The Cauchy Problem for Wave Equations with non Lipschitz Coefficients; Application to Continuation of Solutions of some Nonlinear Wave Equations.

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Le problème de Cauchy pour les équations d'onde à coefficients non Lipschtziens; application au prolongement de solutions d'équations d'ondes non linéaires.

Abstract

In this paper we study the Cauchy problem for second order strictly hyperbolic operators of the form

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\big(a_{j,k}\partial_{y_k}u\big) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f,$$

when the coefficients of the principal part are not Lipschitz continuous, but only "Log-Lipschitz" with respect to all the variables. This class of equation is invariant under changes of variables and therefore suitable for a local analysis. In particular, we show local existence, local uniqueness and finite speed of propagation for the noncharacteristic Cauchy problem. This provides an invariant version of a previous paper of the first author with N.Lerner [6]. We also give an application of the method to a continuation theorem for nonlinear wave equations where the coefficients above depend on u: the smooth solution can be extended as long as it remains Log-Lipschitz.

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On considère le problème de Cauchy pour des équations d'onde strictement hyperboliques:

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\big(a_{j,k}\partial_{y_k}u\big) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f,$$

quand les coefficients de la partie principale sont seulement "Log-Lipschitz" en toutes les variables. Cette classe d'équation est invariante par changement de variables et est donc une classe naturelle pour une étude locale intrinsèque. En particulier, on montre l'existence locale, l'unicité locale et la vitesse finie de propagation pour le problème de Cauchy non caractéristique. donnant une version invariante d'un résultat antérieur du premier auteur avec N.Lerner [6]. Pour les équations non linéaires où les coefficients cidessus dépendent de u, la méthode d' estimations permet de montrer que les solutions régulières se prolongent en solutions régulières aussi longtemps qu'elles restent Log-Lipschitz.

1 Introduction

In this paper we study the well-posedness of the Cauchy problem for second order strictly hyperbolic equations whose coefficients are not Lipschitz continuous:

$$Lu := \sum_{j,k=0}^{n} \partial_{y_j}\big(a_{j,k}\partial_{y_k}u\big) + \sum_{j=0}^{n}\{b_j\partial_{y_j}u + \partial_{y_j}(c_ju)\} + du = f. \tag{1.1}$$

In Section 6, we will present an application of the methods developed for the analysis of the Cauchy problem to nonlinear wave equations, where the various coefficients above depend on u. It is known that the smooth solution can be extended as long as they remains Lipschitz continuous. We prove that this condition can we weakened, and that smooth solution remain smooth as long as they remain Log-Lipschitz. We refer to Section 6 for a precise result and focus now on the analysis of the Cauchy problem.

The question of the well posedness of the Cauchy problem for the wave equation with nonsmooth coefficients has already been studied in the case that the second order part has the special form, in coordinates $y = (t, x)$:

$$\partial_t^2 - \sum_{j,k=1}^{n} \partial_{x_j}\big(a_{j,k}\partial_{x_k}u\big) \tag{1.2}$$

and the Cauchy data are given on the space-like hyperplane $\{t = 0\}$. In this case, when the coefficients depend only on the time variable t, F. Colombini, E. De Giorgi and S. Spagnolo ([5]) have proved that the Cauchy problem is in general ill-posed in C^∞ when the coefficients are only Hölder continuous of order $\alpha < 1$, but is well-posed in appropriate Gevrey spaces. This has been extended to the case where the coefficients are Hölder in time and Gevrey in x ([14, 8]). Moreover, it is also proved in [5] that the Cauchy problem is well posed in C^∞ when the coefficients, which depend only on time, are "Log-Lipschitz" (in short LL) : recall that a function a of variables y is said to be LL on a domain Ω if there is a constant C such that

$$|a(y) - a(y')| \le C|y - y'| \Big(1 + \big|\mathrm{Log}|y - y'|\big|\Big) \tag{1.3}$$

for all y and y' in Ω. In [5], it is proved that for LL coefficients depending only on t and for initial data in the Sobolev spaces $H^s \times H^{s-1}$, the solution satisfies

$$u(t, \cdot) \in H^{s-\lambda t}, \quad \partial_t u(t, \cdot) \in H^{s-1-\lambda t} \tag{1.4}$$

with λ depending only on the LL norms of the coefficients and the constants of hyperbolicity. In particular, there is a loss of smoothness as time evolves and this loss does occur in general when the coefficients are not Lipschitz continuous, and is sharp, as shown in [3].

The analysis of the C^∞ well-posedness has been extended by F. Colombini and N. Lerner ([6]) to the case of equations, still with principal part (1.2), whose coefficients also depend on the space variables x. They show that the Cauchy problem is well-posed if the coefficients are LL in time and C^∞ in x. They also study the problem under the natural assumption of isotropic LL smoothness in (t, x). In this case one has to multiply LL functions with distributions in H^s. This is well defined only when $|s| < 1$. Therefore, one considers initial data in $H^s \times H^{s-1}$ with $0 < s < 1$, noticing that further smoothness would not help. Next, the loss of smoothness (1.4) forces us to limit t to an interval where $0 < s - \lambda t$, yielding only local in time existence theorems. We also refer to [6] for further discussions on the sharpness of LL smoothness.

However, the local uniqueness of the Cauchy problem and the finite speed of propagation for local solutions are not proved in [6]. The main goal of this paper is to address these questions. Classical methods such as convexification, leads one to consider general equations (1.1) with LL coefficients in all variables. However, the meaning of the Cauchy problem for such equations is not completely obvious: as mentioned above, the maximal expected smoothness of the solutions is H^s with $s < 1$ and their traces on the initial manifold are not immediately defined. More importantly, in the general theory of smooth operators, the traces are defined using partial regularity results in the normal direction; in our case, the limited smoothness of the coefficients is a source of difficulties. It turns out that when $s \leq \frac{1}{2}$, one cannot in general define the traces of all the first order derivatives of u, but only the Neumann trace relative to the operator, using a weak formulation of the traces.

Assumption 1.1. *L is a second order operator of the form* (1.1) *on a neighborhood Ω of $\underline{y}$, with coefficients $a_{j,k} \in LL(\Omega)$, b_j and c_j in $C^\alpha(\Omega)$, for some $\alpha \in]\frac{1}{2}, 1[$ and $d \in L^\infty(\Omega)$. Σ is a smooth hypersurface through $\underline{y}$ and L is strictly hyperbolic in the direction conormal to Σ.*

Shrinking Ω if necessary, we assume that Σ is defined by the equation $\{\varphi = 0\}$ with φ smooth and $d\varphi \neq 0$. We consider the one-sided Cauchy problem, say on the component $\Omega_+ = \Omega \cap \{\varphi > 0\}$. We use the Sobolev spaces $H^s(\Omega \cap \{\varphi > 0\}$ for $s \in \mathbb{R}$. As usual, we say that $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$, if for any relatively compact open subset Ω_1 of Ω, the restriction of u to $\Omega_1 \cap \{\varphi > 0\}$ belongs to $H^s(\Omega \cap \{\varphi > 0\})$. Similarly, $u \in H^s_{comp}(\Omega \cap \{\varphi \geq 0\})$ if $u \in H^s(\Omega \cap \{\varphi > 0\})$ has compact support in $\Omega \cap \{\varphi \geq 0\}$.

The adjoint operator

$$L^* v := \sum_{j,k=0}^{n} \partial_{y_k} \big(\overline{a}_{j,k} \partial_{y_j} v \big) - \sum_{j=0}^{n} \{ \overline{c}_j \partial_{y_j} v + \partial_{y_j} (\overline{b}_j v) \} + \overline{d} v \tag{1.5}$$

has the same form as L. For u and v smooth, v compactly supported in $\Omega \cap \{\varphi \geq 0\}$, one has the (formal) identity

$$\big(Lu, v \big)_{L^2(\Omega_+)} - \big(u, L^* v \big)_{L^2(\Omega_+)} = \big(N_\nu u, v \big)_{L^2(\Sigma)} - \big(u, N'_\nu v \big)_{L^2(\Sigma)} \tag{1.6}$$

where

$$\begin{aligned} N_\nu u &= \sum_{j,k} \nu_k (a_{j,k} \partial_j u)_{|\Sigma}, \\ N'_\nu v &= \sum_{j,k} \nu_j (\overline{a}_{j,k} \partial_k v)_{|\Sigma} - \sum_j \nu_j \big((\overline{b}_j + \overline{c}_j) v \big)_{|\Sigma} \end{aligned} \tag{1.7}$$

and $\nu = (\nu_0, \ldots, \nu_d) \neq 0$ is conormal to Σ and the d-integration form on Σ is chosen accordingly. .

Lemma 1.2. *i) For all* $s \in]1-\alpha, 1+\alpha[$ *and* $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$*, all the terms entering in the definition of* Lu *and* L^*u *are well defined as distributions in* $H^{s-2}_{loc}(\Omega \cap \{\varphi \geq 0\})$*.*

ii) For all $s \in]\frac{3}{2}, 1+\alpha[$ *and* $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$*, the traces* $N_\nu u$ *and* $N'_\nu u$ *are well defined in in* $H^{s-\frac{3}{2}}_{loc}(\Sigma \cap \Omega)$*.*

Proof. This is due to multiplicative properties (see [6] and Corollary 3.6):

- If $\sigma \in]-1, 1[$, $a \in LL(\Omega)$ and $v \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$, then $av \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$.
- If $\sigma \in]-\alpha, \alpha[$, $a \in C^\alpha(\Omega)$ and $v \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$, then $av \in H^\sigma_{loc}(\Omega \cap \{\varphi \geq 0\})$. □

Next, we recall that the subspace of functions with compact support in Ω_+ is dense in $H^\sigma(\Omega_+)$ when $|\sigma| < \frac{1}{2}$; moreover, for $0 \leq \sigma < \frac{1}{2}$ and for $u \in H^\sigma(\Omega_+)$ the pairing $(u, v)_{L^2(\Omega)}$ for $v \in L^2$ extends as the duality $\langle u, v\rangle_{H^\sigma \times H^{-\sigma}}$. With this remark in mind, the identity (1.6) holds for smooth functions:

Lemma 1.3. *For* $s \in]\frac{3}{2}, 1+\alpha[$, $u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\})$ *and* $v \in H^s_{comp}(\Omega \cap \{\varphi \geq 0\})$*, there holds*

$$\begin{aligned} \big\langle Lu, v\big\rangle_{H^{-\sigma}\times H^\sigma} - &\big\langle u, L^*v\big\rangle_{H^\sigma \times H^{-\sigma}} \\ &= \big(N_\nu u, D_\Sigma v\big)_{L^2(\Sigma)} - \big(D_\Sigma u, N'_\nu v\big)_{L^2(\Sigma)} \end{aligned} \tag{1.8}$$

with $\sigma = s - \frac{3}{2} \in]0, \frac{1}{2}[$ *and* $D_\Sigma u = u_{|\Sigma}$*.*

Proof. It is sufficient to remark that for $\sigma \in [0, \frac{1}{2}[$, the Green's formula

$$\big\langle \partial_j u, v\big\rangle_{H^{-\sigma}\times H^\sigma} = -\big\langle u, \partial_j v\big\rangle_{H^\sigma\times H^{-\sigma}} + \big(\nu_j D_\Sigma u, D_\Sigma v\big)_{L^2(\Sigma)}$$

is satisfied for $u \in H^{1-\sigma}_{loc}(\Omega \cap \{\varphi \geq 0\})$ and $v \in H^{1-\sigma}_{comp}(\Omega \cap \{\varphi \geq 0\})$ □

Proposition 1.4. *Let* $D(L; H^s) = \{u \in H^s_{loc}(\Omega \cap \{\varphi \geq 0\}) : Lu \in L^2_{loc}(\Omega \cap \{\varphi \geq 0\})\}$*. The operator* N_Σ *and* D_Σ *have unique extensions to* $\bigcup_{s>1-\alpha} D(L; H^s)$ *such that*

i) For all $s \in]1-\alpha, \alpha[$, N_Σ *[resp.* D_Σ*] is continuous from* $D(L; H^s)$ *into* $H^{s-\frac{3}{2}}_{loc}(\Sigma \cap \Omega)$ *[resp.* $H^{s-\frac{1}{2}}_{loc}(\Sigma \cap \Omega)$ *].*

ii) for all $s' \in]1-\alpha, \frac{1}{2}[$ *such that* $s' \leq s$ *and all* $v \in H^{2-s'}_{comp}(\Omega \cap \{\varphi \geq 0\})$ *there holds*

$$\begin{aligned} \big(Lu, v\big)_{L^2} - &\big\langle u, L^*v\big\rangle_{H^{s'}\times H^{-s'}} \\ &= \big\langle N_\nu u, D_\Sigma v\big\rangle_{H^{s-\frac{3}{2}}\times H^{\frac{3}{2}-s}} - \big\langle D_\Sigma u, N'_\nu v\big\rangle_{H^{s-\frac{1}{2}}\times H^{\frac{1}{2}-s}}. \end{aligned} \tag{1.9}$$

This proposition is proved in Section 5. Note that by Lemma 1.2, for $v \in H^{2-s'}_{comp}$, $L^*v \in H^{-s'}_{comp}$ and that $u \in H^{s'}_{loc}$ if $s' \le s$. Moreover, $D_\Sigma v \in H^{\frac{3}{2}-s'}_{comp} \subset H^{\frac{3}{2}-s}_{comp}$ and $N'_\Sigma v \in H^{\frac{1}{2}-s'}_{comp} \subset H^{\frac{1}{2}-s}_{comp}$.

With this Proposition, the Cauchy problem with source term in L^2 and solution in H^s, $s > 1-\alpha$, makes sense.

Theorem 1.5 (Local existence)**.** *Consider $s > 1-\alpha$ and a neigborhood ω of $\underline{y}$ in Σ. Then there are $s' \in]1-\alpha, \alpha[$ and a neighborhood Ω' of $\underline{y}$ in $\mathbb{R}^{1+n}$ such that for all Cauchy data (u_0, u_1) in $H^s(\omega) \times H^{s-1}(\omega)$ near $\underline{y}$ and all $f \in L^2(\Omega' \cap \{\varphi > 0\})$ the Cauchy problem*

$$Lu = f, \quad D_\Sigma u = u_0, \quad N_\Sigma u = u_1, \tag{1.10}$$

has a solution $u \in H^{s'}(\Omega' \cap \{\varphi > 0\})$.

Theorem 1.6 (Local uniqueness)**.** *If $s > 1-\alpha$ and $u \in H^s(\Omega \cap \{\varphi > 0\})$ satisfies*

$$Lu = 0, \quad D_\Sigma u = 0, \quad N_\Sigma u = 0 \tag{1.11}$$

then $u = 0$ on a neighborhood of $\underline{y}$ in $\Omega \cap \{\varphi \ge 0\}$.

Remark 1.7. If the coefficients of the first order term L_1 (see (2.3)) are also LL, the statements above are true with $\alpha = 1$ since the coefficients are then C^α for all $\alpha < 1$. If the b_j are C^α and the c_j are $C^{\tilde{\alpha}}$, the conditions are $1-\tilde{\alpha} < \alpha$ and the limitation on s is $1-\tilde{\alpha} < s$.

Remark 1.8. Theorem 1.6 implies that if u is in H^s and satisfies $Lu = 0$ near $\underline{y}$ and if u vanishes on $\{\varphi < 0\}$, then u vanishes on a neighborhood of $\underline{y}$ (see Section 5.2). Moreover, this local propagation of zero across any space-like manifold implies finite speed of propagation by classical arguments which we do not repeat here. In particular, if $\Omega' \cap \{\varphi \ge 0\}$ is contained in the domain of dependence of ω, there is existence and uniqueness for the Cauchy problem (1.10) in $\Omega' \cap \{\varphi \ge 0\}$.

The proof of these results is given in Section 5 below. Because all the hypotheses are invariant under smooth changes of coordinates, we can assume that in the coordinates $y = (t, x)$, the initial surface is $\{t = 0\}$, and in these coordinates, we prove the existence and uniqueness theorems. We deduce them from similar results on strips $]0, T[\times \mathbb{R}^n$ and there, the main part of the work is to prove good energy estimates for (weak) solutions. In this framework, the results of Theorem 1.5 are improved, by using non isotropic spaces, and by making a detailed account of the loss of spatial smoothness as time evolves, as in [5, 6]. The precise results are stated in section 2 below and are proved in section 4 using the paradifferential calculus of J.-M. Bony, whose LL-version is presented in section 3.

2 The global in space problem

In this section we denote by (t, x) the space-time variables. On $\Omega = [0, T_0] \times \mathbb{R}^n$ consider a second order hyperbolic differential operator

$$Lu = L_2 u + L_1 u + du \tag{2.1}$$

with

$$L_2 = \partial_t a_0 \partial_t + \sum_{j=1}^{n} (\partial_t a_j \partial_{x_j} + \partial_{x_j} a_j \partial_t) - \sum_{j,k=1}^{n} \partial_{x_j} a_{j,k} \partial_{x_k}, \tag{2.2}$$

$$L_1 = b_0 \partial_t + \partial_t c_0 + \sum_{j=1}^{n} (b_j \partial_{x_j} + \partial_{x_j} c_j). \tag{2.3}$$

The coefficients satisfy on $\Omega = [0, T_0] \times \mathbb{R}^n$

$$a_{j,k} = a_{k,j}, \quad a_0,\ a_j,\ a_{j,k} \in L^\infty(\Omega) \cap LL(\Omega), \tag{2.4}$$

$$b_0,\ c_0,\ b_j,\ c_j \in L^\infty(\Omega) \cap C^\alpha(\Omega), \tag{2.5}$$

$$d \in L^\infty(\Omega), \tag{2.6}$$

for some $\alpha \in]\frac{1}{2}, 1[$. Recall that the space LL is defined by (1.3), the semi norm $\|a\|_{LL}$ being the best constant C in (1.3). In addition, for $\alpha \in]0,1[$, C^α denotes the usual Hölder space, equipped with the norm

$$\|a\|_{C^\alpha} = \|a\|_{L^\infty} + \sup_{y \neq y'} \frac{|a(y) - a(y')|}{|y - y'|^\alpha}. \tag{2.7}$$

When $\alpha = 1$, this defines the norm $\|a\|_{Lip}$ in the space of Lipschitz functions.

We assume that L is hyperbolic in the direction dt, which means that there are $\delta_0 > 0$ and $\delta_1 > 0$ such that for all $(t, x, \xi) \in [0, T_0] \times \mathbb{R}^n \times \mathbb{R}^n$

$$a_0(t, x) \geq \delta_0, \qquad \sum_{1 \leq j,k \leq n} (a_{j,k} + \frac{a_j a_k}{a_0}) \xi_j \xi_k \geq \delta_1 |\xi|^2. \tag{2.8}$$

We denote by A_{L^∞}, A_{LL} and B constants such that for all indices

$$\|a_0, a_j, a_{j,k}\|_{L^\infty(\Omega)} \leq A_{L^\infty}, \qquad \|a_0, a_j, a_{j,k}\|_{LL(\Omega)} \leq A_{LL}, \tag{2.9}$$

$$\|b_0, c_0, b_j, c_j\|_{C^\alpha(\Omega)} \leq B, \qquad \|d\|_{L^\infty(\Omega)} \leq B. \tag{2.10}$$

2.1 Giving sense to the Cauchy problem

Consider the vector fields

$$X = a_0 \partial_t + \sum_{j=1}^{n} a_j \partial_{x_j} = a_0 Y. \tag{2.11}$$

Formal computations immediately show that the second order part of L can be written

$$L_2 u = ZXu - \tilde{L}_2 u \tag{2.12}$$

with

$$Zv = \partial_t v + \sum_{j=1}^{n} \partial_{x_j}(\tilde{a}_j v), \qquad \tilde{L}_2 u = \sum_{j,k=1}^{n} \partial_{x_j}\big(\tilde{a}_{j,k} \partial_{x_k} u\big), \tag{2.13}$$

$\tilde{a}_{j,k} = a_{j,k} + a_j a_k / a_0$, and $\tilde{a}_j = a_j / a_0$. Consequently, it follows that

$$Lu = (Z + \tilde{b}_0)(X + c_0)u - \tilde{L}_2 u + \tilde{L}_1 u + \tilde{d} u \tag{2.14}$$

with

$$\tilde{L}_1 u = \sum_{j=1}^{n} \tilde{b}_j \partial_{x_j} u + \sum_{j=1}^{n} \partial_{x_j}(\tilde{c}_j u) \tag{2.15}$$

and

$$\tilde{b}_0 = b_0 / a_0, \quad \tilde{b}_j = b_j - \tilde{b}_0 a_j, \quad \tilde{c}_j = c_j - \tilde{a}_j c_0, \quad \tilde{d} = d - c_0 \tilde{c}_0.$$

The next lemma shows that these identities are rigorous under minimal smoothness assumption on u.

Lemma 2.1. *Suppose that $u \in H^\rho(]0,T[) \times \mathbb{R}^n)$ for some $\rho \in]1-\alpha, \alpha[$. Then cu, Xu and $L_1 u$ belong to $H^{\rho-1}(]0,T[) \times \mathbb{R}^n)$. Moreover $L_2 u$ is well defined as a distribution in $H^{\rho-2}(]0,T[\times \mathbb{R}^n)$.*

Proof. u and its space-time derivatives $(\partial_t u, \partial_{x_j} u)$ belong to $H^{\rho-1}$. Following [6], their multiplication by a bounded LL function belong to the same space (see also Corollary 3.6). This shows that all the individual terms present in the definition of Xu belong to $H^{\rho-1}$ and those occurring in $L_2 u$ and ZXu are well defined in $H^{\rho-2}$ in the sense of distributions.

