On the L^2 well posedness of Hyperbolic Initial Boundary Value Problems

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Abstract

In this paper we give a class of hyperbolic systems, which includes systems with constant mutliplicity but significantly wider, for which the initial boundary value problem (IBVP) with source term and initial and boundary data in L^2, is well posed in L^2, provided that the necessary uniform Lopatinski condition is satisfied. Moreover, the speed of propagation is the speed of the interior problem. In the opposite direction, we show on an example that, even for symmetric systems in the sense of Friedrichs, with variable coefficients and variable multiplicities, the uniform Lopatinski condition is not sufficient to ensure the well posedness of the IBVP in Sobolev spaces.

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1 Introduction

This paper is concerned with the solvability in L^2 of the initial boundary value problem for first order $N \times N$ systems

$$
\begin{cases} Lu := A_0(t,x)\partial_t u + \sum_{j=1}^{d} A_j(t,x)\partial_{x_j} u + B(t,x)u = f \\ u_{|t=0} = u_0, \\ M(t,x')u_{|x_d=0} = g \end{cases} \tag{1.1}
$$

We consider only the case of noncharacteristic boundaries, which means that A_d is invertible when $x_d = 0$. For simplicity, we have assumed here that the boundary is flat and the equation holds for $t > 0$ and $x \in \mathbb{R}^d_+ = \{x_d > 0\}$. We also use the notation $x = (x', x_d)$.

The starting point is the well known theory of hyperbolic symmetric systems in the sense of Friedrichs ([Fr1, Fr2] or e.g. Chapter 3 in [BeSe] for a more recent exposition and more references). If the matrices A_j

are bounded and Lipschitz continuous on $\mathbb{R} \times \overline{\mathbb{R}}^d_+$, hermitian symmetric, if A_0 is definite positive with A_0^{-1} bounded and if the boundary condition is maximal dissipative and uniformly strictly dissipative, then, for all $T > 0$, there is a constant C such that for all $u_0 \in L^2(\mathbb{R}^d)$, $f \in L^1([0,T]; L^2(\mathbb{R}^d_+))$ and $g \in L^2([0,T] \times \mathbb{R}^{d-1})$, the equation (1.1) has a unique solution $u \in C^0([0,T]; L^2(\mathbb{R}^d_+))$ which satisfies

$$
\begin{aligned}
&\big\| u(t) \big\|_{L^2(\mathbb{R}^d_+)} + \big\| u_{|x_d=0} \big\|_{L^2([0,t]\times\mathbb{R}^{d-1})} \le C \big\| u_{|t=0} \big\|_{L^2(\mathbb{R}^d_+)} \\
&\qquad + C \int_0^t \big\| Lu(s) \big\|_{L^2(\mathbb{R}^d_+)} ds + C \big\| M u_{|x_d=0} \big\|_{L^2([0,t]\times\mathbb{R}^{d-1})}.
\end{aligned}
\tag{1.2}
$$

Recall the general scheme of the proof of such theorems. First, one proves *a priori* estimates, which in the case of symmetric systems, follow by integration par parts in

$$
\operatorname{Re} \int_{[0,t]\times\mathbb{R}^d_+} e^{-2\gamma s} \big(Lu(s), u(s) \big) ds dx.
$$

One obtains that there are constants γ_0 and C such that for all smooth enough function u and $\gamma \ge \gamma_0$

$$
\begin{aligned}
&\big\| e^{-\gamma t} u(t) \big\|^2_{L^2(\mathbb{R}^d_+)} + \big\| e^{-\gamma s} u_{|x_d=0} \big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} \le \\
&\qquad C \big\| u_{|t=0} \big\|^2_{L^2(\mathbb{R}^d_+)} + C \big\| e^{-\gamma s} M u_{\ x_d=0} \big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})}. \\
&\qquad + C \int_0^t \big\| e^{-\gamma s} Lu(s) \big\|_{L^2(\mathbb{R}^d_+)} \big\| e^{-\gamma s} u(s) \big\|_{L^2(\mathbb{R}^d_+)} ds.
\end{aligned}
\tag{1.3}
$$

One can replace next $\| e^{-\gamma t} u(t) \|_{L^2}$ by $\sup_{0 \le s \le t} \| e^{-\gamma s} u(s) \|_{L^2}$ which reveals the L^1 norm of $\| e^{-\gamma s} f(s) \|_{L^2}$, and finally, fixing γ and using a rough bound for $e^{\gamma t}$ on $[0,T]$ one obtains (1.2). The second step is to pass from these estimates to an existence and uniqueness theorem. For this part, we refer to [Fr1, Fr2, Rau, BeSe] for details, or to Section 5 where the analysis is carried out in the new context developed in this paper.

We call inequalities of the form (1.2) *semi-group estimates*.

While Friedrichs analysis is based on the semi-group estimates, the alternate approach developed initially by H.Kreiss is based on *resolvent estimates* which can be stated as follows: there are constants C and γ_0 such that for all $\gamma \ge \gamma_0$ and all $u \in C_0^\infty(\overline{\mathbb{R}}^{1+d}_+)$,

$$
\begin{aligned}
&\gamma \big\| u \big\|^2_{L^2(\mathbb{R}^{1+d}_+)} + \big\| u_{|x_d=0} \big\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad \le C \gamma^{-1} \big\| (L + \gamma A_0) u \big\|^2_{L^2(\mathbb{R}^{1+d}_+)} + C \big\| M u_{|x_d=0} \big\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}
\tag{1.4}
$$

Note that the above resolvent estimates are implied by the semi-group estimates, for instance by squaring and integrating in time in (1.3). Wether or not the converse implication holds is one of the points discussed in this paper.

The resolvent estimates are the starting point for establishing the well posed-ness of the boundary value problem in weighted spaces $e^{\gamma t}L^2$ for γ large and then the well posedness of the the initial boundary value problem (1.1) with vanishing initial data u_0, and finally for smooth initial data (see e.g. [Kre, ChPi, Maj]).

The purpose of this paper is to understand better in which conditions these properties remain true. The focus is put on linear problems, but, by differentiating the equation, the maximal estimates above imply similar a-priori estimates in Sobolev spaces, and, using iterative schemes, they ultimately imply the local solvability in time of nonlinear problems. Indeed the strategies that have been developed for instance in [Maj, BeSe, RaMa, Me5], estimating the commutators with the tangential derivatives the via the multiplicative properties of Sobolev spaces or Gagliardo-Nirenberg inequalities, estimating then the normal derivatives by the equation, and finally Picard's iterative schemes, do not use any special property of the system. We do not develop further these aspects, since we do not pretend to novelty there.

In the interior, that is for functions which vanish on the boundary, a necessary condition for the resolvent estimate (1.4) has been given by V.Ivrii and V.Petkov ([IvPe]; see also section 4 in [Me3]): the principal symbol $L_1(t,x,\tau,\xi)$ must admit a *bounded* microlocal symmetrizer $S(t,x,\xi)$: this is a positive definite symmetric matrix, homogeneous of degree 0 in ξ, such that

$$(1.5) \qquad S, S^{-1} \in L^\infty, \qquad \operatorname{Im}\Big(S(t,x,\xi)\sum_{j=1}^{d}\xi_j A_0^{-1}A_j(t,x)\Big) = 0.$$

This property, called *strong uniform hyperbolicity of the symbol* in [Me3], is equivalent to the existence of a constant C such that for all $(t,x) \in]0,T[\times\mathbb{R}^d_+$, $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, $\gamma > 0$ and $u \in \mathbb{C}^N$,

$$(1.6) \qquad |\gamma|\big|u\big| \le C\big|L_1(t,x,\tau - i\gamma,\xi)\big|.$$

From now on, this condition is assumed to be satisfied:

Assumption 1.1. *There is a constant C such that for all $(t,x) \in [0,T]\times\mathbb{R}^d_+$, $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, $\gamma > 0$ and $u \in \mathbb{C}^N$, the inequality* (1.6) *is satisfied.*

Similarly, given a point $(\underline{t}, \underline{x})$ on the boundary, applying the resolvent estimate (1.4) to

$$u\big(\lambda(t-\underline{t}), \lambda(x-\underline{x})\big), \qquad u \in C_0^\infty(\overline{\mathbb{R}}_+^{1+d})$$

with parameter $\lambda\gamma$ in place of γ, and letting λ tend to $+\infty$, one obtains that for $\gamma > 0$:

$$\begin{aligned}(1.7)\qquad &\gamma\big\|u\big\|^2_{L^2(\mathbb{R}_+^{1+d})} + \big\|u_{|x_d=0}\big\|^2_{L^2(\mathbb{R}^d)} \\ &\qquad \le C\gamma^{-1}\big\|(\underline{L}_1 + \gamma A_0)u\big\|^2_{L^2(\mathbb{R}_+^{1+d})} + C\big\|\underline{M}u_{|x_d=0}\big\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

with the same constant C as in (1.4), where $\underline{L}_1$ and M are the operators with constant coefficients frozen at $(\underline{t}, \underline{x})$. Performing a Fourier transform in the tangential variables, one obtains that a necessary condition for (1.4) is that for all (t, x') in the boundary $[0, T] \times \mathbb{R}^{d-1}$, all $(\tau, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$, all $\gamma > 0$ and all $u \in C_0^\infty(\overline{\mathbb{R}}_+)$ one has

$$\begin{aligned}(1.8)\qquad &\gamma\big\|u\big\|^2_{L^2(\mathbb{R}_+)} + \big|u(0)\big|^2 \le \\ &C\gamma^{-1}\big\|(L_1(t, x', 0, i\tau + \gamma, i\xi', \partial_{x_d})u\big\|^2_{L^2(\mathbb{R}_+)} + C\big|M(t, x', 0)u(0)\big|,\end{aligned}$$

Denoting by $\mathbb{C}_- = \{\tau \in \mathbb{C}; \operatorname{Im}\tau < 0\}$, this leads to introduce for (t, x') in the boundary and $\zeta = (\tau, \xi') \in \mathbb{C}_- \times \mathbb{R}^{d-1}$, the space

$$\mathbb{E}^{in}(t, x', \zeta) = L^2(\mathbb{R}_+) \cap \ker L_1(t, x', 0, \tau, \xi', \partial_{x_d}).$$

It is also convenient to extend the definition near the boundary and $\mathbb{E}^{in}(t, x, \zeta)$ is the invariant space of

$$G(t, x, \zeta) = A_d^{-1}(t, x)\big(\tau A_0(t, x) + \sum_{j=1}^{d-1} \xi_j A_j(t, x)\big),$$

associated to eigenvalues in $\mathbb{C}_-$. Hyperbolicity implies that for $\operatorname{Im}\tau < 0$, $G(t, x, \zeta)$ has no real eigenvalues and that the dimension of $\mathbb{E}^{in}$ is constant and equal to the number N^+ of positive eigenvalues of $A_0^{-1}A_d$ (see e.g. [Kre]). The integer N^+ is the correct number of boundary conditions for (1.1) and we assume from now on, that the boundary condition M is a $N^+ \times N$ matrix. More generally, one could consider boundary conditions where M takes its values in a N^+ dimensional vector bundle.

Applied to $u \in \mathbb{E}^{in}$, the estimate (1.8) implies that

$$(1.9)\qquad \forall u \in \mathbb{E}^{in}(t, x', \zeta), \qquad \big|u\big| \le C\big|M(t, x')u\big|$$

with the same constant C, independent of (t, x', ζ). Thus, a necessary condition is that the *uniform Lopatinski condition* must be satisfied (see [Kre]):

Definition 1.2. *The boundary condition M is said to satisfy the uniform Lopatinski condition (in short ULC) for the system L, when there is a constant C such that for $(t, x') \in [0, T] \times \mathbb{R}^{d-1}$ in the boundary and all $\zeta = (\tau, \xi') \in \mathbb{C}_- \times \mathbb{R}^{d-1}$, the estimate* (1.9) *is satisfied.*

For symmetric systems in the sense of Friedrichs, this condition is therefore satisfied for maximal strictly dissipative boundary conditions. However, it is satisfied by a much wider class of boundary conditions, see e.g. [Ma-Os] or other examples below. Another important motivation for considering general boundary condition is the analysis of the stability of multidimensional *shock waves* initiated by A.Majda ([Maj]). H.Kreiss has shown that for strictly hyperbolic systems, the uniform Lopatinski condition implies the a priori resolvent estimate (1.4). In [Ma-Os] and [Maj] it was noticed that Kreiss' proof extended to the case where the so-called *block structure condition* was satisfied and in [Me1] it is shown that this latter condition is satisfied for hyperbolic systems with constant multiplicities. More recently, in [MeZu1], this result has been extended to some cases where the multiplicity varies, with applications to MHD.

At this point, several questions can be raised, and it is the goal of this paper to give them partial answers.

Question 1. To which extent can one push Kreiss construction of symmetrizers? Recall that their existence implies the resolvent estimates (1.4) for the direct problem and for the dual problem, implying the well posed-ness of the boundary value problem in weighted spaces $e^{\gamma t} L^2$ for γ large; next a causality principle follows, showing that if f and g vanish for $t \le t_0$, then the solution also vanishes there. This allows to solve the initial boundary value problem (1.1) with vanishing initial data u_0, and finally for smooth initial data (see e.g. [Kre, ChPi, Maj]).

The obstacle to the construction of Kreiss symmetrizers is the existence of varying multiplicities. In Section 3, we give a reasonable condition which ensures the existence of smooth symmetrizers which extends Kreiss construction.

Definition 1.3. *The system L belongs to the class $\mathcal{M}$ if it is strongly hyperbolic and if near each point of the characteristic variety one of following condition is satisfied :*

i) L is analytically diagonalizable,

ii) denoting by n the conormal to the boundary, either n or $-n$ belongs to the cone of hyperbolic directions for the localized system.

It belongs to the class $s\mathcal{M}$ if in addition it admits a symmetrizer $S(t,x,\xi)$ (1.5) *which is Lipschitz continuous in (t,x) and C^∞ in $\xi \neq 0$.*

We refer to Section 2 for precise definitions and details. Condition *i)* is the geometrical form of the *block structure condition* (see Theorem 3.4 in [MeZu1]). The condition *ii)* extends the Defintion 3.6 in [MeZu1], where it was applied to symmetric systems. We refer to this paper for examples. Near points where the characteristic variety is smooth, the multiplicity is constant, implying that the characteristic variety is analytic with respect to the frequency variables and condition *i)* is satisfied (see Lemma 2.7 below). Moreover, the symmetrizer can be chosen smooth, and even analytic, in ξ. In particular, systems with constant multiplicity belong to the class $s\mathcal{M}$.

Recall that Kreiss' strategy was to construct first *families of symmetrizers*, independently of any boundary condition, and next to show that for all boundary condition which satisfies the ULC, one can select one symmetrizer in the family which makes the boundary conditions strictly dissipative. We call them K-families in Definition 3.5.

Theorem 1.4. *If the system belongs to the class $s\mathcal{M}$, there are K-families of smooth symmetrizers for L.*

If in addition the boundary conditions satisfies the ULC, the boundary value problem is well posed in spaces $e^{\gamma t}L^2$ for γ large enough.

The existence of K-families of symmetrizers implies the *continuity of* $\mathbb{E}^{in}$ up to the boundary $\operatorname{Im}\tau = 0$ (see [MeZu2]), which is a strong limitation at points where the multiplicities of the eigenvalues vary. This question is discussed in Section 2.

Question 2. Is the uniform Lopatinski condition sufficient in general for the validity of the resolvent (1.4)? In the constant-coefficient case, the analysis in [GMWZ] shows that, if the estimate (1.2) (or (1.8)) is satisfied for one boundary matrix M_0 (and then M_0 necessarily satisfies ULC), then it is satisfied for all ULC boundary condition M. This applies to symmetric systems, which admit strictly dissipative boundary condition.

But, in general, the answer to the question is negative:

Theorem 1.5. *There are symmetric hyperbolic systems in the sense of Friedrichs and boundary conditions which satisfy the uniform Lopatinski condition, for which there are families of data bounded in H^s for all s which generate solutions which are not bounded in L^2 on all non trivial interval of time.*

The conclusion is a classical expression of *ill-posedness* of the problem in C^∞. An example is given in Section 6. Of course, it has variable coefficients, variable mutiplicities and the boundary conditions are not dissipative. The strength of the result is that the well posed-ness is ruined not only in L^2 but also in C^∞.

Question 3. What can be said about the local theory, in particular about local uniqueness and finite speed of propagation? We tackle this question under the angle of the invariance of the assumptions by change of time. In Section 2 we prove the following result (see the remark before Theorem 2.17).

Theorem 1.6. *If L is of class $s\mathcal{M}$, the validity of the uniform Lopatinski condition is preserved by any change of time preserving hyperbolicity.*

In particular, this proves that the speed of propagation for the boundary value problem does nor exceed the speed of propagation for the interior problem. This is in sharp contrast with the case of *weakly well posed* problems, where the *weak Lopatinski condition* holds, for which surface waves can propagate faster than interior waves (see [Ben, BeSe, Hör, Gar, Her])

Question 4. Are the semigroup estimates (1.2) satisfied for systems which admit Kreiss symmetrizers? This is easily proved when the system is symmetric, using the obvious energy balance, since the boundary term which involves the L^2 norm of the trace of the solution is controlled by the resolvent estimate (1.4). The general case is much more delicate. A positive answer has been established for strictly hyperbolic systems [Rau] and extended to systems with constant multiplicities [Aud]. An important consequence of this question is the solvability of the initial-boundary value problem (1.1) with *all* data, including u_0, in L^2. In Section 5 we extend the results cited above, using ideas taken from [FrLa1, FrLa2]: the semigroup estimates are proven, assuming the resolvent estimates, and using an holomorphic extension in ξ of the symmetrizer of the Cauchy problem. We make this condition explicit in the next definition (recall Definition 1.3).

Definition 1.7. *A system L in the class $s\mathcal{M}$ is said to belong to the class $a\mathcal{M}$ if the symmetrizer $S(t,x,\xi)$ (1.5) can be chosen Lipschitz continuous in (t,x) and holomorphic in a cone $\{|\mathrm{Im}\,\xi| \leq \delta |\mathrm{Re}\,\xi|\}$ for some $\delta > 0$.*

This condition is trivially satisfied when the system is symmetric in the sense of Friedrichs, since then it is independent of ξ. It is also satisfied when the multiplicities are constant, or more generally when the system is analytically diagonalizable, since then the symmetrizer is explicitly given in the basis of diagonalization. Thus the next theorem extends the result of [Aud] with a completely different method.

Theorem 1.8. *Assume that the coefficients of L and M are $W^{1,\infty}([0,T] \times \mathbb{R}^d_+)$. Suppose that the uniform Lopatinski condition is satisfied and that L is of class $a\mathcal{M}$. Then, for all $f \in L^1([0,T]; L^2(\mathbb{R}^d_+))$, $g \in L^2([0,T] \times \mathbb{R}^{d-1})$ and $u_0 \in L^2(\mathbb{R}^d_+)$, the problem* (1.1) *has a unique solution $u \in C^0([0,T] \times \mathbb{R}^d_+)$. Moreover, there is a constant C such that the semi group estimate* (1.2) *is satisfied.*

Section 5 is devoted to the proof of this results. It uses some paradifferential calculus, in particular for traces, which is presented in Section 4.

2 Symbolic analysis

In this section we extend the known properties of symbols of hyperbolic boundary value problems in two directions, considering variable multiplicities and giving intrinsic definitions which make clear the invariance of these properties under a change of time direction. In particular, it is convenient to treat in a whole the space-time variables $\tilde{x} = (t, x) \in \mathbb{R}^{1+d}$, and accordingly we consider a family of symbols

$$L(a, \tilde{\xi}) = \sum_{j=0}^{d} \xi_j A_j(a) \tag{2.1}$$

where $\tilde{\xi} = (\xi_0, \ldots, \xi_d) \in \mathbb{R}^{1+d}$. The parameter a varies in a compact set $\mathcal{A}$, and the coefficients of the $N \times N$ matrices are supposed to be at least continuous in a. In our analysis, we need the symbols $p(a, \tilde{\xi})$ to be smooth with respect to the frequency variables, but for applications we insist on keeping a limited smoothness in a, typically a C^k regularity with $k \geq 0$, or a $W^{k,\infty}$ regularity with $k \geq 1$. This regularity is kept fixed throughout this section, and to avoid repetition we use just say that a function $p(a, \tilde{\xi})$ is smooth [resp. analytic] if it is C^∞ [resp. real analytic] in $\tilde{\xi}$ and has the given regularity C^k or $W^{k,\infty}$ with respect to $a \in \mathcal{A}$. For example, a family of spaces $\mathbb{E}(a, \tilde{\xi})$ is smooth [analytic] if it admits locally a smooth [analytic] basis.

The symbol $L(a, \cdot)$ is assumed to be strongly hyperbolic in some direction $\underline{\nu}$[1], uniformly with respect to a. Denote by Γ_a the cone of hyperbolic

[1]In the notations of the introduction, $x_0 = t$, $\tilde{\xi}_0 = \tau$ and $\underline{\nu} = dt = (1, 0, \ldots, 0)$.

directions of $L(a,\cdot)$, containing $\underline{\nu}$. Then $L(a,\cdot)$ is strongly hyperbolic in any direction $\nu \in \Gamma_a$ (see e.g. Section 4 in [Me3]). All the estimates below are locally uniform with respect to the parameters (a,ν) with $a \in \mathcal{A}$ and $\nu \in \Gamma_a$. This is made clear by shrinking $\mathcal{A}$ if necessary and choosing a cone with compact basis $\Gamma \subset \bigcap_a \Gamma_a$. The uniform strong hyperbolicity hypothesis can thus be stated as follows (see Proposition 4.4 in [Me3]).

