3) On a :

$$f_n(x) = \sum_{k=0}^{n} \frac{X^k}{k!} \qquad f_n'(x) = \sum_{k=1}^{n} \frac{k X^{k-1}}{k!} = \sum_{k=1}^{n} \frac{X^{k-1}}{(k-1)!} = \sum_{k=0}^{n-1} \frac{X^k}{k!} = f_{n-1}(x).$$

donc $f_n'(x) = f_{n-1}(x)$ (Rem. pour $n \to +\infty$ on retrouve $\exp' = \exp$)

Par absurde, si α racine multiple de $f_n(x)$, alors

$$f_n(\alpha) = f_n'(\alpha) = 0 \iff f_n(\alpha) = f_{n-1}(\alpha) = 0$$

$$\begin{cases} 1 + \alpha + \dfrac{\alpha^2}{2} + \dots + \dfrac{\alpha^{n-1}}{(n-1)!} + \dfrac{\alpha^n}{n!} = 0 \\ 1 + \alpha + \dots + \dfrac{\alpha^{n-1}}{(n-1)!} = 0 \end{cases} \Rightarrow \frac{\alpha^n}{n!} = 0 \Rightarrow \alpha = 0$$

mais 0 n'est racine d'aucun $f_n(x)$.

Ex. 105. 1) $P \in \mathbb{C}[X]$:

si $\alpha \in \mathbb{C}$ t.q. $P(\alpha) - \alpha = 0 \Rightarrow P(\alpha) = \alpha$

$\Rightarrow P(P(\alpha)) - \alpha = P(\alpha) - \alpha = 0$.

2) évident si les racines de $P(x) - X$ sont toutes simples.

3) $P = X^2 + a$

$$P(P(X)) - X = (X^2+a)^2 + a - X = X^4 + 2aX^2 + a^2 + a - X = X^4 + 2aX^2 - X + (a^2+a) = X^4 + 6X^2 - X + 12$$

$$\begin{cases} 2a = 6 \\ a^2 + a = 12 \end{cases} \qquad \begin{cases} a = 3 \\ 9 + 3 = 12 \quad \text{ok} \end{cases} \qquad P = X^2 + 3$$

$\Rightarrow P(x) - X = (X^2 - X + 3) \mid X^4 + 6X^2 - X + 12$

$$\begin{array}{l|l} X^4 + 0X^3 + 6X^2 - X + 12 & X^2 - X + 3 \\ -X^4 + X^3 - 3X^2 & \overline{X^2 + X + 4} \\ \hline \quad X^3 + 3X^2 - X + 12 & \\ \quad -X^3 + X^2 - 3X & \\ \hline \qquad 4X^2 - 4X + 12 & \\ \qquad -4X^2 + 4X - 12 & \\ \hline \qquad\qquad 0 & \end{array}$$

factorisation en irréd dans $\mathbb{R}[X]$.

$\Rightarrow X^4 + 6X^2 - X + 12 = (X^2 - X + 3)(X^2 + X + 4)$

$x = \dfrac{1 \pm \sqrt{1-12}}{2} \in \mathbb{C} \setminus \mathbb{R}$ $\qquad$ $x = \dfrac{-1 \pm \sqrt{1-16}}{2} \in \mathbb{C} \setminus \mathbb{R}$

4) Soit

$$P(X) = \sum_{k=0}^{n} a_k X^k$$

alors
$$\begin{aligned}
P(P(X)) - X &= P(P(X)) - P(X) + P(X) - X \\
&= \sum_{k=0}^{n} a_k (P(X))^k - \sum_{k=0}^{n} a_k X^k + P(X) - X \\
&= \sum_{k=0}^{n} a_k \left((P(X))^k - X^k \right) + P(X) - X \\
&= \cancel{a_0 (1-1)} + a_1 \underbrace{(P(X) - X)}_{P(X)-X} + a_2 \underbrace{\left((P(X))^2 - X^2 \right)}_{P(X)-X} + \dots + a_n \underbrace{\left((P(X))^n - X^n \right)}_{P(X)-X} + \underbrace{P(X) - X}_{P(X)-X}
\end{aligned}$$

donc $P(X) - X$ divise la somme.

5) On essaye d'écrire Q_2 comme $P(P(X)) - X$

$$X^4 + \frac{X^2}{2} - X + \frac{5}{16} = P(P(X)) - X \iff X^4 + \frac{X^2}{2} + \frac{5}{16} = P(P(X))$$

Soit $P(X) = X^2 + a$

$$P(P(X)) = X^4 + 2a X^2 + (a^2 + a)$$

$$\begin{cases} 2a = \frac{1}{2} \\ a^2 + a = \frac{5}{16} \end{cases} \qquad a = \frac{1}{4} \quad \text{ok car} \quad \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$

donc $P(X) = X^2 + \frac{1}{4}$ et $P(X) - X = X^2 - X + \frac{1}{4} \mid Q_2(X)$

or $X^2 - X + \frac{1}{4} = \left(X - \frac{1}{2}\right)^2$, on fait la division

	1	0	$\frac{1}{2}$	-1	$\frac{5}{16}$
$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
	1	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{5}{8}$	/
$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	
	1	1	$\frac{5}{4}$	/	

$$\Rightarrow Q_2(X) = \left(X - \frac{1}{2}\right)^2 \left(X^2 + X + \frac{5}{4}\right)$$

et $\Delta = 1 - 5 < 0$ donc le dernier polynôme est irréductible en $\mathbb{R}[X]$.