Next we recall that the multiplication $(b, u) \mapsto bu$ is continuous from $C^\alpha \times H^s$ to H^s when $|s| < \alpha$. This implies that the terms $b\partial u$ and $\partial(cu)$ that occur in $L_1 u$ and $\tilde{L}_1 u$ belong to $H^{\rho-1}$ since $\rho \in]1-\alpha, \alpha[$.

The last term du is in L^2, thus in $H^{\rho-1}$, since $c \in L^\infty$ and $u \in L^2$.

The identity (2.12) is straightforward from (2.2) since all the algebraic computations make sense by the preceding remarks. □

Next we need partial regularity results in time, showing that the traces of u and Xu at $t = 0$ are well defined, as distributions, for solutions of $Lu = f$. This is based on the remark that this equation is equivalent to the system

$$\begin{cases} Zv + \tilde{b}_0 v = \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d}u + f, \\ Yu + \tilde{c}_0 u = v/a_0 \end{cases} \tag{2.16}$$

with $\tilde{c}_0 = c_0/a_0$. The important remark is that, for this system, the coefficients of ∂_t, both for u and v, are equal to 1, thus smooth. Using the notation $Y = \partial_t + \widetilde{Y}$, $Z = \partial_t + \widetilde{Z}$, the system reads

$$\begin{cases} \partial_t v = -\tilde{Z}v - \tilde{b}_0 v - \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d}u + f, \\ \partial_t u = -\tilde{Y}u + v/a_0. \end{cases} \tag{2.17}$$

Lemma 2.2. *Suppose that $\rho \in]1-\alpha, \alpha[$ and $u \in H^\rho(]0, T[\times \mathbb{R}^n)$ is such that $Lu \in L^1([0, T]; H^{\rho-1}(\mathbb{R}^n))$. Then $u \in L^2([0, T]; H^\rho(\mathbb{R}^n))$ and $\partial_t u \in L^2([0, T]; H^{\rho-1}(\mathbb{R}^n))$. Therefore, $u \in C^0([0, T]; H^{\rho-\frac{1}{2}}(\mathbb{R}^n))$.*

Moreover, $Xu \in L^2([0, T]; H^{\rho-1}(\mathbb{R}^n))$ and $Xu \in C^0([0, T]; H^{\rho-\frac{3}{2}}(\mathbb{R}^n))$.

In particular, the traces $u_{|t=0}$ and $Xu_{|t=0}$ are well defined in $H^{\rho-\frac{1}{2}}(\mathbb{R}^n)$ and $H^{\rho-\frac{3}{2}}(\mathbb{R}^n)$, respectively.

Proof. **a)** We use the spaces $H^{s,s'}$ of Hörmander ([7], chapter 2), which are defined on $\mathbb{R}^{1+n}$ as the spaces of temperate distributions such that their Fourier transform $\hat{u}$ satisfies $(1+\tau^2+|\xi|^2)^{s/2}(1+|\xi|^2)^{s'/2}\hat{u} \in L^2$. The spaces on $[0, T] \times \mathbb{R}^n$ are defined by restriction. In particular, $H^{0,s'}([0, T] \times \mathbb{R}^n) = L^2([0, T]; H^{s'}(\mathbb{R}^n))$. Recall that ∂_{x_j} maps $H^{s,s'}$ to $H^{s,s'-1}$ and that

$$u \in H^{s,s'},\ \partial_t u \in H^{s,s'-1} \quad \Rightarrow \quad u \in H^{s+1,s'-1}. \tag{2.18}$$

b) For $u \in H^\rho$, the first derivatives of u, $\tilde{d}u$, as well as $\tilde{L}_1 u$, Xu and v belong to $H^{\rho-1} = H^{\rho-1,0}$, as well as their multiplication by a LL or C^α coefficient. Thus $\tilde{L}_2 u$ and $\tilde{Z}v$ belong to $H^{\rho-1,-1}$ and

$$\partial_t v = f + g, \quad f = Lu \in L^1(]0, T[; H^{\rho-1}),\ g \in H^{\rho-1,-1}. \tag{2.19}$$

Let

$$v_0(t) = \int_0^t f(t')dt' \in C^0(H^{\rho-1}).$$

In particular, $v \in L^2(]0,T[;H^{\rho-1}) = H^{0,\rho-1} \subset H^{\rho-1,0}$, since $\rho - 1 \leq 0$. Thus, $v - v_0 \in H^{\rho-1,0}$ and $\partial_t(v - v_0) = g \in H^{\rho-1,-1}$. By (2.18) $v - v_0 \in H^{\rho,-1} \subset H^{0,\rho-1}$ since $\rho \geq 0$.

Next, reasoning for fixed time and then taking L^2 norms we note that the multiplication by a LL or C^α function maps $L^2(]0,T[;H^{\rho-1}) = H^{0,\rho-1}$ into itself. Thus, by the second equation of (2.17), $\partial_t u = -\tilde{Y}u + v/a_0 \in H^{0,\rho-1}$. This finishes the proof of the first part of the lemma.

c) In particular, it implies that $v = Xu + b_0 u \in H^{0,\rho-1}$. Thus, $\tilde{Z}v$ and $\tilde{L}_2 u$ which involve multiplication by C^α or LL function, followed by a spatial derivative, belong to $H^{0,\rho-2}$. Therefore, the equation implies that in (2.19) $g \in H^{0,\rho-2}$. Thus applying (2.18) to $v - v_0 \in H^{0,\rho-1}$ implies that $v - v_0 \in H^{1,\rho-2} \subset C^0([0,T];H^{\rho-\frac{3}{2}}(\mathbb{R}^n))$. Since $|\rho - \frac{1}{2}| < \alpha$ and $u \in C^0([0,T];H^{\rho-\frac{1}{2}}(\mathbb{R}^n))$, the product $\tilde{b}_0 u$ belongs to $C^0([0,T];H^{\rho-\frac{1}{2}}(\mathbb{R}^n))$. Since v_0 is also in this space, we conclude that $Xu \in C^0([0,T];H^{\rho-\frac{3}{2}}(\mathbb{R}^n))$. □

Remark 2.3. If $\rho > \frac{1}{2}$, then the multiplication by LL functions maps $H^{\rho-\frac{3}{2}}$ into itself and we can conclude that $\partial_t u \in C^0([0,T];H^{\rho-\frac{3}{2}}(\mathbb{R}^n))$, as well as all the first derivatives of u, so that their traces at $t = 0$ are well defined. When $\rho \leq \frac{1}{2}$, the continuity of $\partial_t u$ is not clear. However, the trace of Xu has an intrinsic meaning, as a consequence of Proposition 1.4 (see Section 5).

Lemma 2.2 allows us to consider the Cauchy problem

$$Lu = f, \quad u_{|t=0} = u_0, \quad Xu_{|t=0} = u_1, \tag{2.20}$$

when $f \in \bigcup_{\rho>-\alpha} L^1([0,T];H^\rho(\mathbb{R}^n))$ and $u \in \bigcup_{\rho>1-\alpha} H^\rho(]0,T[) \times \mathbb{R}^n)$.

2.2 The main results

We first state uniqueness for the Cauchy problem:

Theorem 2.4. *If* $u \in \bigcup_{\rho>1-\alpha} H^\rho(]0,T[) \times \mathbb{R}^n)$ *satisfies*

$$Lu = 0, \quad u_{|t=0} = 0, \quad Xu_{|t=0} = 0 \tag{2.21}$$

then $u = 0$.

As in [5, 6], we prove existence of solutions in Sobolev spaces having orders decreasing in time. The proper definition is given as follows. The operators

$$|D| \quad \text{and} \quad \Lambda := \mathrm{Log}(2+|D|) \tag{2.22}$$

are defined by Fourier transform, associated to the Fourier multipliers $|\xi|$ and $\mathrm{Log}(2+|\xi|)$ respectively.

Definition 2.5. *i) $H^s(\mathbb{R}^n)$ or H^s denotes the usual Sobolev space on $\mathbb{R}^n$. $H^{s+\frac{1}{2}log}$ and $H^{s-\frac{1}{2}log}$ denote the spaces $\Lambda^{-\frac{1}{2}}H^s$ and $\Lambda^{\frac{1}{2}}H^s$ respectively.*

ii) Given parameters σ and λ, we denote by $\mathcal{C}_{\sigma,\lambda}(T)$ the space of functions u such that for all $t_0 \in [0,T]$, $u \in C^0([0,t_0], H^{\sigma-\lambda t_0})$.

iii) $\mathcal{H}_{\sigma\pm\frac{1}{2}log,\lambda}(T)$ denotes the spaces of functions u on $[0,T]$ with values in the space of temperate distributions in $\mathbb{R}^n$ such that

$$(1+|D|)^{\sigma-\lambda t}\Lambda^{\pm\frac{1}{2}}u(t,\cdot) \in L^2([0,T];L^2(\mathbb{R}^n)). \tag{2.23}$$

iv) $\mathcal{L}_{\sigma,\lambda}(T)$ denotes the space of functions u on $[0,T]$ with values in the space of temperate distributions in $\mathbb{R}^n$ such that

$$(1+|D|)^{\sigma-\lambda t}u(t,\cdot) \in L^1([0,T];L^2(\mathbb{R}^n)). \tag{2.24}$$

$\mathcal{C}_{\sigma,\lambda}(T)$ is equipped with the norm

$$\sup_{t\in[0,T]} \|u(t)\|_{H^{\sigma-\lambda t}}. \tag{2.25}$$

The norms in $\mathcal{H}_{\sigma\pm\frac{1}{2}log,\lambda}(T)$ and $\mathcal{L}_{\sigma,\lambda}(T)$ are given by (2.23) and (2.24). Equivalently, $\mathcal{H}_{\sigma\pm\frac{1}{2}log,\lambda}(T)$ and $\mathcal{L}_{\sigma,\lambda}(T)$ are the completions of $C_0^\infty([0,T]\times\mathbb{R}^n)$ for the norms

$$\|u\|_{\mathcal{H}_{\sigma\pm\frac{1}{2}log,\lambda}(T)} = \Big(\int_0^T \|u(t)\|^2_{H^{\sigma-\lambda t\pm\frac{1}{2}log}}dt\Big)^{\frac{1}{2}}. \tag{2.26}$$

and

$$\|u\|_{\mathcal{L}_{\sigma,\lambda}(T)} = \int_0^T \|u(t)\|_{H^{\sigma-\lambda t}}dt. \tag{2.27}$$

Theorem 2.6. *Fix $\theta < \theta_1$ in $]1-\alpha,\alpha[$. Then there are $\lambda>0$ and $K>0$, which depend only on the constants A_{L^∞}, A_{LL}, B, δ_0, δ_1, θ and θ_1, given by (2.8), (2.9) and (2.10), such that for*

$$T = \min\{T_0, \frac{\theta_1-\theta}{\lambda}\} \tag{2.28}$$

$u_0 \in H^{1-\theta}(\mathbb{R}^n)$, $u_1 \in H^{-\theta}(\mathbb{R}^n)$ and $f = f_1 + f_2$ with $f_1 \in \mathcal{L}_{-\theta,\lambda}(T)$ and $f_2 \in \mathcal{H}_{-\theta-\frac{1}{2}log,\lambda}(T)$, the Cauchy problem (2.20), has a unique solution $u \in \mathcal{C}_{1-\theta,\lambda}(T) \cap \mathcal{H}_{1-\theta+\frac{1}{2}log,\lambda}(T)$ with $\partial_t u \in \mathcal{C}_{-\theta,\lambda}(T) \cap \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$. Moreover, it satisfies

$$
\begin{aligned}
&\sup_{0\le t'\le t} \|u(t')\|^2_{H^{1-\theta-\lambda t'}} + \sup_{0\le t'\le t} \|\partial_t u(t')\|^2_{H^{-\theta-\lambda t'}} \\
&+ \int_0^t \Big(\|u(t')\|^2_{H^{1-\theta-\lambda t'+\frac{1}{2}log}} + \|\partial_t u(t')\|^2_{H^{-\theta-\lambda t'+\frac{1}{2}log}} \Big) dt' \\
&\le K \Big\{ \|u_0\|^2_{H^{1-\theta}} + \|u_1\|^2_{H^{-\theta}} \\
&+ \Big(\int_0^t \|f_1(t')\|_{H^{-\theta-\lambda t'}} dt' \Big)^2 + \int_0^t \|f_2(t')\|^2_{H^{-\theta-\lambda t'-\frac{1}{2}log}} dt' \Big\}.
\end{aligned} \tag{2.29}
$$

Note that for $t \in [0,T]$, $1 - \theta - \lambda t \ge 1 - \theta_1 > 1 - \alpha$, so that $f \in L^1([0,T]; H^{-\theta_2})$ with $\theta_1 < \theta_2 < \alpha$. Similarly, $u \in L^2([0,T]; H^{1-\theta_1})$ and $\partial_t u \in L^2([0,T]; H^{-\theta_1})$ implying that $u \in H^{1-\theta_1}([0,T] \times \mathbb{R}^n)$. Therefore, we are in a situation where we have given sense to the Cauchy problem.

Remark 2.7. This is a local in time existence theorem since the life span (2.28) is limited by the choice of λ. Thus the dependence of λ_0 on the coefficient is of crucial importance. In case of Lipschitz coefficients, there is no loss of derivatives; this would correspond to $\lambda = 0$. Using the notations in (2.9) (2.10) and (2.8), the analysis of the proof below shows that there is a function $K_0(\cdot)$ such that one can choose

$$
\lambda = \frac{A_{LL}}{\min\{\delta_0, \delta_1\}} K_0\big(\frac{A_{L^\infty}}{\delta_0}\big), \tag{2.30}
$$

revealing the importance of the LL-norms of the coefficients and the role of the hyperbolicity constant δ_1/δ_0. In particular, it depends only on the second order part of operator L.

Remark 2.8. A closer inspection of the proof, also shows that if the coefficients of the pricipal part of L are $(a_0, a_j, a_{j,k}) = (a'_0 + a''_0, a'_j + a''_j, a'_{j,k} + a''_{j,k})$ with $(a'_0, a'_j, a'_{j,k})$ Lipschitz continous and $(a''_0, a''_j, a''_{j,k})$ Log Lipschitz, with LL norm bounded by A''_{LL}, one can replace A_{LL} by A''_{LL} in the definition of λ. In particular if instead of (1.3) the coefficients satisfy

$$
|a(y) - a(y')| \le C\omega(|y - y'|) \tag{2.31}
$$

with a modulus of continuity ω such that

$$\lim_{\varepsilon \to 0+} \frac{\omega(\varepsilon)}{\varepsilon |\mathrm{Log} \varepsilon|} = 0, \tag{2.32}$$

they can be approximated by Lipschitz functions with errors arbitrarily small in the LL norm. This can be done by usual mollifications, which will preserve the L^∞ bounds A_{L^∞} and keep uniform hyperbolicity constants δ_0 and δ_1. As a consequence, λ can be taken arbitrarily small, yielding global in time existence with arbitrarily small loss of regularity (see Theorem 2.1 in [3] when the coefficients depend only on time).

3 Paradifferential calculus with LL coefficients

In this section we review several known results on paradifferential calculus and give the needed extensions to the case of Log-Lipschitz coefficients.

3.1 The Paley-Littlewood analysis

Introduce $\chi \in C_0^\infty(\mathbb{R})$, real valued, even and such that $0 \leq \chi \leq 1$ and

$$\chi(\xi) = 1 \quad \text{for } |\xi| \leq 1.1\,, \quad \chi(\xi) = 0 \quad \text{for } |\xi| \geq 1.9\,. \tag{3.1}$$

For $k \in \mathbb{Z}$, introduce $\chi_k(\xi) := \chi\big(2^{-k}\xi\big)$, $\widetilde{\chi}_k(x)$ its inverse Fourier transform with respect to ξ and the operators

$$\begin{gathered} S_k u := \widetilde{\chi}_k * u = \chi_k(D_x) u\,, \\ \Delta_0 = S_0, \quad \text{and for } k \geq 1 \quad \Delta_k = S_k - S_{k-1}. \end{gathered} \tag{3.2}$$

We note that Δ_k and S_k are self adjoint. Moreover, by evenness, $\widetilde{\chi}_k$ is real, so that Δ_k and S_k preserve reality. For all temperate distributions u one has

$$u = \sum_{k \geq 0} \Delta_k u\,. \tag{3.3}$$

The next propositions immediately follow from the definitions.

Proposition 3.1. *Consider $s \in \mathbb{R}$. A temperate distribution u belongs to $H^s(\mathbb{R}^n)$ [resp. $H^{s \pm \frac{1}{2} log}$] if and only if*

i) for all $k \in \mathbb{N}$, $\Delta_k u \in L^2(\mathbb{R}^d)$.

ii) the sequence $\delta_k = 2^{ks} \|\Delta_k u\|_{L^2(\mathbb{R}^d)}$ [resp. $\delta_k = (k+1)^{\pm\frac{1}{2}} 2^{ks} \|\Delta_k u\|_{L^2(\mathbb{R}^d)}$] belongs to $\ell^2(\mathbb{N})$.

Moreover, the norm of the sequence δ_k in ℓ^2 is equivalent to the norm of u in the given space.

Proposition 3.2. *Consider $s \in \mathbb{R}$ and $R > 0$. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R}^d)$such that:*

i) the spectrum of u_0 is contained in $\{|\xi| \leq R\}$ and for $k \geq 1$ the spectrum of u_k is contained in $\left\{\frac{1}{R}\,2^k \leq |\xi| \leq R\,2^k\right\}$.

ii) the sequence $\delta_k = 2^{ks}\|u_k\|_{L^2(\mathbb{R}^d)}$ [resp. $\delta_k = (k{+}1)^{\pm\frac{1}{2}}2^{ks}\|\Delta_k u\|_{L^2(\mathbb{R}^d)}$] belongs to $\ell^2(\mathbb{N})$.

Then $u = \sum u_k$ belongs to $H^s(\mathbb{R}^d)$ [resp. $H^{s\pm\frac{1}{2}log}$]. Moreover, the norm of the sequence δ_k in ℓ^2 is equivalent to the norm of u in the given space.

When $s > 0$, it is sufficient to assume that the spectrum of u_k is contained in $\left\{|\xi| \leq R\,2^k\right\}$.

Next we collect several results about the dyadic analysis of LL spaces.

Proposition 3.3. *There is a constant C such that for all $a \in LL(\mathbb{R}^n)$ and all integers $k > 0$*

$$\|\Delta_k a\|_{L^\infty} \leq Ck2^{-k}\|a\|_{LL}. \tag{3.4}$$

Moreover, for all $k \geq 0$

$$\|a - S_k a\|_{L^\infty} \leq C(k+1)\|a\|_{LL} \tag{3.5}$$

$$\|S_k a\|_{Lip} \leq C\Big(\|a\|_{L^\infty} + (k+1)\|a\|_{LL}\Big). \tag{3.6}$$

If $\alpha \in]0,1[$ and $a \in C^\alpha(\mathbb{R}^n)$, then

$$\|\Delta_k a\|_{L^\infty} \leq C2^{-\alpha k}\|a\|_{C^\alpha}. \tag{3.7}$$

Proof. S_k is a convolution operator with $\widetilde{\chi}_k$ which is uniformly bounded in L^1. Thus

$$\|S_k a\|_{L^\infty} \leq C\|a\|_{L^\infty}. \tag{3.8}$$

Moreover, since the integral of $\partial_j\widetilde{\chi}_k$ vanishes

$$\partial_j(S_k a)(x) = \int \partial_j\widetilde{\chi}_k(y)\big(a(x-y) - a(x)\big)dy.$$

Using the LL smoothness of a yields

$$\|\nabla S_k a\|_{L^\infty} \leq C(k+1)\|a\|_{LL}. \tag{3.9}$$

This implies (3.6). The proof of (3.4) is similar (cf [6]). The third estimate is classical. □

3.2 Paraproducts

Following J.-M. Bony ([2]), for $N \geq 3$ one defines the para-product of a and u as

$$T_a^N u = \sum_{k=N}^{\infty} S_{k-N} a \ \Delta_k u \tag{3.10}$$

The remainder $R_a^N u$ is defined as

$$R_a^N u = au - T_a^N u. \tag{3.11}$$

The next proposition extends classical results (see [2, 13]) to the case of LL coefficients and Log Sobolev spaces.

Proposition 3.4. *i) For $a \in L^\infty$ and $s \in \mathbb{R}$, T_a^N continuously maps H^s to H^s and $H^{s\pm\frac{1}{2}log}$ to $H^{s\pm\frac{1}{2}log}$. Moreover, the operator norms are uniformly bounded for s in a compact set.*

ii) If $a \in L^\infty \cap LL$ and $N' \geq N \geq 3$, $T_a^N - T_a^{N'}$ maps $H^{s+\frac{1}{2}log}$ into $H^{s+1-\frac{1}{2}log}$, for all $s \in \mathbb{R}$.

iii) If $a \in L^\infty \cap LL$, $N \geq 3$ and $s \in]0,1[$, R_a^N maps $H^{-s+\frac{1}{2}log}$ into $H^{1-s-\frac{1}{2}log}$, and

$$\|R_a^N u\|_{H^{1-s-\frac{1}{2}log}} \leq C\|a\|_{LL}\|u\|_{H^{-s+\frac{1}{2}log}} \tag{3.12}$$

with C uniformly bounded for s in a compact subset of $]0,1[$.

Proof. The first statement is an immediate consequence of (3.8) and Propositions 3.1 and 3.2.