Assumption 2.1. *Γ is a closed convex cone in $\mathbb{R}^{1+d}\backslash 0$ which is contained in Γ_a for all $a \in \mathcal{A}$ and there is a constant C such that for all $a \in \mathcal{A}$, all $\tilde{\xi} = \mathrm{Re}\,\tilde{\xi} + i\mathrm{Im}\,\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma \subset \mathbb{C}^{1+d}\backslash 0$ and all $u \in \mathbb{C}^N$;*

$$\big|\mathrm{Im}\,\tilde{\xi}\big||u| \leq C\big|L(a,\tilde{\xi})u\big|. \tag{2.2}$$

In the framework of Section 1, (2.2) is just an extension of the esimate (1.6) to imaginary frequencies in a conical neighborhood of the given time direction dt, considering (t,x) as the parameters and Proposition 4.4 in [Me3] shows that this extension is legitimate.

Remark 2.2. Changing $\tilde{\xi}$ to $-\tilde{\xi}$, we see that the estimate (2.2) is satisfied as well when $\mathrm{Im}\,\tilde{\xi} \in \Gamma$.

Together with L we consider boundary operator

$$M(a)u_{|x_n=0} \tag{2.3}$$

where $x_n = n\cdot x$ and $n \in \mathbb{R}^{1+d}\backslash\{0\}$ is the inner conormal vector to the boundary of $\Omega = \{x_n > 0\}$[2]. More intrinsically, the data is $\mathbb{K}(a) = \ker M(a) \subset \mathbb{C}^N$. We assume that

Assumption 2.3. *For all $a \in \mathcal{A}$, the boundary matrix $L(a,n)$ is invertible, $\mathbb{K}$ is a bundle over $\mathcal{A}$ of class W and $\dim \mathbb{K}(a) = N_-$, the number of negative eigenvalues of $L(a,\underline{\nu})^{-1}L(a,n)$.*

2.1 Localization and microhyperbolicity

The (real) characteristic variety of $L(a,\cdot)$ is

$$\mathcal{C}_a = \{\tilde{\xi} \in \mathbb{R}^{1+d}\backslash\{0\},\ \det L(a,\tilde{\xi}) = 0\}.$$

We denote by $\mathcal{C}$ the set of $(a,\tilde{\xi})$ with $a \in \mathcal{A}$ and $\tilde{\xi} \in \mathcal{C}_a$. At $(a,\tilde{\xi}) \in \mathcal{C}$, invariant data are the kernel and the image of $L(a,\tilde{\xi})$. Denoting by

[2] In the notations of the introduction, $n = dx_d = (0,0,\ldots,1)$, and $x_n = x_d$.

$\iota_{a,\xi}$ the embedding $\ker L(a,\tilde\xi) \mapsto \mathbb{C}^N$ and by $\varpi_{a,\xi}$ the projection $\mathbb{C}^N \mapsto \mathbb{C}^N/\mathrm{range} L(a,\tilde\xi)$, the *localized symbol at* $(a,\tilde\xi)$ is

$$L_{a,\tilde\xi}(\tilde\eta) = \varpi_{a,\xi} L(a,\tilde\eta)\iota_{a,\xi}.$$

It acts from $\ker L(a,\tilde\xi)$ to $\mathbb{C}^N/\mathrm{range} L(a,\tilde\xi)$. The characteristic variety of $L_{a,\tilde\xi}$ is denoted by $\mathcal{C}_{a,\tilde\xi} \subset \mathbb{R}^{1+d}$.

Strong hyperbolicity implies that for $\tilde\xi \in \mathcal{C}_a$ one has

$$p(\tilde\eta) := \det\Big(L(a, \underline{\tilde\xi} + \tilde\eta)\Big) = O\big(|\tilde\eta|)^m\big) \tag{2.4}$$

where $m = \dim\ker L(a,\tilde\xi)$ is the order of the root $\tau = 0$ of $p(\tilde\xi + \tau\nu) = 0$. The limit

$$p_0(\tilde\eta) = \lim_{\varepsilon\to 0} \varepsilon^{-m} p(\varepsilon\tilde\eta) \tag{2.5}$$

exits and is homogeneous of degree m. Moreover, (2.2) implies that p_0 is hyperbolic any direction $\nu \in \Gamma_a$ (see. Lemma 8.7.2 in [Hör]). Denoting by by $\Gamma_{a,\tilde\xi}$ the cone of hyperbolic directions for $L_{a,\tilde\xi}$ containing $\underline{\nu}$ this means that

$$\forall (a,\tilde\xi) \in \mathcal{C}, \qquad \Gamma \subset \Gamma_a \subset \Gamma_{a,\tilde\xi}.$$

Following the terminology of [Hör] (see [KK] for the original definition) $\Gamma_{a,\tilde\xi}$ is the cone of *microhyperbolic directions* near $\underline{\tilde\xi}$. Moreover, the strong form of hyperbolicity is preserved. This is the content of the next proposition.

Proposition 2.4. *Let* $\underline{\tilde\xi} \in \mathcal{C}_{\underline{a}}$ *of multiplicity* m *and let* Γ' *be a closed convex subcone of* $\Gamma_{\underline{a},\underline{\tilde\xi}}$. *Then there is a neighborhood* $\mathscr{V}$ *of* $(\underline{a}, \underline{\tilde\xi})$ *in* $\mathcal{A} \times \mathbb{C}^{1+d}$ *and there are* $\gamma_1 > 0$, $r > 0$ *and* C *such that :*

i) for $(a,\xi) \in \mathscr{V}$ *with* $\mathrm{Im}\,\tilde\xi \in -\Gamma'$ *with* $|\tilde\xi| \le \gamma_1$,

$$\forall u \in \mathbb{C}^N, \qquad |\mathrm{Im}\,\tilde\xi||u| \le C\big|L(a,\tilde\xi)u|. \tag{2.6}$$

ii) for $(a,\xi) \in \mathscr{V}$ *with* $|\tilde\xi| \le \gamma_1$ *and for* $\theta \in \Gamma'$ *with* $|\theta| = 1$, *the polynomial in* s, $\det L(a, \tilde\xi + s\theta)$ *has exactly* m *roots counted with their multiplicities contained in the disc* $\{|s| \le r\}$. *Moreover, if* $\mathrm{Im}\,\tilde\xi \in -\Gamma$ *all of them have a positive imaginary part, and if* $\mathrm{Im}\,\tilde\xi = 0$ *all of them are real .*

We first prove the following lemma.

Lemma 2.5. *There is a neighborhood* $\mathscr{V}$ *of* $(\underline{a}, \underline{\tilde\xi})$ *and there are* $\gamma_1 > 0$, $r > 0$ *and* $K > 0$*, such that for all matrix* B *with* $|B| \le K$ *and all* $(a, \xi) \in \mathscr{V}$ *with* $\gamma = |\mathrm{Im}\,\tilde\xi| \le \gamma_1$, $\tilde\eta \in \mathbb{R}^{1+d}$ *with* $|\tilde\eta| \le r$*:*

$$\mathrm{Im}\,\tilde\xi \in -\Gamma,\ \tilde\eta \in \Gamma' \quad \Rightarrow \quad \det\big(A(a, \tilde\xi - i\tilde\eta) + \gamma B\big) \neq 0.$$

Proof. **a)** Consider the polynomial in s

$$p(b,s) = \det\big(L(a, \tilde\xi - i\gamma\nu - is\theta) + \gamma B\big)$$

Where $\tilde\xi \in \mathbb{R}^{1+d}$, $\nu \in \Gamma$, $\theta \in \Gamma'$ with $|\nu| = |\theta| = 1$ and b stands fo $(a, \tilde\xi, \nu, \theta, B, \gamma)$. The assumption (2.2) implies that for all matrix B with $|B| < M = 1/C$, all real $\tilde\xi$ and all $\gamma > 0$, $L(a, \tilde\xi - i\gamma\nu) + \gamma B$ is invertible. Therefore, $p(b, \cdot)$ has no root on the imaginary axis when $\gamma > 0$.

b) When $(a, \tilde\xi) = (\underline{a}, \underline{\tilde\xi})$ and $\gamma = 0$, $p(b,s) = (-is)^m p_0(\tilde\theta) + O(s^{m+1})$ where p_0 was introduced at (2.5). Because $p_0(\theta) \neq 0$ and the set of θ is compact, as well as the sets of B and ν, there is a real neighborhood $\mathscr{V}_\mathbb{R}$ of $(\underline{a}, \underline{\tilde\xi})$ and there are γ_1 and $r > 0$ such that for $(a, \xi) \in \mathscr{V}_\mathbb{R}$, $|\gamma| \le \gamma_1$ and $|B| \le K$, $p(b, \cdot)$ has exactly m roots counted with their multiplicity in the open disc $D := \{|s| < r\}$ and no root in $r \le s \le 2r$.

c) When $(a, \tilde\xi) = (\underline{a}, \underline{\tilde\xi})$, $B = 0$ and $\gamma > 0$, $q(\gamma, \sigma) = (-i\gamma)^{-m} p(b, \gamma\sigma)$ is a polynomial in σ. It extends to $\gamma = 0$ and at $\gamma = 0$, $q(0, \sigma) = p_0(\nu + \sigma\theta)$ is a polynomial of degree m in σ. Because both ν and θ belong to the cone $\Gamma_{\underline{a}, \underline{\tilde\xi}}$ of hyperbolicity of p_0, $q(0, \sigma) = 0$ has only real negative roots (see e.g. [Gar] or Lemma 8.7.3 in [Hör]). By compactness in ν and θ, there are $R > R_1 > 0$ such that these roots remain in $\{-R \le \sigma \le -R_1\}$. By continuity, for γ small and positive, $q(\gamma, \sigma)$ has m roots in $|\sigma| \le 2R$ which all satisfy $\mathrm{Re}\,\sigma < -\frac{1}{2}R_1$. This shows that for $(a, \tilde\xi) = (\underline{a}, \underline{\tilde\xi})$, $B = 0$ and $\gamma > 0$ small, $p(b,s)$ has m roots in $\{|s| \le \gamma 2R, \mathrm{Re}\,s < 0\}$.

Decreasing γ_1 if necessary, we can assume that $2\gamma_1 R \le r$, and this shows that for $\gamma \in]0, \gamma_1]$ and $(a, \tilde\xi) = (\underline{a}, \underline{\tilde\xi})$, $B = 0$, the m roots of $p(b, \cdot)$ in the disc D, are located in $D_- = \{s \in D, \mathrm{Re}\,s < 0\}$.

By a) and b), there are no root in ∂D_- for $(a, \xi) \in \mathscr{V}$, $|B| \le K$ and $\gamma \in]0, \gamma_1]$. Therefore, the number of roots in D_- is constant and independent of b when $\gamma > 0$, if we have chosen, as we can, $\mathscr{V}_\mathbb{R}$ connected. Hence $p(b,s)$ has no roots in $\{|s| \le r, \mathrm{Re}\,s \ge 0\}$ when $\gamma > 0$ and the lemma is proved. $\square$

Proof of Proposition 2.4. Lemma 2.5 implies that for all $(a, \xi) \in \mathscr{V}$, all with $\mathrm{Im}\,\tilde\xi \in -\Gamma$ and $|\mathrm{Im}\,\tilde\xi| \le \gamma_1$ and all $\tilde\eta \in \Gamma'$ with $|\tilde\eta| \le r$, $L(a, \tilde\xi + i\tilde\eta)$ is invertible and

$$|\mathrm{Im}\,\tilde\xi|\,|L(a, \tilde\xi + i\tilde\eta)^{-1}| \le 1/K. \tag{2.7}$$

Because $\Gamma_{\underline{a},\underline{\tilde\xi}}$ is open and because Γ and Γ' are closed convex cones, there is $\varepsilon > 0$ such that for

$$\tilde\eta \in \Gamma', \ \tilde\xi \in \Gamma, \ |\tilde\xi| \le \varepsilon|\tilde\eta| \quad \Rightarrow \quad \tilde\eta + \tilde\xi \in \Gamma''$$

where Γ'' is another closed subcone of $\Gamma_{\underline{a},\underline{\tilde\xi}}$ which contains Γ' in its interior. Choose $\nu \in \Gamma$ with $|\nu| = 1$. There is a neighborhood $\mathscr{V}_1$ of $(\underline{a},\underline{\tilde\xi})$ and there is $\gamma_2 > 0$ such that for $(a, \mathrm{Re}\,\tilde\xi - i\tilde\eta) \in \mathscr{V}_1$ with $\eta \in \Gamma'$ and $|\eta| \le \gamma_2$, one has $(a, \mathrm{Re}\,\tilde\xi - i\varepsilon|\tilde\eta|\nu) \in \mathscr{V}$, $-\varepsilon|\tilde\eta|\nu \in \Gamma$, $\big| - \varepsilon|\tilde\eta|\nu\big| \le \gamma_1$ and $\tilde\eta^1 = \tilde\eta - \varepsilon|\tilde\eta|\nu \in \Gamma_{\underline{a},\underline{\tilde\xi}}$ with $|\tilde\eta^1| \le r$. Thus the estimate (2.7) which is valid on Γ'' implies that

$$\varepsilon|\tilde\eta|\, |L(a, \mathrm{Re}\,\tilde\xi - i\tilde\eta)^{-1}| \le 1/K$$

and (2.6) follows with $C = 1/(\varepsilon K)$.

Part b) of the proof of the lemma above implies that for $(a, \tilde\xi)$ close to $(\underline{a}, \underline{\tilde\xi})$ and θ of length 1 in Γ', $\det L(a, \tilde\xi + s\theta) = 0$ has exactly m roots in s in the disc $\{|s| < r\}$. Part c) says they are in $\mathrm{Im}\, s > 0$ when $\mathrm{Im}\,\tilde\xi \in -\Gamma$.

If $\mathrm{Im}\,\tilde\xi = 0$, then (2.6) shows that the roots are located in $\mathrm{Im}\, s \le 0$. Now we note that the assumption satisfied by (ν, θ) are also satisfied by $(-\nu, -\theta)$ and therefore, shrinking the neighborhoods if necessary, the m roots of $\det L(a, \tilde\xi - s\theta)$ in the disc of radius r have also nonpositive imaginary part, therefore the m roots of $\det L(a, \xi + s\tilde\theta)$ in the disc are real. This proves finishes the proof of *ii*). □

2.2 Smooth modes and the class $\mathcal{M}$

Recall that the characteristic variety $\mathcal{C}$ has been defined in the first lines of Section 2.1.

Definition 2.6. *$\mathcal{C}$ is said to be smooth at $(\underline{a}, \underline{\tilde\xi})$ if there is a neighbohood $\mathscr{V}$ of this point in $\mathcal{A} \times \mathbb{R}^{1+d}$ and a smooth function φ on $\mathscr{V}$, such that $d_{\tilde\xi}\varphi(a, \tilde\xi) \ne 0$ and $\mathcal{C} \cap \mathscr{V} = \{(a, \tilde\xi) \in \mathscr{V} : \varphi(a, \tilde\xi) = 0\}$.*

Lemma 2.7. *Suppose that $\mathcal{C}$ is smooth at $(\underline{a}, \underline{\tilde\xi})$ and given locally by the equation $\varphi = 0$. Then one can choose φ analytic in $\tilde\xi$ and*

i) The characteristic variety of $L_{\underline{a},\underline{\tilde\xi}}$ is the hyperplane $\{\tilde\eta \cdot d_{\tilde\xi}\varphi(\underline{a}, \underline{\tilde\xi}) = 0\}$.

ii) There is a neighborhood $\mathscr{V}$ of $(\underline{a}, \underline{\tilde\xi})$, and an analytic family of spaces $\mathbb{E}(a, \tilde\xi)$ on $\mathscr{V}$, such that $\mathbb{E}(a, \tilde\xi) = \ker L(a, \tilde\xi)$ for all $(a, \tilde\xi) \in \mathcal{C} \cap \mathscr{V}$.

In particular, the dimension of $\ker L(a, \tilde\xi)$ is constant for $(a, \tilde\xi) \in \mathcal{C} \cap \mathscr{V}$.

Proof. Consider the polynomials p and p_0 as in (2.4) and (2.5). Consider $\nu \in \Gamma$ and choose an hyperplane H such that $\mathbb{R}^{1+d} = \mathbb{R}\nu \oplus H$. By hyperbolicity and continuity of the roots of polynomials, the roots in s of $p_0(\eta + s\nu) = 0$ are the limits of s_ε where $p(\varepsilon\eta + \varepsilon s_\varepsilon) = 0$ for sequences $\varepsilon \to 0$. Thus the characteristic the set $\{p_0 = 0\}$ is the tangent space to $\mathcal{C}$ at $(\underline{a}, \underline{\tilde{\xi}})$ and this proves i).

Moreover, $\nu \cdot \partial_{\tilde{\xi}}\varphi(\underline{a}, \underline{\tilde{\xi}}) \neq 0$ since ν is a direction of hyperbolicity, and thus non characteristic, for $L_{(\underline{a}, \underline{\tilde{\xi}})}$. By the implicit function theorem, there are neighborhoods $\mathscr{V}$ of $(\underline{a}, \underline{\tilde{\xi}})$ and $\mathscr{V}_1$ of $(\underline{a}, 0)$ and a smooth function $\lambda(a, \eta)$ on $\mathscr{V}_1$ such that

$$\mathcal{C} \cap \mathscr{V} = \{(a, \underline{\tilde{\xi}} + \eta + s\nu),\ \xi \in H, s \in \mathbb{R},\ s + \lambda(a, \eta) = 0\}. \tag{2.8}$$

In particular, for $(a, \eta) \in \mathscr{V}_1$, $-\lambda(a, \eta)$ is the unique eigenvalue close to 0 of $L(a, \nu)^{-1} L(a, \underline{\tilde{\xi}} + \eta)$ and this eigenvalue is semi-simple because of Assumption 2.1. Thus λ is analytic in η and the corresponding eigenspace $\mathbb{E}_\lambda(a, \eta)$ depends analytically on η. This proves ii). □

Definition 2.8. *L is said to be smoothly [analytically] diagonalizable at $(\underline{a}, \underline{\tilde{\xi}}) \in \mathcal{C}$ if there is a neighbohood $\mathscr{V}$ of this point in $\mathcal{A} \times \mathbb{R}^{1+d}$, smooth [analytic] functions φ_j on $\mathscr{V}$, and smooth [analytic] families of spaces $\mathbb{E}_j(a, \tilde{\xi})$ on $\mathscr{V}$, such that*

i) $\varphi_j(\underline{a}, \underline{\tilde{\xi}}) = 0$ and $d_{\tilde{\xi}}\varphi_j(a, \tilde{\xi}) \neq 0$ on $\mathscr{V}$,

ii) $\mathcal{C} \cap \mathscr{V} = \bigcup \mathcal{C}_j$ where $\mathcal{C}_j = \{(a, \tilde{\xi}) \in \mathscr{V},\ \varphi_j(a, \tilde{\xi}) = 0\}$,

iii) the $\mathbb{E}_j(a, \tilde{\xi})$ are in direct sum,

iv) for all $(a, \tilde{\xi}) \in \mathcal{C} \cap \mathscr{V}$, $\ker L(a, \tilde{\xi})$ is the direct sum of the $\mathbb{E}_k(a, \tilde{\xi})$ for those indices k such that $(a, \tilde{\xi}) \in \mathcal{C}_k$.

Fix $\nu \in \Gamma$ and H as before. ν is not characteristic for the localized symbol and, shrinking ω, there are smooth [analytic] functions λ_j for $(a, \xi) \in \mathcal{A} \times H$ close to $(\underline{a}, 0)$:

$$\mathcal{C}_j \cap \omega = \{(a, \underline{\tilde{\xi}} + \xi + s\nu),\ \xi \in H, s \in \mathbb{R},\ s + \lambda_j(a, \xi) = 0\}. \tag{2.9}$$

Hence, the $-\lambda_j(a, \xi)$ are the eigenvalue close to 0 of $L(a, \nu)^{-1} L(a, \underline{\tilde{\xi}} + \xi)$. They are semi-simple because of the strong hyperbolicity.

Remark 2.9. This condition is very restrictive at non smooth points of $\mathcal{C}$. It is not satisfied in the example of MHD or non-isotropic Maxwell equations, as shown in [MeZu2]. Indeed a strong motivation for [MeZu2] and the present paper is to go beyond this condition. However, it is important to make is explicit for two reasons:

- It is an almost necessary and sufficient condition for the validity of the *block structure condition* (see [MeZu2]) which is the key structural assumption for the construction of Kreiss-symmetrizers, see Section 3 below. Moreover, the definition above is intrinsic and in particular, this shows that the block structure condition is preserved by change of time.

- When all the $\mathcal{C}_j$, of codimension one, cross on a analytic submanifold Σ of codimension 2, then, after a block reduction, we are left, locally, with the spectral analysis of a matrix of the form $\lambda(\sigma)\mathrm{Id} + A(\sigma, \eta)$ where $\sigma \in \Sigma$, $A(\sigma, 0) = 0$ and η is a single variable transversal to Σ. In this case, one can expect to be able to follow analytically in η both the eigenvalues close to zero and associated eigenvectors of A.