Ex. 106. a)

$$\begin{array}{l|l} x^5 + 3x^4 + x^3 + x^2 + 3x + 1 & x^4 + 2x^3 + x + 2 \\ -x^5 - 2x^4 + 0x^3 - x^2 - 2x & \overline{x + 1} \\ \hline \quad x^4 + x^3 + 0x^2 + x + 1 & \\ \quad -x^4 - 2x^3 + 0x^2 - x - 2 & \\ \hline \quad\quad -x^3 + 0x^2 + 0x - 1 & \end{array}$$

$$\begin{array}{l|l} x^4 + 2x^3 + 0x^2 + x + 2 & -x^3 + 0x^2 + 0x - 1 \\ -x^4 \qquad\qquad -x & \overline{-x - 2} \\ \hline \quad 2x^3 + 0x^2 + 0x + 2 & \\ \quad -2x^3 \qquad\qquad -2 & \\ \hline & \end{array}$$

on a :

$$x^5 + 3x^4 + x^3 + x^2 + 3x + 1 = (x+1)(x^4 + 2x^3 + x + 2) + \boxed{(-x^3 - 1)}$$

$$x^4 + 2x^3 + x + 2 = (-x-2)(-x^3-1)$$

donc $\operatorname{pgcd}(A, B) = x^3 + 1$ et

$$-x^3 - 1 = A - (x+1)B \Rightarrow x^3 + 1 = (-1)\cdot A(x) + (x+1)\cdot B(x)$$

Ex. 107. Solutions de $\boxed{P_1 A + Q_1 B = C}$ (*) : en $\mathbb{K}[x]$

$\exists (P_1, Q_1)$ t.q. $P_1 A + Q_1 B = C \Leftrightarrow \operatorname{pgcd}(A,B) \mid C$.

En effet : $(\Rightarrow)$: $\operatorname{pgcd}(A,B) \mid A$, $\operatorname{pgcd}(A,B) \mid B \Rightarrow \operatorname{pgcd}(A,B) \mid C$

$(\Leftarrow)$: soit $C = C_1 \operatorname{pgcd}(A,B)$, $A = A_1 \operatorname{pgcd}(A,B)$, $B = B_1 \operatorname{pgcd}(A,B)$

$P_1 A + Q_1 B = C$ a solutions $\Leftrightarrow$

$P_1 A_1 \operatorname{pgcd}(A,B) + Q_1 B_1 \operatorname{pgcd}(A,B) = C_1 \operatorname{pgcd}(A,B) \Leftrightarrow$

$P_1 A_1 + Q_1 B_1 = C_1$ a solutions.

or, $\operatorname{pgcd}(A_1, B_1) = 1$ et $\exists R(x), S(x)$ t.q. $RA_1 + SB_1 = 1$

(algorithme d'Euclide + relation de Bézout) $\Rightarrow$

$$(C_1 R)A_1 + (C_1 S)B_1 = C_1 \Rightarrow \underbrace{C_1 R}_{P_1} \underbrace{\operatorname{pgcd}(A,B)\, A_1}_{A} + \underbrace{C_1 S}_{Q_1} \underbrace{\operatorname{pgcd}(A,B)\, A_2}_{B} = C$$

Les autres solutions :

Soit $\begin{cases} P_1 A + Q_1 B = C \\ P_2 A + Q_2 B = C \end{cases}$ alors $(P_1 - P_2)A + (Q_1 - Q_2)B = 0$

$$(P_1 - P_2)A = (Q_2 - Q_1)B$$

on divise par $\operatorname{pgcd}(A,B)$ et on trouve

$$(P_1 - P_2) A' = (Q_2 - Q_1) B'$$

donc $\begin{cases} \cancel{P_1 P_2}\, A' \mid (Q_2 - Q_1) B' \\ B' \mid (P_1 - P_2) A' \end{cases}$ mais $\mathrm{pgcd}(A', B') = 1$ donc $\begin{cases} A' \mid Q_2 - Q_1 \\ B' \mid P_1 - P_2 \end{cases}$

donc $\begin{cases} A' T = Q_2 - Q_1 \\ B' U = P_1 - P_2 \end{cases}$ c-à-d $\begin{cases} Q_2 = Q_1 + T \dfrac{A}{\mathrm{pgcd}(A, B)} \\ P_2 = P_1 - U \dfrac{B}{\mathrm{pgcd}(A, B)} \end{cases}$ pour $T, U \in \mathbb{K}[X]$

vice-versa, $\left(\cancel{Q_1 + C} \; P_1 - U \dfrac{B}{\mathrm{pgcd}(A,B)} , Q_1 + T \dfrac{A}{\mathrm{pgcd}(A,B)} \right)$ est solution de (*) si

$$\left(P_1 - U \frac{B}{\mathrm{pgcd}(A,B)} \right) A + \left(Q_1 + T \frac{A}{\mathrm{pgcd}(A,B)} \right) B = C$$

$$\Leftrightarrow \underbrace{P_1 A + Q_1 B}_{\overset{\shortparallel}{C}} - U \frac{B}{\mathrm{pgcd}(A,B)} A + T \frac{A}{\mathrm{pgcd}(A,B)} B = C$$

$$\Leftrightarrow \frac{AB}{\mathrm{pgcd}(A,B)} (T - U) = 0$$

On suppose $A \neq 0$, $B \neq 0$ sinon la question est triviale

donc $T - U = 0$ c-à-d $T = U$.

En conclusion, si (P_1, Q_1) est solution de $P_1 A + Q_1 B = C$, toutes les solutions sont données par (P_2, Q_2) où

$$\begin{cases} P_2 = P_1 - T \dfrac{B}{\mathrm{pgcd}(A,B)} \\ Q_2 = Q_1 + T \dfrac{A}{\mathrm{pgcd}(A,B)} \end{cases} \quad \forall\, T \in \mathbb{K}[X].$$