Next, $T_a^N u - T_a^{N'} u = \sum_k v_k$ with $v_k = (S_{k-N}a - S_{k-N'}a) \ \Delta_k u$. By Proposition 3.3

$$\|v_k\|_{L^2} \leq C(k+1)2^{-k}\|\Delta_k u\|_{L^2}.$$

With Proposition 3.2, this implies *ii*).

To prove *iii*) we can assume that $N = 3$. Then

$$R_a u = \sum_{k\geq 3} \Delta_k a \ S_{k-3} u + \sum_k \sum_{|k-j|\leq 2} \Delta_j a \Delta_k u. \tag{3.13}$$

If $u \in H^{-s+\frac{1}{2}log}$, then

$$\|\Delta_j u\|_{L^2} \leq \frac{2^{js}}{\sqrt{j+1}}\varepsilon_j$$

with $\{\varepsilon_j\} \in \ell^2$. We note that the sequence

$$\widetilde{\varepsilon}_k = \sum_{j\le k} \frac{\sqrt{k+1}}{\sqrt{j+1}} 2^{(j-k)s} \varepsilon_j \tag{3.14}$$

is also in ℓ^2 with

$$\|\widetilde{\varepsilon}_k\|_{\ell^2} \le C\|\varepsilon_j\|_{\ell^2}$$

with C uniformly bounded when s in a compact subset of $]0, +\infty]$. Thus

$$\|S_{k-3}u\|_{L^2} \le \frac{2^{ks}}{\sqrt{k+1}} \varepsilon'_k$$

with $\{\varepsilon'_k\} \in \ell^2$. Therefore,

$$\|\Delta_k a \; S_{k-3}u\|_{L^2} \le C\sqrt{k+1}\; 2^{(s-1)k}\varepsilon'_k.$$

Proposition 3.2 implies that the first sum in (3.13) belongs to $H^{1-s-\frac{1}{2}log}$.

Similarly,

$$\Big\| \sum_{|k-j|\le 2} \Delta_j a \Delta_k u \Big\|_{L^2} \le C\sqrt{k+1}\; 2^{(s-1)k}\varepsilon''_k.$$

with $\{\varepsilon''_k\} \in \ell^2$. Now the spectrum of $\Delta_j a \Delta_k u$ is contained in the ball $\{|\xi| \le 2^{k+3}\}$; because $1-s > 0$, Proposition 3.2 implies that the second sum in (3.13) also belongs to $H^{1-s-\frac{1}{2}log}$, and the norm is uniformly bounded when s remains in a compact subset of $[0, 1[$. □

Remark 3.5. By $ii)$ we see that the choice of $N \ge 3$ is essentially irrelevant in our analysis, as in [2]. To simplify notation, we make a definite choice of N, for instance $N = 3$, and use the notation T_a and R_a for T_a^N and R_a^N.

Corollary 3.6. *The multiplication* $(a, u) \mapsto au$ *is continuous from* $(L^\infty \cap LL) \times H^{s+\delta log}$ *to* $H^{s+\delta log}$ *for* $s \in]-1, 1[$ *and* $\delta \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$.

Proof. (see [6]) Property $iii)$ says that R_a is smoothing by almost one derivative in negative spaces, and therefore, for all $\sigma \in]-1, 1[$ it maps H^σ to $H^{\sigma'}$ for all $\sigma' > \max\{\sigma, 0\}$ such that $\sigma' < \min\{\sigma + 1, 1\}$. Combining this observation with $i)$, the corollary follows. □

In particular, we note the following estimate

$$\|au\|_{H^{s+\frac{1}{2}log}} \le C\big(\|a\|_{L^\infty}\|u\|_{H^{s+\frac{1}{2}log}} + \|a\|_{LL}\|u\|_{H^s}\big). \tag{3.15}$$

Proposition 3.7. *Consider $q = \sqrt{(1+|\xi|^2)}$ and $\psi(\xi)$ a symbol of degree m on $\mathbb{R}^n$. Denote by $Q = \sqrt{(1-\Delta)}$ and Ψ the associated operators. If $a \in L^\infty \cap LL$, then the commutator $[Q^{-s}\Psi, T_a]$ maps $H^{-s+\frac{1}{2}log}$ into $H^{1-m-\frac{1}{2}log}$ and*

$$\|[Q^{-s}\Psi, T_a]u\|_{H^{1-m-\frac{1}{2}log}} \le C\|a\|_{LL}\ \|u\|_{H^{-s+\frac{1}{2}log}} \tag{3.16}$$

with C uniformly bounded for $s \in [0,1]$ and ψ in a bounded set.

Proof. We use Theorem 35 of [4], which states that if H is a Fourier multiplier with symbol h of degree 0 and if a is Lipschitzean, then

$$\|[H,a]\partial_{x_j}u\|_{L^2} \le C\|\nabla_x a\|_{L^\infty}\ \|u\|_{L^2}.$$

For $k>0$, writing $\Delta_k u$ as sum of derivatives, this implies that

$$\|[H,a]\Delta_k u\|_{L^2} \le C2^{-k}\|\nabla_x a\|_{L^\infty}\ \|\Delta_k u\|_{L^2}. \tag{3.17}$$

with C independent of k and H, provided that the symbol h remains in a bounded set of symbols of degree 0.

We now proceed to the proof of the proposition. Since Ψ and Q commute with Δ_k, one has

$$[Q^{-s}\Psi, T_a]u = \sum_{k\ge 3}[Q^{-s}\Psi, S_{k-3}a]\Delta_k u. \tag{3.18}$$

Moreover, since the spectrum of $S_{k-3}a\Delta_k u$ is contained in the annulus $2^{k-1} \le |\xi| \le 2^{k+2}$, it follows that

$$[Q^{-s}\Psi, S_{k-3}a]\Delta_k = 2^{k(m-s)}[H_k, S_{k-3}a]\Delta_k \tag{3.19}$$

where the symbol of H_k is

$$h_k(\xi) = 2^{k(s-m)}q^{-s}(\xi)\psi(\xi)\varphi(2^{-k}\xi)$$

and φ supported in a suitable fixed annulus. Note that the family $\{h_k\}$ is bounded in the space of symbols of degree 0, uniformly in k, $s \in [0,1]$ and ψ in a bounded set of symbols of degree m. By (3.17), it follows that

$$\|[H_k, S_{k-3}a]\Delta_k u\|_{L^2} \le C2^{k(m-s-1)}\|\nabla S_{k-3}a\|_{L^\infty}\ \|\Delta_k u\|_{L^2}.$$

Together with (3.9) and Proposition 3.1, this implies that for $u \in H^{-s+\frac{1}{2}log}$,

$$\|[Q^{-s}\Psi, S_{k-3}a]\Delta_k u\| \le C(k+1)\|a\|_{LL}\ \|\Delta_k u\|_{L^2}.$$

Using Proposition 3.2, the estimate (3.16) follows. □

Proposition 3.8. *If $a \in L^\infty \cap LL$ is real valued, then $\big(T_a - (T_a)^*\big)\partial_{x_j}$ and $\partial_{x_j}\big(T_a - (T_a)^*\big)$ map $H^{s+\frac{1}{2}log}$ into $H^{s-\frac{1}{2}log}$ and satisfy*

$$\begin{aligned} &\|\big(T_a - (T_a)^*\big)\partial_{x_j} u\|_{H^{s-\frac{1}{2}log}} \le C\|a\|_{LL}\ \|u\|_{H^{s+\frac{1}{2}log}}, \\ &\|\partial_{x_j}\big(T_a - (T_a)^*\big) u\|_{H^{s-\frac{1}{2}log}} \le C\|a\|_{LL}\ \|u\|_{H^{s+\frac{1}{2}log}}. \end{aligned} \tag{3.20}$$

Proof. The $S_k a$ are real valued, since a is real, and the Δ_k are self adjoint, thus

$$(T_a)^* u = \sum_{k=3}^{\infty} \Delta_k\big((S_{k-3}a)\, u\big).$$

Therefore, one has

$$\big(T_a - (T_a)^*\big) = \sum [S_{k-3}a, \Delta_k] = \sum [S_{k-3}a, \Delta_k]\Psi_k$$

where Ψ_k is a Fourier multiplier with symbol $\psi_k = \psi(2^{-k}\xi)$ and ψ is supported in a suitable annulus. Using again [4] (see (3.17)) yields

$$\|[S_{k-3}a, \Delta_k]\partial_{x_j}\Psi_k u\|_{L^2} \le C(k+1)\|a\|_{LL}\|\Psi_k u\|_{L^2},$$

and a similar estimate when the derivative is on the left of the commutator. Since the spectrum of $[S_{k-3}a, \Delta_k]\Psi_k u$ is contained in a annulus of size $\approx 2^k$, this implies (3.20). □

Proposition 3.9. *If a and b belong to $L^\infty \cap LL$, then $\big(T_a T_b - T_{ab}\big)\partial_{x_j}$ maps $H^{s+\frac{1}{2}log}$ into $H^{s-\frac{1}{2}log}$ and*

$$\begin{aligned} &\|\big(T_a T_b - T_{ab}\big)\partial_{x_j} u\|_{H^{s-\frac{1}{2}log}} \\ &\qquad \le C\Big(\|a\|_{LL}\|b\|_{L^\infty} + \|b\|_{LL}\|a\|_{L^\infty}\Big)\ \|u\|_{H^{s+\frac{1}{2}log}}. \end{aligned} \tag{3.21}$$

Proof. By Proposition 3.4, it is sufficient to prove the estimate for any paraproduct T^N. One has

$$T_a^N T_b^N \partial_{x_j} u = \sum_{k\ge N}\sum_{l\ge N} S_{k-N}a\ \Delta_k\Big(S_{l-N}b\ \Delta_l\partial_{x_j} u\Big).$$

In this sum, terms with $|l-k| \le 2$ vanish, because of the spectral localization of $S_{l-N}b\,\Delta_l\partial_{x_j}$. The commutators $[\Delta_k, S_{l-N}b]$ contribute to terms which are estimated as in (3.18):

$$\|[\Delta_k, S_{l-N}b]\Delta_l\partial_{x_j} u\|_{L^2} \le C(k+1)\|b\|_{LL}\ \|\Delta_l u\|_{L^2}.$$

If N is large enough, the spectrum of the corresponding term is contained in a annulus of size $\approx 2^k$ and hence the commutators contribute to an error term in (3.21). Therefore, it is sufficient to estimate

$$\sum_{k\geq N}\sum_{l\geq N}\Big(S_{k-N}aS_{l-N}b - S_{k-N}(ab)\Big)\Delta_k\,\Delta_l\partial_{x_j}u. \tag{3.22}$$

Again, only terms with $|l-k|\leq 2$ contribute to the sum. Using (3.5), one has

$$\begin{aligned}\|a - S_{k-N}a\|_{L^\infty} &\leq C(k+1)2^{-k}\|a\|_{LL},\\ \|b - S_{l-N}b\|_{L^\infty} &\leq C(k+1)2^{-k}\|b\|_{LL},\\ \|ab - S_{k-N}(ab)\|_{L^\infty} &\leq C(k+1)2^{-k}\|ab\|_{LL}.\end{aligned}$$

Thus

$$\begin{aligned}\|S_{k-N}aS_{l-N}b - &S_{k-N}(ab)\|_{L^\infty}\\ &\leq C(k+1)2^{-k}\big(\|a\|_{LL}\|b\|_{L^\infty} + \|a\|_{L^\infty}\|b\|_{LL}\big).\end{aligned}$$

Since the terms in the sum (3.22) have their spectrum in annuli of size $\approx 2^k$, this implies that this sum belongs to $H^{0-\frac{1}{2}log}$ when $u\in H^{0+\frac{1}{2}log}$, with an estimate similar to (3.21). □

3.3 Positivity estimates

The paradifferential calculus sketched above is well adapted to the analysis of high frequencies but does not take into account the low frequencies. For instance, the positivity of the function a does not imply the positivity of the operator T_a in L^2, only the positivity up to a smoothing operator. However, in the derivation of energy estimates, such positivity results are absolutely necessary. To avoid a separate treatment of low frequencies, we introduce *modified paraproducts* for which positivity results hold (we could also introduce weighted paraproducts as in [10, 11, 12]).

Consider a nonnegative integer ν and define the modified paraproducts

$$P^\nu_a u = \sum_{k=0}^{\infty} S_{\max\{\nu,k-3\}}a\ \Delta_k u = S_\nu a S_{\nu+2}u + \sum_{k=\nu}^{\infty} S_k a\ \Delta_{k+3}u. \tag{3.23}$$

Then

$$P^\nu_a u - T_a u = \sum_{k=0}^{\nu+2}\ \sum_{j=\max\{0,k-2\}}^{\nu} \Delta_j a\ \Delta_k u \tag{3.24}$$

and

$$au - P_a^\nu u = \sum_{j=\nu+1}^{\infty} \Delta_j a \; S_{j+2} u. \tag{3.25}$$

The difference (3.24) concerns only low frequencies, and therefore the results of Propositions 3.7, 3.8 and 3.9 are valid if one substitutes P_a^ν in place of T_a, at the cost of additional error terms. In particular, (3.24) and (3.25) immediately imply the following estimates:

Lemma 3.10. *i) There is a constant C such that for all ν, $a \in L^\infty$ and all $u \in L^2$,*

$$\|(P_a^\nu - T_a)\partial_{x_j} u\|_{L^2} + \|\partial_{x_j}(P_a^\nu - T_a)u\|_{L^2} \le C2^\nu \|a\|_{L^\infty} \|u\|_{L^2}. \tag{3.26}$$

ii) There is a constant C_0 such that for all ν for all $a \in LL$ and all $u \in L^2$,

$$\|au - P_a^\nu u\|_{L^2} \le C_0 \nu 2^{-\nu} \|a\|_{LL} \, \|u\|_{L^2}. \tag{3.27}$$

We will also use the following extension of Proposition 3.8:

Proposition 3.11. *If $a \in L^\infty \cap LL$ is real valued, then $\big(P_a^\nu - (P_a^\nu)^*\big)\partial_{x_j}$ and $\partial_{x_j}\big(P_a^\nu - (P_a^\nu)^*\big)$ map $H^{0+\frac{1}{2}log}$ into $H^{0-\frac{1}{2}log}$ and*

$$\begin{aligned} &\|\big(P_a^\nu - (P_a^\nu)^*\big)\partial_{x_j} u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\big(\|u\|_{H^{0+\frac{1}{2}log}} + \nu\|u\|_{L^2}\big), \\ &\|\partial_{x_j}\big(P_a^\nu - (P_a^\nu)^*\big)u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\big(\|u\|_{H^{0+\frac{1}{2}log}} + \nu\|u\|_{L^2}\big). \end{aligned} \tag{3.28}$$

Proof. One has

$$\big(P_a^\nu - (P_a^\nu)^*\big)\partial_{x_j} u = [S_\nu a, S_{\nu+2}]\partial_{x_j} u + \sum_{k\ge\nu} [S_k a, \Delta_{k+3}]\partial_{x_j} u.$$

The sum over k is treated exactly as in the proof of Proposition 3.8 and contibutes to the same error term. Using again Theorem 35 of [4], the L^2 norm of the first term is estimated by

$$C\|\nabla_x S_\nu a\|_{L^\infty}\|u\|_{L^2} \le C(\nu+1)\|a\|_{LL}\|u\|_{L^2}$$

and contibutes to the second error term in (3.28). When the derivative is on the left, the proof is similar. □

Moreover, a comparison of $P_a^\nu u$ with au immediately implies the following positivity estimate.

Corollary 3.12. *There is a constant c_0, such that for any positive LL-function a such that $\delta = \min a(x) > 0$, all ν such that $\nu 2^{-\nu} \le c_0\delta/\|a\|_{LL}$, and $u \in L^2(\mathbb{R}^n)$,*

$$\operatorname{Re}\big(P_a^\nu u, u\big)_{L^2} \ge \frac{\delta}{2}\|u\|_{L^2}^2. \tag{3.29}$$

Here, $(\cdot,\cdot)_{L^2}$ denotes the scalar product in L^2. This estimate extends to vector valued functions u and matrices a, provided that a is symmetric and positive.

3.4 The time dependent case

In the sequel we will consider functions of $(t,x) \in [0,T] \times \mathbb{R}^n$, considered as functions of t with values in various spaces of functions of x. In particular we denote by T_a the operator acting for each fixed t as $T_{a(t)}$:

$$(T_a u)(t) = \sum_{k=3}^{\infty} S_{k-3}(D_x)a(t)\ \Delta_k(D_x)u(t). \tag{3.30}$$

The Propositions 3.4, 3.7, 3.8 and 3.9 apply for each fixed t. There are similar definitions for the modified paraproducts P_a^ν; further, Lemma 3.10 and Corollary 3.12 apply for fixed t.

When a is a Lipschitz function of t, the definition (3.30) immediately implies that

$$[\partial_t, T_a] = T_{\partial_t a}, \qquad [\partial_t, P_a^\nu] = P_{\partial_t a}^\nu. \tag{3.31}$$

When a is only Log Lipschitz this formula does not make sense, since $\partial_t a$ is not defined as a function. The idea, already used in [5, 6], is that it is sufficient to commute ∂_t with time regularization of a. In our context, this simply means that in (3.30), we will replace the term $S_{k-3}a$, which is a spatial regularization of a, by a space-time regularization, namely $S_{k-3}a_k$ where a_k is a suitable time mollification of a. We now give the details for P^ν, as we will need them in the next section.

Introduce the mollifiers

$$\jmath_k(t) = 2^k \jmath(2^k t) \tag{3.32}$$

where $\jmath \in C_0^\infty(\mathbb{R})$ is non negative, with integral over $\mathbb{R}$ equal to 1.

Definition 3.13. *Given $a \in L^\infty \cap LL([0,T_0] \times \mathbb{R}^n)$, define*

$$a_k(t,x) = \jmath_k *_t \tilde{a} = \int \jmath_k(t-s)\tilde{a}(s,x)ds \tag{3.33}$$

where $\tilde{a}$ is the LL extension of a given by

$$\tilde{a}(t,x) = a(0,x), \quad t \le 0, \qquad \tilde{a}(t,x) = a(T_0,x), \quad t \ge T_0. \tag{3.34}$$

Next, for fixed t, the operator $\widetilde{P}^\nu_{a(t)}$ is defined by

$$\widetilde{P}^\nu_{a(t)} u = S_\nu a_\nu S_{\nu+2} u + \sum_{k=\nu}^{\infty} S_k a_k \ \Delta_{k+3} u. \tag{3.35}$$

We denote by $\widetilde{P}^\nu_a$ the operator acting on functions of (t,x) by $(\widetilde{P}^\nu_a u)(t) = \widetilde{P}^\nu_{a(t)} u(t)$.

Proposition 3.14. *Let $a \in L^\infty \cap LL([0,T_0] \times \mathbb{R}^n)$. Then for each $t \in [0,T_0]$, the operators $R_1(t) = (P^\nu_{a(t)} - \widetilde{P}^\nu_{a(t)})\partial_{x_j}$, $R_2(t) = \partial_{x_j}(P^\nu_{a(t)} - \widetilde{P}^\nu_{a(t)})$, $R_3(t) = \big((\widetilde{P}^\nu_{a(t)})^* - \widetilde{P}^\nu_{a(t)}\big)\partial_{x_j}$, $R_4(t) = \partial_{x_j}\big((\widetilde{P}^\nu_{a(t)})^* - \widetilde{P}^\nu_{a(t)}\big)$, and $R_5(t) = [D_t, \widetilde{P}^\nu_a](t)$ map $H^{0+\frac{1}{2}log}$ into $H^{0-\frac{1}{2}log}$ and there is a constant C such that for all $t \in [0,T_0]$ and for $k = 1,\dots,5$,*

$$\|R_k u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\big(\|u\|_{H^{0+\frac{1}{2}log}} + \nu\|u\|_{L^2}\big). \tag{3.36}$$

Proof. **a)** First, we recall several estimates from [6]. For $a \in LL([0,T_0] \times \mathbb{R}^n)$ the difference $a - a_k$ satisfies

$$|a(t,x) - a_k(t,x)| \le C(k+1)2^{-k}\|a\|_{LL}, \tag{3.37}$$

$$|\partial_t a_k(t,x)| \le C(k+1)\|a\|_{LL}. \tag{3.38}$$

with C independent of t and x. In particular, we note that

$$\|S_k(a(t) - a_k(t))\|_{L^\infty} \le C(k+1)2^{-k}\|a\|_{LL}. \tag{3.39}$$

b) In accordance with (3.35), for $l = 1,2,5$, we split R_l in two terms

$$R_l(t)u = B_l u + H_l u, \quad H_l u = \sum_{k \ge \nu} w_k \tag{3.40}$$

with $B_l u$ spectrally supported in the ball of radius $2^{\nu+4}$ and with w_k spectrally supported in an annulus $|\xi| \approx 2^k$. For R_1,

$$B_1 u = S_\nu(a(t) - a_\nu(t))\ S_{\nu+2}\partial_{x_j} u, \quad w_k = S_k(a(t) - a_k(t))\ \Delta_{k+3}\partial_{x_j} u.$$

With (3.39), this implies that

$$\|B_1 u\|_{L^2} \le C(\nu+1)\|a\|_{LL}\|u\|_{L^2}$$

and

$$\|w_k\|_{L^2} \le C(k+1)\|a\|_{LL}\|\Delta_{k+3}u\|_{L^2},$$

implying that

$$\|H_1 u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\|u\|_{H^{0+\frac{1}{2}log}}.$$

For R_2, the analysis is similar. One has

$$B_2 u = \partial_{x_j}\big(S_\nu(a(t)-a_\nu(t))\ S_{\nu+2}u\big), \quad w_k = \partial_{x_j}\big(S_k(a(t)-a_k(t))\ \Delta_{k+3}u\big).$$

Thanks to the spectral localization, the estimates for B_2u and w_k are the same as in the case of R_1, implying that

$$\|B_2 u\|_{L^2} \le C(\nu+1)\|a\|_{LL}\|u\|_{L^2} \tag{3.41}$$

$$\|H_2 u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\|u\|_{H^{0+\frac{1}{2}log}}. \tag{3.42}$$

c) For $k=5$ we write (3.40) with

$$B_5 u = S_\nu(\partial_t a_\nu(t))\ \Delta_{\nu+2}u, \quad w_k = S_k(\partial_t a_k(t))\ \Delta_{k+3}u.$$

Thus the estimates (3.38) imply

$$\|B_5 u\|_{L^2} \le C(\nu+1)\|a\|_{LL}\|u\|_{L^2}$$
$$\|H_5 u\|_{H^{0-\frac{1}{2}log}} \le C\|a\|_{LL}\|u\|_{H^{0+\frac{1}{2}log}}.$$

c) One has

$$R_3(t) = R_1(t) + R_2^*(t) + \big((P^\nu_{a(t)})^* - P^\nu_{a(t)}\big)\partial_{x_j}.$$

The third term is estimated in Proposition 3.11. The operators R_1 and $R_2^* = B_2^* + H_2^*$ are estimated in part b), implying that R_3 satisfies (3.36) for $k=3$. The proof for $R_4 = R_3^* = R_1^* + R_2 + \partial_{x_j}\big((P^\nu_{a(t)})^* - P^\nu_{a(t)}\big)$ is similar.