At regular point $(a, \tilde{\xi}) \in \mathcal{C}$, the localized operator has the form

$$L_{a,\tilde{\xi}}(\eta) = \tilde{\eta} \cdot d\varphi(a, \tilde{\xi})\ J$$

where $\{\varphi = 0\}$ is the local equation of $\mathcal{C}$ and J an isomorphism from $\ker L(a, \tilde{\xi})$ to $\mathbb{C}^N/\mathrm{range}L(a, \tilde{\xi})$. The vector field H_φ with symbol $\tilde{\eta} \cdot d\varphi$ determines the propagation of singularities. In presence of a boundary, this depends on the position of H_φ relatively to that boundary : tangent, incoming or outgoing. That is $\partial_n \varphi = n \cdot d\varphi = 0$, > 0 or < 0 (assuming as we may that $\underline{\nu} \cdot d\varphi > 0$). In the first case, the classical terminology is that the mode $\tilde{\xi}$ is *glancing*, and in the other cases that it is *hyperbolic*. Another formulation is that n is characteristic for $L_{a,\tilde{\xi}}$, $n \in \Gamma_{a,\tilde{\xi}}$ or $-n \in \Gamma_{a,\tilde{\xi}}$. These three properties make sense in general and we are led to the following definition.

Definition 2.10. *Given the domain $\Omega = \{n \cdot x > 0\}$, $(a, \tilde{\xi}) \in \mathcal{C}$ is said hyperbolic incoming [resp. outgoing] if $n \in \Gamma_{a,\tilde{\xi}}$ [resp. $-n \in \Gamma_{a,\tilde{\xi}}$].*

In this case, the boundary value problem for the localized operator needs full [resp. no] boundary conditions and no precise analysis of the singularities of $\mathcal{C}$ near $(a, \tilde{\xi})$ is needed. According to the discussion before Proposition 2.4 a more correct terminology would be to say that the mode is *microhyperbolic*.

The condition that n is characteristic for $L_{a,\tilde{\xi}}$ also makes sense in general. However, in contrast with the situation at smooth points, in the general case, there is a gap between this condition and the hyperbolicity.

If L is smoothly diagonalizable near $(a, \tilde{\xi})$, the characteristic variety is singular as soon as there are different sheets $\mathcal{C}_j$. But at these points the localized operator has a particular structure: it is block diagonal (see [MeZu1] and below) with blocks $H_{\varphi_j} J_j$. Each of the H_{φ_j} can be glancing, incoming or outgoing, but the analysis can be carried on because of the strong decoupling of these modes.

Summing up, the technical motivation for introducing of the class $\mathcal{M}$ in as in Definition 1.3 is to rule out the difficult case where the localized operator is not hyperbolic and cannot be decoupled into a diagonal system of vector fields which can be handled separately. There are other and more profound motivations that are explained in the sequel.

2.3 The incoming bundle, block decomposition

The Fourier-Laplace analysis of the boundary value problem relies on the spectral properties of the matrix

$$G(a, \tilde{\xi}) = L(a, n)^{-1} L(a, \tilde{\xi})$$

for complex $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$, in particular in the limit $\operatorname{Im} \tilde{\xi} \to 0$.

For $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$, the hyperbolicity implies that $G(a, \tilde{\xi}))$ has no eigenvalues on the real axis. The *incoming space* $\mathbb{E}^{in}(a, \tilde{\xi})$ is defined as the invariant space of $G(a, \tilde{\xi})$ associated to the eigenvalues in $\{\operatorname{Im} \lambda < 0\}$. $\mathbb{E}^{in}(a, \tilde{\xi})$ is holomorphic in $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$, and in particular, the dimension of $\mathbb{E}^{in}$ is constant.

If $n \in \Gamma$ [resp. $-n \in \Gamma$], then one can choose above $\tilde{\xi} = -in$ [resp. $\tilde{\xi} = n$] and since $G(a, n) = \mathrm{Id}$, $\dim \mathbb{E}^{in} = N$ [resp. $\dim \mathbb{E}^{in} = 0$]. Hence, for all $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$, $\mathbb{E}^{in}(a, \tilde{\xi}) = \mathbb{C}^N$ [resp. $\mathbb{E}^{in}(a, \tilde{\xi}) = \{0\}$.

So we now exclude these trivial cases and assume that

$$n \notin \pm\Gamma. \tag{2.10}$$

We first show that $\mathbb{E}^{in}$ only depends on the tangential frequencies.

Lemma 2.11. *If $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma$, then for all complex number s such that $\tilde{\xi} + sn \in \mathbb{R}^{1+d} - i\Gamma$, one has*

$$\mathbb{E}^{in}(a, \tilde{\xi} + sn) = \mathbb{E}^{in}(a, \tilde{\xi}). \tag{2.11}$$

Proof. Because Γ is a convex cone, for all $t \in [0,1]$, $\tilde{\xi} + tsn \in \mathbb{R}^{1+d} - i\Gamma$ and the eigenvalues of $G(a, \tilde{\xi} + tsn)$ do not cross the real axis. Because the invariant spaces of $G(a, \tilde{\xi} + tsn) = G(a, \tilde{\xi}) + ts\mathrm{Id}$ do not depend on t, this implies that the invariant space associated to the eigenvalues in $\{\operatorname{Im} \lambda < 0\}$ is constant. □

Consider the projection $\varpi : \mathbb{R}^{1+d} \mapsto \mathbb{R}^{1+d}/\mathbb{R}n = T^*\partial\Omega$ and its complex extension $\mathbb{C}^{1+d} \mapsto \mathbb{C}^{1+d}/\mathbb{C}n = \mathbb{C} \otimes T^*\partial\Omega$. The image by ϖ of $\mathbb{R}^{1+d} - i\Gamma$ is $T^*\partial\Omega - i\Gamma^\flat$ where $\Gamma^\flat = \varpi\Gamma$ is a closed convex cone in $T^*\partial\Omega\backslash\{0\}$. The invariance (2.11) legitimates the definition of $\mathbb{E}^{in}$ for frequencies for $\zeta \in T^*\partial\Omega - i\Gamma^\flat$:

$$\mathbb{E}^{in}(a,\zeta) = \mathbb{E}^{in}(a,\tilde{\xi}), \qquad \tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma, \quad \varpi\tilde{\xi} = \zeta. \tag{2.12}$$

Another important remark is that for $\alpha \in \mathbb{C}\backslash\{0\}$, $G(a,\alpha\tilde{\xi}) = \alpha G(a,\tilde{\xi})$ and therefore they have the same invariant spaces. Therefore, by continuity,

$$\mathbb{E}^{in}(a,\alpha\tilde{\xi}) = \mathbb{E}^{in}(a,\tilde{\xi}) \tag{2.13}$$

as long as $\operatorname{Im}\tilde{\xi} \in \Gamma$ and $\operatorname{Im}(\alpha\tilde{\xi}) \in \Gamma$, since the set of $\alpha \in \mathbb{C}$ such that $\operatorname{Im}(\alpha\xi) \in \Gamma$ is an open convex cone which contains 1. Introduce the open set

$$\mathscr{Z} = \{\alpha\tilde{\xi},\ \operatorname{Im}\tilde{\xi} \in -\Gamma, \alpha \in \mathbb{C}\backslash\{0\}\} \subset \mathbb{C}^{1+d}\backslash\{0\} \tag{2.14}$$

and its projection $\mathscr{Z}^\flat = \varpi\mathscr{Z} \subset \mathbb{C}^{1+d}/\mathbb{C}n \approx \mathbb{C} \otimes T^*\partial\Omega$

$$\mathscr{Z}^\flat = \{\zeta, \exists\alpha \in \mathbb{C}\backslash\{0\}\ :\ \operatorname{Im}\alpha\zeta \in -\Gamma^\flat\} \tag{2.15}$$

This set is conic and stable by multiplication by complex numbers $\neq 0$, but is *not* convex. It does not contain 0. Moreover, if $\alpha\tilde{\xi} = \beta\tilde{\eta} \in \mathscr{Z}$, with $\operatorname{Im}\tilde{\xi}$ and $\operatorname{Im}\tilde{\eta}$ in $-\Gamma$, then $\tilde{\eta} = \alpha/\beta\tilde{\xi}$ and by (2.13) , $\mathbb{E}^{in}(a,\tilde{\xi}) = \mathbb{E}^{in}(a,\tilde{\eta})$. Therefore, this legitimates the definition

$$\mathbb{E}^{in}(a,\zeta) = \mathbb{E}^{in}(a,\alpha\zeta), \qquad \operatorname{Im}\alpha\zeta \in -\Gamma^\flat \tag{2.16}$$

for $\zeta \in \mathscr{Z}^\flat$, and the property (2.13) is satisfied on $\mathscr{Z}$.

An important issue is to understand the structure of the bundle $\mathbb{E}^{in}$ in the limit $\operatorname{Im}\zeta \to 0$.

Though this is not necessary, we simplify the exposition by choosing $\mathcal{Z} \subset \mathbb{R}^{1+d}$ an hyperplane which does not contain n. We identify $\mathcal{Z}$ to $T^*\partial\Omega$ considering the projection $\tilde{\varpi}$ from $\mathbb{R}^{1+d} \to \mathcal{Z}$ which corresponds to the decomposition $\tilde{\xi} = \zeta + \xi_n n \in \mathcal{Z} \oplus \mathbb{R}n$. The complex cotangent space is identified with $\mathcal{Z}^{\mathbb{C}} = \mathcal{Z} + i\mathcal{Z}$. In these coordinates, $\Gamma^\flat = \tilde{\varpi}\Gamma \subset \mathcal{Z}$ and $\mathbb{E}^{in}(a,\zeta)$ is defined for $\zeta \in \mathcal{Z} - i\Gamma^\flat$. We denote by $\overline{\Gamma}^\flat = \Gamma^\flat \cup \{0\}$ the closure of $\Gamma^\flat$ in $\mathcal{Z}$.

Fix $\underline{\zeta} \in \mathcal{Z} - i\overline{\Gamma}$. We study the spectral decomposition of $G(a,\zeta)$ for (a,ζ) close to $(\underline{a},\underline{\zeta})$. Consider the *distinct* complex eigenvalues $\underline{\mu}_k$, $k \in$

$\{1, \dots, \underline{k}\}$ of $G(\underline{a}, \underline{\xi})$. The invariant spaces of $G(\underline{a}, \underline{\zeta})$ associated to $\underline{\mu}_k$ can be holomorphically continued on a neighborhood $\mathscr{V}$ of $(\underline{a}, \underline{\zeta})$ in $\mathcal{A} \times \mathcal{Z}^{\mathbb{C}}$. Denote by $\mathbb{E}_k(a, \zeta)$ these spaces. Taking holomorphic basis, this yields a block reduction

$$G(a, \zeta) = W^{-1}(a, \zeta)\mathrm{diag}(G_k(a, \zeta))W(a, \zeta). \tag{2.17}$$

where the W and G_k are holomorphic in ζ and the spectrum of $G_k(\underline{a}, \underline{\tilde{\xi}})$ is reduced to $\{\underline{\mu}_k\}$. If $\mathrm{Im}\,\zeta \in \Gamma^\flat$ the eigenvalues of $G_k(a, \tilde{\xi})$ are not real and the invariant subspace $\mathbb{E}^{in}_k(a, \tilde{\xi})$ associated to eigenvalues in $\mathrm{Im}\,\lambda < 0$ is well defined and holomorphic on this domain.

Case 1. If $\underline{\zeta} \in \Gamma^\flat$, none of the eigenvalues $\underline{\mu}_k$ is real and for (a, ζ) in a complex neighborhood of $(\underline{a}, \underline{\zeta})$, $\mathbb{E}^{in}(a, \zeta) = \mathbb{E}_k(a, \zeta)$ [resp. $\mathbb{E}^{in}(a, \zeta) = \{0\}$] if $\mathrm{Im}\,\underline{\mu}_k < 0$ [resp. $\mathrm{Im}\,\underline{\mu}_k < 0$]

Case 2. Suppose now that $\mathrm{Im}\,\underline{\zeta} = 0$.

Subcase 2.1. If $\underline{\mu}_k \notin \mathbb{R}$, then again there is a complex neighborhood $\mathscr{V}$ of $(\underline{a}, \underline{\zeta})$ such that for $(a, \zeta) \in \mathscr{V}$ with $\mathrm{Im}\,\zeta \in \Gamma^\flat$, $\mathbb{E}^{in}(a, \zeta) = \mathbb{E}_k(a, \zeta)$ [resp. $\mathbb{E}^{in}(a, \zeta) = \{0\}$] if $\mathrm{Im}\,\underline{\mu}_k < 0$ [resp. $\mathrm{Im}\,\underline{\mu}_k > 0$]. In particular, $\mathbb{E}^{in}$ has an holomorphic extension to $\mathscr{V}$, which is $\mathbb{E}_k$ or $\{0\}$.

Subcase 2.1. Suppose now that $\underline{\mu}_k \in \mathbb{R}$. This means that $\underline{\tilde{\xi}}^k = \underline{\zeta} - \underline{\mu}_k n$ belongs to the real characteristic variety $\mathcal{C}$. We consider first the case where it is an hyperbolic point in the sense of Definition 2.10.

Proposition 2.12. *If $(\underline{a}, \underline{\tilde{\xi}}^k) \in \mathcal{C}$ is hyperbolic incoming [resp. outgoing] in the sense of Definition 2.10, then there is a complex neighborhood $\mathscr{V}$ such that for $(a, \zeta) \in \mathscr{V}$ with $\mathrm{Im}\,\zeta \in \Gamma^\flat$, $\mathbb{E}^{in}(a, \zeta) = \mathbb{E}_k(a, \zeta)$ [resp. $\mathbb{E}^{in}(a, \zeta) = \{0\}$]. In particular, $\mathbb{E}^{in}$ has an holomorphic extension to $\mathscr{V}$, which is $\mathbb{E}_k$ or $\{0\}$.*

Proof. For (a, ζ) near $(\underline{a}, \underline{\zeta})$, the invariant space of $G(a, \zeta)$ for eigenvalues close to $\underline{\mu}_k$ is the invariant space of $G(a, \tilde{\xi})$ for eigenvalues close to zero for $(a, \tilde{\xi})$ close to $(\underline{a}, \underline{\tilde{\xi}}^k)$. Suppose that $n \in \Gamma_{\underline{a}, \underline{\tilde{\xi}}^k}$. We apply by Proposition 2.4 with Γ' a cone containing Γ and n and $\theta = n$. For $(a, \tilde{\xi})$ close to $(\underline{a}, \underline{\tilde{\xi}}^k)$ and $\tilde{\xi} \in \Gamma$, $\det G(a, \tilde{\xi} + sn) = 0$ has m_k roots near 0 and they all belong to $\{\mathrm{Im}\, s > 0\}$. Thus $\det G(a, \tilde{\xi})$ has m_k eigenvalues counted with multiplicities near 0 and they all belong to $\{\mathrm{Im}\,\mu < 0\}$. Projecting on $\mathcal{Z}$ gives the result.

If $-n \in \Gamma_{\underline{a}, \underline{\tilde{\xi}}^k}$, then the roots in $\mathrm{Im}\, s < 0$ and the eigenvalues in $\mathrm{Im}\,\mu > 0$ implying that $\mathbb{E}^{in} = \{0\}$. □

Next we consider the case where the system is analytically diagonalizable near $(\underline{a}, \underline{\tilde{\xi}}^k)$.

Proposition 2.13. *Suppose that L is analytically diagonalizable near $(\underline{a}, \underline{\tilde{\xi}}^k)$. Then there is a neighborhood $\mathscr{V}$ of $(\underline{a}, \underline{\zeta})$, such that $\mathbb{E}_k^{in}(a, \zeta)$ has a continuous extension to $\mathscr{V} \cap \mathcal{Z} - i\overline{\Gamma}^\flat$.*

Proof. It is proved in [MeZu2] (see also Remark 3.8 below)that if a matrix $\tilde{G}(p, \zeta, \gamma)$ with parameters p, frequencies $\zeta \in \mathbb{R}^d$ and $\gamma > 0$, satisfies the block structure condition, then the incoming space $\tilde{\mathbb{E}}^{in}(p, \zeta, \gamma)$ has a continuous extension to $\gamma = 0$. We apply this property to $G_k(a, \zeta - \underline{\mu}_k n - i\gamma\nu)$, with $\nu \in \Gamma$ of length 1, considering $p = (a, \nu)$ as the parameters. This implies that the limit

$$\tilde{\mathbb{E}}^{in}(a, \zeta, \nu) = \lim_{\gamma \to 0} \mathbb{E}_k^{in}(a, \zeta - \underline{\mu}_k n - i\gamma\nu)$$

exists and the convergence is locally uniform in (a, ν, ζ). It remains to show that the limit is independent of ν. This is clear from the proof in [MeZu2], since the limit is explicit in terms of $\partial_n^l e_j(\underline{a}, \underline{\tilde{\xi}}^k)$ where the $e_j(a, \tilde{\xi})$ are analytic eigenvectors of diagonalization of L. $\square$

Corollary 2.14. *If the system L belongs to the class $s\mathcal{M}$, the bundle $\mathbb{E}^{in}(a, \zeta)$ has a continuous extension to $\mathcal{A} \times (\mathcal{Z}\backslash\{0\} - i\overline{\Gamma}^\flat)$.*

2.4 The Lopatinski condition

We consider boundary conditions (2.3) satisfying Assumption 2.3. The invariant datum is the kernel of the boundary condition $\mathbb{K}(a) = \ker M(a) \subset \mathbb{E}$ with $\dim \mathbb{K} = N - N_+$. The Lopatinski determinant $D(a, \zeta)$ is the angle between $\mathbb{K}$ and $\mathbb{E}^{in}(a, \zeta)$ or

$$D(a, \tilde{\xi}) = \left| \det(\mathbb{K}(a), \mathbb{E}^{in}(a, \zeta)) \right| \tag{2.18}$$

where the determinant is computed by taking orthonormal bases in each space. $D(a, \zeta)$ does not depend on the choice of theses bases. It depends only on the choice of a scalar product on $\mathbb{E}$. The invariance property (2.13) shows that the natural domain of definition of D is $\mathscr{X}^\flat$, which is larger that than $T^*\partial\Omega - i\Gamma^\flat$. In particular, we note that, for $\alpha \in \mathbb{C}\backslash\{0\}$, if ζ and $\alpha\zeta$ both belong to $T^*\partial\Omega - i\Gamma^\flat$ then

$$D(a, \zeta) = D(a, \alpha\zeta). \tag{2.19}$$

Given a time direction $\nu \in \Gamma$, the weak Lopatinski condition is that $\mathbb{E}^{in}(a, \zeta - i\gamma\nu) \cap \mathbb{K}(a) = \{0\}$, or equivalently that $D(a, \zeta - i\gamma\nu^\flat) \neq 0$, for all $(a, \zeta) \in \mathcal{A} \times T^*\partial\Omega$ and $\gamma > 0$. The strong Lopatinski condition is that there is a constant C such that

$$\forall (a, \zeta) \in \mathcal{A} \times T^*\partial\Omega,\ \forall \gamma > 0 \forall u \in \mathbb{E}^{in}(a, \zeta - i\gamma\nu^\flat), \quad |u| \leq C|M(a)u|.$$

This is equivalent to the condition that there is a constant $c > 0$ such that

$$\forall (a, \zeta) \in \mathcal{A} \times T^*\partial\Omega,\ \forall \gamma > 0, \quad D(a, \zeta - i\gamma\nu^\flat) > c. \tag{2.20}$$

Locally there are holomorphic versions of D:

Lemma 2.15. *For all $(\underline{a}, \underline{\zeta}) \in \mathcal{A} \times \mathcal{Z}^\flat$, there are neighborhoods of $\underline{a}$ and $\underline{\zeta}$, there is a function $\ell(a, \zeta)$ continuous in a and holomorphic in ζ and there is a constant $C > 1$ such that on ω*

$$\frac{1}{C}|\ell(a, \zeta)| \leq D(a, \zeta) \leq C|\ell(a, \zeta)|. \tag{2.21}$$

Proof. One can fix an orthonormal basis $\{e_j\}$ of $\mathbb{E}^{in}(\underline{a}, \underline{\zeta})$. For (a, ζ) in a neighborhood of $(\underline{a}, \underline{\zeta})$, the image of this basis by $\Pi(a, \zeta)$ is a basis of $\mathbb{E}^{in}(a, \zeta)$. Together with a continuous basis $\{f_k\}$ of $\mathbb{K}(a)$, we can form the determinant

$$\ell(a, \zeta) = \det\big(\Pi(a, \zeta)e_1, \ldots, f_1, \ldots,\big)$$

which is holomorphic in ζ and $D(a, \zeta) = \sigma(a, \zeta)|\ell(a, \zeta)|$ where $\sigma(\underline{a}, \underline{\zeta}) = 1$. ☐

Remark 2.16. The function ℓ can be globalized using analytic continuation and the property that $T^*\partial\Omega - i\Gamma^\flat$ is contractible. However, when dealing with the *uniform* Lopatinski condition, we think that the geometric definition (2.18) is more adapted. For instance, if L is of class $\mathcal{M}$, D has a continuous extension to $T^*\partial\Omega\backslash\{0\}$, while the holomorphic version ℓ may have no.

Theorem 1.6 is a consequence of Corollary 2.14 and of the next result:

Theorem 2.17. *If the bundle $\mathbb{E}^{in}(a, \zeta)$ has a continuous extension to $\mathcal{A} \times (T^*\partial\Omega\backslash\{0\} - i\overline{\Gamma}^\flat)$ and if the uniform Lopatinski condition is satisfied in the direction $\underline{\nu}$, then for any closed subcone Γ_1 contained in the interior of Γ, there exists a constant $c > 0$ such that*

$$\forall (a, \zeta) \in \mathcal{A} \times (T^*\partial\Omega - i\Gamma_1^\flat), \quad D(a, \zeta) > c. \tag{2.22}$$

In particular, the uniform Lopatinski condition is satisfied in all direction $\nu \in \Gamma_1$.

