coefficients, on trouve tout de suite $a_n = \alpha n a_n \Rightarrow \alpha = \frac{1}{n}$

donc $P = \left(\frac{1}{n}X + \beta\right)P' \Leftrightarrow nP = (X + n\beta)P'$ ~~où~~ donc $a = -n\beta$

$n P(X) = (X-a) P'(X)$ soit $X - a = Y$

$n P(Y + a) = Y P'(Y+a)$ soit $Q(Y) = P(Y+a)$

$n Q(Y) = Y Q'(Y)$ $Q'(Y) = \big(P(Y+a)\big)' = P'(Y+a)$

$\Rightarrow Q(Y) = \lambda Y^n \Rightarrow P(Y+a) = \lambda Y^n \Rightarrow P(X) = \lambda (X-a)^n$.

<u>Ex. 89.</u> $A, B \in \mathbb{Q}[X]$. Si $B \mid A$ dans $\mathbb{Q}[X] \Rightarrow$ évidemment $B \mid A$ dans $\mathbb{R}[X]$.

Vice versa : $A(X) = B(X) Q(X)$, $Q(X) \in \mathbb{R}[X]$

j'écris : $a_0 + a_1 X + a_2 X^2 + \dots + a_{n+m} X^{n+m} = (q_0 + q_1 X + \dots + q_n X^n)(b_0 + b_1 X + \dots + b_m X^m)$

j'égale les coefficients :

$$\begin{cases} a_0 = q_0 b_0 \longrightarrow a_0, b_0 \in \mathbb{Q} \Rightarrow q_0 \in \mathbb{Q} \\ a_1 = q_0 b_1 + q_1 b_0 \longrightarrow a_1, b_1, q_0, b_0 \in \mathbb{Q} \Rightarrow q_1 \in \mathbb{Q} \\ a_2 = q_0 b_2 + q_1 b_1 + q_2 b_0 \longrightarrow a_2, q_0, b_2, q_1, b_1, b_0 \in \mathbb{Q} \Rightarrow q_2 \in \mathbb{Q} \\ \dots \end{cases}$$

et ainsi de suite, $Q(X) \in \mathbb{Q}[X]$.

<u>Ex. 90.</u>

$$\begin{array}{rrrrr|l} 2X^4 & +0X^3 & -3X^2 & +0X & +1 & X^3+X^2-X-1 \\ \hline -2X^4 & -2X^3 & +2X^2 & +2X & & 2X-2 \\ & -2X^3 & -X^2 & +2X & +1 & \\ & 2X^3 & +2X^2 & +2X & -2 & \\ \hline & & X^2 & +0X & -1 & \end{array}$$

$$\begin{array}{rrrr|l} X^3 & +X^2 & -X & -1 & X^2-1 \\ \hline -X^3 & & +X & & X+1 \\ & X^2 & & -1 & \\ & -X^2 & & +1 & \\ \hline & & & & \end{array}$$

$2X^4 - 3X^2 + 1 = (2X-2)(X^3+X^2-X-1) + X^2 - 1$

$X^3 + X^2 - X - 1 = (X+1)(X^2-1)$

$\Rightarrow$ pgcd $= X^2 - 1$

<u>Ou sinon :</u> $P(1) = 2 - 3 + 1 = 0$ $Q(1) = 0$

$P(-1) = 2 - 3 + 1 = 0$ $Q(-1) = 0$

donc $(X^2-1) \mid \text{pgcd}(P, Q)$

$Q = (X+1)^2 (X-1)$

	2	0	−3	0	1
1		2	2	−1	−1
	2	2	−1	−1	/
−1		−2	0	1	
	2	0	−1	/	
−1		−2	2		
	2	−2	1		

~~Q~~	1	1	−1	−1
1		1	2	1
	1	2	1	/
−1		−1	−1	
	1	1	/	

$\Rightarrow$ ~~R~~ ~~G~~ -1 n'est pas racine double de P donc

pgcd $(P, Q) = (X+1)(X-1)$.