This finishes the proof of the Proposition. □

Lemma 3.15. *There is a constant C_0 such that for any $a \in LL([0,T_0]\times\mathbb{R}^n)$, $u \in L^2(\mathbb{R}^n)$, $\nu \ge 0$ and all $t \in [0,T_0]$, one has*

$$\|a(t)u - \widetilde{P}^\nu_{a(t)}u\|_{L^2} \le C_0 \nu 2^{-\nu}\|a\|_{LL}\,\|u\|_{L^2}. \tag{3.43}$$

Proof. We have

$$au - \widetilde{P}^{\nu}_{a} u = (a - S_\nu a_\nu) S_{\nu+2} u + \sum_{k=\nu}^{\infty} (a - S_k a_k)\ \Delta_{k+3} u.$$

Combining (3.5) and (3.39), we see that

$$\|a(t) - S_k a_k(t)\|_{L^\infty} \le C k 2^{-k} \|a\|_{LL}.$$

This implies (3.43). □

The lemma immediately implies the following positivity estimate.

Corollary 3.16. *There is a constant c_0, such that for any positive LL-function a such that $\delta = \min a(t,x) > 0$, all ν such that $\nu 2^{-\nu} \le c_0 \delta / \|a\|_{LL}$, and $u \in L^2(\mathbb{R}^n)$,*

$$\operatorname{Re}\big(P^{\nu}_{a(t)} u, u\big)_{L^2(\mathbb{R}^n)} \ge \frac{\delta}{2} \|u\|^2_{L^2(\mathbb{R}^n)}. \tag{3.44}$$

The same result holds for vector valued functions u and definite positive square matrices a.

Finally, we quote the following commutation result which will be needed in the next section.

Proposition 3.17. *Suppose that $a \in LL([0,T_0] \times \mathbb{R}^n)$. Then $\Lambda^{\frac12} [\widetilde{P}^{\nu}_{a(t)}, \Lambda^{\frac12}]$ and $[\widetilde{P}^{\nu}_{a(t)}, \Lambda^{\frac12}] \Lambda^{\frac12}$ are bounded in L^2 and satisfy*

$$\begin{aligned}\|\Lambda^{\frac12} [\widetilde{P}^{\nu}_{a(t)}, \Lambda^{\frac12}] u\|_{L^2} &+ \|[\widetilde{P}^{\nu}_{a(t)}, \Lambda^{\frac12}] \Lambda^{\frac12} u\|_{L^2} \\ &\le C\big(\nu^2 2^{-\nu} \|a\|_{LL} + \nu \|a\|_{L^\infty}\big) \|u\|_{L^2}.\end{aligned}$$

Proof. Thanks to the spectral localization, the low frequency part $S_\nu a_\nu S_{\nu+2}$ in $\widetilde{P}^{\nu}_{a}$ contributes to terms whose L^2 norm is bounded by

$$C\nu \|u\|_{L^2}.$$

The commutator with the high frequency part reads

$$\sum_{k \ge \nu} [\Lambda^{\frac12}, S_k a_k] \Delta_{k+3} u.$$

We argue as in the proof of Proposition 3.7 and write

$$[\Lambda^{\frac12}, S_k a_k] \Delta_{k+3} = (k+1)^{\frac12}\, [H_k, S_k a_k] \Delta_{k+3} \tag{3.45}$$

where the symbol of H_k is $h_k(\xi) = (k+1)^{-\frac{1}{2}}(\mathrm{Log}(2+|\xi|))^{\frac{1}{2}}\varphi(2^{-k}\xi)$ and φ is supported in a suitable fixed annulus. Note that the family $\{h_k\}$ is bounded in the space of symbols of degree 0. By (3.17), one has

$$\|[H_k, S_k a(t)]\Lambda^{\frac{1}{2}}\Delta_{k+3}u\|_{L^2} \leq C(k+1)2^{-k}\|\nabla_x S_k a_k(t)\|_{L^\infty}\|\Delta_k u\|_{L^2}.$$

Since $\nabla_x S_k a_k = (\nabla_x S_k a) * \jmath_k$, its L^∞ norm is bounded by $Ck\|a\|_{LL}$. Adding up, and using the spectral localization, these terms contribute a function whose L^2 norm is bounded by $C\nu^2 2^{-\nu}\|a\|_{LL}\|u\|_{L^2}$.

When $\Lambda^{\frac{1}{2}}$ is on the left of the commutator, the analysis is similar. □

4 Proof of the main results

4.1 The main estimate

We consider the operator (2.1) with coefficients which satisfy (2.4), (2.5) and (2.6). We fix $\theta < \theta_1$ in $]1-\alpha, \alpha[$, and with λ to be chosen later, we introduce the notation

$$s(t) = \theta + t\lambda. \tag{4.1}$$

Recall that

$$T = \min\{T_0, \frac{\theta_1 - \theta}{\lambda}\}. \tag{4.2}$$

Note that for $t \in [0, T]$, $s(t)$ remains in $[\theta, \theta_1] \subset]1-\alpha, \alpha[$.

We will consider solutions of the Cauchy problem

$$Lu = f, \quad u_{|t=0} = u_0, \quad Xu_{|t=0} = u_1 \tag{4.3}$$

with

$$u \in \mathcal{H}_{1-\theta+\frac{1}{2}log,\lambda}(T), \qquad \partial_t u \in \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T), \tag{4.4}$$

$$u_0 \in H^{1-\theta}(\mathbb{R}^n), \quad u_1 \in H^{-\theta}(\mathbb{R}^n), \tag{4.5}$$

$$f = f_1 + f_2, \quad f_1 \in \mathcal{L}_{-\theta,\lambda}(T), \quad f_2 \in \mathcal{H}_{-\theta-\frac{1}{2}log,\lambda}(T), \tag{4.6}$$

Note that if u and f satisfy (4.4) and (4.6), then

$$u \in L^2([0,T]; H^{1-\theta_1}), \quad \partial_t u \in L^2([0,T]; H^{-\theta_1}), \tag{4.7}$$

$$f \in L^1([0,T]; H^{-\theta_2}) \tag{4.8}$$

for all $\theta_2 \in]\theta_1, \alpha[$, so that the meaning of the Cauchy condition is clear.

The main step in the proof of Theorem 2.6 is the following:

Theorem 4.1. *There is a $\lambda_0 \geq 0$ of the form* (2.30) *such that for $\lambda \geq \lambda_0$ there is a constant K such that: for all f, u_0 and u_1 satisfying* (4.5) (4.6)*, and all u satisfying* (4.4) *solution of the Cauchy problem* (4.3)*, then*

$$u \in \mathcal{C}_{1-\theta,\lambda}(T), \quad \partial_t u \in \mathcal{C}_{-\theta,\lambda}(T) \tag{4.9}$$

and u satisfies the energy estimate (2.29)*.*

This theorem contains two pieces of information : first an energy estimate for smooth u, see Propositions 4.3 and 4.4. By a classical argument, smoothing the coefficients and passing to the limit, this estimate allows for the construction of weak solutions, see Section 5.2. The second piece of information contained in the theorem is a "weak=strong" type result showing that for data as in the theorem, any (weak) solution u satisfying (4.4) is the limit of smooth (approximate) solutions, in the norm given by the left hand side of the energy estimate, implying that u satisfies the additional smoothness (4.9) and the energy estimate. This implies uniqueness of weak solutions.

The idea is to get an energy estimate by integration by parts, from the analysis of

$$2\mathrm{Re}\langle Lu, e^{-2\gamma t}(1-\Delta_x)^{-s(t)} Xu\rangle \tag{4.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product in $\mathbb{R}^n$ extended to the Hermitian symmetric duality $H^\sigma \times H^{-\sigma}$ for all $\sigma \in \mathbb{R}$, and Δ_x denote the Laplace operator on $\mathbb{R}^d$. This extends the analysis of [6] where $X = \partial_t$. The parameter γ is chosen at the end to absorb classical error terms (present for Lipschitz coefficients) while the parameter λ which enters in the definition of $s(t)$, is chosen to absorb extra error terms coming from the loss of smoothness of the coefficients.

To prove Theorem 4.1, the first idea would be to mollify the equation. However, the lack of smoothness of the coefficients does not allow us to use this method directly and we cannot prove that the weak solutions are limits of exact smooth solutions. Instead, the idea is to write the equation as a system (2.16) for (u, v) and mollify this system. This leads to the consideration of the equations:

$$\begin{cases} Zv + \tilde{b}_0 v = \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d} u + f, \\ Yu + \tilde{c}_0 u = v/a_0 + g. \end{cases} \tag{4.11}$$

In this form, the commutator of spatial mollifiers with ∂_t are trivial, and we can prove that weak solutions of (4.11) are limits of smooth solutions, $(u^\varepsilon, v^\varepsilon)$ with $g^\varepsilon \neq 0$, which thus do not correspond to exact solutions u^ε of (4.3).

Notations. It is important for our purpose to keep track of the dependence of the various constants on the Log-Lipschitz norms. In particular we will use the notations δ_0, δ_1 of (2.8) and A_{LL}, A_{L^∞}, B of (2.9) (2.10). To simplify the exposition, we will denote by C, K_0 and K constants which may vary from one line to another, C denoting universal constants depending only on the paradifferential calculus; K_0 depending also on A_{L^∞}/δ_0; K, still independent of the parameters (γ, ε), but dependent also on δ_0, δ_1, θ_0, θ_1 and the various norms of the coefficients.

4.2 Estimating v

First, we give estimates that link v and $\partial_t u$.

Lemma 4.2. *Suppose that u satisfies* (4.4). *Then $v = Xu + c_0 u$ belongs to the space $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T) \subset L^2([0,T]; H^{-\theta_1})$ and for almost all t,*

$$\begin{aligned} \|v(t)\|_{H^{-s(t)+\frac{1}{2}log}} \leq & C A_{L^\infty}\big(\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} + \|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}}\big) \\ & + C(A_{LL}+B)\big(\|u(t)\|_{H^{1-s(t)}} + \|\partial_t u(t)\|_{H^{-s(t)}}\big), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}} \leq & K_0 \|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} + \frac{C}{\delta_0}\|v(t)\|_{H^{-s(t)+\frac{1}{2}log}} \\ & + K\big(\|u(t)\|_{H^{1-s(t)}} + \|v(t)\|_{H^{-s(t)}}\big). \end{aligned} \tag{4.13}$$

There are similar estimates in the spaces H^s without the $\frac{1}{2}log$.

If in addition $Lu = f$ with f satisfying (4.6), *then $\partial_t v \in L^1([0,T]; H^{-1-\theta_1})$.*

Proof. **a)** First, we note that the multiplication $(a, u) \mapsto au$ is continuous from $(L^\infty \cap LL)([0,T] \times \mathbb{R}^n) \times \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$ to $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$. Indeed, the corresponding norm estimate of the product is clear for smooth u, from (3.15) integrated in time. The claim follows by density. In particular, this shows that $a_0\partial_t u$ and the $a_j \partial_{x_j} u$ belong to $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$. Similarly, the estimate

$$\|bu(t)\|_{H^{-s(t)+\frac{1}{2}log}} \leq C\|bu(t)\|_{H^{1-s(t)}} \leq C\|b\|_{C^\alpha}\|u(t)\|_{H^{1-s(t)}} \tag{4.14}$$

implies that $c_0 u \in \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$. Therefore $v \in \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$ and the estimate (4.12) holds. The proof of (4.13) is similar, noting that

$$\partial_t u = \frac{1}{a_0} v - \sum_{j=1}^{d} \frac{a_j}{a_0}\partial_{x_j} u - \frac{c_0}{a_0} u.$$

b) As in the proof of Lemma 2.2, we see that the equation implies that

$$\partial_t v = f - \sum_{j=1}^{d} \partial_{x_j}(\tilde{a}_j v) - \tilde{b}_0 v + \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d} u.$$

The conservative form of $\tilde{L}_2$ and the multiplicative properties above show that

$$\partial_{x_j}(\tilde{a}_j v),\ \tilde{L}_2 u \in \mathcal{H}_{-\theta-1+\frac{1}{2}log,\lambda}(T) \subset L^2([0,T]; H^{-1-\theta_1}).$$

Similarly, $\tilde{L}_1 u$ and $\tilde{b}_0 v$ belong to $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T)$ and thus to $L^2([0,T]; H^{-\theta_1})$. The last term $\tilde{u}$ is in L^2. Therefore, $\partial_t v - f \in L^2([0,T]; H^{-1-\theta_1})$. Since $f \in L^1([0,T]; H^{-\theta_2})$ for $\theta_2 \in]\theta_1, \alpha[$, the lemma follows. □

Next, we give a-priori estimates in the space $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}(T) \cap \mathcal{C}_{-\theta,\lambda}(T)$ for smooth solutions of

$$(Z + \tilde{c}_0) v = \varphi, \quad v_{|t=0} = v_0. \tag{4.15}$$

We define the operators

$$(Qv)(t) = (1-\Delta_x)^{-s(t)/2} v(t), \quad (Q_\gamma v)(t) = e^{-\gamma t}(Qv)(t). \tag{4.16}$$

Proposition 4.3. *Suppose that* $v \in L^2([0,T]; H^1)$ *and* $\partial_t v \in L^1([0,T]; L^2)$. *Then the functions* $v_\gamma(t) := Q_\gamma v$ *belong to* $C^0([0,T], L^2)$ *and satisfy*

$$\begin{aligned} &\|v_\gamma(t)\|_{L^2}^2 + 2\int_0^t \|(\gamma + \lambda\Lambda)^{1/2} v_\gamma(t')\|_{L^2}^2 dt' \\ \leq & 2\int_0^t \langle (Z+\tilde{c}_0) v(t'), Q_\gamma^2(t') v(t')\rangle \, dt' + \|v_\gamma(0)\|_{L^2}^2 \\ &+ \int_0^t F(t') dt' \end{aligned} \tag{4.17}$$

with

$$F(t') \leq K_0 \frac{A_{LL}}{\delta_0} \|e^{-\gamma t'} \Lambda^{1/2} v(t')\|_{H^{-s(t')}}^2 + K \|v(t')\|_{H^{-s(t')}}^2. \tag{4.18}$$

Proof. **a)** Since $v \in L^2([0,T]; H^1)$ and $\partial_t v \in L^1([0,T]; L^2)$, we have

$$\partial_t Q_\gamma v = Q_\gamma \partial_t v - (\gamma + \lambda\Lambda) Q_\gamma v \in L^1([0,T]; L^2) \tag{4.19}$$

as immediately seen using the spatial Fourier transform. Moreover, $v_\gamma = Q_\gamma v \in C^0([0,T]; L^2)$ and satisfies the following identity

$$\|v_\gamma(t)\|^2_{L^2} - \|v_\gamma(0)\|^2_{L^2} = 2\mathrm{Re}\int_0^t \langle \partial_t Q_\gamma v, Q_\gamma v\rangle\, dt'.$$

Thus,

$$\begin{aligned} 2\mathrm{Re}\int_0^t \langle \partial_t v, Q_\gamma^2 v\rangle\, dt' &= 2\mathrm{Re}\int_0^t \langle Q_\gamma \partial_t v, Q_\gamma v\rangle\, dt' \\ &= \|v_\gamma(t)\|^2_{L^2} - \|v_\gamma(0)\|^2_{L^2} + 2\int_0^t \|(\gamma+\lambda\Lambda)^{1/2} v_\gamma(t')\|^2_{L^2} dt' \end{aligned} \tag{4.20}$$

b) Next we consider the terms $\partial_{x_j}(\tilde a_j v)$. We note that they belong to $L^2([0,T]; H^{-\sigma})$ for all $\sigma > 0$. In particular, since $s(t) \geq \theta > 0$, we note that the pairing

$$\langle \partial_{x_j}(\tilde a_j v), Q_\gamma^2 v\rangle$$

is well defined. We give an estimate for

$$2\mathrm{Re}\int_0^t \langle \partial_{x_j}(\tilde a_j v), Q_\gamma^2 v\rangle dt',$$

using the decomposition

$$\tilde a_j v = T_{\tilde a_j} v + R_{\tilde a_j} v.$$

By Proposition 3.4 it follows

$$\|R_{\tilde a_j} v(t)\|_{H^{1-s(t)-\frac{1}{2}log}} \leq C\|\tilde a_j\|_{LL}\|v(t)\|_{H^{-s(t)+\frac{1}{2}log}}$$

since $s(t) \in [\theta, \theta_1] \subset]0,1[$. Moreover,

$$\|Q_\gamma^2 v(t)\|_{H^{s(t)+\frac{1}{2}log}} \leq Ce^{-2\gamma t}\|v(t)\|_{H^{-s(t)+\frac{1}{2}log}}.$$

Thus

$$\begin{aligned} |\langle \partial_{x_j} R_{\tilde a_j} v(t), Q^2_{\gamma,\varepsilon} v(t)\rangle| &\leq \|R_{\tilde a_j} v(t)\|_{H^{1-s(t)-\frac{1}{2}log}}\ \|Q^2_{\gamma,\varepsilon} v(t)\|_{H^{s(t)+\frac{1}{2}log}} \\ &\leq C\|\tilde a_j\|_{LL} e^{-2\gamma t}\|v(t)\|^2_{H^{-s(t)+\frac{1}{2}log}}. \end{aligned}$$

It remains to consider

$$\begin{aligned} \mathrm{Re}\langle \partial_{x_j} T_{\tilde a_j} v, Q_\gamma^2 v\rangle &= \mathrm{Re}\langle Q_\gamma \partial_{x_j} T_{\tilde a_j} v, Q_\gamma v\rangle \\ &= \mathrm{Re}\langle \partial_{x_j} T_{\tilde a_j} Q_\gamma v, Q_\gamma v\rangle + \mathrm{Re}\langle \partial_{x_j}[Q_\gamma, T_{\tilde a_j}] v, Q_\gamma v\rangle. \end{aligned}$$

Note that these computations make sense because $v(t) \in H^1$ and all the pairings are well defined. Proposition 3.7 implies that

$$\|\langle \partial_{x_j}[Q_\gamma, T_{\tilde{a}_j}]v(t)\|_{0-\frac{1}{2}log} \leq Ce^{-\gamma t}\|\tilde{a}_j\|_{LL}\|v(t)\|_{-s(t)+\frac{1}{2}log}$$

and therefore

$$|\langle \partial_{x_j}[Q_\gamma, T_{\tilde{a}_j}]v(t), Q_\gamma v(t)\rangle| \leq C\|\tilde{a}_j\|_{LL}e^{-2\gamma t}\|v(t)\|^2_{H^{-s(t)+\frac{1}{2}log}}. \tag{4.21}$$

Next, for $v_\gamma(t) \in H^{2-\theta_1}$, we have

$$\begin{aligned} 2\text{Re}\langle \partial_{x_j}T_{\tilde{a}_j}v_\gamma, v_\gamma\rangle &= \text{Re}\langle(\partial_{x_j}T_{\tilde{a}_j} - T^*_{\tilde{a}_j}\partial_{x_j})v_\gamma, v_\gamma\rangle \\ &= \text{Re}\langle(T_{\tilde{a}_j} - T^*_{\tilde{a}_j})\partial_{x_j}v_\gamma, v_\gamma\rangle + \text{Re}\langle[\partial_{x_j}, T_{\tilde{a}_j}]\partial_{x_j}v_{\gamma,}, v_\gamma\rangle. \end{aligned}$$

Using Propositions 3.8 and 3.7, one can bound both terms by the right hand side of (4.21). Adding up, we have proved that

$$\left|2\text{Re}\int_0^t \langle \partial_{x_j}(\tilde{a}_j v), Q^2_\gamma v\rangle\, dt'\right| \leq C\|\tilde{a}_j\|_{LL}\int_0^t \|e^{-\gamma t'}\Lambda^{1/2}v(t')\|^2_{H^{-s(t)}}dt'.$$

c) The zero-th order term is clearly a remainder, and the multiplicative properties imply that

$$|\langle \tilde{c}_0 v(t), Q^2_\gamma v(t)\rangle \leq K\|v(t)\|^2_{H^{-s(t)}}.$$

d) We note that

$$\begin{aligned} \|a_j/a_0\|_{LL} &\leq \|a_j\|_{LL}\|\|1/a_0\|_{L^\infty} + \|a_j\|_{L^\infty}\|\|1/a_0\|_{LL} \\ &\leq \frac{A_{LL}}{\delta_0} + \frac{A_{L^\infty}A_{LL}}{\delta_0^2} \leq 2\frac{A_{L^\infty}A_{LL}}{\delta_0^2}, \end{aligned}$$

since $\delta_0 \leq a_0 \leq A_{L^\infty}$. Using identity (4.20) and the estimates of parts b) and c), implies (4.17) and so the proof of the Lemma is complete. □

4.3 Estimating $\nabla_x u$

We now get estimates of $\nabla_x u$ from the analysis of

$$-2\text{Re}\langle \widetilde{L}_2 u, Q^2_\gamma X u\rangle = -\sum_{j,k=1}^n 2\text{Re}\langle \partial_{x_j}(\tilde{a}_{j,k}\partial_{x_k}u), Q^2_\gamma X u\rangle \tag{4.22}$$

Proposition 4.4. *Suppose that* $u \in L^2([0,T];H^2)$ *with* $\partial_t u \in L^2([0,T];H^1)$. *Then* $u_\gamma := Q_\gamma u \in C^0([0,T],H^1)$ *and*

$$
\begin{aligned}
&\frac{1}{2}\delta_0\delta_1\|\nabla_x u_\gamma(t)\|^2_{L^2} + \int_0^t \delta_0\delta_1\|(\gamma+\lambda\Lambda)^{1/2}\nabla_x u_\gamma(t')\|^2_{L^2}dt' \\
&\leq -2\mathrm{Re}\int_0^t \langle \widetilde{L}_2 u, Q_\gamma^2 v\rangle\, dt' + CA^2_{L^\infty}\|\nabla_x u_\gamma(0)\|^2_{L^2} + \int_0^t E(t')dt',
\end{aligned}
\tag{4.23}
$$

where

$$
\begin{aligned}
|E(t)| \leq\ & \\
& K_0 A_{LL}A_{L^\infty}e^{-2\gamma t}\big(\|u(t)\|^2_{H^{1-s(t)+\frac{1}{2}log}} + \frac{1}{\delta_0^2}\|Xu(t)\|^2_{H^{-s(t)+\frac{1}{2}log}}\big) \\
& + Ke^{-2\gamma t}\big(\|u(t)\|^2_{H^{1-s(t)}} + \|Xu(t)\|^2_{H^{-s(t)}}\big).
\end{aligned}
\tag{4.24}
$$

To simplify the exposition, we note here that all the dualities $\langle\cdot,\cdot\rangle$ written below make sense, thanks to the smoothness assumption on u. This will not be repeated at each step. Moreover, in the proof below, we assume that u itself is smooth (in time).