Proof. Again, it is convenient to use a parametrization $\mathcal{Z}$ of $T^*\partial\Omega$. A consequence of the assumption is that D has a continuous extension to $\mathcal{A} \times (T^*\partial\Omega\backslash\{0\} - i\overline{\Gamma}^\flat)$ and this extension is bounded from below by c. For $a \in \mathcal{A}$, $\zeta \in \mathcal{Z}$ with $|\zeta| = 1$ and $\nu \in \Gamma_1$ with $|\nu| = 1$, consider the function

$$f_{a,\zeta,\nu}(s) = D(a, \zeta - is\nu) \tag{2.23}$$

which is defined for $\mathrm{Re}\, s \geq 0$.

a) We show that there is $R > 0$ such that for all (a, ζ, ν)

$$|s| \geq R \quad \Rightarrow \quad f_{a,\zeta,\nu}(s) \geq c/2. \tag{2.24}$$

Indeed, $|\mathrm{Im}\, \zeta/s| \leq 1/R$ and therefore if R is large $\mathrm{Im}\,(s^{-1}\zeta - i\nu) \in -\Gamma$. Then for such s, the invariance property (2.19) implies that

$$f_{a,\zeta,\nu}(s) = D(a, s^{-1}\zeta - i\nu). \tag{2.25}$$

The uniform Lopatinski condition implies that $D(a, \zeta) \geq c$ for real frequencies $\zeta \in \Gamma^\flat$. For such ζ, the invariance property (2.19), which can be extended by continuity, implies that $D(a, -i\zeta) = D(a, \zeta) \geq c$. Hence, by continuity and compactness, $D(a, \zeta - i\nu) \geq c/2$ when $\nu \in \Gamma_1^\flat$ and $|\zeta|$ is so small enough. With (2.25), this implies (2.24).

b) The assumption implies that for real frequencies with $|\zeta| = 1$, $D(a, \zeta) \geq c$. Hence, by continuity, there is $\varepsilon > 0$ such that

$$|s| \leq R, \ \mathrm{Re}\, s \leq \varepsilon, \quad \Rightarrow \quad f_{a,\zeta,\nu}(s) \geq c/2. \tag{2.26}$$

c) On the compact domain $\{|s| \leq R, \mathrm{Re}\, s \geq \varepsilon\}$, $f_{a,\zeta,\nu}$ is proportional to an holomorphic function, which is bounded from below on the boundary. Thus the number of zeros in this domain in independent of the parameters. When $\nu = \underline{\nu}$, the assumption is that there are no roots, so that $f_{a,\zeta,\nu}$ never vanishes on this domain. By compactness, it is uniformly bounded from below and the theorem follows. □

3 Tangential symmetrizers

The goal of this section is to prepare the proof Theorem 1.4 with the construction of Kreiss symmetrizers. We first review their general approach and the new piece is added at Theorem 3.12. The time direction is fixed and we use the notations $(\tau, \xi) = (\tau, \xi', \xi_d)$ of the introduction. We denote by $\zeta = (\tau, \xi')$ the tangential frequencies. We consider

$$G(a, \zeta) = A_d(a)^{-1}\big(\tau A_0(a) + \sum_{j=1}^{d-1} \xi_j A_j(a)\big). \tag{3.1}$$

The parameter a varies in $\mathcal{A}$; as in the previous section a level of smoothness with respect to a for functions or symbols is fixed, and not repeated in the statements. By homogeneity we can assume that $\zeta \in S^d_- = \{(\tau, \xi') \in \mathbb{C} \times \mathbb{R}^{d-1}, |\tau|^2 + |\xi'|^2 = 1, \operatorname{Im} \tau < 0\}$. The incoming space $\mathbb{E}^{in}(a, \zeta)$ is defined for $\operatorname{Im} \tau < 0$ and the uniform Lopatinski condition is satisfied if and only if there is a constant C_0 such that

$$\forall (a, \zeta) \in \mathcal{A} \times S^{d+},\ \forall u \in \mathbb{E}^{in}(a, \zeta) \quad |u| \le C_0 \big|M(a)u\big|. \tag{3.2}$$

Definition 3.1. *A bounded symmetrizer on $\Omega = \omega \times U \subset \mathcal{A} \times S^d_-$, is a smooth matrix $S(a, \zeta)$ on Ω, such that there are C, $c > 0$ such that for all $(a, \zeta) \in \Omega$,*

$$S(a, \zeta) = S^*(a, \zeta), \tag{3.3}$$

$$|S(a, \zeta)| \le C, \tag{3.4}$$

$$\operatorname{Im} S(a, \zeta) G(p, \zeta) \ge c |\operatorname{Im} \tau| \mathrm{Id}, \tag{3.5}$$

It is a Kreiss symmetrizer for the boundary condition M if in addition, there are positive constants C_1 and c_1 such that

$$S(a, \zeta) + C_1 M^*(a) M(a) \ge c_1 \mathrm{Id}. \tag{3.6}$$

The symmetrizer is continuous [smooth], if it extends continuously [smoothly] to $\omega \times \overline{U} \subset \omega \times \overline{S}^d_-$.

Remark 3.2. Changing the constants, one can replace (3.6) by

$$S(a, \zeta) \ge c_1 \mathrm{Id} \quad \text{on } \ker M(a). \tag{3.7}$$

Theorem 1.4 is a consequence of the following two results:

Theorem 3.3. *Under the assumptions of Theorem* 1.4*, there is a smooth Kreiss symmetrizer.*

Theorem 3.4. *If there is a smooth Kreiss symmetrizer, the maximal resolvent estimates* (1.4) *are satisfied.*

The remaining part of this section is devoted to the proof of the first theorem. The second is proved in [Kre, Maj, ChPi] when the coefficients are smooth in (t, x) and for instance in [Me5] when the coefficients are Lipschitz.

3.1 The general strategy

The holomorphic regularity in τ is forgotten. In [Kre], H.Kreiss constructs near each point $(a, \zeta) \in \mathcal{A} \times \overline{S}^d_-$, *families of symmetrizers* S^κ which are independent of the boundary conditions, such that the negative cone of S^κ is an arbitrarily small conic neighborhood of $\mathbb{E}^{in}$. Next, he uses the uniform Lopatinski condition to choose the parameter: because $\ker M$ does not intersect $\mathbb{E}^{in}$, it is contained in the positive cone of S^κ for κ large enough, implying (3.6). The construction of the S^κ is performed locally, and we sum up the main intermediate step in the following definition:

Definition 3.5. *Let $(\underline{a}, \underline{\zeta}) \in \mathcal{A} \times \overline{S}^d_-$. Consider a family of symmetrizers S^κ on $\omega^\kappa \times U^\kappa_-$ where the ω^κ are neighborhoods of of $\underline{a}$ and $U^\kappa_- = U^\kappa \cap S^d_-$ where the U^κ are neighborhoods of $\underline{\zeta}$ in S^d. It is called a K-family near $(\underline{a}, \underline{\zeta})$ if there is a space $\underline{\mathbb{E}}$ of dimension N^+ and a projector $\underline{\Pi}$ on $\underline{\mathbb{E}}$ such that for all $(a, \zeta) \in \omega^\kappa \times U^\kappa_-$ and for all κ,*

$$S^\kappa(a, \zeta) \geq m(\kappa)\underline{\Pi}'^*\underline{\Pi}' - \underline{\Pi}^*\underline{\Pi} \tag{3.8}$$

where $\underline{\Pi}' = \mathrm{Id} - \underline{\Pi}$ and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$.

Note that the constants C and c in (3.4) (3.5) may (and do in general) depend on κ.

Remark 3.6. If S is continuous at $(\underline{a}, \underline{\zeta})$, or has a continuous extension at this point when $\operatorname{Im} \underline{\tau} = 0$, shrinking the neighborhoods if necessary and changing the parameters, it is sufficient to verify (3.8) at $(\underline{a}, \underline{\zeta})$.

Remark 3.7. The choice of the projector $\underline{\Pi}$ is arbitrary, if one accepts to modify the S^κ. If $\tilde{\Pi}$ is another projector on $\underline{\mathbb{E}}$, then $\tilde{\Pi}\underline{\Pi} = \underline{\Pi}$, $\tilde{\Pi}'\underline{\Pi} = 0$ and $\tilde{\Pi}' = \tilde{\Pi}'\underline{\Pi}'$. Hence,

$$|\tilde{\Pi}'u| = |\tilde{\Pi}'\underline{\Pi}'u| \leq \tilde{C}|\underline{\Pi}'u|, \qquad |\underline{\Pi}u| \leq C(|\tilde{\Pi}'u| + |\tilde{\Pi}u|).$$

with $C = |\underline{\Pi}$ and $\tilde{C} = |\tilde{\Pi}|$. Thus,

$$m(\kappa)|\underline{\Pi}'u|^2 - |\underline{\Pi}u|^2 \geq (m(\kappa)/\tilde{C}^2 - 2C^2)|\tilde{\Pi}'u|^2 - 2C^2|\tilde{\Pi}u|^2.$$

Therefore, changing S^κ to $\tilde{S}^\kappa = \frac{1}{2}C^{-2}S^\kappa$ we see that (3.8) for S^κ and $\underline{\Pi}$ implies similar estimates for $\tilde{S}^\kappa$ and $\tilde{\Pi}$, with $\tilde{m}(\kappa) = m(\kappa)/2C^2\tilde{C}^2 - 1$. In particular, we can always choose $\underline{\Pi}$ to be the orthogonal projector on $\underline{\mathbb{E}}$ for a given scalar product in $\mathbb{C}^N$.

Remark 3.8 (see [MeZu2]). Any symmetrizer $S(a,\zeta)$ is necessarily negative definite on $\mathbb{E}^{in}(a,\zeta)$ for $\operatorname{Im}\zeta < 0$ implying that for $(a,\zeta) \in \omega^\kappa \times U^\kappa_-$ and $u \in \mathbb{E}^{in}(a,\zeta)$

$$m(\kappa)\big|\underline{\Pi}' u\big|^2 \leq \big|\underline{\Pi} u\big|^2.$$

Therefore, the space $\mathbb{E}^{in}(a,\zeta)$ has a limit as $(a,\zeta) \to (\underline{a},\underline{\zeta})$ in $\mathcal{A} \times S^d_-$ and this limit is $\underline{\mathbb{E}}$:

$$\underline{\mathbb{E}} = \lim_{(a,\zeta,\gamma)\to(\underline{a},\underline{\zeta})} \mathbb{E}^{in}(a,\zeta)$$

This shows that $\underline{\mathbb{E}}$ is unique. Denoting by $\mathbb{E}^{in}(a,\zeta)$ this limit when $\zeta \in \partial S^d_- = S^{d-1}$ is real, the same analysis shows that the family $\mathbb{E}^{in}(a,\zeta - i\gamma\nu)$ is a Cauchy sequence for the uniform convergence on $\mathcal{A} \times S^{d-1}$ implying that the following limit is uniform in $(a,\zeta) \in \mathcal{A} \times S^{d-1}$

$$\mathbb{E}^{in}(a,\zeta) = \lim_{\gamma\to 0} \mathbb{E}^{in}(a,\zeta - i\gamma\nu).$$

Lemma 3.9. *Suppose that S^κ is a K-family of symmetrizers on $\omega^\kappa \times U^\kappa_-$. Then for any boundary condition M which satisfies the uniform Lopatinski condition, S^κ is a Kreiss symmetrizer for κ large enough.*

Proof. The Lopatinski condition and Remark 3.8 imply that there is a constant C_0 such that

$$|\underline{\Pi} u| \leq C_0 |M\underline{\Pi} u| \leq C_0|Mu| + C_0|M|\,|\underline{\Pi}' u|.$$

Thus,

$$|u|^2 \leq 2|\underline{\Pi} u|^2 + 2|\underline{\Pi}' h|^2 \leq 6C_0^2|Mu|^2 + 6C_0^2|M|^2|\underline{\Pi}' u|^2 - |\underline{\Pi} u|^2.$$

and, for $m(\kappa) \geq 6C_0|M|^2$, (3.6) follows, with $C_1 = 6C_0^2$ and $c_1 = 1$. □

Proposition 3.10. *Suppose that for all $(\underline{a},\underline{\zeta}) \in \overline{\omega} \times \overline{S}^d_-$, there are neigborhoods $\omega^\kappa \times U^\kappa$ of $(\underline{a},\underline{\zeta})$ and a K-family of bounded [resp. smooth] symmetrizers $S^\kappa(p,\zeta)$ on $\omega^\kappa \times U^\kappa_-$. Then for any boundary condition M which satisfies the uniform Lopatinski condition, there is a bounded [resp. smooth] Kreiss symmetrizer for the boundary value problem (L,M).*

Proof. By Lemma 3.9, all $(\underline{a},\underline{\zeta}) \in \overline{\omega} \times \overline{S}^d_-$, has a neighborhood $\omega \times U$ such that there is a bounded [resp. smooth] symmetrizer S on $\omega \times U_-$. Therefore there is a finite covering of $\mathcal{A}\omega \times \overline{S}^d_-$ by open sets $\omega_j \times U_j$ and Kreiss symmetrizers S_j on $\omega_j \times U_{j,+}$. Consider a a partition of unity $1 = \sum \chi_j$ with χ^κ_j supported in $\omega_j \times U_j$. Then $\Sigma = \sum_j \chi_j S_j$ is a Kreiss symmetrizer, wich is bounded [resp. smooth] on $\mathcal{A} \times S^d_-$. □

3.2 Construction of K-families of symmetrizers

Let $(\underline{a}, \underline{\zeta}) \in \mathcal{A} \times \overline{S}^d_-$. To construct symmetrizers, we use the smooth diagonal block reduction 2.17 of G on a neighborhood $\omega \times U$ of $(\underline{a}, \underline{\zeta})$:

For $\operatorname{Im} \tau < 0$, we denote by $\mathbb{E}_k^{in}(a, \zeta)$ the invariant subspace of G_k associated to eigenvalues in $\{\operatorname{Im} \mu < 0\}$. Thus,

$$\mathbb{E}^{in}(a, \zeta) = W^{-1}(a, \zeta)\Big(\bigoplus_k \mathbb{E}_k^{in}(a, \zeta)\Big). \tag{3.9}$$

It is sufficient to construct K-families for each block separately:

Lemma 3.11. *Suppose that for all k, S_k^κ is a K-family of bounded [smooth] symmetrizers for G_k near $(\underline{a}, \underline{\zeta})$. There are K-families of bounded [smooth] symmetrizers S^κ for G near $(\underline{a}, \underline{\zeta})$.*

Proof. Taking finite intersection, we can find common neighborhoods $\omega^\kappa \times U^\kappa$ for the different G_k. Relabeling the families S_k^κ, we can also assume that they satisfy (3.8) with the same $m(\kappa)$ and by Remark 3.7 that the projectors $\underline{\Pi}_k$ are the orthogonal projectors on $\underline{\mathbb{E}}_k$.

Then $S^\kappa = W^* \mathrm{diag}(S_k^\kappa) W$ is a family of bounded [smooth] symmetrizers for G and for $u = W^{-1}(u_1, \ldots, u_{\underline{k}})^t$, there holds

$$(S^\kappa u, u) = \sum (S_k^\kappa u_k, u_k) \geq m(\kappa) \sum |\underline{\Pi}'_k u_k|^2 - \sum |\underline{\Pi}_k u_k|^2.$$

Let $\underline{\Pi} = W^{-1} \mathrm{diag}(\underline{\Pi}_k) W$. It is a projector on $\underline{\mathbb{E}}$ and

$$|W|^{-2}(S^\kappa u, u) \geq \tilde{m}(\kappa) |\underline{\Pi}' u|^2 - |\underline{\Pi} u|^2$$

with $\tilde{m} = m/(|W^{-1}|^2 |W|^2)$. Therefore, $|W|^{-2} S^\kappa$ is a K-family near $(\underline{a}, \underline{\zeta})$. $\square$

The construction of S_k^κ for blocks G_k is already made in several cases (see [Kre, BeSe, Me5]).

- First, when the spectrum of $G_k(\underline{a}, \underline{\zeta})$ does not intersect the real line and this is always the case when $\operatorname{Im} \underline{\tau} \neq 0$;

- When $\underline{\tau}$ is real and the spectrum of $G_k(\underline{a}, \underline{\zeta})$ contains real eigenvalues, we can split further the blocks to consider only the case where this spectrum is limited to a single eigenvalue $\underline{\mu}_k$. In this case, $\underline{\tilde{\xi}} = (\underline{\zeta}, -\underline{\mu}_k) \in \mathbb{R}^{1+d} \backslash \{0\}$ is characteristic for $L(\underline{a}, \cdot)$. If the characteristic manifold is smooth near $(\underline{a}, \underline{\tilde{\xi}})$ or more generally if L is smoothly diagonalizable near this point, then G_k satisfies the *block structure* condition and Kreiss construction applies (see [Maj, MeZu2, Me5]).

Therefore, to finish the proof of Theorem 3.3, we only have to construct K families when the block G_k is associated to an hyperbolic characteristic point $\underline{\tilde{\xi}}$ in the sense of Defintion 2.10.

3.3 Symmetrizers for hyperbolic blocks

Consider $(\underline{a}, \underline{\zeta}) \in \mathcal{A} \times \mathbb{R}^d \backslash \{0\}$ and an invariant block G_k near this point, such that the spectrum of $G_k(\underline{a}, \underline{\zeta})$ is $\{\underline{\mu}_k\}$. Denote by $\underline{\tilde{\xi}} = (\underline{\zeta}, -\underline{\mu}_k) \in \mathcal{C}_{\underline{a}}$.

Theorem 3.12. *Assume that the system L admits a smooth symmetrizer $\mathcal{S}(a, \xi)$. If $\underline{\tilde{\xi}}$ is an hyperbolic point in the sense of Definition* 2.10*, then there are families of Kreiss symmetrizers for the block G_k.*

The main part of the construction is made in the following

Lemma 3.13. *Suppose that $\underline{\tilde{\xi}}$ is incoming [resp. outgoing]. There is a smooth symmetrizer $S(a, \zeta)$ such that*

$$S = S^* \gg 0, \qquad SG_k = (SG_k)^*, \tag{3.10}$$

and

$$\operatorname{Re} SG_k(a, \underline{\nu}) \gg 0; \qquad \big[resp. \ \ \operatorname{Re} SG_k(a, \underline{\nu}) \ll 0 \ \ \big]. \tag{3.11}$$

Here the notation $S \gg 0$ means that the matrix S is positive definite.

Proof. **a)** Because n is not characteristic for the linearized symbol $L_{\underline{a}, \underline{\tilde{\xi}}}$,

$$\ker L(\underline{a}, \underline{\tilde{\xi}}) \cap \operatorname{range} L(a, n)^{-1} L(\underline{a}, \underline{\tilde{\xi}}) = \{0\}$$

implying that $\underline{\mu}_k$ is a semi simple eigenvalue of $G(\underline{a}, \underline{\zeta})$, with multplicity $m_k = \dim \ker L(\underline{a}, \underline{\tilde{\xi}})$.

By Proposition 2.4, for $\nu \in \Gamma$ of length 1 one has for a in a neighborhood of $\underline{a}$, (ζ, ξ_d) in a neighborhood of $\underline{\tilde{\xi}}$, $\gamma \geq 0$ and $\operatorname{Re} s > 0$ small :

$$(s + \gamma)|u| \leq C \big| G(a, \zeta - i\gamma\nu) + (\xi_d - is\mathrm{Id})u \big| \tag{3.12}$$

This remains true for the block G_k. Moreover, Proposition 2.4 also asserts that $G(a, \zeta - i\gamma\nu) + \xi_d \mathrm{Id}$ has m_k eigenvalues close to 0, which are real when $\gamma = 0$. They must be the eigenvalues of G_k, and therefore, for (a, ζ) in a neighborhood of $(\underline{a}, \underline{\zeta})$, $G_k(a, \zeta)$ has only real eigenvalues. The estimate (3.12) implies that they are semi simple and that the eigenprojectors are uniformly bounded.

b) The existence of a smooth symmetrizer implies that there is a smooth *full symmetrizer* $\tilde{\mathcal{S}}(a, \tilde{\xi})$ (see [FrLa1, FrLa2] [Me3]). It is a smooth matrix $\tilde{\mathcal{S}}$ such that

$$\tilde{\mathcal{S}}L = (\tilde{\mathcal{S}}L)^*, \qquad \mathrm{Re}\, \tilde{\mathcal{S}}(a, \tilde{\xi})L(a, \tilde{\nu}) \gg 0 \quad \text{on} \quad \ker L(a, \tilde{\xi}).$$

Thus $\mathscr{S}_1(a, \tilde{\xi}) = \tilde{S}(a, \xi)L(a, n)$ is a full symmetrizer for $G(a, \tilde{\xi})$ and this can be transported in the block decomposition (2.17). Therefore, for (a, ζ, ξ_d) in a neighborhood of $(\underline{a}, \underline{\tilde{\xi}})$ there is a smooth full symmetrizer for $G_k(a, \zeta) + \xi_d \mathrm{Id}$.

With a), we are now in position to apply Theorem 6.5 of [Me3] to conclude that there is a smooth symmetrizer $S_k(a, \zeta)$ for $G_k(a, \zeta)$, satisfying (3.10). Moreover, the construction in [Me3] implies that $\underline{S} = S(\underline{a}, \underline{\zeta}) = \tilde{\mathcal{S}}_1(\underline{a}, \underline{\tilde{\xi}})$.

c) It is sufficient to prove the third property(3.11) for $a = \underline{a}$. It is also proved in [Me3] that $\mathcal{S}(\underline{a}, \underline{\tilde{\xi}})$ is a Friedrichs symmetrizer for the localized operator $L_{\underline{a}, \underline{\tilde{\xi}}}$. A version of the localized operator is

$$L'(\zeta', \xi'_d) = \zeta' \cdot \nabla_\zeta G_k(\underline{a}, \underline{\zeta}) + \xi'_d \mathrm{Id}$$

and $\underline{S}_k = S_k(\underline{a}, \underline{\zeta})$ is a Friedrichs symmetrizer for L'. In particular, $\underline{S}'_k L'(\theta)$ is definite positive for all direction θ in the cone of hyperbolicity of L' containing n. In particular this is true for $\nu \in \Gamma$ in the incoming case and for $\nu \in -\Gamma$ in the outgoing case and (3.11) follows. This finishes the proof of the lemma. □

Proof of Theorem 3.12. When the mode is incoming, we choose $S^\kappa_k = -\rho S_k$ for some $\rho > 0$ such that the property (3.8) is satisfied. and $\underline{\mathbb{E}} = \mathbb{E}_k(\underline{a}, \underline{\zeta})$. 5 Because $G(a, \zeta - i\gamma\nu) = G(a, \zeta) - i\gamma\partial_\tau G(a, \zeta) + O(\gamma^2)$, we see that

$$\mathrm{Im}\, S^\kappa_k G = \gamma\rho \mathrm{Re}\, S_k \partial_\tau G + O(\gamma^2)$$

therefore the property (3.5) is satisfied if γ is small enough.

When the mode is outgoing, we choose $S^\kappa_k = \kappa S_k$ and $\underline{\mathbb{E}} = \{0\}$. Again, (3.5) is satisfied for γ small and (3.8) is satisfied. □

4 Para-differential estimates

To prove Theorem 1.8 we use different pseudo or para-differential calculi. In this section we present the technical results which will be needed. On the one hand, we consider *tangential* operators, with symbols $a(t,x,\tau,\xi')$ such as Kreiss symmetrizers. On the other hand, we deal with *spatial* operators with symbols $a(t,x,\xi',\xi_d)$ such as symmetrizers for L. Combining these two approaches is one of the major technical difficulty in the analysis of non symmetric initial boundary value problem. In this section, we gather several estimates which will be used in the proof of Therorem 5.8

4.1 Paradifferential calculi

We give here some definitions and notations and we refer for instance to Chapter 5 in [Me2] for details.

The spatial para-differential operators we consider are associated to symbols which belong to classes denoted by $\mathbf{\Gamma}_0^m$ and $\mathbf{\Gamma}_1^m$. Given an interval $I \subset \mathbb{R}$, a symbol $a(t,x,\xi)$ defined on $I \times \mathbb{R}^d \times \mathbb{R}^d$ belongs to $\mathbf{\Gamma}_0^m$ if it is C^∞ in ξ and for all $\alpha \in \mathbb{N}^d$ there is C_α such that for all ξ

$$\left\| \partial_\xi^\alpha a(\,\cdot\,,\cdot,\,\xi) \right\|_{L^\infty(I\times\mathbb{R}^d)} \le C_\alpha (1+|\xi|)^{m-|\alpha|}.$$

It belongs to $\mathbf{\Gamma}_1^m$ if in addition the first derivatives $\partial_{t,x} a$ belong to $\mathbf{\Gamma}_0^m$. Next, $\mathbf{\Sigma}_k^m$ is the set of symbols $\sigma(t,x,\xi) \in \mathbf{\Gamma}_k^m$ which satisfies the *spectral condition* that their Fourier transform with respect to the x-variables, $\hat\sigma(t,\eta,\xi)$, is supported in $\{|\eta| \le \varepsilon(1+|\xi|)$ for some $\varepsilon < 1$.

The para-differential operator T_a is by definition the pseudodifferential operator

$$T_a = \sigma_a(t,x,D_x)$$

with symbol

$$\sigma_a(t,x,\xi) = \int G(x-y,\xi) a(t,y,\xi) dy \tag{4.1}$$

and

$$G(y,\xi) = (2\pi)^{-d} \int e^{iy\eta} \chi(\eta,\xi) d\eta \tag{4.2}$$

where χ is a C^∞ function supported in $\{|\eta| \le \varepsilon(1+|\xi|)$, equal to 1 on $\{|\eta| \le \varepsilon_1(1+|\xi|)$, for some $0 < \varepsilon_1 < \varepsilon < 1$ and such that

$$\left| \partial_\eta^\beta \partial_\xi^\alpha \chi(\eta,\xi) \right| \le C_\alpha (1+|\xi|)^{-|\alpha|-|\beta|}.$$

The symbol σ_a and the quantization T_a depend on the choice of the cut-off function χ, but if χ_1 and χ_2 satisfy the spectral condition, the difference between the two symbols σ_a^1 and σ_a^2 belong to Σ_0^{m-1} if $a \in \mathbf{\Gamma}_0^m$ so that the two operators of order m T_a^1 and T_a^2 differ by an operator of order $m-1$ (see [Me2]). All the results below do not depend on the choice of the cutoff function χ.

The tangential quantization is defined similarly, permuting the role of t and x_d. Tangential symbols are functions of (t, x, ζ) where $\zeta = (\tau, \xi')$ denote the tangential frequencies. Using the notation $z = (t, x')$ for the tangential variables, we see tangential symbols as functions of (t, x, ζ) or (x_d, z, ζ). To avoid confusion, when necessary, we will note T^{tg} the corresponding quantization.

Remark 4.1. The Kreiss symmetrizers are associated to the operators $L((t, x, \partial_T + \gamma, D_x)$ which depend on the parameter γ. They are tangential pseudo or para-differential operators and their symbols are functions of $(t, x, \tau, \xi', \gamma)$. The proof of the energy estimates (1.4) called resolvent estimates in the introduction relies on a pseudo or para-differential calculus *with parameter* γ, see [Kre, ChPi, MeZu1]. We do not give details here, as we do not use this calculus.

4.2 A microlocal Cauchy problem

We first give a para-differential version of the classical symmetrizable hyperbolic Cauchy problem.

Proposition 4.2. *Consider a matrix of symbols $G \in \mathbf{\Gamma}_1^1$. Assume that there is a matrix $S \in \mathbf{\Gamma}_1^0$ such that $S = S^*$ is uniformly definite positive and $SG = (SG)^*$. Then, for $u_0 \in L^2$ and $f \in L^2([0,T] \times \mathbb{R}^d)$ the Cauchy problem*

$$\partial_t u + iT_G u = f, \qquad u_{|t=0} = u_0 \tag{4.3}$$

has a solution $u \in C^0([0,T]; L^2(\mathbb{R}^d))$

Sketch of proof. First, we modify the symbol S into

$$\tilde{S}(t, x, \xi) = S(t, x, \xi) + \lambda (1 + |\xi|^2)^{-1} \in \mathbf{\Gamma}_1^0 \tag{4.4}$$

with λ large enough so that the operator $\mathbf{S} = \operatorname{Re} T_{\tilde{S}}$ is definite positive in L^2. Considering the energy $\big(\mathbf{S}u, u\big)_{L^2}$ and computing its times derivative, the symbolic calculus implies the following estimate (see e.g. Theorem 7.1.3 and Chapter 7 in [Me2]) :

Lemma 4.3. *There is a constant C such that for all $u \in C^1([0,T]; H^1(\mathbb{R}^d))$ one has*

$$
\begin{aligned}
\|u(t)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|u(0)\|^2_{L^2(\mathbb{R}^d)} &+ \int_0^t \|u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \\
&+ \int_0^t \mathrm{Re}\, \big(\mathbf{S}(\partial_t u + iT_G u)(t'), u(t')\big)_{L^2(\mathbb{R}^d)} dt'.
\end{aligned} \tag{4.5}
$$

The adjoint of T_G is $T_{G^*} + R$ where $R(t)$ is bounded in L^2, uniformly in t. The symbol $(S^*)^{-1}$ is a symmetrizer for G^*, and therefore there are similar estimates for the backward Cauchy problem for $-i(T_G)^*$. By duality, this implies the existence of a solution $u \in L^2([0,T] \times \mathbb{R}^d)$ of (4.3). A variant of Friedrichs' lemma, still using the symbolic para-differential calculus, implies that this solution is strong, thus belongs to $C^0([0,T]; L^2(\mathbb{R}^d))$ and satisfies (4.5). □

4.3 Estimates of traces

Operators of the form

$$
P_a u = (T_a u)_{|x_d = 0} \tag{4.6}
$$

will occur in the analysis.

For fixed t, T_a is bounded from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ when $a \in \mathbf{\Gamma}_0^{-1}$ and from $H^1(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ when $a \in \mathbf{\Gamma}_0^0$. Hence,

Lemma 4.4. *i) If $a \in \mathbf{\Gamma}_0^{-1}$, then P_a is bounded from $L^2([0,T] \times \mathbb{R}^d)$ to $L^2([0,T]; H^{\frac{1}{2}}(\mathbb{R}^{d-1}))$.*

ii) If $a \in \mathbf{\Gamma}_0^0$, P_a is bounded from $L^2([0,T]; H^1(\mathbb{R}^d))$ to $L^2([0,T]; H^{\frac{1}{2}}(\mathbb{R}^{d-1}))$ thus to $L^2([0,T] \times \mathbb{R}^{d-1})$.

When a is of degree 0 conditions must be imposed to be able to define the trace of $T_a u$ when $u \in L^2$. We will assume that

$$
a\big(t, x, (0, \dots, 0, \xi_d)\big) = 0. \tag{4.7}
$$

The next proposition states that under this condition, the trace is well defined when $u \in L^2$. The idea is to replace a by $a^0 = a(t, x', 0, \xi)$, that is to freeze the value of x_d at $x_d = 0$, and next to replace the quantization T_a by $T_a^0 = \sigma_a^0(t, x', D_x)$ where

$$
\sigma_a^0(t, x', \xi) = \int G^0(x' - y', \xi')) a(t, y', 0, \xi) dy' \tag{4.8}
$$

with the modified mollification kernel

$$G^0(y',\xi') = (2\pi)^{-d} \int e^{iy'\eta'} \chi\big((\eta',0),(\xi',0)\big) d\eta'. \tag{4.9}$$

It acts only in the variables x' and on the frequency side, the cut off is made at $|\eta'| \le \varepsilon(1+|\xi'|)$, independently of ξ_d. We prove that for $u \in L^2$ the trace $P_a^0 u = (T_a^0 u)_{|x_d=0}$ is well defined and that $P_a - P_a^0$ is a bounded operator in L^2.

Proposition 4.5. *If $a \in \mathbf{\Gamma}_1^0$ satisfies (4.7) then P_a is bounded from L^2 to $L^2([0,T], H^{-\frac{1}{2}}(\mathbb{R}^{d-1}))$. Moreover, $P_a - P_a^0$ is bounded from L^2 to L^2.*

The remaining part of the Section 4.3 is mainly devoted to the proof of this proposition. To simplify the exposition, we delete t from the notations below since it appears as a parameter and the L^2 integrability in time over $[0,T]$ follows from the uniformity of the estimates at each fixed t.

First, we note that if $a \in \mathbf{\Gamma}_1^0$ satisfies (4.7) there are symbols $a_j \in \mathbf{\Gamma}_1^{-1}$ such that

$$a(x,\xi) = \sum_{j=1}^{d-1} a_j(x,\xi)\xi_j + a_0(x,\xi). \tag{4.10}$$

Indeed, we can take $a_0 = a$, $a_j = 0$ in the domain $\{|\xi| \le 1\}$; in the domain $\{|\xi'| \le 2|\xi_d|\}$ this is a consequence of a Taylor expansion in ξ' and in the domain $\{|\xi'| \ge |\xi_d|\}$ (4.10) is true with $a_j = a\xi_j/(1+|\xi'|^2)$, $a_0 = a/(1+|\xi'|^2)$. Next one can glue the different pieces by a partition of unity.

Using Lemma 4.4 and that $T_{a_j\xi_j} - \frac{1}{i}\partial_{x_j} T_{a_j}$ is bounded from L^2 to H^1, we see that for $u \in L^2$ the trace of $T_a u$ belongs to $H^{-\frac{1}{2}}$ and the first statement of the proposition is proved.

Next, we compare P_a and P_a^0. First, we note that

$$a(x,\xi) = a^0(x',\xi) + x_d b(x,\xi), \qquad b \in \mathbf{\Gamma}_0^0. \tag{4.11}$$

Lemma 4.6. *If $b \in \mathbf{\Gamma}_0^0$, then $T_{x_d b} - x_d T_b$ is bounded from L^2 to H^1. In particular, the trace operator $P_{x_d b}$ is bounded from L^2 to L^2.*

Proof. From (4.1) we see that $\rho = \sigma_{x_d b} - x_d \sigma_b$ is given by

$$\rho(t,x,\xi) = \int (x_d - y_d) G(x-y,\xi) b(y,\xi) dy.$$

Note that

$$y_d G(y,\xi) = i \int e^{iy\eta} \partial_{\eta_d} \chi(\eta,\xi) d\eta$$

and $\partial_{\eta_d}\chi$ is a symbol of degree -1 supported in a cone $|\eta| \le \varepsilon(1+|\xi)$. Therefore ρ satisfies the spectral condition and the estimates of $\mathbf{\Gamma}_0^{-1}$. Therefore $\rho \in \mathbf{\Sigma}_0^{-1}$ and $\rho(x, D_x)$ is bounded from L^2 to H^1. □

Hence, to prove the Proposition, it remains to study $P_a^0 - P_{a^0}$. We first compare the two symbols σ_a^0 and σ_{a^0}.

Lemma 4.7. *Suppose that $a = b\xi_j$ with $b \in \mathbf{\Gamma}_1^{-1}$ and $1 \le j \le d-1$. Then the symbol $\rho(x', \xi) = \sigma_a^0 - \sigma_{a^0}$ satisfies for $|\alpha| \le 1$ and all β,*

$$\left|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\rho(x', \xi)\right| \lesssim (1+|\xi|)^{-1+|\alpha|}(1+|\xi'|)^{-|\beta|}. \tag{4.12}$$

Proof. Because a^0 does not depend on x_d,

$$\begin{aligned}\sigma_{a^0}(x, \xi) &= \int G(x'-y', x_d - y_d, \xi)a(y', 0, \xi)dy \\ &= \int G^1(x'-y', \xi)a(y', 0, \xi)dy = \sigma_{a^0}(x', 0, \xi)\end{aligned}$$

where

$$G^1(y', \xi) = (2\pi)^{-d}\int e^{iy'\eta'}\chi(\eta', 0, \xi)d\eta'.$$

Therefore,

$$\rho(x', \xi) = \xi_j \int H(x'-y', \xi)b(y', 0, \xi)dy',$$

where

$$H(y', \xi) = (2\pi)^{-d}\int e^{iy'\eta'}\theta(\eta', \xi)d\eta', \quad \theta(\eta', \xi) = \chi(\eta', 0, \xi) - \chi(\eta', 0, \xi', 0).$$

The cut off function θ is supported in $\{\varepsilon_1(1+|\xi'|) \le |\eta'| \le \varepsilon(1+|\xi|)\}$.

For all fixed ξ_d, consider $p_{\xi_d}(x', \xi') = (1+|\xi|^2)^{\frac{1}{2}}b(x', \xi', \xi_d)$ as a symbol in (x', ξ'). They form a uniformly bounded family in $\mathbf{\Gamma}_1^0(\mathbb{R}^{d-1})$. Let

$$q_{x_d}(x', \xi) = \xi_j \int H(x'-y', \xi)p_{x_d}(y', \xi)dy'.$$

They are bouned in in $\mathbf{\Gamma}_1^0(\mathbb{R}^{d-1})$. Moreover, since θ is supported in $\{|\eta'| \ge \varepsilon_1(1+|\xi'|)\}$, the support of their Fourier transform in x' is contained in this set and by Bernstein inequality

$$\|q_{\xi_d}(\cdot, \xi')\|_{L^\infty} \lesssim (1+|\xi'|)^{-1}\|\nabla_{x'}q_{\xi_d}(\cdot, \xi')\|_{L^\infty}$$

implying that they are bounded in $\mathbf{\Gamma}_0^{-1}(\mathbb{R}^{d-1})$. Since $\rho(x',\xi',\xi_d) = (1+|\xi|^2)^{-1} q_{\xi_d}(x',\xi')$, we conclude that

$$|\partial^\beta_{\xi'}\rho(x',\xi)| \lesssim (1+|\xi|)^{-1}(1+|\xi'|)^{-|\beta|}.$$

Since θ is supported in $\{|\eta'| \leq \varepsilon(1+|\xi|)\}$ this implies bounds for the x' derivative and the lemma is proved. □

Combining (4.10) and the lemmas above, the next result finishes the proof of Proposition 4.5.

Lemma 4.8. *Suppose that $\rho(x',\xi)$ satisfies (4.12). Then the mapping $u \mapsto v = \big(\rho(x',D_x)u\big)_{|x_d=0}$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{d-1})$.*

Proof. Denote by $\tilde{u}(x',\xi_d)$ the partial Fourier transform of u with respect to the variable x_d. Then

$$v(x') = \int_{\mathbb{R}} \rho(x',D_{x'},\xi_d)\tilde{u}(x',\xi_d)d\xi_d. \tag{4.13}$$

Next use a dyadic partition of unity in the ξ' variables, which yields a decomposition

$$\tilde{u}(x',\xi_d) = \sum \tilde{u}_k(\xi',\xi_d) \tag{4.14}$$

so that

$$v = \sum v_k, \qquad v_k(x') = \int_{\mathbb{R}} w_k(x',\xi_d)d\xi_d$$

with

$$w_k(x',\xi_d) = \rho(x',D'_x,\xi_d)\tilde{u}_k(x',\xi_d) = \rho_k(x',D'_x,\xi_d)\tilde{u}_k(x',\xi_d)$$

where the ρ_k are localized in $|\xi'| \approx 2^k$ and satisfy, uniformly on k, for all $|\alpha| \leq 1$ and all β:

$$\big|\partial^\alpha_{x'}\partial^\beta_{\xi'}\rho_k(x',\xi)\big| \lesssim (|\xi_d| + 2^k)^{|\alpha|-1}2^{-k|\beta|}. \tag{4.15}$$

In particular, the symbols $p_{k,\xi_d}(x',\xi') := (|\xi_d| + 2^k)\rho_k(x',\xi',\xi_d)$ satisfy for all β

$$\big|\partial^\beta_{\xi'}p_{k,\xi_d}(x',\xi)\big| \leq 2^{-k|\beta|}.$$

uniformly in k and ξ_d. Moreover, they are supported in $\{|\xi'| \approx 2^k\}$. Hence the operators $p_{k,\xi_d}(x',D_{x'})$ are uniformly bounded in $L^2(\mathbb{R}^{d-1})$ (see e.g. Lemma 4.3.3 in [Me2]). Therefore

$$\big\|w_k(\cdot,\xi_d)\big\|_{L^2(\mathbb{R}^{d-1})} \lesssim (2^k + |\xi_d|)^{-1}\varepsilon_k(\xi_d) \tag{4.16}$$

with $\varepsilon_k(\xi_d) = \|\tilde{u}_k(\cdot,\xi_d)\|_{L^2}$ Similarly, applying the same estimates with ρ replaced by $(i\xi_j + \partial_{x_j})\rho$, implies that

$$\big\|w_k(\cdot,\xi_d)\big\|_{H^1(\mathbb{R}^{d-1})} \lesssim \varepsilon_k(\xi_d). \tag{4.17}$$

Consider dyadic partition of unity $Q_j(D_{x'})$ so that

$$v = \sum_j Q_j v, \qquad Q_j v = \sum_k \int Q_j w_k(x',\xi_d)dx_d.$$

Then

$$\big\|Q_j w_k(\cdot,\xi_d)\big\|_{L^2} \lesssim \min\Big(\big\|w_k(\cdot,\xi_d)\big\|_{L^2}, 2^{-j}\big\|w_k(\cdot,\xi_d)\big\|_{H^1(\mathbb{R}^{d-1})}\Big)$$

so that

$$\big\|Q_j w_k(\cdot,\xi_d)\big\|_{L^2} \lesssim (2^j + 2^k + |\xi_d|)^{-1}\varepsilon_k(\xi_d).$$

Hence

$$\|Q_j v\|_{L^2} \leq \sum_k \int \|Q_j w_k(\cdot,\xi_d)\|_{L^2} d\xi_d \leq \sum_k \int \frac{\varepsilon_k(\xi_d)d\xi_d}{2^j + 2^k + |\xi_d|}$$

and

$$\begin{aligned}\|Q_j v\|_{L^2}^2 &\leq \left(\sum_k \int \varepsilon_k^2(\xi_d)d\xi_d\right)\left(\sum_k \int \frac{d\xi_d}{(2^j + 2^k + |\xi_d|)^2}\right)\\ &\lesssim \big\|\tilde{u}\big\|_{L^2}^2 \sum_k (2^j + 2^k)^{-1} \lesssim j2^{-j}\big\|\tilde{u}\big\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\big\|v\big\|_{L^2}^2 \lesssim \sum_j \big\|Q_j v\big\|_{L^2}^2 \lesssim \big\|\tilde{u}\big\|_{L^2}^2 = \big\|u\big\|_{L^2}^2$$

and the lemma is proved. □

We end this section with a lemma which we will need later on.