Proof. **a)** We first perform several reductions. Using *iii)* of Proposition 3.4, one shows that

$$
\langle \partial_{x_j}(\tilde{a}_{j,k}\partial_{x_k}u), Q_\gamma^2 Xu\rangle = \langle \partial_{x_j}(T_{\tilde{a}_{j,k}}\partial_{x_k}u), Q_\gamma^2 Xu\rangle + E_1
$$

with

$$
|E_1(t)| \leq C\|\tilde{a}_{j,k}\|_{LL}\|\partial_{x_k}u(t)\|_{H^{-s(t)+\frac{1}{2}log}}\|Q_\gamma^2 Xu(t)\|_{H^{s(t)+\frac{1}{2}log}}.
\tag{4.25}
$$

Since $\|\tilde{a}_{j,k}\|_{LL} \leq K_0 A_{LL} \leq K_0 A_{LL}A_{L^\infty}/\delta_0$, E_1 satisfies (4.24). Similarly,

$$
\begin{aligned}
\langle \partial_{x_j}(T_{\tilde{a}_{j,k}}\partial_{x_k}u), Q_\gamma^2 Xu\rangle &= \langle \partial_{x_j}Q_{\gamma,\varepsilon}T_{\tilde{a}_{j,k}}\partial_{x_k}u, Q_\gamma Xu\rangle + \\
&= \langle \partial_{x_j}T_{\tilde{a}_{j,k}}\partial_{x_k}Q_\gamma u, Q_\gamma Xu\rangle + E_2
\end{aligned}
$$

where E_2 also satisfies (4.25), and hence (4.24).

b) Next we write

$$
Xu = T_{a_0}\partial_t u + \sum T_{a_j}\partial_{x_j}u + r
$$

and

$$
\begin{aligned}
\|r(t)\|_{H^{1-s(t)-\frac{1}{2}log}} \leq CA_{LL}\big(\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} + \|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}}\big) \\
+CB\|u(t)\|_{H^{1-s(t)}}.
\end{aligned}
$$

Therefore, r contributes to an error term $E_3 = \langle \partial_{x_j} T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, Q_\gamma r \rangle$ such that

$$|E_3(t)| \leq e^{-2\gamma t} K_0 A_{L^\infty} \|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} \|r(t)\|_{H^{1-s(t)-\frac{1}{2}log}}.$$

Using (4.13) in the estimate of r, we see that

$$|E_3(t)| \leq e^{-2\gamma t} K_0 A_{L^\infty} A_{LL} \|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} \Big(\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} + \frac{1}{\delta_0} \|Xu(t)\|_{H^{-s(t)+\frac{1}{2}log}} + K\|u(t)\|_{H^{1-s(t)}} + K\|Xu(t)\|_{H^{-s(t)}} \Big)$$

and hence satisfies (4.24).

c) Consider now the term

$$\begin{aligned}
\langle \partial_{x_j} T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, Q_\gamma T_{a_0} \partial_t u \rangle &= -\langle T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_j} Q_\gamma T_{a_0} \partial_t u \rangle \\
&= -\langle T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, T_{a_0} \partial_{x_j} Q_\gamma \partial_t u \rangle + E_4 \\
&= -\langle (T_{a_0})^* T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_j} Q_\gamma \partial_t u \rangle + E_4 \\
&= -\langle T_{a_0} T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_j} Q_\gamma \partial_t u \rangle + E_4 + E_5 \\
&= -\langle T_{a_0 \tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_j} Q_\gamma \partial_t u \rangle + E_4 + E_5 + E_6
\end{aligned}$$

where E_4, E_5 and E_6 are estimated by Proposition 3.7, 3.8 and 3.9 respectively. They all satisfy

$$|E_k(t)| \leq C e^{-2\gamma t} A \|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} \|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}}.$$

with $A = \|\tilde{a}_{j,k}\|_{LL} \|a_0\|_{L^\infty} + \|\tilde{a}_{j,k}\|_{L^\infty} \|a_0\|_{LL} \leq K_0 A_{L^\infty} A_{LL}$. Again using (4.13) to replace $\partial_t u$ by Xu, one shows that these errors satisfy (4.24).

Similarly

$$\langle \partial_{x_j} T_{\tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, Q_\gamma T_{a_l} \partial_{x_l} u \rangle = -\langle T_{a_l \tilde{a}_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_l} \partial_{x_j} Q_\gamma u \rangle + E_7$$

where E_7 satisfies

$$|E_7(t)| \leq C e^{-2\gamma t} K_0 A_{L^\infty} A_{LL} \|u(t)\|^2_{H^{1-s(t)+\frac{1}{2}log}} \tag{4.26}$$

thus (4.24).

d) Introduce the notation

$$w_j = \partial_{x_j} Q_\gamma u. \tag{4.27}$$

Because $\tilde{a}_{j,k} = \tilde{a}_{k,j}$, we have

$$\begin{aligned}\mathrm{Re}\langle T_{a_l \tilde{a}_{j,k}} w_k, \partial_{x_l} w_j\rangle + \mathrm{Re}\langle T_{a_l \tilde{a}_{k,j}} w_j, \partial_{x_l} w_k\rangle \\ = \mathrm{Re}\langle ((T_{a_l \tilde{a}_{j,k}})^* \partial_{x_l} - \partial_{x_l} T_{a_l \tilde{a}_{j,k}}) w_k, w_j\rangle := E_8\end{aligned}$$

Using Propositions 3.8 and 3.7, one shows that E_8 satisfies

$$|E_8(t)| \le C \|a_l \tilde{a}_{j,k}\|_{LL} \|w_j(t)\|_{H^{0+\frac{1}{2}log}} \|w_k(t)\|_{H^{0+\frac{1}{2}log}}$$

and therefore E_8 also satisfies (4.26) thus (4.24).

e) It remains to consider the sum

$$S := \mathrm{Re} \sum_{j,k=1}^{n} \langle T_{b_{j,k}} \partial_{x_k} Q_\gamma u, \partial_{x_j} Q_\gamma \partial_t u\rangle \tag{4.28}$$

with $b_{j,k} = a_0 \tilde{a}_{j,k} = a_0 a_{j,k} + a_j a_k$. By the strict hyperbolicity assumption (2.8), it follows for all $\xi \in \mathbb{R}^n$

$$\sum_{j,k=1}^{n} b_{j,k}(t,x) \xi_j \xi_k \ge \delta_0 \delta_1 |\xi|^2.$$

Therefore, we can use Corollary 3.16. Since $\|b_{j,k}\|_{LL} \le 2 A_{L^\infty} A_{LL}$, there exists an integer ν, with

$$\frac{2^\nu}{\nu} \approx \frac{A_{L^\infty} A_{LL}}{\delta}, \tag{4.29}$$

such that for all $t \in [0, T_0]$ and $(w_1, \ldots, w_n)$ in $L^2(\mathbb{R}^n)$, the following estimate is satisfied

$$\mathrm{Re} \sum_{j,k=1}^{n} \langle P^\nu_{b_{j,k}(t)} w_k, w_j\rangle \ge \frac{\delta_0 \delta_1}{2} \|w\|^2_{L^2} \tag{4.30}$$

From now on we fix such a ν and use the notation P_b in place of P^ν_b.

Using Lemma 3.10 and Proposition 3.14, we see that

$$\|\partial_{x_j} T_{b_{j,k}} w_k - \partial_{x_j} \widetilde{P}_{b_{j,k}} w_k\|_{H^{0-\frac{1}{2}log}} \le C \|b_{j,k}\|_{LL} \Big(\|w_k\|_{H^{0+\frac{1}{2}log}} + K \|w_k\|_{L^2} \Big)$$

Therefore

$$S = \mathrm{Re}\sum_{j,k=1}^{n}\langle \widetilde{P}_{b_{j,k}}\partial_{x_k}Q_\gamma u, \partial_{x_j}Q_\gamma \partial_t u\rangle + E_9$$

where

$$|E_9(t)| \leq Ce^{-2\gamma t}\|b_{j,k}\|_{LL}\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}}\|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}} + e^{-2\gamma t}\nu K\|u(t)\|_{H^{1-s(t)}}\|\partial_t u(t)\|_{H^{-s(t)+\frac{1}{2}log}}.$$

Using (4.13), implies that E_9 satisfies (4.24).

Next, we use Proposition 3.14 to replace $\partial_{x_j}\widetilde{P}_{b_{j,k}}$ by $\frac{1}{2}\partial_{x_j}(\widetilde{P}_{b_{j,k}} + (\widetilde{P}_{b_{j,k}})^*)$ at the cost of an error E_{10} similar to E_9.

At this stage, we commute Q_γ and ∂_t as in (4.19). Using the notation (4.27), yields

$$\begin{aligned}2S = &\sum_{j,k=1}^{n}\mathrm{Re}\langle(\widetilde{P}_{b_{j,k}} + (\widetilde{P}_{b_{j,k}})^*)w_k, \partial_t w_j\rangle \\ &+ \gamma\sum_{j,k=1}^{n}\mathrm{Re}\langle(\widetilde{P}_{b_{j,k}} + (\widetilde{P}_{b_{j,k}})^*)w_k, w_j\rangle \\ &+ \lambda\sum_{j,k=1}^{n}\mathrm{Re}\langle(\widetilde{P}_{b_{j,k}} + (\widetilde{P}_{b_{j,k}})^*)w_k, \Lambda w_j\rangle + 2E_9 + 2E_{10}.\end{aligned} \tag{4.31}$$

We denote by S^1, S^2 and S^3 the sums on the right hand side.

f) The symmetry $b_{j,k} = b_{k,j}$ implies the identity

$$S^1 = \frac{d}{dt}\sum_{j,k=1}^{n}\mathrm{Re}\langle\widetilde{P}_{b_{j,k}}w_k, w_j\rangle + E_{11}$$

where

$$E_{11} = \sum_{j,k=1}^{n}\mathrm{Re}\langle[\widetilde{P}_{b_{j,k}}, \partial_t]w_k, w_j\rangle$$

is estimated using Proposition 3.14:

$$\begin{aligned}|E_{11}(t)| &\leq C\|b_{j,k}\|_{LL}\big(\|w(t)\|_{H^{0+\frac{1}{2}log}} + \nu\|w\|_{L^2}\big)\|w(t)\|_{H^{0+\frac{1}{2}log}} \\ &\leq Ce^{-2\gamma t}\|b_{j,k}\|_{LL}\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} \\ &\qquad\qquad \big(\|u(t)\|_{H^{1-s(t)+\frac{1}{2}log}} + \nu\|u(t)\|_{H^{1-s(t)}}\big)\end{aligned}$$

and therefore satisfies (4.24). Moreover,

$$\mathrm{Re}\langle \widetilde{P}_{b_{j,k}} w_k, \Lambda w_j\rangle = \mathrm{Re}\langle \widetilde{P}_{b_{j,k}} \Lambda^{\frac12} w_k, \Lambda^{\frac12} w_j\rangle + \mathrm{Re}\langle \Lambda^{\frac12}[\Lambda^{\frac12}, \widetilde{P}_{b_{j,k}},] w_k, w_j\rangle$$

$$\mathrm{Re}\langle (\widetilde{P}_{b_{j,k}})^* w_k, \Lambda w_j\rangle = \mathrm{Re}\langle \Lambda^{\frac12} w_k, \widetilde{P}_{b_{j,k}} \Lambda^{\frac12} w_j\rangle + \mathrm{Re}\langle w_k, [\widetilde{P}_{b_{j,k}}, \Lambda^{\frac12}]\Lambda^{\frac12} w_j\rangle.$$

We use Proposition 3.17 to estimate the commutators and

$$S^3 = 2\sum_{j,k=1}^{n} \mathrm{Re}\langle \widetilde{P}_{b_{j,k}} \Lambda^{\frac12} w_k, \Lambda^{\frac12} w_j\rangle + E_{12}$$

where

$$|E_{12}(t)| \leq K\|w(t)\|^2_{L^2} \leq K\|u(t)\|^2_{H^{1-s(t)}}.$$

Summing up, we have shown that up to an error which satisfies (4.24), the quantity (4.22) under consideration is equal to

$$\begin{aligned}\frac{d}{dt}\sum_{j,k=1}^{n} \mathrm{Re}\langle \widetilde{P}_{b_{j,k}} w_k, w_j\rangle &+ \gamma\sum_{j,k=1}^{n} 2\mathrm{Re}\langle \widetilde{P}_{b_{j,k}} w_k, w_j\rangle \\ &+ \lambda\sum_{j,k=1}^{n} 2\mathrm{Re}\langle \widetilde{P}_{b_{j,k}} \Lambda^{\frac12} w_k, \Lambda^{\frac12} w_j\rangle.\end{aligned} \tag{4.32}$$

By (4.30), the last two sums are larger than or equal to $\delta_0\delta_1\|w(t)\|^2_{L^2}$ and $\delta_0\delta_1\|w(t)\|^2_{H^{0+\frac12 log}}$, respectively. Similarly, integrating the first term between 0 and t and using (4.30) gives control of $\frac{\delta_0\delta_1}{2}\|w(t)\|_{L^2}$, finishing the proof of (4.23). □

4.4 A-priori estimates for the solutions of (4.11)

The proof of Theorem 4.1 is based on a-priori estimates for smooth solutions of the system (4.11).

Theorem 4.5. *There are $\lambda_0 \geq 0$ of the form* (2.30) *and γ_0 such that for $\lambda \geq \lambda_0$ and $\gamma \geq \gamma_0$ the following is true:*

for all $u \in L^2([0,T]; H^2)$ and $v \in L^2([0,T]; H^1)$ with $\partial_t u \in L^2([0,T]; H^1)$ and $\partial_t v \in L^1([0,T]; L^2)$ and for all parameters λ, γ and all $t \leq T$, the *following holds:*

$$
\begin{aligned}
&\sup_{0\le t'\le t} e^{-2\gamma t'}\Big(\frac{1}{2}\delta_0\delta_1\|u(t')\|^2_{H^{1-s(t')}} + \|v(t')\|^2_{H^{-s(t')}}\Big)\\
&+\delta_0\delta_1\int_0^t e^{-2\gamma t'}(\lambda\|u(t')\|^2_{H^{1-s(t')+\frac{1}{2}log}} + \gamma\|u(t')\|^2_{H^{1-s(t')}})dt'\\
&+\int_0^t e^{-2\gamma t'}(\lambda\|v(t')\|^2_{H^{-s(t')+\frac{1}{2}log}} + \gamma\|v(t')\|^2_{H^{-s(t')}})dt' \qquad (4.33)\\
&\le CA^2_{L^\infty}\|u(0)\|^2_{H^{1-\theta}} + \|v(0)\|^2_{H^{-\theta}} + 2\mathrm{Re}\int_0^t \langle f, Q^2_\gamma v\rangle\, dt'\\
&+K\int_0^t e^{-2\gamma t'}\|g(t')\|_{1-s(t)-\frac{1}{2}log}\|u(t')\|_{1-s(t)+\frac{1}{2}log}dt',
\end{aligned}
$$

with $f = Zv+\tilde b_0 v-\tilde L_2 u+\tilde L_1 u+\tilde d u \in L^1([0,T];H^{\alpha'-1})$, $g = Yu+\tilde c_0 u - v/a_0 \in L^2([0,T];H^{\alpha'})$ *for all* $\alpha' < \alpha$.

Proof. We compute the integral over $[0,t]$ of $\mathrm{Re}\langle f, Q^2_\gamma v\rangle$. Proposition 4.3 takes care of the first term $2\mathrm{Re}\langle Zv+\tilde b_0 v, Q^2_\gamma v\rangle$. We split the second term into three pieces

$$
\langle \tilde L_2 u, Q^2_\gamma v\rangle = \langle \tilde L_2 u, Q^2_\gamma X u\rangle - \langle \tilde L_2 u, Q^2_\gamma(a_0 g)\rangle + \langle \tilde L_2 u, Q^2_\gamma(c_0 u)\rangle
$$

and use Proposition 4.4 for the first piece. The multiplicative properties imply that

$$
\begin{aligned}
|\langle \tilde L_2 u(t), Q^2_\gamma(a_0 g)(t)\rangle| \le& K\|g(t)\|_{1-s(t)-\frac{1}{2}log}\|\tilde L_2 u(t)\|_{-1-s(t)+\frac{1}{2}log}\\
\le& K\|g(t)\|_{1-s(t)-\frac{1}{2}log}\|u(t)\|_{1-s(t)+\frac{1}{2}log},
\end{aligned}
$$

and

$$
\begin{aligned}
|\langle \tilde L_2 u(t), Q^2_\gamma(c_0 u)(t)\rangle| \le& K\|u(t)\|_{1-s(t)}\|\tilde L_2 u(t)\|_{-1-s(t)}\\
\le& K\|u(t)\|^2_{1-s(t)}.
\end{aligned}
$$

Next, using the multiplicative properties stated in Corollary 3.6 for the products $\tilde b_j\partial_{x_j}u$ and $\partial_{x_j}(\tilde c_j u)$, and the embedding $L^2\subset H^{-s}$ for the term $\tilde d u$, we see that

$$
\|(\widetilde L_1 u + \tilde d u)(t)\|_{H^{-s(t)}} \le K\|u(t)\|_{H^{1-s(t)}}.
$$

Thus

$$
\begin{aligned}
|\langle(\tilde L_1+\tilde d)u(t), Q^2_\gamma v(t)\rangle| \le& K\|u(t)\|_{1-s(t)}\|v(t)\|_{-s(t)}\\
\le& K\big(\|u(t)\|^2_{1-s(t)} + \|v(t)\|^2_{-s(t)}\big).
\end{aligned}
$$

Proposition 4.4 gives an estimate of $\nabla_x u$. We also need an estimate for u. The identity (4.20) applied to u yields

$$e^{-2\gamma t}\|u(t)\|^2_{H^{-s(t)}} + \int_0^t e^{-2\gamma t'}(\lambda\|u(t')\|^2_{H^{-s(t')+\frac{1}{2}log}} + \gamma\|u(t')\|^2_{H^{-s(t')}})dt'$$
$$= \|u_\gamma(0)\|^2_{H^{-s(0)}} + 2\mathrm{Re}\int_0^t \langle\partial_t u, Q_\gamma^2 u\rangle dt'.$$

Next, we use the inequality

$$|\langle\partial_t u, Q_\gamma^2 u\rangle| \le C\big(\|u(t)\|^2_{H^{1-s(t)}} + \|\partial_t u(t)\|^2_{H^{-1-s(t)}}\big).$$

In addition, we note that the second equation in (4.11) yields

$$\|\partial_t u(t)\|_{H^{-1-s(t)}} \le K\big(\|v(t)\|^2_{H^{-s(t)}} + \|u(t)\|^2_{H^{-s(t)}}\big) + \|g(t)\|^2_{H^{-1-s(t)}}.$$