Lemma 4.9. *If $a \in \mathbf{\Gamma}_1^0$ satisfies (4.7) then P_a^0, and hence P_a, are bounded from L^2 to $H^{-\frac{1}{2}}$ and from $\langle D_{x'}\rangle^{-\frac{1}{2}}L^2$ to L^2. In particular, there is a constant C such that for all $u \in \langle D_{x'}\rangle^{-\frac{1}{2}}L^2(\mathbb{R}^d)$;*

$$\big\|P_a^0 u\big\|_{L^2(\mathbb{R}^{d-1})} \leq C\big\|\langle D_{x'}\rangle^{\frac{1}{2}}u\big\|_{L^2(\mathbb{R}^d)}. \tag{4.18}$$

Here we have used the notation $\langle D_{x'}\rangle = (1-\Delta_{x'})^{\frac{1}{2}}$ which is associated to the symbol $\langle \xi'\rangle = (1+|\xi'|^2)^{\frac{1}{2}}$.

Proof. Because of (4.10), the symbol σ_a^0 satisfies

$$\left|\partial_{\xi'}^{\beta}\sigma_a^0(x',\xi',\xi_d)\right| \lesssim |\xi'|(\langle\xi'\rangle + |\xi_d|)^{-1}\langle\xi'\rangle^{-|\beta|}.$$

It also satisfies the para-differential spectral localization in the tangent variables x'. This implies that $\sigma_a^0(\cdot,\cdot,\xi_d)$ and $\xi_d\sigma_a^0(\cdot,\cdot,\xi_d)$ are bounded families of para-differential symbols in the class $\mathbf{\Sigma}_1^0$ and $\mathbf{\Sigma}_1^1$ respectively on $\mathbb{R}^{d-1}$. Therefore for all v in $L^2(\mathbb{R}^{d-1})$ and all $\xi_d \in \mathbb{R}$,

$$\begin{aligned}&\left\|\sigma_a^0(x',D_x',\xi_d)v\right\|_{L^2(\mathbb{R}^{d-1})} \lesssim \left\|v\right\|_{L^2(\mathbb{R}^{d-1})},\\ &|\xi_d|\left\|\sigma_a^0(x',D_x',\xi_d)v\right\|_{H^{-1}(\mathbb{R}^{d-1})} \lesssim \left\|v\right\|_{L^2(\mathbb{R}^{d-1})}.\end{aligned} \tag{4.19}$$

Introduce a dyadic partition of unity in $\mathbb{R}^{d-1}$ so that

$$\tilde{u}(x',\xi_d) = \sum_j Q_j(D_{x'})\tilde{u}(x',\xi_d)$$

where $\tilde{u}$ denotes the partial Fourier transform of u in the variable x_d. The spectrum in ξ' of $Q_j\tilde{u}$ is contained in $|\xi'| \approx 2^j$. Then $f_j = \sigma_a^0(x',D_x',\xi_d)Q_j\tilde{u}$ has also its spectrum in a larger annulus but still of order $|\xi'| \approx 2^j$. The estimates (4.19) imply that

$$\left\|f_j(\cdot,\xi_d)\right\|_{L^2} \lesssim (1+2^{-j}|\xi_d|)^{-1}\varepsilon_j(\xi_d)$$

where $\varepsilon_j(\xi_d) = \|Q_j\tilde{u}(\cdot,\xi_d)\|_{L^2}$. Hence

$$v = P_a^0 u = \sum_j v_j, \qquad v_j(x') = \int f_j(x',\xi_d)d\xi_d$$

and

$$\left\|v_j\right\|_{L^2} \lesssim \int (1+2^{-j}|\xi_d|)^{-1}\varepsilon_j(\xi_d)d\xi_d \lesssim 2^{j/2}\|\varepsilon_j\|_{L^2(\mathbb{R})}.$$

Because the v_j are spectrally supported in annuli $|\xi'| \approx 2^j$ one has

$$\|v\|_{L^2}^2 \lesssim \sum \left\|v_j\right\|_{L^2}^2 \lesssim \sum_j 2^j\|\varepsilon_j\|_{L^2(\mathbb{R})}^2 \approx \|\langle D_x'\rangle^{\frac{1}{2}}w\|_{L^2(\mathbb{R}^d)}^2$$

and the lemma is proved. □

4.4 Energy balance for the IBVP

Consider a system $L = \partial_t + \sum A_j \partial_{x_j}$ with Lipschitz coefficients. We assume that it admits a symmetrizer $S(t, x, \xi)$, which is Lipschitz continuous in (t, x), homogeneous of degree 0 and C^∞ in $\xi \neq 0$.

For such L, the Cauchy problem is well posed in L^2, see e.g.Theorem 7.1.3 in [Me2] (see also Proposition 4.2 above and its proof). Following ideas from [FrLa2], we want to obtain an inequality similar to the energy estimate of the Cauchy problem, using the same symmetrizer, but now on the domain $\{x_d \geq 0\}$. Non local boundary terms occur, but in contrast with [FrLa2], we analyze them *assuming that we already have a control of the traces* $\|u_{|x_d=0}\|_{L^2}$, for instance given by a preliminary use of Kreiss symmetrizers.

The L^2 estimate for the Cauchy problem is proved using the energy $(\mathbf{S}u, u)_{L^2}$, where $\mathbf{S} = \operatorname{Re} T_{\widetilde{S}}$ and $\widetilde{S}$ a low frequency modification of the symbol S :

$$\tilde{S}(t, x, \xi) = \theta(\xi) S(t, x, \xi) + \lambda (1 + |\xi|^2)^{-1} \in \mathbf{\Gamma}^0_1, \tag{4.20}$$

with $1 - \theta$ compactly supported and $\theta = 0$ near the origin, and λ large enough so that the operator $\mathbf{S} = \operatorname{Re} T_{\tilde{S}}$ is definite positive in L^2. Using the approximation $\|Lu - (\partial_t + iT_A)u\|_{L^2} \lesssim \|u\|_{L^2}$ where A denotes here the symbol $\sum \xi_j A_j(t, x)$ (Theorem 5.2.9 in [Me2]), and the symbolic calculus as recalled in Proposition 4.2, one obtains that for $u \in C^1([0, T]; H^1(\mathbb{R}^d))$ one has

$$\big\|u(t)\big\|^2_{L^2(\mathbb{R}^d)} \lesssim \big\|u(0)\big\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \operatorname{Re}\big(\mathbf{S}Lu(t'), u(t')\big)_{L^2(\mathbb{R}^d)} dt'. \tag{4.21}$$

In the remaining part of this section we analyze how this estimate is modified when it is applied on the half space $\{x_d \geq 0\}$.

Note that there is no restriction in assuming that the symbol of the symmetrizer is even in ξ. An important element is the value of S on the conormal to the boundary and we introduce

$$S_\infty(t, x) = S(t, x, (0, \ldots, 0, 1)). \tag{4.22}$$

Proposition 4.10. *Suppose that $L = \partial_t + \sum A_j \partial_{x_j}$ has Lipschitz coefficients on $[0, T] \times \mathbb{R}^d$ and admits a symmetrizer $S(t, x, \xi)$, which is Lipschitz continuous in (t, x), homogeneous of degree 0, C^∞ and even in $\xi \neq 0$. Then, there is a constant C such that for $u \in C^\infty_0([0, T] \times \overline{\mathbb{R}}^d_+)$ one has*

$$\begin{aligned}\big\|u(t)\big\|^2_{L^2(\mathbb{R}^d_+)} \lesssim & \big\|u(0)\big\|^2_{L^2(\mathbb{R}^d_+)} + \big\|u\big\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} + \big\|Lu\big\|^2_{L^2([0,t]\times\mathbb{R}^d_+)} \\ & + \big\|u_{|x_d=0}\big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \big\|P_{S_1}\tilde{u}\big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})}\end{aligned} \tag{4.23}$$

where P_{S_1} is the trace operator associated to the symbol $S_1 = \theta(S - S_\infty)$ and $\tilde u$ denotes the extension of u by 0 for $x_d < 0$.

Proof. Consider $u \in C_0^\infty([0,T] \times \overline{\mathbb{R}}^d_+)$ and introduce $f = Lu \in C_0^\infty([0,T] \times \overline{\mathbb{R}}^d_+)$ and $g = u_{|x_d=0} \in C_0^\infty([0,T] \times \mathbb{R}^{d-1})$. Let $\chi \in C^\infty(\mathbb{R})$ with support in $]0,\infty[$ and equal to 1 on $[1,\infty[$. Let $\chi_\varepsilon(x_d) = \chi(x_d/\varepsilon)$ and $u_\varepsilon = \chi_\varepsilon u$. Then

$$Lu_\varepsilon = f_\varepsilon + f^1_\varepsilon, \qquad f_\varepsilon = \chi_\varepsilon f, \qquad f^1_\varepsilon = \chi'_\varepsilon A_d u = \varepsilon^{-1}\chi'(x_d/\varepsilon) A_d u.$$

We apply the energy estimate (4.21) to u_ε and pass to the limit. The difficulty is concentrated in the term

$$I_\varepsilon = I_{\mathbf{S}}(f^1_\varepsilon, u_\varepsilon) = \int_0^t \big(\mathbf{S} f^1_\varepsilon(t'), u_\varepsilon(t')\big)_{L^2(\mathbb{R}^d)} dt'. \tag{4.24}$$

The proposition will follow from the estimate

$$\begin{aligned}\limsup_{\varepsilon\to 0} |I_\varepsilon| \lesssim \big\|u\big\|^2_{L^2([0,t]\times\mathbb{R}^d)} + \big\|f\big\|^2_{L^2([0,t]\times\mathbb{R}^d)} \\ + \big\|g\big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})} + \big\|P^0_{S_1}\tilde u\big\|^2_{L^2([0,t]\times\mathbb{R}^{d-1})}.\end{aligned} \tag{4.25}$$

a) For u smooth, $f^1_\varepsilon = g_1\chi'_\varepsilon + h^1_\varepsilon$ with $g_1 = A_d u_{|x=0}$ and $\|h^1_\varepsilon\|_{L^2} \lesssim \varepsilon^{\frac12}(\|u(t)\|_{L^2} + \|\partial_x u(t)\|_{L^2})$. Therefore is sufficient to prove (4.25) for $I_S(h_\varepsilon, u_\varepsilon)$ with $h_\varepsilon := g_1\chi'_\varepsilon$.

b) The spatial Fourier transform of h_ε is $\widehat h_\varepsilon(t) = \widehat{\chi}'(\varepsilon\xi_d)\widehat g_1(t,\xi')$. Since $\chi' \in \mathcal{S}(\mathbb{R})$,

$$\big\|h_\varepsilon\big\|_{L^2([0,T];H^{-1}(\mathbb{R}^d))} \lesssim \|g\|_{L^2}. \tag{4.26}$$

Recalling the definition (4.20), note that $T_{\theta S} - (T_{\theta S})^*$ and hence $\mathbf{S} - (T_{\theta S})^*$ are of degree -1. This implies that

$$\big\|\mathbf{S} h_\varepsilon - T^*_{\theta S} h_\varepsilon\big\|_{L^2} \lesssim \big\|g\big\|_{L^2}.$$

Therefore we are reduced to prove (4.25) with

$$I_\varepsilon = \tilde I_S(h_\varepsilon, u_\varepsilon) = \int_0^t \big(h_\varepsilon, T_{\theta S} u_\varepsilon\big)_{L^2(\mathbb{R}^d)} dt'. \tag{4.27}$$

With S_∞ defined by (4.22), we split S into $S_\infty + (S - S_\infty)$ and we study each term separately.

c) The operator $T_{\theta S_\infty} - S_\infty$ is of degree -1, implying that $T_{\theta S_\infty} u_\varepsilon - S_\infty u_\varepsilon$ is bounded in $L^2([0,T];H^1)$ by $\|u\|_{L^2}$. Therefore

$$\tilde{I}_{S_\infty}(h_\varepsilon, u_\varepsilon) = \int_0^t \big(g_1\chi'_\varepsilon, S_\infty \chi_\varepsilon u\big) dt' + O\big(\|g\|_{L^2}\|u\|_{L^2}\big)$$

For u smooth,

$$\lim_{\varepsilon\to 0} \int_0^t \big(g_1\chi'_\varepsilon, S_\infty \chi_\varepsilon u\big) dt' = \frac{1}{2}\big(g_1, S_{\infty|x_d=0} g\big)_{L^2([0,T]\times\mathbb{R}^{d-1})}$$

and therefore

$$\limsup_{\varepsilon\to 0} \tilde{I}_{S_\infty}(h_\varepsilon, u_\varepsilon) \lesssim \|g\|^2 + \|u\|^2. \tag{4.28}$$

d) We now show, using the notation (4.6), that

$$\lim_{\varepsilon\to 0} \tilde{I}_{S-S_\infty}(h_\varepsilon, u_\varepsilon) = \big(g_1, P_{S_1}\tilde{u}\big)_{L^2([0,T]\times\mathbb{R}^{d-1})} \tag{4.29}$$

and this will finish the proof of the proposition.

Using that S is homogeneous of degree 0 and even, we can write

$$S_1(t,x,\xi) = \theta(\xi)(S - S_\infty)(t,x,\xi) = \sum_{j=1}^{d-1} S_{1,j}(t,x,\xi)\xi_j \tag{4.30}$$

where the $S_{1,j}$ are of degree -1. Hence,

$$\big\|T_{S_1}\chi_\varepsilon u\big\|_{L^2([0,T],H^1)} \lesssim \sum_{j=1}^{d-1} \big\|\partial_{x_j} u\big\|_{L^2}.$$

Moreover, $T_{S_1}\chi_\varepsilon u \to T_{S_1}\tilde{u}$ in $L^2([0,T],H^1)$ and the trace on $\{x_d = 0\}$ is well defined. Using (4.26) and the convergence $g_1\chi'_\varepsilon \to g_1 \otimes \delta_{|x_d=0}$ in $L^2([0,T],H^{-1})$, this implies that

$$\big(g_1\chi'_\varepsilon, T_{S_1}u_\varepsilon\big) \to \big(g_1, (T_{S_1}\tilde{u})_{|x_d=0}\big)$$

that is (4.30) and the proposition is proved. □

The main difficulty is now to estimate the trace $P_{S_1}\tilde{u}$ in L^2. By Proposition 4.5 and Lemma 4.9, we only know that

$$\big\|P_{S_1}\tilde{u}\big\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \le C\big\|\langle D_{x'}\rangle^{\frac{1}{2}} u\big\|_{L^2([0,T]\times\mathbb{R}^d_+)}. \tag{4.31}$$

The idea which is also taken from [FrLa2] is the following: the definition of P_{S_1} involves taking a trace on $x_d = 0$, which on the Fourier side means an integration in ξ_d. The idea is to shift the integration path to the complex domain, using the holomorphy of the Fourier transform of $\tilde u$ in $\operatorname{Im}\xi_d < 0$ and an *assumed holomorphy* in ξ_d of the symbol.

Proposition 4.11. *Suppose that $S(t,x,\xi)$ admits a bounded holomorphic extension in the cone $\{|\operatorname{Im}\xi| \le \delta|\xi|\}$ for some $\delta > 0$. Then, there are constant C and $\delta_1 > 0$ such that for all $\delta' \in [0,\delta_1]$ and all $u \in C_0^\infty([0,T] \times \overline{\mathbb{R}}^d_+)$*

$$\begin{aligned}\big\|P_{S_1}\tilde u\big\|_{L^2([0,T]\times\mathbb{R}^{d-1})} \le\\ C\Big(\big\|\langle D_{x'}\rangle^{\frac12} e^{-\delta' x_d\langle D_{x'}\rangle} u\big\|_{L^2([0,T]\times\mathbb{R}^d_+)} + \big\|u\big\|_{L^2([0,T]\times\mathbb{R}^d_+)}\Big).\end{aligned} \tag{4.32}$$

Proof. To simplify notations, we omit the variable t which is just a parameter. Note that when $\delta' = 0$, (4.32) is simply (4.31).

Introduce

$$S_2 = \theta(\xi',0)\big(S - S_\infty\big)$$

Since $1-\theta$ is compactly supported, say in $\{|\xi| \le R\}$, we note that $\theta(\xi) - \theta(\xi',0) = 0$ when $|\xi'| \ge R$. Hence $S_1 - S_2$ is supported in $|\xi'| \le R$ and thus of order -1 by (4.30). Therefore it is sufficient to prove the estimate for P_{S_2}. According to Proposition 4.5, we can also replace P_{S_2} by $P^0_{S_2}$ and $v = P^0_{S_2}\tilde u$ is given by

$$v(x') = \int e^{ix'\cdot\xi'}\sigma(x',\xi',\xi_d)\hat u(\xi)d\xi$$

where $\hat u$ is the spatial Fourier transform of $\tilde u$ and

$$\sigma(x',\xi',\xi_d) = \int G^0(x'-y',\xi')S_2(y',0,\xi',\xi_d)dy'.$$

By assumption, $S - S_\infty$, and thus S_2, have holomorphic extensions in ξ_d to the domain $\{|\operatorname{Im}\xi_d| \le \delta|\xi|\}$ and this extension vanishes when $\xi' = 0$. Hence, by homogeneity,

$$\big|(S - S_\infty)(x,\xi)\big| \lesssim \frac{|\xi'|}{|\xi|}. \tag{4.33}$$

Therefore since σ vanishes for ξ' small, σ, has an holomorphic extension in ξ_d to a domain $\{\operatorname{Im}\xi_d \le \delta_1\langle|\xi'|\rangle\}$. Moreover, because u is supported in

$\{x_d \geq 0\}$, its Fourier transform $\hat{u}$ is holomorphic in $\{\operatorname{Im}\xi_d < 0\}$. Therefore, one can shift the integration path in ξ_d to $\mathbb{R} - i\delta\langle\xi'\rangle$

$$\begin{aligned} v(x') &= \int e^{ix'\cdot\xi'}\sigma(x',\xi',\xi_d - i\delta\langle\xi'\rangle)\hat{u}(\xi',\xi_d - i\delta\langle\xi'\rangle)d\xi \\ &= \int e^{ix'\cdot\xi'}\rho(x',\xi',\xi_d)\hat{w}(\xi',\xi_d)d\xi \\ &= \int \rho(x',D'_x,\xi_d)\tilde{w}(x',\xi_d)d\xi_d \end{aligned}$$

with $\rho(x',\xi',\xi_d) = \sigma(x',\xi',\xi_d - i\delta\langle\xi'\rangle)$, $w = e^{-\delta' x_d\langle D_{x'}\rangle}u$, and $\tilde{w}$ denoting its partial Fourier transform in the variable x_d. We conclude by applying (4.18) to ρ and w. □

4.5 Elliptic estimates

The last ingredient in the proof of Theorem 1.8 is to estimate the L^2 norm of $\langle D_{x'}\rangle^{\frac{1}{2}}e^{-\delta' x_d\langle D_{x'}\rangle}u$, *using again the equation* satisfied by u. Again the idea is taken from [FrLa2]. Microlocally, one can choose δ' such that $v = e^{-\delta' x_d\langle D_{x'}\rangle}u$ satisfies an *elliptic* equation, which reduces the problem to the proof of elliptic estimates, which we now recall.

Consider a system

$$\partial_{x_d}u + iT^{\rm tg}_A u = f \tag{4.34}$$

where A is a matrix with coefficients in $\mathbf{\Gamma}^1_1$.

Proposition 4.12. *Suppose that the spectrum of $A(t,x,\zeta)$ is contained in $|\operatorname{Im}\lambda| \geq c|\zeta|$. Then there is a constant C such that*

$$\big\|\langle D_z\rangle^{\frac{1}{2}}u\big\|_{L^2} \leq C\big(\big\|u\big\|_{L^2} + \big\|f\big\|_{L^2} + \big\|u_{|x_d=0}\big\|_{L^2}\big)$$

Proof. This is a special case of the tangential analysis (see e.g. [Kre, ChPi, Me5]). The assumption implies that the matrix A is conjugated to a block diagonal matrix with blocks $A_\pm$ having their spectrum in $\{\pm\operatorname{Im}\lambda \geq c|\zeta|\}$. Each block has a symmetrizer, and there is a symmetrizer $S = S^*$ such that $\operatorname{Im} SA \geq c'|\zeta|$ (see e.g. Section 8.1.3 in [Me5]). □

We will use the following extension of this estimate.

Proposition 4.13. *Suppose that the spectrum of $A(t,x,\zeta)$ is contained in $|\operatorname{Im}\lambda - \delta|\zeta|| \geq c|\zeta|$ for some $\delta \in [0,1]$ and $c > 0$. Then there is a constant C such that*

$$\big\|\langle D_z\rangle^{\frac{1}{2}}e^{-\delta\langle D_z\rangle x_d}u\big\|_{L^2} \leq C\big(\big\|u\big\|_{L^2} + \big\|f\big\|_{L^2} + \big\|u_{|x_d=0}\big\|_{L^2}\big)$$

Proof. The symbols in $\zeta \mapsto e^{-\delta x_d \langle\zeta\rangle}$ form a bounded family of symbols of degree zero. Therefore, the commutator $\left[e^{-\delta\langle D_z\rangle x_d}, T_A\right]$ are uniformly bounded in L^2. This shows that $v = e^{-\delta\langle D_z\rangle x_d} u$ satisfies

$$\left\|\partial_{x_d} v + i(T_A^{\mathrm{tg}} - i\delta\langle D_z\rangle)v\right\|_{L^2} \le C(\left\|u\right\|_{L^2} + \left\|f\right\|_{L^2}).$$

The symbol of $T_A^{\mathrm{tg}} - i\delta\langle D_z\rangle$ is $A(t,x,\zeta) - i\delta\langle\zeta\rangle$ and its spectrum is contained in $|\mathrm{Im}\,\lambda| \ge c'|\zeta|$. Hence one can apply Proposition 4.12 to v and the estimate follows since $v_{|x_d=0} = u_{|x_d=0}$. □

5 Semi group estimates and the IBVP in L^2

The goal of this section is to solve the initial boundary value problem (1.1) and to complete the proof of Theorem 1.8. We first review the analysis of the boundary value problem and next show what has to be added to treat initial data in L^2.

5.1 The main steps

Consider the hyperbolic system L on $\{x_d \ge 0\}$

$$L = A_0(t,x)\partial_t + \sum_{j=1}^{d} A_j(t,x)\partial_{x_j} + B(t,x) \tag{5.1}$$

together with boundary conditions M on $\{x_d = 0\}$. The adjoint operator L^* is

$$L^* = -(A_0(t,x))^*\partial_t - \sum_{j=1}^{d} A_j^*(t,x)\partial_{x_j} - B_1(t,x) \tag{5.2}$$

where $B_1 = -B^* + \partial_t A_0 + \sum \partial_j A_j^*$. There are adjoint boundary conditions M' for L^* such that for all smooth enough functions u and v on $[a,b]\times\mathbb{R}^d_+$:

$$\begin{aligned}(Lu,v) - (u,L^*v) &= (Mu_{|x_d=0}, M_1 v_{|x_d=0}) - (M_1' u_{|x_d=0}, M' v_{|x_d=0}) \\ &\quad + (u_{|t=b}, v_{|t=b}) - (u_{|t=a}, v_{|t=a}).\end{aligned} \tag{5.3}$$

for some boundary matrices M_1 and M_1'. Here $(\cdot,\cdot)$ denotes the L^2 scalar products on the appropriate domains. The formula extends to unbounded time intervals. The matrices M_1, M', M_1' are not unique but the invariant key property is that

$$\ker M' = (A_d \ker M)^{\perp}. \tag{5.4}$$

Introduce the spaces $L^2_\gamma = e^{\gamma t} L^2$ with norms $\|u\|_{L^2_\gamma} = \|e^{-\gamma t} u\|^2_{L^2}$ and note that, in the identity (5.3), $(\cdot, \cdot)$ can be understood as well as the duality $L^2_\gamma \times L^2_{-\gamma}$. We suppose here that the resolvent estimates have already been proven and take them as an assumption.

Assumption 5.1. *The following a priori estimates are valid : for $\gamma \geq \gamma_0$ and smooth u:*

$$\gamma \|u\|^2_{L^2_\gamma} + \|u_{|x_d=0}\|^2_{L^2_\gamma} \lesssim \gamma^{-1} \|Lu\|^2_{L^2_\gamma} + \|Mu_{|x_d=0}\|^2_{L^2_\gamma} \tag{5.5}$$

and

$$\gamma \|v\|^2_{L^2_{-\gamma}} + \|v_{|x_d=0}\|^2_{L^2_{-\gamma}} \lesssim \gamma^{-1} \|L^* u\|^2_{L^2_{-\gamma}} + \|M' v_{|x_d=0}\|^2_{L^2_{-\gamma}}. \tag{5.6}$$

By Theorem 1.4, this assumption is satisfied for systems in the class $s\mathcal{M}$.

Proposition 5.2 ([Kre, ChPi])**.** *Under Assumption 5.1 the boundary value problem*

$$Lu = f, \quad Bu_{|x_d=0} = g \tag{5.7}$$

is well posed in L^2_γ for $\gamma \geq \gamma_0$.

Indeed, (5.6) implies that (5.7) has a weak solution in L^2_γ. By tangential smoothing and Friedrichs Lemma, this solution is a strong solution and therefore satisfies (5.5). In particular, this implies uniqueness of weak solution. Moreover, the causality principle is satisfied: if f and g vanish for $t \leq t_0$, then u also vanishes for $t \leq t_0$.

We now consider the initial-boundary value problem

$$Lu = f, \quad Mu_{|x_d=0} = g, \quad u_{|t=0} = u_0. \tag{5.8}$$

Proposition 5.3. *The problem* (5.8) *is well posed in $L^2_{\gamma_0}$ when $u_0 = 0$.*

Proof. Existence is obtained by extending f and g by 0 for $t < 0$. Then there is a solution $u \in L^2_{\gamma_0}$ and the causality principle implies that it vanishes when $t < 0$. Therefore, its trace $u_{|t=0}$ also vanishes. Note that the trace is well defined in $H^{-\frac{1}{2}}_{loc}$ since the equation is non characteristic in time. Uniqueness follows in the same way : if $f = 0$ and $g = 0$, the extension $\tilde{u}$ of u by 0 in the past is a weak solution of $L\tilde{u} = 0$, $Mu = 0$ and therefore vanishes. □

This result settles the question of uniqueness of solutions for (5.8). The existence part is easy when the data vanish on a neighborhood of the edge $\{t = x_d = 0\}$.

Proposition 5.4. *If* $u_0 \in L^2$, $f \in L^2_{\gamma_0}$ *and* $g \in L^2_{\gamma_0}$ *vanish on a neighborhood of* $\{t = x_d = 0\}$, *then* (5.8) *has a unique solution* $u \in L^2_{\gamma_0}$.

Moreover, if in addition u_0, f *and* g *belong to* H^1, *the solution* u *also belongs to* H^1.

Proof. Extend u_0 and f by 0 for $x_d \le 0$ and solve the Cauchy problem $Lv = \tilde{f}$, $v_{|t=0} = \tilde{u}_0$. Then there is a unique solution $v \in C^0([0,1]; L^2(\mathbb{R}^d)$, which by finite speed of propagation vanishes for $x_d \le 0$ and $t \le t_0$ for some $t_0 > 0$. We solve the problem for $w = u - \chi v$, where $\chi(t)$ is supported in $[0, t_0[$ and $\chi(0) = 1$:

$$Lw = (1-\chi)f - A_0 \partial_t \chi v, \quad M w_{|x_d=0} = g, \quad w_{|t=0} = 0.$$

Indeed, by Proposition 5.3 there is a solution $w \in L^2_{\gamma_0}$.

The H^1 smoothness is proved similarly taking H^1 extensions of u_0 and f , which vanish near the edge and an H^1 extension of g which vanish in the past. □

The difficult part of the proof is now to prove estimates for u independent of the neighborhood where the data vanish. We prove them under the following assumption:

Assumption 5.5. *L admits a symmetrizer $S(t, x, \xi)$ which is Lipschitz continuous in (t, x), and real analytic in ξ.*

Theorem 5.6. *Under Assumptions* 5.1 *and* 5.5, *there is a constant* C *such that for all smooth* u_0, f *and* g *which vanish on a neighborhood of* $\{t = x_d = 0\}$, *the unique* H^1 *solution of* (5.8) *satisfies*

$$\begin{aligned} \|u(t)\|_{L^2} + \|u_{|x_d=0}\|_{L^2([0,t]\times\mathbb{R}^{d-1})} &\le C \\ \Big(\|u_0\|_{L^2} + \|g\|_{L^2([0,t]\times\mathbb{R}^{d-1})} &+ \int_0^t \|f(s)\|_{L^2[(\mathbb{R}^d_+)} ds \Big). \end{aligned} \tag{5.9}$$

By density-continuity, the mapping $(u_0, f, g) \mapsto u$ uniquely extends to $u_0 \in L^2$, $f \in L^1([0,T]; L^2)$ and $g \in L^2$. Then $u \in C^0([0,T]; L^2)$, is a weak solution of (5.8). Since uniqueness is already known, the theorem above implies the next corollary and hence Theorem 1.8.

Corollary 5.7. *Under Assumptions* 5.1 *and* 5.5, *for all* $u_0 \in L^2(\mathbb{R}^d_+)$, $f \in L^1([0,T]; L^2)$ *and* $g \in L^2([0,T] \times \mathbb{R}^{d-1})$, *there is a unique* $u \in C^0([0,T]; L^2(\mathbb{R}^d_+))$ *solution of* (5.8) *on* $[0,T] \times \mathbb{R}^d_+$. *Moreover,* u *satisfies* (5.9).

5.2 The main estimate

We reduce the proof of the estimate (5.9) to a simpler one:

Theorem 5.8. *Under Assumption* 5.5, *for* $u \in H^1(\mathbb{R}^{1+d}_+)$ *with support in* $t \in [0,2]$ *one has*

$$\big\|u(t)\big\|_{L^2(\mathbb{R}^d_+)} \lesssim \|Lu\|_{L^2} + \|u\|_{L^2} + \|u_{|x_d=0}\|_{L^2}. \tag{5.10}$$

Taking this theorem for granted, we show that it implies Theorem 5.6. Because $(SA_0)^{-1}(A_0^*)^{-1}$ is a symmetrizer for L^* , one has similar estimates for the adjoint problem:

$$\big\|v(t)\big\|_{L^2(\mathbb{R}^d_+)} \lesssim \|L^*v\|_{L^2} + \|v\|_{L^2} + \|v_{|x_d=0}\|_{L^2}. \tag{5.11}$$

Corollary 5.9. *Consider the backward initial boundary value problem for* $t \leq 1$

$$L^*\Phi = \varphi, \quad M'\varphi_{|x_d=0} = \psi, \quad \Phi_{|t=1} = 0. \tag{5.12}$$

Then

$$\|\Phi_{|t=0}\|_{L^2} + \|\Phi\|_{L^2([0,1]\times\mathbb{R}^d_+)} + \|\Phi_{|x_d=0}\|_{L^2} \lesssim \|\varphi\|_{L^2[0,1]\times\mathbb{R}^d_+)} + \|\psi\|_{L^2}. \tag{5.13}$$

By duality, this implies the following estimate for the direct problem:

Proposition 5.10. *There is a constant* C *such that for data vanishing on a neighborhood of the edge, the solutions of* (5.8) *satisfy*

$$\|u_{|t=1}\|_{L^2} + \|u\|_{L^2([0,1]\times\mathbb{R}^d_+)} + \|u_{|x_d=0}\|_{L^2([0,1]\times\mathbb{R}^{d-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^d_+)} + \|f\|_{L^2([0,1]\times\mathbb{R}^d_+)} + \|g\|_{L^2([0,1]\times\mathbb{R}^{d-1})}. \tag{5.14}$$

Proof. By duality, the corollary implies that

$$\|u\|_{L^2([0,1]\times\mathbb{R}^d_+)} + \|u_{|x_d=0}\|_{L^2} \lesssim \|u_0\|_{L^2} + \|f\|_{L^2([0,1]\times\mathbb{R}^d_+)} + \|g\|_{L^2}. \tag{5.15}$$

To get the missing term, that is the L^2 norm of $u_{|t=1}$, it is now sufficient to apply the direct estimate of Theorem 5.8 to $v = tu$, since $Lv = tLu + A_0u$ is now controlled in L^2. □

We pass from the estimate for time 1 to estimates for all time $t \in]0,1]$ by scaling. For $\varepsilon \in]0,1]$ consider the systems

$$L_\varepsilon(t,x,\partial_t,\partial_x) = A_0(\varepsilon t,\varepsilon x)\partial_t + \sum A_j(\varepsilon t,\varepsilon x)\partial_{x_j} + \varepsilon E(\varepsilon t,\varepsilon x) \tag{5.16}$$

and the boundary conditions

$$M_\varepsilon(t,x) = M(\varepsilon t,\varepsilon x). \tag{5.17}$$

Lemma 5.11. *If u is a solution of (5.8), then $u_\varepsilon(t,x) = u(\varepsilon t,\varepsilon x)$ safisfies*

$$L_\varepsilon u_\varepsilon = \varepsilon f_\varepsilon, \quad B_\varepsilon u_{\varepsilon|x_d=0} = g_\varepsilon, \quad u_{\varepsilon|t=0} = u_{0,\varepsilon} \tag{5.18}$$

where f_ε, g_ε and $u_{0,\varepsilon}$ are deduced from f, g and u_0 by the scaling.

Proof. One has

$$A_j(\varepsilon t,\varepsilon x)(\partial_j u_\varepsilon)(t,x) = \varepsilon (A_j\partial_j u)(\varepsilon t,\varepsilon x)$$

and similar formulas for the traces. □

We note that the Assumptions 5.1 are satisfied for all $\varepsilon \in]0,1]$, with uniform constants:

Lemma 5.12. *The boundary value problems $(L_\varepsilon, B_\varepsilon)$ satisfy the estimates (5.5) and (5.6) with constants independent of $\varepsilon \in]0,1]$, for $\gamma \geq \varepsilon\gamma_0$.*

Proof. With $\gamma' = \gamma/\varepsilon \geq \gamma_0$, the direct estimates are immediate consequences of the scaling identities

$$\gamma \|u_\varepsilon\|^2_{L^2_\gamma} = \varepsilon^{-d}\gamma'\|u\|^2_{L^2_{\gamma'}}, \qquad \gamma^{-1}\|\varepsilon f_\varepsilon\|^2_{L^2_\gamma} = \varepsilon^{-d}(\gamma')^{-1}\|u\|^2_{L^2_{\gamma'}}$$

and

$$\|u_{\varepsilon|x_d=0}\|^2_{L^2_\gamma} = \varepsilon^{-d}\|u_{|x_d=0}\|^2_{L^2_{\gamma'}}, \qquad \|g_\varepsilon\|^2_{L^2_\gamma} = \varepsilon^{-d}\|g\|^2_{L^2_{\gamma'}}$$

The adjoint operator $(L_\varepsilon)^*$ is the scaled operator $(L^*)_\varepsilon$ deduces from L^* as in (5.16). Similarly, $B'_\varepsilon = B'(\varepsilon t,\varepsilon x)$ are dual boundary conditions so that the estimates for L^*_ε follow by the same scaling argument. □

In the same vein, considering the symmetrizers $S_\varepsilon(t,x,\xi) = S(\varepsilon t,\varepsilon x,\xi)$ implies that the Assumption 5.5 is satisfied for L_ε, with uniform constants.

Proposition 5.13. *There is a constant C such that for data vanishing on a neighborhood of the edge, the solutions of* (5.8) *are continuous in time with values in L^2 satisfy for $t \in [0,1]$*

$$\|u(t)\|_{L^2}+\|u_{|x_d=0}\|_{L^2([0,t]\times\mathbb{R}^{d-1})} \lesssim \|u_0\|_{L^2(\mathbb{R}^d_+)} + \|f\|_{L^2([0,t]\times\mathbb{R}^d_+)} + \|g\|_{L^2([0,t]\times\mathbb{R}^{d-1}}. \tag{5.19}$$

Proof. The estimates at time t_1 follows from (5.14) applied to u_ε with $\varepsilon = t_1$. When the data are H^1, the solution is H^1 and therefore continuous in time with values in L^2. Therefore, by density the solutions belong to $C^0([0,1];L^2)$. □

This is almost the desired estimate (5.9), except for the norm of f. It remains to replace the L^2 norm above by an $L^1([0,1],L^2)$ norm. For that we split the problem into two pieces :

$$Lu = f, \quad Mu_{|x_d=0} = 0, \quad u_{|t=0} = 0. \tag{5.20}$$

and

$$Lu = 0, \quad Mu_{|x_d=0} = g, \quad u_{|t=0} = u_0. \tag{5.21}$$

By linearity, it is sufficient to prove (5.9) for the solution of each problem separately. For the second equation, this follows directly from Poposition 5.13 and it remains to prove (5.9) for the solution of (5.20). We show that it follows from (5.19) using Duhamel's principle.

Proposition 5.14. *There is a family of bounded operators $\mathcal{E}(t,s)$ from $L^2(\mathbb{R}^d_+)$ to $L^2(\mathbb{R}^d_+)$, for $0 \le s \le t \le 1$, such that for all $s \in [0,1[$, $u(t) = \mathcal{E}(t,s)u_0$ is the unique solution in $C^0([s,1],L^2(\mathbb{R}^d_+)$ of*

$$Lu = 0, \quad Mu_{|x_d=0} = 0, \quad u_{|t=s} = u_0.$$

In particular, for all $u_0 \in L^2(\mathbb{R}^d_+)$, $t \mapsto \mathcal{E}(t,s)u_0$ belongs to $C^0([s,1],L^2(\mathbb{R}^d_+))$.

Moreover, for all $u_0 \in L^2(\mathbb{R}^d_+)$, $s \mapsto \mathcal{E}(t,s)u_0$ belongs to $C^0([0,t],L^2_w(\mathbb{R}^d_+)$ where $L^2_w(\mathbb{R}^d_+)$ denotes the space L^2 equipped with the weak topology.

Proof. Clearly, what we have done before for the initial time $t=0$ is true for all initial time $t=s$. Thus, Proposition 5.13 implies that when $u_0 \in L^2$ vanishes near the boundary, there is a unique solution $u \in C^0([s,1];L^2)$ which satisfies

$$\|u(t)\|_{L^2} \le C\|u_0\|_{L^2}$$

The operator $u_0 \mapsto u$ extends by density to $u_0 \in L^2$ implying the first part of the proposition. The second follows by duality : the corresponding operator $\mathcal{F}(t, s')$ for the backward transposed problem is defined for $0 \le t \le s' \le 1$ and $v = \mathcal{F}(\cdot, s')v_0$ solves

$$L^* v = 0, \quad M' u_{|x_d=0} = 0, \quad u_{|t=s'} = v_0.$$

In particular, $t \mapsto \mathcal{F}(t, s')v$ is continuous from $[0, s']$ to $L^2(\mathbb{R}^d_+)$. The duality relation (5.3) shows that $\mathcal{E}(t, s) = \mathcal{F}(s, t)^*$ and therefore $s \mapsto \big(\mathcal{E}(t, s)u_0, v_0\big)$ is continuous for all u_0 and v_0 in L^2. □

Lemma 5.15. *For f smooth, vanishing in a neighborhood of the edge, the solution of* (5.20) *is given by Duhamel's principle:*

$$u(t) = \int_0^t \mathcal{E}(t, s) f(s) ds. \tag{5.22}$$

Proof. Note that for $f \in C^0([a, 1]; L^2(\mathbb{R}^d_+))$, $s \mapsto \mathcal{E}(t, s)f(s)$ is continuous from $[0, t]$ to L^2_w so that the integral (5.22) makes sense. Denote it by $\tilde{u}(t)$.

For $\psi \in H^1(\mathbb{R}^d_+)$ vanishing near $x_d = 0$, let $\Psi(\cdot) = \mathcal{F}(\cdot, t)\psi$ which is a H^1 solution on $[0, t] \times \mathbb{R}^d_+$ of

$$L^*\Psi = 0, \quad B'\Psi_{|x_d=0} = 0, \quad \Psi(t) = \psi.$$

Then

$$\begin{aligned}(\tilde{u}(t), \psi) &= \int_0^t (\mathcal{E}(t, s)f(s), \psi) ds = \int_0^t (f(s), \mathcal{F}(s, t)\psi) ds \\ &= \big(Lu, \Psi\big)_{L^2([0,t]\times\mathbb{R}^d_+)} = (u(t), \psi)\end{aligned}$$

where the last equality follows from (5.3), which is satisfied since u is H^1. Hence $\tilde{u}(t) = u(t)$ and the lemma is proved. □

Using the estimates of Proposition 5.13 for $\mathcal{E}(\cdot, s)f(s)$ and integrating them in s implies

Corollary 5.16. *For f smooth, vanishing in a neighborhood of the edge, the solution of* (5.20) *satisfies*

$$\|u(t)\|_{L^2} + \|u_{|x_d=0}\|_{L^2([0,t]\times\mathbb{R}^{d-1})} \lesssim \int_0^t \|f(s)\|_{L^2[(\mathbb{R}^d_+)} ds \tag{5.23}$$

This finishes the proof of Theorem 5.6.

5.3 Proof of Theorem 5.8

Recall that we are considering a function $u \in H^1(\mathbb{R}^{1+d}_+)$, supported in $t \in [0, 2]$. We can paralinearize the operator $A_d^{-1}L = \partial_{x_d} + G$ and write

$$\partial_{x_d} u + T_{iG} u = f, \tag{5.24}$$

where T_{iG} denotes the tangential paradifferential operator of symbol $iG(t, x, \zeta)$ with $\zeta = (\tau, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$, and f satisfies

$$\|f\|_{L^2} \lesssim \|Lu\|_{L^2} + \|u\|_{L^2}. \tag{5.25}$$

The proof relies on microlocal estimates which are stated in Propositions 5.17 and 5.18 below. We glue the different pieces using a finite partition of unity

$$1 = \chi_0(\zeta) + \sum_{k=1}^{n} \chi_k(\zeta) \tag{5.26}$$

with χ_0 is supported in $\{|\xi'| \leq c|\ \tau|\}$ while the χ_k for $k \geq 1$ are supported in $\{|\tau| \leq 2c^{-1}|\xi'|\}$. Let $u_k = \chi_k(D_z)u$, where $z = (t, x')$. We will estimate the L^2 norm of each $u_k(t)$ separately, using different methods according to $k = 0$ or $k \geq 1$. Note that

$$\partial_{x_d} u_k + T_{iG\tilde{\chi}_k} u_k = f_k, \tag{5.27}$$

where f_k satisfies (5.25) and $\tilde{\chi}_k$ is equal to one on the support of χ_k. Note also that u_k is not any more supported in $t \in [0, 2]$, but u_k has an H^1 norm for $t \notin [-1, 3]$ ccontrolled by the L^2 norm of u. In particular

$$\left\| u_{k|t=-1} \right\|_{L^2} + \left\| u_{k|t=3} \right\|_{L^2} \lesssim \left\| u \right\|_{L^2}. \tag{5.28}$$

We prove that one can choose the partition so that the u_k satisfy

$$\|u_k(t)\|_{L^2} \lesssim \|u_k\|_{L^2} + \|f_k\|_{L^2} + \|u_{k|x_d=0}\|_{L^2} + \|u_k(-1)\|_{L^2}. \tag{5.29}$$

Theorem 5.8 is a consequence of the following two results:

Proposition 5.17. *One can choose $c > 0$ such that if χ_0 is supported in $\{|\xi'| \leq c|\tau|\}$ then $u_0 = \chi_0(D_z)u$ satisfies* (5.29)*.*

Proposition 5.18. *For all $\underline{z} = (\underline{t}, \underline{x}')$ and all $\underline{\zeta} = (\underline{\tau}, \underline{\xi}') \in \mathbb{R}^d$ with $|\underline{\xi}'| = 1$, there is a conical neighborhood of $(\underline{z}, \underline{\zeta})$ such that if χ_k is supported in this neighborhood, $u_k = \chi_k u$ satisfies* (5.29)*.*

Indeed, by compactness, one can then choose a partition of unity 5.26 such that the estimate (5.29) is satisfied for all $u_k = \chi_k(D_z)u$.