We add the various estimates and use Propositions 4.3 and 4.4 to obtain a final estimate. On the left hand side we have

$$\sup_{0\le t'\le t} e^{-2\gamma t'}\Big(\frac{1}{2}\delta_0\delta_1\|u(t')\|^2_{H^{1-s(t')}} + \|v(t')\|^2_{H^{-s(t')}}\Big) \tag{4.34}$$

$$+\gamma\int_0^t e^{-2\gamma t'}\big(\delta_0\delta_1\|u(t')\|^2_{H^{1-s(t')}} + \|v(t')\|^2_{H^{-s(t')}}\big)dt' \tag{4.35}$$

$$+\lambda\int_0^t e^{-2\gamma t'}\big(\delta_0\delta_1\|u(t')\|^2_{H^{1-s(t')+\frac{1}{2}log}} + \|v(t')\|^2_{H^{-s(t')+\frac{1}{2}log}}\big)dt'. \tag{4.36}$$

On the right hand side, we find the initial data

$$CA^2_{L^\infty}\|u(0)\|^2_{H^{1-s(0)}} + \|v(0)\|^2_{H^{-s(0)}}, \tag{4.37}$$

the contribution of f

$$2\mathrm{Re}\int_0^t \langle f(t'), Q_\gamma v(t')\rangle dt', \tag{4.38}$$

an estimated contribution of g

$$K\int_0^t e^{-2\gamma t'}\|g(t')\|_{1-s(t)-\frac{1}{2}log}\|u(t')\|_{1-s(t')+\frac{1}{2}log}dt', \tag{4.39}$$

and two types of "remainders":

$$K_0 A_{LL} A_{L^\infty} \int_0^t e^{-2\gamma t'} \big(\|u(t')\|^2_{H^{1-s(t')+\frac{1}{2}log}} + \frac{1}{\delta_0^2} \|v_\gamma(t')\|^2_{H^{-s(t')+\frac{1}{2}log}} \big) dt' \tag{4.40}$$

and

$$K \int_0^t e^{-2\gamma t'} \big(\|u_\gamma(t')\|^2_{H^{1-s(t')}} + \|v(t')\|^2_{H^{-s(t')}} \big) dt'. \tag{4.41}$$

If

$$\lambda \geq 2K_0 \frac{A_{LL} A_{L^\infty}}{\delta_0 \delta_1} \quad \text{and} \quad \lambda \geq 2K_0 \frac{A_{LL} A_{L^\infty}}{\delta_0^2} \tag{4.42}$$

the term in (4.40) can be absorbed by (4.36). Note that this choice of λ is precisely the choice announced in (2.30), with a new function K_0 of A_{L^∞}/δ_0. Finally, if γ is large enough, the term (4.41) is absorbed by (4.35), finishing the proof of the main estimate (4.33). □

4.5 Proof of Theorem 4.1

From now on, we assume that $\lambda \geq \lambda_0$ and $\gamma \geq \gamma_0$ are fixed, so that the estimate (4.33) holds. Consider u, f, u_0 and u_1 satisfying the equation (4.3) and the smoothness assumptions (4.4), (4.5), (4.6). Consider $v = Xu + c_0 u$, which by Lemma 4.2 satisfies

$$v \in \mathcal{H}_{-\theta+\frac{1}{2}log}, \quad \partial_t v \in L^1([0,T]; H^{-1-\theta_1}), \quad v_{|t=0} = v_0 \in H^{-\theta}, \tag{4.43}$$

with $v_0 = a_{0|t=0} u_1 + \sum a_{j|t=0} \partial_{x_j} u_0 + c_{0|t=0} u_0$. In particular, (u, v, f) and $g = 0$ satisfy (4.11).

We mollify u and v and introduce, for $\varepsilon > 0$,

$$u_\varepsilon = J_\varepsilon u, \quad v_\varepsilon = J_\varepsilon v \quad \text{with} \quad J_\varepsilon = (1 - \varepsilon \Delta_x)^{-1}. \tag{4.44}$$

For all $\varepsilon > 0$, (4.4) and (4.43) imply that

$$\begin{aligned} u_\varepsilon &\in L^2([0,T], H^2), & \partial_t u_\varepsilon &\in L^2([0,T], H^1), \\ v_\varepsilon &\in L^2([0,T], H^1), & \partial_t v_\varepsilon &\in L^1([0,T], L^2), \end{aligned}$$

(see (4.7)). Moreover, using the spatial Fourier transform, one immediately sees that u_ε converges to u in $\mathcal{H}_{1-\theta,\lambda}(T)$ and v_ε converges to v in $\mathcal{H}_{-\theta,\lambda}(T)$.

Define

$$\begin{aligned} f_\varepsilon &= Z v_\varepsilon + \tilde{b}_0 v_\varepsilon - \tilde{L}_2 u_\varepsilon + \tilde{L}_1 u_\varepsilon + \tilde{d} u_\varepsilon, \\ g_\varepsilon &= Y u_\varepsilon + \tilde{c}_0 u_\varepsilon - v_\varepsilon / a_0. \end{aligned}$$

Lemma 4.6. *Assumptions* (4.4) *and* (4.6) *imply that* $f_\varepsilon = f_{1,\varepsilon} + f_{2,\varepsilon}$ *with* $f_{1,\varepsilon} \to f_1$ *in* $\mathcal{L}_{-\theta,\lambda}(T)$ *and* $f_{2,\varepsilon} \to f_2$ *in* $\mathcal{H}_{-\theta-\frac{1}{2}log,\lambda}(T)$. *Moreover,* $g_\varepsilon \to 0$ *in* $\mathcal{H}_{1-\theta-\frac{1}{2}log,\lambda}(T)$.

Taking this lemma for granted, we finish the proof of Theorem 4.1. We use the estimate (4.33) for $(u_\varepsilon, v_\varepsilon)$, together with the estimates

$$|\langle f_\varepsilon(t), Q_\gamma^2 v(t)\rangle| \leq Ce^{-2\gamma t}\big(\|f_{1,\varepsilon}(t)\|_{H^{-s(t)}}\|v_\varepsilon(t)\|_{H^{-s(t)}} \\ + \|f_{2,\varepsilon}(t)\|_{H^{-s(t)-\frac{1}{2}log}}\|v_\varepsilon(t)\|_{H^{-s(t)+\frac{1}{2}log}}\big)$$

and

$$\Big|\int_0^t \langle f_\varepsilon, Q_\gamma^2 v\rangle dt'\Big| \leq C\Big(\int_0^t e^{-\gamma t'}\big(\|f_{1,\varepsilon}(t')\|_{H^{-s(t')}}dt'\Big) \sup_{0\leq t'\leq t} e^{-\gamma t'}\|v_\varepsilon(t')\|_{H^{-s(t')}} \\ + C\Big(e^{-2\gamma t'}\|f_{2,\varepsilon}(t')\|^2_{H^{-s(t')-\frac{1}{2}log}}dt'\Big)^{\frac{1}{2}}\Big(e^{-2\gamma t'}\|v_\varepsilon(t')\|^2_{H^{-s(t')+\frac{1}{2}log}}\big)dt'\Big)^{\frac{1}{2}}.$$

This implies that there is a K such that for all $\varepsilon > 0$, one has

$$\sup_{0\leq t'\leq t}\|u_\varepsilon(t')\|^2_{H^{1-s(t')}} + \sup_{0\leq t'\leq t}\|v_\varepsilon(t')\|^2_{H^{-s(t')}} \\ + \int_0^t \Big(\|u_\varepsilon(t')\|^2_{H^{1-s(t')+\frac{1}{2}log}} + \|v_\varepsilon(t')\|^2_{H^{-s(t')+\frac{1}{2}log}}\Big)dt'$$

(4.45)

$$\leq K\Big\{\|u_\varepsilon(0)\|^2_{H^{1-s(0)}} + \|v_\varepsilon(0)\|^2_{H^{-s(0)}} + \int_0^t \|g_\varepsilon(t')\|^2_{H^{1-s(t)-\frac{1}{2}log}}dt' \\ + \Big(\int_0^t \|f_{1,\varepsilon}(t')\|_{H^{-s(t')}}dt'\Big)^2 + \int_0^t \|f_{2,\varepsilon}(t')\|^2_{H^{-s(t')-\frac{1}{2}log}}dt'\Big\}.$$

In addition, there are similar estimates for the differences $(u_\varepsilon - u_{\varepsilon'}, v_\varepsilon - v_{\varepsilon'})$. Since $u_\varepsilon(0) = J_\varepsilon u_0$ and $v_\varepsilon(0) = J_\varepsilon v_0$ converge to u_0 and v_0 in $H^{1-s(0)}$ and $H^{-s(0)}$, respectively, the estimate implies that u_ε is a Cauchy sequence in $\mathcal{H}_{1-\theta,\lambda}(T)$ and in $C^0([0,t]; H^{1-s(t)})$ for all $t \in [0,T]$. Therefore, the limit u in $\mathcal{H}_{1-\theta,\lambda}(T)$ also belongs to $\mathcal{C}_{1-\theta,\lambda}(T)$. Similarly, v_ε is a Cauchy sequence in $\mathcal{H}_{-\theta,\lambda}(T)$ and in $C^0([0,t]; H^{-s(t)})$ for all $t \in [0,T]$ and $v \in \mathcal{C}_{1-\theta,\lambda}(T)$. In addition, we can pass to the limit in (4.45) proving that

$$
\begin{aligned}
&\sup_{0\le t'\le t}\|u(t')\|^2_{H^{1-s(t')}} + \sup_{0\le t'\le t}\|v(t')\|^2_{H^{-s(t')}} \\
&+\int_0^t \Big(\|u(t')\|^2_{H^{1-s(t')+\frac{1}{2}log}} + \|v(t')\|^2_{H^{-s(t')+\frac{1}{2}log}}\Big)dt' \\
&\le K\Big\{\|u_0\|^2_{H^{1-s(0)}} + \|v_0\|^2_{H^{-s(0)}} \\
&\qquad + \Big(\int_0^t \|f_1(t')\|_{H^{-s(t')}}dt'\Big)^2 + \int_0^t \|f_2(t')\|^2_{H^{-s(t')-\frac{1}{2}log}}dt'\Big\}.
\end{aligned}
\tag{4.46}
$$

Using the equation $Yu + \tilde{c}_0 u = v/a_0$ and the estimate (4.13) of Lemma 4.2 to bound the time derivative $\partial_t u$, we see that $\partial_t u \in \mathcal{C}_{-\theta,\lambda}(T)$ and that the energy estimate (2.29) is satisfied.

Therefore, it remains only to prove the lemma.

Proof of Lemma 4.6. By assumption (4.6), $f = f_1 + f_2$ and $J_\varepsilon f_1 \to f_1$ in $\mathcal{L}_{-\theta,\lambda}(T)$ and $J_\varepsilon f_2 \to f_2$ in $\mathcal{H}_{-\theta-\frac{1}{2}log,\lambda}(T)$. Therefore, it is sufficient to prove that the commutators

$$
\begin{aligned}
&[Z, J_\varepsilon]v, \quad [\tilde{L}_2, J_\varepsilon]u \\
&[\tilde{b}_0, J_\varepsilon]v, \quad [\tilde{L}_1, J_\varepsilon]u, \quad [\tilde{d}, J_\varepsilon]u,
\end{aligned}
$$

converge to 0 in $\mathcal{H}_{-\theta,\lambda}(T)$ and that the commutators

$$
[Y, J_\varepsilon]u, \quad [\tilde{c}_0, J_\varepsilon]u, \quad [1/a_0, J_\varepsilon]v
$$

converge to 0 in $\mathcal{H}_{1-\theta,\lambda}(T)$. We note that J_ε commutes with ∂_t in Z and Y. Thanks to (4.4) (4.43) and to the conservative form of Z and $\tilde{L}*_2$, we see that there are four types of commutators to consider :

$$
\begin{aligned}
&[a, J_\varepsilon]w \to 0 \ \text{ in } \mathcal{H}_{1-\theta,\lambda}(T), \quad \text{when} \\
&a \in L^\infty \cap LL([0,T]\times\mathbb{R}^d), \quad w \in \mathcal{H}_{-\theta,\lambda}(T),
\end{aligned}
\tag{4.47}
$$

$$
\begin{aligned}
&[b, J_\varepsilon]w \to 0 \ \text{ in } \mathcal{H}_{-\theta,\lambda}(T), \quad \text{when} \\
&b \in C^\alpha([0,T]\times\mathbb{R}^d), \quad w \in \mathcal{H}_{-\theta,\lambda}(T),
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
&[c, J_\varepsilon]w \to 0 \ \text{ in } \mathcal{H}_{1-\theta,\lambda}(T), \quad \text{when} \\
&c \in C^\alpha([0,T]\times\mathbb{R}^d), \quad w \in \mathcal{H}_{1-\theta,\lambda}(T),
\end{aligned}
\tag{4.49}
$$

$$
(4.50) \qquad \begin{aligned} &[d, J_\varepsilon]w \to 0 \ \text{ in } \mathcal{H}_{-\theta,\lambda}(T), \quad \text{when} \\ &d \in L^\infty([0,T] \times \mathbb{R}^d), \quad w \in \mathcal{H}_{1-\theta,\lambda}(T). \end{aligned}
$$

The first commutators $[a, J_\varepsilon] = [T_a, J_\varepsilon] + R_a, J_\varepsilon - J_\varepsilon R_a$ are uniformly bounded from $\mathcal{H}_{-\theta,\lambda}(T)$ to $\mathcal{H}_{1-\theta,\lambda}(T)$: this is true for the first term by Proposition 3.7, since the J_ε form a bounded family of operators of degree 0; for the last two terms, this follows from Proposition 3.4. Moreover, $[a, J_\varepsilon]w \to 0$ in $L^2([0,T]; H^\sigma)$ for all $\sigma < 1$, and thus also in $\mathcal{H}_{1-\theta,\lambda}$, when w is smooth and $a \in L^\infty \cap LL$. By density, this implies (4.47).

For the commutators (4.48), we note that they are uniformly bounded from $\mathcal{H}_{-\theta,\lambda}(T)$ to $\mathcal{H}_{-\theta,\lambda}(T)$. This is true for both terms bJ_ε and $J_\varepsilon b$ since $s(t)$ remains in a compact subset of $[0, \alpha[$. Because $[b, J_\varepsilon]w$ converges to zero in $L^2([0,T]; H^\sigma)$ for all $\sigma < \alpha$, when w is smooth and $b \in L^\infty \cap LL$, the convergence in (4.48) follows. The proof for (4.49) is similar.

Finally, we note that $[d, J_\varepsilon]w \to 0$ in $L^2([0,T] \times \mathbb{R}^d)$, hence in $\mathcal{H}_{-\theta,\lambda}(T)$ when $d \in L^\infty([0,T] \times \mathbb{R}^d)$ and $w \in L^2([0,T] \times \mathbb{R}^d)$, thus in particular when $w \in \mathcal{H}_{1-\theta,\lambda}(T)$. □

4.6 Existence and uniqueness

Proof of Theorem 2.4.

Assume that $u \in H^s(]0,T[\times\mathbb{R}^n)$ with $s \in]1-\alpha, \alpha[$, $T \le T_0$, and satisfies

$$
(4.51) \qquad Lu = 0, \quad u_{|t=0} = 0, \quad Xu_{|t=0} = 0.
$$

We want to prove that $u = 0$.

Fix $\theta < \theta_1$ in $]1-\alpha, \alpha[$ with $1 - \theta < s$. Let λ and T' be the parameter and time associated to them by Theorem 4.1. Note that they depend only on θ, θ_1, the norms A_{L^∞} and A_{LL} in (2.9) and the constants of hyperbolicity δ_0 and δ_1 in (2.8).

From Lemma 2.2, we know that $u \in L^2([0,T]; H^s(\mathbb{R}^n))$ and $\partial_t u \in L^2([0,T]; H^{s-1}(\mathbb{R}^n))$ and therefore, on $[0,T'] \times \mathbb{R}^n$, $u \in \mathcal{H}_{1-\theta+\frac{1}{2}log,\lambda}$ and $\partial_t u \in \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}$ since $s > 1 - \theta - \lambda t$. By Theorem 4.1, u satisfies the energy estimate (2.29) on $[0,T']$, and since the right hand side vanishes, $u = 0$ for $t < T'$. By a finite number of iterations, u vanishes for $t < T$. □

Proof of Theorem 2.6.

On $[0,T_0] \times \mathbb{R}^d$, the coefficients of L_2 can be approximated in L^∞ and $C^{\alpha'}$ for all $\alpha' < 1$ by C^∞ functions which are uniformly bounded in L^∞ and in LL, in such a way that the hyperbolicity condition (2.8) remains satisfied.

Similarly, the coeffficients of L_1 can be approximated in L^∞ and $C^{\alpha'}$ for all $\alpha' < \alpha$ by smooth functions which are uniformly bounded in C^α. Further, the coefficient c can be approximated in L^2_{loc} by functions uniformly bounded in L^∞. This defines operators L^ε with C^∞ coefficients which satisfy (2.8), (2.9) and (2.10) uniformly in ε and converge to the coefficients of L in the sense described above.

We fix the parameter $\lambda \geq \lambda_0$, where λ_0 is given by Theorem 4.1. Recall that T is then given by (4.2). Consider Cauchy data $u_0 \in H^{1-\theta}$ and $u_1 \in H^{-\theta}$ and a source term $f = f_1 + f_2$ with $f_1 \in \mathcal{L}_{-\theta,\lambda}(T)$ and $f_2 \in \mathcal{H}_{-\theta-\frac{1}{2}log,\lambda}$. We can approximate these data in the corresponding spaces by C^∞ functions u_0^ε, u_1^ε , f_1^ε and f_2^ε, compactly supported in x. The strictly hyperbolic problems with smooth coefficients and smooth data

$$L^\varepsilon u^\varepsilon = f_1^\varepsilon + f_2^\varepsilon, \quad u^\varepsilon{}_{|t=0} = u_0^\varepsilon, \quad X^\varepsilon u^\varepsilon{}_{|t=0} = u_1^\varepsilon \tag{4.52}$$

have a unique smooth solution u^ε, compactly supported in x.

By Theorem 4.1, the energy estimate (2.29) is satisfied with a constant K independent of ε. Therefore the family $\{u^\varepsilon\}$ is bounded in $\mathcal{H}_{1-\theta+\frac{1}{2}log,\lambda}$, thus in $L^2([0,T], H^{1-\theta_1})$ and the families $\{\partial_t u^\varepsilon\}$ and $\{X^\varepsilon u^\varepsilon\}$ are bounded in $\mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}$, hence in $L^2([0,T], H^{-\theta_1})$. Therefore, extracting a subsequence if necessary, u^ε converges to a limit u, weakly in $L^2([0,T], H^{1-\theta_1})$ and in $H^1([0,T], H^{-\theta_1})$. Moreover, $u \in \mathcal{H}_{1-\theta+\frac{1}{2}log,\lambda}$ and $\partial_t u \in \mathcal{H}_{-\theta+\frac{1}{2}log,\lambda}$. There is no difficulty in passing to the limit in the equation in the sense of distributions: all the products are well defined and involve one strong and one weak convergence. Thus $Lu = f$.

The weak convergence in $L^2([0,T], H^{1-\theta_1}) \cap H^1([0,T], H^{-\theta_1})$ implies the strong convergence in $C^0([0,T]; H^{-\theta_1}_{loc})$ and therefore the convergence of $u^\varepsilon_{|t=0}$ to $u_{|t=0}$ in $H^{-\theta_1}_{loc}$. Therefore, $u_{|t=0} = u_0$.

Using the equation as in Lemma 2.2, we prove that the family $v^\varepsilon = X^\varepsilon u^\varepsilon + c_0^\varepsilon u^\varepsilon$, which converges weakly to $v = Xu + c_0 u$, is bounded in $L^2([0,T], H^{-\theta_1}) \cap H^1([0,T], H^{-1-\theta_1})$. Thus $v^\varepsilon_{|t=0}$ converges to $v_{|t=0}$ in $H^{-\theta_1}_{loc}$. Hence $v_{|t=0} = u_1 + c_{0|t=0} u_0$ implying that $X u_{|t=0} = u_1$.

By Theorem 4.1 the solution u also belong to $\mathcal{C}_{1-\theta,\lambda}$ with $\partial_t u \in \mathcal{C}_{1-\theta,\lambda}$ and satisfies the energy estimate (2.29). □

5 Local results

We consider the equation (1.1) together with an initial hypersurface Σ satisfying Assumption 1.1. This section contains the proofs of Proposition 1.4 and Theorems 1.5 and 1.6.

5.1 Change of coordinates. Traces

Consider a smooth change of variables $y = \chi(\tilde{y})$ and for a function u let $\tilde{u}$ denote $u \circ \chi$. Then

$$(\partial_{y_j} u) \circ \chi = \sum_k \psi'_{j,k}\, \partial_{\tilde{y}_k} \tilde{u} = \sum_k \partial_{\tilde{y}_k} \big(\psi'_{j,k} \tilde{u}\big) - \big(\sum_k \partial_{\tilde{y}_k} \psi'_{j,k}\big) \tilde{u}$$

with $\psi'_{j,k} = (\partial_{y_j} \psi_k) \circ \chi\big)$ and $\psi = \chi^{-1}$. Thus

$$\widetilde{Lu} = \widetilde{L} \tilde{u} \tag{5.1}$$

where $\widetilde{L}$ has the same form as L and satisfies Assumption 1.1.