5.3.1 The case $|\xi'| \leq c|\tau|$

The proof of Proposition 5.17 is based on an extension of u_0 to $\{x_d < 0\}$. We first make a block reduction of the symbol $G(t,x,\zeta)$ for large τ. If c is small enough, the cone $\{|\xi'| \leq c|\tau|\}$ is contained in the interior of the cone $\Gamma^\flat$ defined in Section 2 and therefore, by Proposition 2.4 :

Lemma 5.19. *If c is small enough, the eigenvalues of $G(t,x,\zeta)$ in $\{|\xi'| \leq c|\tau|\}$ are real and split in two groups, located in $\{\pm\lambda \geq c|\tau|\}$ respectively.*

As a corollary, there is a smooth microlocal block reduction

$$\tilde\chi_0(\zeta)G(t,x,\zeta) = V^{-1}G_1V, \qquad G_1 = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix} \tag{5.30}$$

where the eigenvalues of $G_\pm$ are located in $\{|\lambda| \geq c_1|\tau|\}$ with $\pm\lambda\tau > 0$.

If $\chi(D_z)$ is supported in $\Gamma^\flat$ and $u_0 = \chi(D_z)u$, on can split

$$v := T_V u_0 = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$$

satisfies

$$\partial_{x_d} v_\pm + T_{iG_\pm} v_\pm = f_\pm \tag{5.31}$$

with

$$\begin{aligned} \|f_\pm\|_{L^2} + \|v_\pm\|_{L^2} + &\|v_{\pm|x_d=0}\|_{L^2} \lesssim \\ &\|u_0\|_{L^2} + \|f_0\|_{L^2} + \|u_{0|x_d=0}\|_{L^2}. \end{aligned} \tag{5.32}$$

Moreover, the blocks $G_\pm$ are strongly hyperbolic and admit smooth symmetrizers. Therefore, we are in position to apply Proposition 4.2 with x_d as time variable, on a small but fixed interval $[-X, 0]$ and then truncate the solution for $x_d < X/2$. Therefore, one can extend $f_\pm$ and $v_\pm$ to $\{x_d < 0\}$ so that (5.31) and (5.32) remain satisfied. Denoting by $\tilde v$ the extension of v, we see that $\tilde u_0 = T_{V^{-1}}\tilde v$ satisfies

$$\|L\tilde u_0\|_{L^2} + \|\tilde u_0\|_{L^2} + \lesssim \|u_0\|_{L^2} + \|f_0\|_{L^2} + \|u_{0|x_d=0}\|_{L^2}. \tag{5.33}$$

Moreover, on $\{x_d > 0\}$, $\tilde u_{0|x_d>0} - u_0 = (T_{V^{-1}}T_V - \mathrm{Id})u_0$ and by the symbolic calculus

$$\int_0^\infty \left\|\tilde u_{0|x_d>0}(\cdot, x_d) - u_0(\cdot, x_d)\right\|^2_{H^1(\mathbb{R}^d_+)} dx_d \lesssim \|u_0\|^2_{L^2}$$

and therefore

$$\sup_t \|\tilde u_{0|x_d>0}(t,\cdot) - u_0(t,\cdot)\|^2_{L^2(\mathbb{R}^d_+)} \lesssim \|\tilde u_0\|^2_{L^2} \tag{5.34}$$

Using the energy estimate for the Cauchy problem implies that for $t \in [0,T]$

$$\|\tilde u_0(t)\|_{L^2} \le \|L\tilde u_0\|_{L^2([-1,T]\times\mathbb{R}^d)} + \|\tilde u_0\|_{L^2([-1,T]\times\mathbb{R}^d)} \tag{5.35}$$

and Proposition 5.17 follows from (5.33) and (5.34).

5.3.2 The case $|\tau| \le C|\xi'|$

Let c be chosen as in Proposition 5.17. We now consider the remaining frequencies $|\tau| \le C|\xi'|$ with $C = 1/c$. The idea is to use Propositions 4.10 and 4.11 to estimate $\|u_k(t)\|_{L^2}$. Let $\delta_1 > 0$ be given by Proposition 4.11 and let $\alpha_1 = \delta_1/(2+C)$, so that

$$|\tau| \le (1+C)|\xi' l \quad \Rightarrow \quad \alpha_1\langle\zeta\rangle \le \delta_1\langle\xi'\rangle. \tag{5.36}$$

We fix a point $\underline{\tilde x} = (\underline{t}, \underline{x})$ and $\underline{\zeta} = (\underline{\tau}, \underline{\xi}')$ with $|\underline{\tau}| \le C|\underline{\xi}'|$ We assume as we may that $|\underline{\zeta}| = 1$. The spectrum of $G(\underline{\tilde x}, \underline{\zeta})$ is made of at most N isolated eigenvalues. Denote by μ_k the distinct values of their imaginary part. Then, there is $\alpha \in [0, \alpha_1]$ such that $\inf|\mu_k - \alpha| \ge \alpha_1/N$. Therefore, with $c = \frac{1}{2}\alpha_1/N$, there is a conical neighborbhood of $(\underline{\tilde x}, \underline{\zeta})$ such that for $(\tilde x, \zeta)$ in this neighborhood, the spectrum of $G(\tilde x, \zeta)$ is contained in $\big|\mathrm{Im}\,\lambda - \alpha|\zeta|\big| \ge c|\zeta|$.

We choose χ supported in this neighborhood and set $v = \chi(\tilde x, D_z)u$. Then

$$\big\|\partial_{x_d} v + iT_{\tilde G} v\big\|_{L^2} \lesssim \|u\|_{L^2} + \|f\|_{L^2},$$

where $\tilde G$ is an extension of G outside the given neighborhood such that $\tilde G$ satisfies the spectral property everywhere. Hence, the elliptic estimate of Proposition 4.13 implies that

$$\big\|\langle D_z\rangle^{\frac12} e^{-\alpha\langle D_z\rangle x_d} v\big\|_{L^2} \lesssim \big\|u\big\|_{L^2} + \big\|f\big\|_{L^2} + \big\|u_{|x_d=0}\big\|_{L^2}.$$

Using (5.36), this implies that

$$\big\|\langle D_{x'}\rangle^{\frac12} e^{-\delta x_d\langle D_{x'}\rangle} v\big\|_{L^2} \lesssim \big\|u\big\|_{L^2} + \big\|f\big\|_{L^2} + \big\|u_{|x_d=0}\big\|_{L^2}.$$

with $\delta = (2+C)\alpha \in [0, \delta_1]$. Hence the estimate (5.29) for v follows from the energy estimates of Propositions 4.10 and 4.11, and the proof of Theorem 5.8 is now complete.

6 Counterexamples

In this section we prove Theorem 1.5. We give an example of an ill posed initial boundary value problem for a 6×6 symmetric system with boundary conditions which satisfy the uniform Lopatinski condition. This example can be seen as well as a transmission problem for a symmetric 3×3 system. The example is in dimension $d = 3$, the space variables are denoted by (x, y, z) and the boundary is $\{x = 0\}$. The dual variables are (ξ, η, ζ). The eigenvalues have variable multiplicities on the manifold $\xi = \eta = y = 0$, $\zeta \neq 0$.

Consider in $\mathbb{R}^{1+3}$

$$L_\varepsilon = \begin{pmatrix} \partial_t - \varepsilon\partial_x & \partial_y & y\partial_z \\ \partial_y & \partial_t + \varepsilon\partial_x & 0 \\ y\partial_z & 0 & \partial_t + \varepsilon\partial_x \end{pmatrix} = \mathrm{Id}\partial_t + \varepsilon J\partial_x + A\partial_y + yB\partial_z \tag{6.1}$$

With $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$, consider on $\{x > 0\}$ the doubled system

$$L_{\varepsilon_1} U_1 = 0, \quad L_{\varepsilon_2} U_2 = 0 \tag{6.2}$$

together with boundary conditions on $\{x = 0\}$ of the form

$$\mathcal{B}U := \begin{pmatrix} u_2 \\ v_1 \\ w_1 \end{pmatrix} - M \begin{pmatrix} u_1 \\ v_2 \\ w_2 \end{pmatrix} = 0, \qquad \text{where} \quad U_j = \begin{pmatrix} u_j \\ v_j \\ w_j \end{pmatrix}. \tag{6.3}$$

We choose M of the form

$$M = \begin{pmatrix} c & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \tag{6.4}$$

The system is symmetric. The form (6.3) is well adapted to the diagonal 1-D system $L(\partial_t, \partial_x, 0, 0)$ since then the boundary condition prescribes the incoming components in terms of the outgoing ones. This 1-D analysis also shows that when $\|M\| < 1$, the system is maximal strictly dissipative. In particular, if M is of the form (6.4), the boundary condition is dissipative if and only if $|c| \leq 1$. The uniform Lopatinski condition is satisfied on a wider range of c:

Proposition 6.1. *When* $|c| < 2$, *the boundary conditions* (6.3) (6.4) *satisfy the uniform Lopatinski condition for the system* (6.2).

This result will be proved later on. Taking it for granted, Theorem 6 follows from the next result :

Theorem 6.2. *When $c \in]-2,-1[$, the initial boundary value problem is strongly ill posed in the sense that there are families U_λ of smooth solutions of $LU = 0$ on $\{t \geq 0, x \geq 0\}$, $\mathcal{B}U = 0$ on $\{t \geq 0, x = 0\}$ such that*
i) the $U_\lambda(0,\cdot)$ are bounded in $H^s(\mathbb{R}^3_+)$ for all s,
ii) for all time $t > 0$, the $U_\lambda(t,\cdot)$ are not bounded in $L^2(\mathbb{R}^3_+)$.

Remark 6.3. Since the U_λ are smooth up to the boundary on the initial surface, the *compatibility conditions* are satisfied at infinite order on the edge $\{t = x = 0\}$. We do not make them explicit nor comment more on this point here.

To prove the theorem, we first construct exact solutions of $L_\varepsilon U_\varepsilon = 0$. Consider the basis

$$e_1 = \begin{pmatrix} 0 \\ -1 \\ i \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}. \tag{6.5}$$

Lemma 6.4. *Let $\mu > 0$ and $\gamma > 0$ satisfy $\mu^2 - \gamma^2 = 1$ and for $\varepsilon \in \{-1, +1\}$ let*

$$\delta = \varepsilon\mu - \gamma = \frac{1}{\gamma + \varepsilon\mu}. \tag{6.6}$$

For all $\zeta > 0$, introduce

$$\Phi(t, x, y, z, \zeta) = \sqrt{\zeta}(\gamma t - \mu x) + i\zeta z - \frac{1}{2}\zeta y^2 \tag{6.7}$$

Then

$$U_\varepsilon(t, x, y, z, \zeta) = e^\Phi\big(\sqrt{\zeta} y e_1 + \delta e_2\big) \tag{6.8}$$

is an exact solution of $L_\varepsilon U_\varepsilon = 0$.

Proof. One has

$$\zeta^{-\frac{1}{2}} e^{-\Phi} L_\varepsilon U_\varepsilon = \zeta y^2 F_2 + \zeta^{\frac{1}{2}} y F_1 + F_0$$

with

$$\begin{aligned} F_2 &= (-A + iB)e_1 = 0, \\ F_1 &= (-A + iB)\delta e_2 + (\gamma \mathrm{Id} - \varepsilon\mu J)e_1 = (\delta + \gamma - \varepsilon\mu)e_1, \\ F_0 &= Ae_1 + (\gamma \mathrm{Id} - \varepsilon\mu J)\delta e_2 = \big((\gamma + \varepsilon\mu)\delta - 1\big)e_2. \end{aligned}$$

where we have used that

$$(-A+iB)e_1 = 0, \quad (-A+iB)e_2 = e_1, \quad Ae_1 = -e_2,$$
$$Je_1 = e_1, \quad Je_2 = -e_2.$$

Therefore, the conditions on the parameters imply that $L_\varepsilon U_\varepsilon = 0$ □

Lemma 6.5. *Let $\mu > 0$ and $\gamma > 0$ satisfy $\mu^2 - \gamma^2 = 1$. Let U_1 and U_2 be defined by* (6.8) *with $\delta_j = \varepsilon_j\mu - \gamma$, with $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$ as above. Then $L_{\varepsilon_j} U_j = 0$ and the boundary condition* (6.3) (6.4) *is satisfied if $c = \delta_2/\delta_1$.*

Proof. On the boundary

$$U_1 = e^\Phi \begin{pmatrix} \delta_1 \\ -Y \\ iY \end{pmatrix}, \qquad U_2 = e^\Phi \begin{pmatrix} \delta_2 \\ -Y \\ iY \end{pmatrix},$$

with $Y = y\sqrt{\zeta}$. Therefore,

$$e^{-\Phi}\mathcal{B}U = \begin{pmatrix} \delta_2 \\ -Y \\ iY \end{pmatrix} - \begin{pmatrix} c & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \delta_1 \\ -Y \\ iY \end{pmatrix} = 0$$

when $c = \delta_2/\delta_1$. □

Corollary 6.6. *Let $\mu > 0$ and $\gamma > 0$ satisfy $\mu^2 - \gamma^2 = 1$ and let $c = \delta_2/\delta_1$. Then the initial boundary value problem for* (6.2) (6.3) (6.4) *is strongly ill posed.*

Proof. Consider for λ large

$$U_\lambda(t,x,y,z) = e^{-\lambda^\rho} \int U(t,x,y,z,\zeta)\varphi(\zeta-\lambda)d\zeta$$

where $\varphi \in C_0^\infty(]1,\infty[)$ and $\rho < \frac{1}{2}$. It satisfies $LU_\lambda = 0$ and the boundary condition $\mathcal{B}U_\lambda = 0$, for all time. In particular, the compatibility conditions at the edge $\{t = x = 0\}$ are satisfied.

At $t = 0$ the phase Φ is $-\sqrt{\zeta}\mu x - \frac{1}{2}\zeta y^2 + i\zeta z$ and for all s,

$$\left\|U_\lambda(0,\cdot)\right\|_{H^s(\mathbb{R}^3_+)} = O(1)$$

and similar estimates are true for $t < 0$ since the factor of t in the phase is positive. On the other hand, for $t > 0$ the phase has the amplification factor $\gamma\sqrt{\zeta}t$, and for $\lambda \gg t^{-2/(1-2\rho)}$,

$$\left\|U_\lambda(t,\cdot)\right\|_{L^2(\mathbb{R}^3_+)} \gtrsim e^{\gamma\sqrt{\lambda}t/2}.$$

□

Remark 6.7. Note that the blow up also occurs in $L^2(B\cap\mathbb{R}^3_+)$ for any ball $B\subset\mathbb{R}^3$ centered at the origin.

Proof of Theorem 6.2. For $c\in]-2,-1[$, one can choose $\mu>0$ and $\gamma>0$ such that $\mu^2=1+\gamma^2$ and

$$c=\delta_2/\delta_1=-\frac{\mu+\gamma}{\mu-\gamma}$$

and the theorem follows. □

Proof of Proposition 6.1. For the symbolic analysis, y is a parameter independent of η and and we can replace $y\zeta$ by ζ, which we do below. Clearly, this is where the commutative calculus for symbols diverges from the non-commutative calculus for differential operators.

a) We compute the spaces $\mathbb{E}^{in}(\tau,\eta,\zeta)$ when Im <0. Due to the form of the equation, it is the space of

$$\begin{pmatrix}U_1\\U_2\end{pmatrix}\qquad\text{with}\quad U_j\in\mathbb{E}^{in}_{\varepsilon_j}(\tau,\eta,\zeta),$$

where $\mathbb{E}^{in}_{\varepsilon_j}$ is the incoming space associated with L_{ε_j}. Recall from Section 2 that they are the invariant spaces associated to characteristic values ξ lying in the half plane $\operatorname{Im}\xi>0$.

For L_ε, the characteristic equations for the eigenvectors are

$$\left\{\begin{array}{l}(-\varepsilon\xi+\tau)u+\eta v+\zeta w=0\\(\varepsilon\xi+\tau)v+\eta u=0\\(\varepsilon\xi+\tau)w+\zeta u=0\end{array}\right.\tag{6.9}$$

Introduce polar coordinates for (η,ζ):

$$\eta=\rho\cos\theta,\quad\zeta=\rho\sin\theta.$$

The characteristic determinant is $(\tau+\varepsilon\xi)(\tau^2-\xi^2-\rho^2)$. The characteristic frequencies are $-\varepsilon\tau$ and $\pm\sqrt{\tau^2-\rho^2}$. They are distinct and simple when $\operatorname{Im}\tau<0$.

An eigenvector for $-\varepsilon\tau$ is

$$R_0=\begin{pmatrix}0\\-\sin\theta\\\cos\theta\end{pmatrix}.$$

Denote by ξ the square root of $\tau^2 - \rho^2$ which is located in $\mathrm{Im}\,\xi > 0$. Eigenvectors associated to the characteristic frequency ξ are

$$R = \begin{pmatrix} \alpha \\ \cos\theta \\ \sin\theta \end{pmatrix} \qquad \text{and} \qquad R' = \begin{pmatrix} 1 \\ \beta\cos\theta \\ \beta\sin\theta \end{pmatrix}$$

with

$$\alpha = \frac{-\rho}{-\varepsilon\xi + \tau}, \qquad \beta = \frac{-\rho}{\varepsilon\xi + \tau}.$$

Of course, they are parallel to each other, since $\alpha = 1/\beta$. Depending on the sign of ε, we use one or the other form, depending on the sign of the imaginary part of the denominators.

Consider the case $\varepsilon = -1$. Since $\mathrm{Im}\,\tau < 0$, $\mathrm{Im}\,(-\varepsilon\tau) < 0$ and thus $\mathbb{E}^{in}_\varepsilon$ has dimension one, associated to the characteristic value ξ. Because $\mathrm{Im}\,(\varepsilon\xi + \tau) < 0$ we choose R' as a generator and hence

$$\mathbb{E}^{in}_{-1} = \mathbb{C}\begin{pmatrix} 1 \\ \beta\cos\theta \\ \beta\sin\theta \end{pmatrix}, \qquad \beta = \frac{-\rho}{\tau - \xi}.$$

When $\varepsilon = +1$, $\mathrm{Im}\,(-\varepsilon\tau) > 0$ and $\mathbb{E}^{in}_\varepsilon$ has dimension two, associated to the characteristic frequencies $-\tau$ and ξ. Because $\mathrm{Im}\,(-\varepsilon\xi + \tau) < 0$, we choose by R_0 and R as generators and

$$\mathbb{E}^{in}_{+1} = \mathbb{C}\begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} \alpha \\ \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \alpha = \frac{-\rho}{\tau - \xi}.$$

Combining the two cases, we conclude that for the symbol of the doubled system (6.2), $\mathbb{E}^{in}$ has dimension three and is generated by

$$E_0 = \begin{pmatrix} R_0 \\ 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \tag{6.10}$$

with

$$R_0 = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}, \qquad R_1 = \begin{pmatrix} a \\ \cos\theta \\ \sin\theta \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 \\ a\cos\theta \\ a\sin\theta \end{pmatrix}.$$

with

$$a = \frac{-\rho}{\tau - \xi}, \qquad \text{where} \quad \xi^2 = \tau^2 - \rho^2,\ \mathrm{Im}\,\xi > 0. \tag{6.11}$$

The definition of $\mathbb{E}^{in}$ extends to the limit case $\mathrm{Im}\,\tau = 0$, provided that $|\tau|^2 + \rho^2 \neq 0$, choosing the correct limit for ξ.

b) Applying the boundary conditions $\mathcal{B}$ to the basis of $\mathbb{E}^{in}$ yields

$$\mathcal{B}E_0 = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix},$$

$$\mathcal{B}E_1 = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix} - M \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -ca \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\mathcal{B}E_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - M \begin{pmatrix} 0 \\ a\cos\theta \\ a\sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2}ae^{i\theta} \\ \frac{i}{2}ae^{i\theta} \end{pmatrix}.$$

Because the three vectors in (6.10) are uniformly independent, the uniform Lopatinski condition is satisfied if and only if the modulus of the Lopatinski determinant $\Delta = \det(\mathcal{B}E_0, \mathcal{B}E_1, \mathcal{B}E_2)$ is bounded from below by a positive constant. One has

$$\Delta = \det \begin{pmatrix} 0 & -ca & 1 \\ -\sin\theta & \cos\theta & -\frac{1}{2}ae^{i\theta} \\ \cos\theta & \sin\theta & \frac{i}{2}ae^{i\theta} \end{pmatrix}$$

and

$$\Delta = -1 + \frac{1}{2}ca^2e^{i\theta}(\cos\theta - i\sin\theta) = -1 + \frac{1}{2}ca^2.$$

Recall the following elementary result.

Lemma 6.8. *The image of* $\{\operatorname{Im}\tau < 0, \rho \in \mathbb{R}\}$ *by the mapping* (6.11) *is* $D := \{|a| < 1\}$.

It implies that $|\Delta| \geq 1 - \frac{1}{2}|c|$ and the proposition is proved. □

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