If $\nu(y)$ is conormal to Σ, then $\tilde{\nu}(\tilde{y}) = {}^t\chi'(y)\nu(\chi(\tilde{y}))$ is conormal to $\widetilde{\Sigma} = \chi^{-1}(\Sigma)$. Using the notations (1.7), for smooth functions, the Neumann traces associated to (L, ν) and $(\widetilde{L}, \tilde{\nu})$, are linked by the relation

$$(N_\nu u) \circ \chi = \widetilde{N}_{\tilde{\nu}} \tilde{u}. \tag{5.2}$$

The Green's formula (1.9) can be transported by χ, taking into account the Jacobian factors:

$$\big(f, g\big)_{L^2(\Omega_+)} = \big(\tilde{f}, J\tilde{g}\big)_{L^2(\widetilde{\Omega}_+)} \tag{5.3}$$

with $J = |\det \chi'|$. This relations extends to the duality $H^s \times H^{-s}$ for $|s| < \frac{1}{2}$. In particular, comparing the Green's formula for L and $\widetilde{L}$ tested on smooth functions implies that :

$$(\widetilde{L})^*(J\tilde{v}) = J\widetilde{L^* v} \tag{5.4}$$

$$\widetilde{N}'_{\tilde{\nu}}(J\tilde{v}) = J_\Sigma \widetilde{N'_\nu v} \tag{5.5}$$

where J_Σ is the Jacobian of $\chi_{|\widetilde{\Sigma}}$.

As a corollary, the statement of Proposition 1.4 is invariant by smooth changes of variables and therefore can be proved in any suitable system of coordinates.

Proof of Proposition 1.4.

a) Uniqueness. We prove that if $u_0 \in H^{s-\frac{1}{2}}_{loc}$ and $u_1 \in H^{s-\frac{3}{2}}_{loc}$ satisfy

$$\big\langle u_1, D_\Sigma v \big\rangle_{H^{s-\frac{3}{2}} \times H^{\frac{3}{2}-s}} - \big\langle u_0, N'_\Sigma v \big\rangle_{H^{s-\frac{1}{2}} \times H^{\frac{1}{2}-s}} = 0$$

for all $s' \in]1-\alpha, \frac{1}{2}[$ such that $s' \leq s$ and all $v \in H^{2-s'}_{comp}(\Omega \cap \{\varphi \geq 0\})$, then $u_0 = u_1 = 0$.

It is sufficient to prove that for v_0 and v_1 in $C_0^\infty(\Omega \cap \Sigma)$, there is $v \in C_0^{1+\alpha}(\Omega)$ such that $v_{|\Sigma} = v_0$ and $N'_\nu v = v_1$. This can be done in local coordinates $y = (t, x)$ where $\Sigma = \{t = 0\}$ and this amounts to solve

$$v_{|t=0} = v_0, \qquad \partial_t v_{|t=0} = g_0 v_1 + \sum g_j \partial_{x_j} v_0 + h v_0$$

where $g_0, \dots, g_d$ belong to $LL(\Omega)$ and h belongs to $C^\alpha(\Omega)$.

b) Existence. According to the discussion above we may assume that we are working in coordinates $y = (t, x)$ such that $\underline{y} = (0, 0)$ and $\Sigma = \{t = 0\}$. The conormal direction is $\nu = \lambda(x) dt$ and the Neumann trace for smooth functions is:

$$N_\nu u = \lambda\, (Xu)_{|t=0}$$

where X is the vector field (2.11).

Let $s \in]1 - \alpha, \alpha[$. For $u \in H^s_{loc}(\Omega \cap \{t \geq 0\})$ such that $Lu \in L^2(\Omega_+)$. Local versions of Lemmas 2.1 and 2.2 imply that for $T > 0$ small and ω relatively compact in $\Omega \cap \{t = 0\}$

$$u \in H^{1,s-1}(]0, T[\times\omega)), \qquad Xu \in H^{1,s-2}(]0, T[\times\omega)). \tag{5.6}$$

Indeed, the proofs are identical, using local multiplicative properties and local versions of the spaces $H^{s,s'}$. The trace operator $w \mapsto w_{|t=0}$ has a unique extension as a bounded operator from $H^{1,\sigma}(]0, T[\times\omega)$ to $H^{\sigma+\frac{1}{2}}(\omega)$. Therefore, the traces $u_{|t=0}$ and $Xu_{|t=0}$ are well defined in $H^{s-\frac{1}{2}}$ and $H^{s-\frac{3}{2}}$ respectively. We show that, in these coordinates, Green's formula (1.9) holds with

$$D_\Sigma u = u_{|t=0}, \qquad N_\nu u = \lambda(Xu)_{|t=0}. \tag{5.7}$$

This follows immediately by integration by parts, the only difficulty is to check that at each step we have enough smoothness to justify the computations. We sketch here the main points of the discussion. First, recall that for $w \in H^{1,\sigma}$ and $v \in H^{1,-1-\sigma}$ compactly supported in $[0, T[\times\omega$:

$$\big\langle \partial_t w, v \big\rangle_{L^2(\sigma)} = -\big\langle w, \partial_t v \big\rangle_{L^2(\sigma+1)} + \big\langle w_{|t=0}, v_{|t=0} \big\rangle_{H^{\sigma+\frac{1}{2}} \times H^{-\frac{1}{2}-\sigma}}, \tag{5.8}$$

and for $w \in H^{0,\sigma+1}$ and $v \in H^{0,-\sigma}$ compactly supported in $[0, T[\times\omega$:

$$\big\langle \partial_{x_j} w, v \big\rangle_{L^2(\sigma)} = -\big\langle w, \partial_{x_j} v \big\rangle_{L^2(\sigma+1)} \tag{5.9}$$

where $\langle \cdot\,, \cdot \rangle_{L^2(\sigma)}$ denotes the duality $L^2([0, T], H^\sigma) \times L^2([0, T]; H^{-\sigma})$ and the traces are taken in the sense indicated above.

Consider $v \in H^{2-s'}_{comp}([0,T[\times\omega)$, where $s' \in]1-\alpha, \frac{1}{2}[$ with $s' \leq s$. Using (5.6), (5.8), (5.9) and the multiplicative properties of functions in LL-functions and H^σ, one obtains that

$$\begin{aligned}\big\langle ZXu, v\big\rangle_{L^2(s-2)} &= \big\langle Xu, Z^*v\big\rangle_{L^2(s-1)} + \big\langle Xu_{|t=0}, v_{|t=0}\big\rangle_{H^{s-\frac{3}{2}}\times H^{\frac{3}{2}-s}},\\ &= \big\langle Yu, \overline{a}_0 Z^*v\big\rangle_{L^2(s-1)} + \big\langle Xu_{|t=0}, v_{|t=0}\big\rangle_{H^{s-\frac{3}{2}}\times H^{\frac{3}{2}-s}},\end{aligned}$$

(recall the definitions (2.11) and (2.13) of X, Y and Z). Let $w = \overline{a}_0 Z^* v \in H^{1-s'}$. Because $1-s' \geq 0$ and $s' \leq s$, $w \in H^{0,1-s'} \subset H^{0,1-s}$. Therefore,

$$\big\langle \tilde{a}_j \partial_{x_j} u, w\big\rangle_{L^2(s-1)} = -\big\langle u, \partial_{x_j}(\overline{\tilde{a}_j}\, v)\big\rangle_{L^2(s)}$$

The term $\langle \partial_t u, w\rangle$ is more delicate since $\partial_t w \in H^{-s'}$ and $s' > 0$. However, as in Lemma 1.3, one can use the duality $H^{s'}(\{t \geq 0\}) \times H^{-s'}(\{t \geq 0\})$ for $0 \leq s' < \frac{1}{2}$ and for $u \in H^{1,s-1}$ and $w \in H^{1-s'}_{comp}$, (5.8) can be extended as

$$\big\langle \partial_t u, w\big\rangle_{L^2(s-1)} = -\big\langle u, \partial_t w\big\rangle_{H^{s'}\times H^{-s'}} + \big\langle u_{|t=0}, w_{|t=0}\big\rangle_{H^{s+\frac{1}{2}}\times H^{\frac{1}{2}-s}}, \tag{5.10}$$

noticing that the trace $w_{|t=0}$ belongs to $H^{\frac{1}{2}-s'} \subset H^{\frac{1}{2}-s}$.

Repeated use of (5.9) implies that for the tangential second order part $\widetilde{L}_2$ defined in (2.13), there holds

$$\big\langle \widetilde{L}_2 u, v\big\rangle_{L^2(s-2)} = \big\langle u, (\widetilde{L}_2)^* v\big\rangle_{L^2(s)}.$$

First order terms are treated similarly, and summing up we get that

$$\begin{aligned}&\big(Lu,v\big)_{L^2(\{t>0\}} - \big\langle u, L^*v\big\rangle_{H^{s'}(\{t>0\})\times H^{-s'}(\{t>0\})}\\ &\quad = \big\langle Xu_{|t=0}, v_{|t=0}\big\rangle_{H^{s-\frac{3}{2}}\times H^{\frac{3}{2}-s}} - \big\langle u_{|t=0}, X'v_{|t=0}\big\rangle_{H^{s-\frac{1}{2}}\times H^{\frac{1}{2}-s}}\end{aligned} \tag{5.11}$$

In the computations above, the underlying measure in $\{t=0\}$ is the Lebesque measure dx. The surface measure associate to the conormal λdt as in (1.6) is $\lambda^{-1}dx$. This proves that the identity (1.6) is proved with D_Σ and N_ν given by (5.7), as claimed. □

5.2 Local existence

Choose Φ, a smooth map from $\mathbb{R}^{1+n}$ to Ω, with $\Phi(y) = y$ on a smaller neighborhood Ω_1 and $\Phi(y) = 0$ for y large enough. Changing the coefficients acoording to the rule $a^\sharp(y) = a(\Phi(y))$ we obtain an operator $L^\sharp$ which coincides with L on Ω_1, satisfies the regularity conditions (2.4) to (2.6), and the hyperbolicity conditions (2.8) globally on $\mathbb{R}^{1+n}$.

Fix $s > 1-\alpha$. Without loss of generality for the statement of Theorem 1.5, we can assume that $s < \alpha$. We are going to apply Theorem 2.6 to the operator $L^\sharp$ with $\theta = 1-s \in]1-\alpha, \alpha[$. Choosing $\theta_1 \in]\theta, \alpha[$, this theorem provides us with λ and $T = (\theta_1 - \theta)/\lambda$. We fix $\Omega' = \Omega_1 \cap \{|t| < T\}$.

Suppose that u_0 and u_1 are Cauchy data in $H^s(\omega)$ and $H^{s-1}(\omega)$ respectively, on a neighborhood ω of 0 in $\mathbb{R}^n$. There are restrictions to ω of functions $u_0^\sharp \in H^s(\mathbb{R}^n)$ and $u_1^\sharp \in H^{s-1}(\mathbb{R}^n)$ respectively. Suppose that $f \in L^2(\Omega' \cap \{t > 0\})$. We extend it, for instance by 0, to $f^\sharp \in L^2([0,T] \times \mathbb{R}^n)$. By Theorem 2.6, the Cauchy problem

$$L^\sharp u^\sharp = f^\sharp, \quad u^\sharp_{|t=0} = u_0^\sharp, \quad (X^\sharp u^\sharp)_{|t=0} = u_1^\sharp \tag{5.12}$$

has a solution $u^\sharp$ on $[0,T] \times \mathbb{R}^n$, which belongs in particular to $L^2([0,T]; H^{s_1})$ with $s_1 = 1 - \theta_1$ and such that $\partial_t u \in L^2([0,T]; H^{s_1 - 1})$. In particular, $u^\sharp \in H^{s_1}([0,T] \times \mathbb{R}^n)$ and by restriction to Ω' defines a solution of (1.10).

5.3 Local uniqueness

To prove Theorem 1.6, we first reduce the problem to proving a theorem of propagation of zero across the surface $\{t = 0\}$.

Lemma 5.1. *Suppose that $s > 1 - \alpha$ and $u \in H^s(\Omega \cap \{t > 0\})$ satisfies*

$$Lu = 0, \quad u_{|t=0} = 0, \quad Xu_{|t=0} = 0. \tag{5.13}$$

Then the extension u_e of u by 0 for $t < 0$ satisfies

$$u_e \in H^s \quad \text{and} \quad Lu_e = 0 \tag{5.14}$$

on a neighborhood Ω_1 of 0.

Proof. If the coefficients were smooth, this would be immediate. We check that we have enough smoothness to extend the result to our case.

We can assume that $\Omega =]-T, T[\times \omega$. From Lemma 2.2 (localized in space) we know that $u \in L^2([0,T]; H^s_{loc}(\omega))$, thus its extension $u_e \in L^2([-T,T]; H^s_{loc}(\omega))$. Moreover, $\partial_t u \in L^2([0,T]; H^{s-1}_{loc}(\omega))$ and by assumption $u_{|t=0} = 0$. Therefore, $\partial_t u_e$ is the extension of $\partial_t u$ by 0 and thus belongs to $L^2([-T,T]; H^{s-1}_{loc}(\omega))$. In particular, $u_e \in H^s_{loc}(]-T,T[\times \omega)$.

Let $v = Xu + c_0 u \in L^2([0,T]; H^{s-1}_{loc}(\omega))$ and let $v_e \in L^2([-T,T]; H^{s-1}_{loc}(\omega))$ denote its extension by 0. The first step implies that Xu_e is the extension of Xu and therefore $v_e = Xu_e + c_0 u_e$. Write the equation as

$$\partial_t v = P(u, v) \tag{5.15}$$

where P involves only spatial derivatives (see (2.17)). Morever, we have seen in the proof of Lemma 2.2 that $P(u,v) \in L^2([0,T]; H^{s-2}_{loc}(\omega))$. Since by assumption the trace of v vanishes, this implies that $\partial_t v_e$ is the extension by 0 of $\partial_t v$, thus the extension of $P(u,v)$, that is $P(u_e, v_e)$. Since $v_e = X u_e + c_0 u_e$, this means that u_e satisfies the equation on $\Omega =]-T, T[\times \omega$. □

We now finish the proof of Theorem 1.6. We suppose that $u \in H^s(\Omega \cap \{t > 0\})$ satisfies (5.13), with $s > 1 - \alpha$ and we denote by u_e its extension by 0 for $t < 0$. We use the classical convexification method, and consider the change of variables

$$(t,x) \mapsto (\tilde{t}, \tilde{x}) \quad \tilde{t} = t + |x|^2, \ \tilde{x} = x, \tag{5.16}$$

which maps the past $\{t < 0\}$ to $\{\tilde{t} < |\tilde{x}|^2\}$. Thus there is $T_0 > 0$ such that the function $\tilde{u}$ deduced from u_e is defined for $\tilde{t} < T_0$ and vanishes for $\tilde{t} < |\tilde{x}|^2$. Moreover, decreasing T_0 if necessary, the operator $\tilde{L}$ deduced from L is defined on a neighborhood $\tilde{\Omega}$ of the origin which contains the closed lens $\overline{D} = \{|\tilde{x}|^2 \le t \le T_0\}$ and $\tilde{L}\tilde{u} = 0$ on $\tilde{\Omega} \cap \{t < T_0\}$. Now we extend the coefficients of $\tilde{L}$, as above, and obtain a new operator $L^\sharp$, defined on $\mathbb{R}^{1+n}$, satisfying the assumptions of section 2, and equal to $\tilde{L}$ on a neighborhood of $\overline{D}$. Therefore, on $]-\infty, T_0[\times \mathbb{R}^n$

$$L^\sharp \tilde{u} = 0, \quad \tilde{u} \in H^s, \quad \tilde{u}_{|\{\tilde{t} < |\tilde{x}|^2\}} = 0. \tag{5.17}$$

Since $\tilde{u}$ vanishes in the past, the traces $\tilde{u}_{|t=-\varepsilon}$ and $X^\sharp \tilde{u}_{|t=-\varepsilon}$ vanish for all $\varepsilon > 0$. Therefore, Theorem 2.4 applied to the Cauchy problem for $L^\sharp$ with initial time $-\varepsilon$ implies that $\tilde{u} = 0$ for all $(\tilde{t}, \tilde{x})$ such that $\tilde{t} < T_0$. Hence $u = 0$ on a neighborhood of the origin.

6 Application : a blow-up criterion for nonlinear equations

6.1 Statement of the result

In coordinates $y = (t,x)$, we consider a nonlinear wave equation:

$$\begin{aligned} &\partial_t\big(a_0(u)\partial_t u\big) + \sum_{j=1}^n \partial_t\big(a_j(u)\partial_{x_j} u\big) + \partial_{x_j}\big(a_j(u)\partial_t u\big) \\ &\qquad - \sum_{j,k=1}^n \partial_{x_j}\big(a_{j,k}(u)\partial_{x_k} u\big) + \partial_t\big(b_0(u)\big) + \sum_{j=1}^n \partial_{x_j}(b_j(u)) = F(u). \end{aligned} \tag{6.1}$$

Assumption 6.1. *The coefficients are smooth functions of $u \in \mathbb{R}$. Morever, for all fixed u, the polynomial $a_0\tau^2 + 2\sum a_j \tau \xi_j - \sum a_{j,k}\xi_j\xi_k$ is strictly hyperbolic in the direction dt.*

The Cauchy problem for (6.1) with initial data

$$u_{|t=0} = u_0, \qquad \partial_t u_{|t=0} = u_1, \tag{6.2}$$

is well posed for $u_0 \in H^s(\mathbb{R}^n)$ and $u_1 \in H^{s-1}(\mathbb{R}^n)$ when $s > \frac{n}{2} + 1$. The solution u belongs to $C^0([0,T], H^s) \cap C^1([0,T]; H^{s-1})$. By uniqueness, there is a maximal time of existence T^* and $u \in C^0([0,T^*[; H^s) \cap C^1([0,T^*[; H^{s-1})$. Moreover, there is a classical blow-up criterion for the creation of singularities:

Theorem 6.2. *For $s > \frac{n}{2} + 1$ and data $u_0 \in H^s$, $u_1 \in H^{s-1}$, if the maximal time of existence is finite, then*

$$\sup_{0 \le t < T^*} \big\| u(t) \big\|_{L^\infty} + \big\| \partial_{t,x} u(t) \big\|_{L^\infty} = +\infty. \tag{6.3}$$

See e.g. [1] for an extensive discussion of blow-up for solutions of wave equations or [9] for general first order quasilinear systems. Our goal is to show that one can replace the Lipschitz norm in (6.3) by a LL -norm.

Theorem 6.3. *For $s > \frac{n}{2} + 1$ and data $u_0 \in H^s$, $u_1 \in H^{s-1}$, if $T^* < +\infty$, then*

$$\sup_{0 \le t < T^*} \big\| u \big\|_{L^\infty([0,t]\times\mathbb{R}^n)} + \big\| u \big\|_{LL([0,t]\times\mathbb{R}^n)} = +\infty. \tag{6.4}$$

The proof of Theorem 6.2 is based on the estimate :

Theorem 6.4. *For $s > \frac{n}{2} + 1$, $M \in \mathbb{R}$ and $T_0 > 0$ given, there is a constant C, such that if $T \le T_0$ and $u \in C^0([0,T[; H^s) \cap C^1([0,T[; H^{s-1})$ is a solution of (6.1) such that*

$$\sup_{0 \le t < T} \big\| u(t) \big\|_{L^\infty} + \big\| \partial_{t,x} u(t) \big\|_{L^\infty} \le M \tag{6.5}$$

then

$$\sup_{0 \le t < T} \big\| u(t) \big\|_{H^s} + \big\| \partial_t u(t) \big\|_{H^{s-1}} \le C\big(\big\| u_0 \big\|_{H^s} + \big\| u_1 \big\|_{H^{s-1}} \big). \tag{6.6}$$

Similarly, the proof of Theorem 6.3 is based on the following estimate :

Theorem 6.5. *For $s > \frac{n}{2} + 1$ and $K \in \mathbb{R}$, there are constants $T_1 > 0$, C_1 and λ such that such that if $u \in C^0([0,T[; H^s) \cap C^1([0,T[; H^{s-1})$ is a solution of* (6.1) *such that*

$$\sup_{0 \le t < T} \|u\|_{L^\infty([0,t]\times\mathbb{R}^n)} + \|u\|_{LL([0,t]\times\mathbb{R}^n)} \le K \tag{6.7}$$

then, for $t < \min\{T, T_1\}$,

$$\|u(t)\|_{H^{s-\lambda t}} + \|\partial_t u(t)\|_{H^{s-1-\lambda t}} \le C_1\big(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\big). \tag{6.8}$$

Proof of Theorem 6.3 assuming Theorem 6.5.

It is sufficient to prove that if $u \in C^0([0,T[; H^s) \cap C^1([0,T[; H^{s-1})$ satisfies (6.7) then,

$$\sup_{0 \le t < T} \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} < +\infty. \tag{6.9}$$

implying that the solution can be continued after T.

Fix $s_1 \in]1 + \frac{n}{2}, s[$. Decreasing T_1, we can assume that $T_1 \le (s - s_1)/\lambda$. Then (6.8) and the Sobolev imbedding theorem imply that

$$\sup_{0 \le t < T_1} \|u(t)\|_{L^\infty} + \|\partial_{t,x} u(t)\|_{L^\infty} \le C(K)\big(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\big),$$

where $C(K)$ depends only on K. Therefore, Theorem 6.4 implies that

$$\sup_{0 \le t < T_1} \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \le C\big(K, \|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}\big).$$

The important point is that T_1 depends only on K. One can repeat the analysis for the Cauchy problem with initial time T_1' arbitrarily close to T_1, and after a finite number of iterations , this implies (6.9). $\square$

6.2 Proof of the nonlinear estimate

We write the equation as a system

$$\begin{aligned} \partial_t v + \sum_{j=1}^{n} \partial_{x_j}\big(\tilde{a}_j(u)v\big) - \sum_{j,k=1}^{n} \partial_{x_j}\big(\tilde{a}_{j,k}\partial_{x_k} u\big) \\ = -\sum_{j=1}^{n} \partial_{x_j}\big(\tilde{b}_j(u)\big) + F(u) \end{aligned} \tag{6.10}$$

$$\partial_t u + \sum_{j=1}^{n} \tilde{a}_j(u)\partial_{x_j} u + \tilde{b}_0(u) = v/a_0 \tag{6.11}$$

with

$$\tilde{a}_j = a_j/a_0, \quad \tilde{a}_{j,k} = a_{j,k} + \tilde{a}_j a_k, \quad \tilde{b}_0 = b_0/a_0, \quad \tilde{b}_j = b_j - a_j \tilde{b}_0.$$

Introduce a dyadic partition of unity *in space*, $\sum \Delta_k(D_x) = \mathrm{Id}$, as in (3.3). The first step in the proof Theorem 6.5 is an estimate of $\Delta_k \partial_t u$ and $\Delta_k v$:

Proposition 6.6. *There is a constant $C(T, K)$ such that if $u \in C^0([0, T[; H^s) \cap C^1([0, T[; H^{s-1})$ is a solution of (6.1) which satisfies (6.7), then for all $k \geq 0$ and $t \in [0, T[$:*

$$\big\| S_k \partial_t u(t) \big\|_{L^\infty} + \big\| S_k v(t) \big\|_{L^\infty} \leq C(T, K)(k + 1). \tag{6.12}$$

This estimate is proved in the next subsection. Taking it for granted, we finish the proof of (6.8).

We use the para-differential calculus introduced in Section 3. The para-linearization procedure is based upon the following result.

Lemma 6.7. *Given $s > s_1 > 0$, there is a constant C such that for $\sigma \in [s_1, s]$, $a \in H^{\sigma + \frac{1}{2} log}(\mathbb{R}^n)$ and v such that*

$$\sup_{k \geq 0} (k + 1)^{-1} \big\| S_k v \big\|_{L^\infty} \leq K \tag{6.13}$$

there holds $av - T_a v \in H^{\sigma - \frac{1}{2} log}(\mathbb{R}^n)$ and

$$\big\| av - T_a v \big\|_{H^{\sigma - \frac{1}{2} log}} \leq CK \big\| a \big\|_{H^{\sigma + \frac{1}{2} log}} \tag{6.14}$$

Proof. There holds $av - T_a v = \sum w_k$ with $w_k = \Delta_k a S_{k+2} v$. The spectrum of w_k is contained in the ball $\{|\xi| \leq 2^{k+4}\}$ and

$$\big\| w_k \big\|_{L^2} \leq CK(k + 1)^{\frac{1}{2}} 2^{-ks} \varepsilon_k \big\| a \big\|_{H^{s + \frac{1}{2} log}}$$

with $\{\varepsilon_k\}_k$ in the unit ball of ℓ^2. □

We also use the following nonlinear estimates :

Lemma 6.8. *Suppose that $u \in H^{\sigma + \frac{1}{2} log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and a is a C^∞ function on $\mathbb{R}$ such that $a(0) = 0$. Then $a(u) \in H^{s + \frac{1}{2} log}(\mathbb{R}^n)$ and*

$$\big\| a(u) \big\|_{H^{\sigma + \frac{1}{2} log}} \leq C\big(\|u\|_{L^\infty}\big) \big\| a \big\|_{H^{\sigma + \frac{1}{2} log}}. \tag{6.15}$$

Proof of Theorem 6.5.

Consider a solution $u \in C^0([0,T[;H^s) \cap C^1([0,T[;H^{s-1})$ of (6.1) satisfying (6.5). Fix $s_1 \in]\frac{d}{2}+1, s[$. We show that there is are constants C and C_1, which depends only on K, such that for all $\lambda > 0$ and $t \leq \min\{T, \frac{s-s_1}{\lambda}\}$:

$$\begin{aligned}
&\|u(t)\|^2_{H^{s-\lambda t}} + \|\partial_t u(t)\|^2_{H^{s-1-\lambda t}} \\
&\quad + \lambda \int_0^t \Big(\|u(t')\|^2_{H^{s-\lambda t+\frac{1}{2}log}} + \|\partial_t u(t')\|^2_{H^{s-1-\lambda t+\frac{1}{2}log}}\Big) dt' \\
&\quad \leq C\Big(\|u(0)\|^2_{H^s} + \|\partial_t u(0)\|^2_{H^{s-1}}\Big) \\
&\qquad + C_1 \int_0^t \Big(\|u(t')\|^2_{H^{s-\lambda t+\frac{1}{2}log}} + \|\partial_t u(t')\|^2_{H^{s-1-\lambda t+\frac{1}{2}log}}\Big) dt'.
\end{aligned} \tag{6.16}$$

Choosing $\lambda \geq C_1$, this implies (6.8).

a) We use the para-differential calculus as in Section 4. In addition to the quantization T_a we use the modified operators P_a^ν (3.23). Using Corollary 3.12, we can fix ν and $\delta > 0$ depending only on K, such that for all $t \in [0,T]$ and $w = (w_1, \ldots, w_n) \in C_0^\infty(\mathbb{R}^n)$:

$$\mathrm{Re} \sum_{j,k=1}^n \big(P^\nu_{b_{j,k}(t)} w_k, w_j\big)_{L^2} \geq \delta \|w\|^2_{L^2} \tag{6.17}$$

with $b_{j,k} = a_0(u)\tilde{a}_{j,k}(u)$.

From now on we fix such a ν and use the notation P_b in place of P_b^ν. Lemma 3.10 can be extended to all values of σ and there is a constant C such that for all $t \in [0,T]$ and $\sigma \in [s_1, s]$:

$$\|(P_a - T_a)w\|_{H^{\sigma+1}} \leq C 2^\nu \|a\|_{L^\infty} \|w\|_{H^\sigma}. \tag{6.18}$$

Similarly

$$\|(P_a - T_a)\partial_{x_j}\|_{H^{\sigma \pm \frac{1}{2}log}} \leq C 2^\nu \|a\|_{L^\infty} \|w\|_{H^{\sigma \pm \frac{1}{2}log}}. \tag{6.19}$$

Using Proposition 3.3 for the spatial derivatives $\partial_{x_j} u$ and Proposition 6.6 for v, we deduce from the lemmas above that

$$v = P_{a_0}\partial_t u + \sum_{j=1}^n P_{a_j}\partial_{x_j} u + g \tag{6.20}$$

where a_j stands for $a_j(u)$ and

$$\|g(t)\|_{H^{s-\lambda t-\frac{1}{2}log}} \leq C(K)\Big(\|\partial_t u(t)\|_{H^{s-1-\lambda t+\frac{1}{2}log}} + \|u(t)\|_{H^{s-\lambda t+\frac{1}{2}log}}\Big). \tag{6.21}$$

In particular, this implies that

$$\big\|v(t)\big\|_{H^{s-1-\lambda t}} \le C(K)\Big(\big\|\partial_t u(t)\big\|_{H^{s-1-\lambda t}} + \big\|u(t)\big\|_{H^{s-\lambda t}}\Big), \tag{6.22}$$

$$\begin{aligned}&\big\|v(t)\big\|_{H^{s-1-\lambda t+\frac{1}{2}log}} \\ &\qquad \le C(K)\Big(\big\|\partial_t u(t)\big\|_{H^{s-1-\lambda t+\frac{1}{2}log}} + \big\|u(t)\big\|_{H^{s-\lambda t+\frac{1}{2}log}}\Big).\end{aligned} \tag{6.23}$$

Similarly,

$$\partial_t u + \sum_{j=1}^{n} P_{\tilde a_j}\partial_{x_j} u = P_{a_0^{-1}} v + g_1 \tag{6.24}$$

where $\tilde a_j = \tilde a_j(u)$ and

$$\big\|g_1(t)\big\|_{H^{s-\lambda t-\frac{1}{2}log}} \le C(K)\Big(\big\|u(t)\big\|_{H^{s-\lambda t+\frac{1}{2}log}} + \big\|v(t)\big\|_{H^{s-1-\lambda t+\frac{1}{2}log}}\Big). \tag{6.25}$$

With (6.35), this implies that g_1 also satisfies an estimate similar to (6.21).

Another consequence is that

$$\big\|\partial_t u(t)\big\|_{H^{s-1-\lambda t}} \le C(K)\Big(\big\|v(t)\big\|_{H^{s-1-\lambda t}} + \big\|u(t)\big\|_{H^{s-\lambda t}}\Big), \tag{6.26}$$

$$\begin{aligned}&\big\|\partial_t u(t)\big\|_{H^{s-1-\lambda t+\frac{1}{2}log}} \\ &\qquad \le C(K)\Big(\big\|v(t)\big\|_{H^{s-1-\lambda t+\frac{1}{2}log}} + \big\|u(t)\big\|_{H^{s-\lambda t+\frac{1}{2}log}}\Big).\end{aligned} \tag{6.27}$$

In the same vein,

$$\partial_t v + \sum_{j=1}^{n} \partial_{x_j} P_{\tilde a_j} v - \sum_{j,k=1}^{n} \partial_{x_j} P_{\tilde a_{j,k}} \partial_{x_k} u = f \tag{6.28}$$

with

$$\big\|f(t)\big\|_{H^{s-1-\lambda t-\frac{1}{2}log}} \le C(K)\big\|u(t)\big\|_{H^{s+\lambda t+\frac{1}{2}log}}. \tag{6.29}$$

b) Multiply the equation (6.28) by $(1-\Delta_x)^{2(s-1-\lambda t)} v$ and integrate over $\mathbb{R}^n$. Using Proposition 3.8 to bound the terms $\big(P_{\tilde a_j}\partial_{x_j} v, (1-\Delta_x)^{2(s-1-\lambda t)} v\big)_{L^2}$, implies that

$$
(6.30)\qquad \begin{aligned}
&\frac{d}{dt}\big\|v(t)\big\|^2_{H^{s-1-\lambda t}} + 2\lambda\big\|v(t)\big\|^2_{H^{s-1-\lambda t+\frac{1}{2}log}} \\
&-2\mathrm{Re}\big(\sum_{j,k=1}^{n} \partial_{x_j} P_{\tilde{a}_{j,k}} \partial_{x_k} u, (1-\Delta_x)^{2(s-\lambda t)} v\big)_{L^2} \\
&\qquad \leq C\big\|f(t)\big\|_{H^{s-1-\lambda-\frac{1}{2}log}}\big\|v(t)\big\|_{H^{s-1-\lambda+\frac{1}{2}log}} \\
&\qquad\qquad + C\big\|v(t)\big\|^2_{H^{s-1-\lambda t+\frac{1}{2}log}}.
\end{aligned}
$$

where C depends only on K.

c) Multiply the equation (6.11) and $(1-\Delta_x)^{2(s-\lambda t)}\sum \partial_{x_j} P_{\tilde{a}_{j,k}} \partial_{x_k} u$. Using Propositions 3.8 and 3.9,

$$
\begin{aligned}
-\mathrm{Re}\big(\sum_{j,k=1}^{n} \partial_{x_j} P_{\tilde{a}_{j,k}} \partial_{x_k} u, (1-\Delta_x)^{2(s-\lambda t)} v\big)_{L^2} \\
=\mathrm{Re}\big(\sum_{j,k=1}^{n} P_{a_0\tilde{a}_{j,k}} \partial_{x_k} u, (1-\Delta_x)^{2(s-\lambda t)} \partial_t \partial_{x_j} u\big)_{L^2} + E
\end{aligned}
$$

where

$$
E(t) \leq C(K)\Big(\big\|\partial_t u(t)\big\|^2_{H^{s-1-\lambda t+\frac{1}{2}log}} + \big\|u(t)\big\|^2_{H^{s-\lambda t+\frac{1}{2}log}}\Big).
$$

By Lemmas 6.10 and 6.11 below, the coefficients $b_{j,k}(u) = a_0(u)\tilde{a}_{j,k}(u)$ satisfy estimates similar to (6.12)

$$
\big\|S_k \partial_t b_{j,k}\big\|_{L^\infty} \leq (k+1)C(K).
$$

Therefore $P_{\partial_t b_{j,k}}$ is of order $\mathrm{Log}(D)$ and

$$
\begin{aligned}
2\mathrm{Re}\big(\sum_{j,k=1}^{n} P_{a_0\tilde{a}_{j,k}} \partial_{x_k} u, (1-\Delta_x)^{2(s-\lambda t)} \partial_t \partial_{x_j} u\big)_{L^2} \\
= \frac{d}{dt}\mathrm{Re}\big(\sum_{j,k=1}^{n} P_{b_{j,k}} w_k, w_j\big)_{L^2} + 2\lambda\big(\sum_{j,k=1}^{n} P_{b_{j,k}} \tilde{w}_k, \tilde{w}_j\big)_{L^2} + E_1
\end{aligned}
$$

where E_1 satisfies an estimate similar to E, $w_j = (1-\Delta_x)^{(s-\lambda t)}\partial_{x_j} u$ and $\tilde{w}_j = \big(\ln(1-\Delta_x)\big)^{\frac{1}{2}} w_j$.

Substituting these estimates in (6.30), integrating between 0 and t and using (6.17) implies (6.16) and the theorem follows. □

Notes on the proof of Theorem 6.4.

The proof of this theorem is quite similar, but much simpler, using the para-differential calculus with no logarithmic loss when the coefficients are Lipschitz continuous. □

6.3 Proof of Proposition 6.6

For a C^1 function a bounded with bounded derivatives on $[0,T]\times\mathbb{R}^n$, introduce the norm:

$$\big\|a\big\|_{\mathcal{Z}} = \sup_{k\geq 0}(k+1)^{-1}\big\|S_k(D_x)a\big\|_{L^\infty([0,T]\times\mathbb{R}^n)}. \tag{6.31}$$

Lemma 6.9. *There is a constant C such that if $\mathrm{a}=\{a_j,a_{j,k},b_j,c\}$ is a set of $C^1\cap W^{1,\infty}$ functions on $[0,T]\times\mathbb{R}^n$ satisfying*

$$\partial_t^2 a_0 = \sum_{j=1}^{d}\partial_{x_j}\big(\partial_t a_j + b_j\big) + \sum_{j,k=1}^{n}\partial_{x_j}\partial_{x_k}a_{j,k} + \partial_t b_0 + c, \tag{6.32}$$

then

$$\big\|\partial_t a_0\big\|_{\mathcal{Z}} \leq C\Big(\big\|\mathrm{a}\big\|_{L^\infty([0,T]\times\mathbb{R}^n)} + \big\|\mathrm{a}\big\|_{LL([0,T]\times\mathbb{R}^n)}\Big). \tag{6.33}$$

Proof. **a)** Introducing a partition of unity, it is sufficient to prove the result when the functions are defined and compactly supported in $[0,+\infty[$ and $]-\infty,T]$. The two cases are similar, so we assume that the functions are defined for $t\geq 0$.

Consider the extension operator

$$P_0 a(t,x) = \begin{cases} a(t,x) & t\geq 0,\\ \alpha a(-t,x)+\beta a(-2t,x)+\gamma a(-3t,x), & t\leq 0\end{cases} \tag{6.34}$$

with

$$\alpha+\beta+\gamma=1,\quad \alpha+2\beta+3\gamma=-1,\quad \alpha+4\beta+9\gamma=1,$$

so that P_0a, ∂_tP_0a and $\partial_t^2P_0a$ are continuous at $t=0$ when a is C^2 on $\{t\geq 0\}$. Moreover

$$\partial_tP_0a = P_1\partial_t a,\quad \partial_tP_1b = P_2\partial_t b,$$

where P_1 and P_2 are similar extension operators. Then, the equation (6.32) can be extended to $\mathbb{R}^{1+n}$, with P_0a_0 in place of a_0, P_1a_j in place of a_j, $P_2a_{j,k}$

in place of $a_{j,k}$ etc. Because the extensions operators preserve continuity at $t=0$, there is a constant C such that

$$\|P_l a\|_{L^\infty(\mathbb{R}^{1+d})} \leq C\|a\|_{L^\infty(\{t\geq 0\})}, \qquad \|P_l a\|_{LL(\mathbb{R}^{1+d})} \leq C\|a\|_{LL(\{t\geq 0\})}.$$

Hence it is sufficient to prove the lemma when the functions are defined on $\mathbb{R}^{1+d}$, which we now assume.

b) In addition to the partition of unity $\mathrm{Id} = \sum \Delta_k(D_x)$ consider a similar partition of unity *in time*: $\mathrm{Id} = \sum \Delta'_p(D_t)$. By Proposition 3.3,

$$\|S'_p S_p \partial_t a_0\|_{L^\infty(\mathbb{R}^{1+d})} \leq C(p+1)\|a\|_{LL(\mathbb{R}^{1+d})}.$$

Similarly, for $q > p$ there holds

$$\begin{aligned} &\|\Delta'_q S_p \partial_t \partial_{x_j} a_j\|_{L^\infty} \leq C(q+1)2^p \|a_j\|_{LL}, \\ &\|\Delta'_q S_p \partial_{x_k} \partial_{x_j} a_{j_k}\|_{L^\infty} \leq C(p+1)2^p \|a_{j,k}\|_{LL}. \end{aligned}$$

Finally, using the equation (6.32) and similar estimates for the other functions, we see that for $q > p$:

$$\|\Delta'_q S_p \partial_t^2 a_0\|_{L^\infty(\mathbb{R}^{1+d})} \leq CKq2^p.$$

For $q > p$, the spectral localization of Δ'_q implies that

$$\|\Delta'_q \partial_t S_p a_0\|_{L^\infty} \leq C2^{-q}\|\Delta'_q \partial_t^2 S_p a_0\|_{L^\infty} \leq CKq2^{p-q}.$$

Therefore, writing that $S_p\partial_t a_0 = S'_p S_p \partial_t a_0 + \sum_{q>p} \Delta'_q S_p \partial_t a_0$ and adding the estimates above, one obtains (6.33). □

To complete the proof of Proposition 6.6 we need the following estimates:

Lemma 6.10. *Let F be s smooth function on $\mathbb{R}$ and let $a \in W^{1,\infty}([0,T] \times \mathbb{R}^n)$. Then $F(a) \in W^{1,\infty}([0,T]\times\mathbb{R}^n)$ and*

$$\|F(a)\|_{LL} \leq C(\|a\|_{L^\infty})\,\|a\|_{LL}. \tag{6.35}$$

Lemma 6.11. *Let $a \in W^{1,\infty}([0,T]\times\mathbb{R}^n)$ and $b \in L^\infty([0,T]\times\mathbb{R}^d)$. Then*

$$\|ab\|_{\mathcal{Z}} \leq C\big(\|a\|_{L^\infty} + \|a\|_{LL}\big)\|b\|_{\mathcal{Z}}. \tag{6.36}$$

Proof. The proof of (6.35) is immediate from the definition of the LL seminorm.

To prove (6.36) write

$$S_k(ab) = S_k(S_{k+2}aS_{k+4}b) + \sum_{p\geq k+3} \sum_{|q-p|\leq 2} S_k(\Delta_p a \Delta_q b).$$

The first term satisfies

$$\|S_k(S_{k+2}aS_{k+4}b)\|_{L^\infty} \leq C\|S_{k+2}a\|_{L^\infty}\|S_{k+4}b\|_{L^\infty} \leq C(k+1)\|a\|_{L^\infty}\|b\|_{\mathcal{Z}}.$$

Next, note that for $|p-q|\leq 2$,

$$\|S_k(\Delta_p a\Delta_q b)\|_{L^\infty} \leq C\|\Delta_p a\|_{L^\infty}\|\Delta_q b\|_{L^\infty} \leq C(p+1)^2 2^{-p}\|a\|_{LL}\|b\|_{\mathcal{Z}}.$$

Adding up for $p\geq k+3$, this implies (6.36). □

Proof of Proposition 6.6.

Let A_0, A_j and $A_{j,k}$ be smooth functions on $\mathbb{R}$, vanishing at the origin, with derivative equal to a_0, a_j and $a_{j,k}$ respectively. Then for $C^0([0,T];H^s)\cap C^1([0,T];H^{s-1})$ solutions the equation (6.1) reads:

$$\begin{aligned} \partial_t^2 A_0(u) + \sum_{j=1}^n 2\partial_t\partial_{x_j}A_j(u) - \sum_{j,k=1}^n \partial_{x_j}\partial_{x_k}A_{j,k}(u) \\ \partial_t\big(b_0(u)\big) + \sum_{j=1}^n \partial_{x_j}(b_j(u)) = F(u). \end{aligned} \tag{6.37}$$

By Lemma 6.10, the $A_0(u)$, $A_j(u)$ and $A_{j,k}(u)$ are C^1 and their L^∞ and LL norms are bounded by $C(K)$. Therefore, by Lemma 6.9 there is a constant $C(T,K)$ such that

$$\|\partial_t A_0(u)\|_{\mathcal{Z}} \leq C(T,K).$$

Since

$$\partial_t u = \frac{1}{a_0(u)}\partial_t A_0(u),$$

Lemma 6.11 implies that

$$\|\partial_t u\|_{\mathcal{Z}} \leq C(T,K).$$

Proposition 3.3 implies that

$$\|\partial_{x_j}u\|_{\mathcal{Z}} \leq C\|u\|_{LL}.$$

Therefore, with Lemma 6.11 this implies that v also satisfies the estimate (6.12) and the proof is now complete. □

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